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Lattice Nomenclature Survey from LGA to Modern LBM

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Abstract
Lattice configuration is a core parameter in Lattice-Boltzmann (LB) methods, both from theoretical and implementation standpoints. As LB methods have progressed over the past decades, a variety of lattice configurations have been proposed and referred to according to a plurality of lattice nomenclature systems that usually include the Euclidean *space dimensionality*, the lattice *velocity count* and, in fewer instances, the *discretization order* in their format. This work surveys lattice nomenclature systems, or lattice naming schemes, along the history of LB methods, starting from their Lattice Gas Automata (LGA) predecessor method, up to the present time. Findings include multiple lattice categories, competing naming standards, ambiguous names particularly in higher-order models, naming systems of varying model parameter scopes, and lack of unambiguous naming schemes even for space-filling, Bravais lattice types.

Keywords
lattice-Boltzmann stencils — lattice-Boltzmann models — lattice nomenclature systems — high-order lattices.

Highlights
Surveys lattice nomenclature systems in the history of LBM: from the LGA to current times — Discusses lattice categories, competing standards, naming ambiguity, model parameter scope.

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1. Introduction

Historically, the lattice-Boltzmann (LB) method had its origins in the frame of Lattice-Gas Automata [54], and has been intensely developed since its inception [10, 17, 56, 48]. One important conceptual and implementation parameter of LB methods is the employed *lattice stencil*—understood as the lattice geometry, velocity set, weights, and scale parameters [42, 53, 52], although some authors may include in the stencil designation additional modeling elements, such as the relaxation time scheme [49].

Both LB and LGA methods can be implemented on a variety of lattices, and historically many such lattices (along with their corresponding names or naming systems) have been developed. This work presents a LB literature survey focused on lattice naming schemes, or model nomenclature systems, from its Lattice-Gas Automata (LGA) predecessor method until the present time, in a somewhat chronological timeline.

2. Lattice Nomenclature Survey

2.1 Lattice-Gas Automata Lattice Designations

Some LGA lattices were named with *acronyms* after its first proposers, such as the 'HPP' one [33], after Hardy, de Pazzis, and Pomeau [37, 36, 80], or geometry-based *acronyms*, such as the 'HLG' one [33], which stands for 'hexagonal lattice gas,' later on referred to as the 'FHP' one [32, 80], after Frisch, Hasslacher, and Pomeau. Another geometry-based lattice of the time is the 'FCHC' one [32], which stands for 'face-centered-hypercubic' model, due to d'Humières, Lallemand, and Frisch.

Later on designations such as 'FHP + 3 rest particles' and 'FCHC + 3 rest particles' also appeared [13], as well as suffixes such as '-I', '-III', and '-IV' after 'FHP', for alternative collision rules [7, 13, 16].

2.2 Early Lattice-Boltzmann Years

Inception Period:

LB methods adhered to LGA lattice nomenclature in its inception period, as witnessed by reference [54] in 1988 and by subsequent references [44, 45] in 1989, by [9, 11, 14, 83] in 1990, and by [23, 26, 31, 35, 79] in 1991, to cite a few.

Early 1990's:

It seems that Qian [71] (apud [67, p. 235]) was the one to introduce, in 1990, the 'DdQb' lattice naming scheme for LB methods—in which d is the lattice *Euclidean dimensionality* and b is the lattice *velocity count*, as in D1Q3, D2Q9, and D3Q15, etc. [68, 69]—that seems to be the most prevalent lattice naming system to date, although notable exceptions appear long after the paper [72] came out in 1991.

As far as increasing lattice velocity counts go, the relationship between mesoscopic lattice *symmetry* and resulting macroscopic description *isotropy* has been established from early in the history of LGA methods [37, 36], in two [33] and in three Euclidean dimensions, the latter requiring the lattice to include links beyond nearest neighbors [87, pp. 473, 490], hence particle velocities with unequal magnitudes.

Moreover, Koelman [47] had proposed matching discrete velocity moments up to a certain order n with the d-dimensional continuous Boltzmann distribution, since only those moments influence the macroscopic flow behavior; such procedure would yield values for lattice velocity *weights* W_α. The proposed criteria were deemed more stringent than previously well-known symmetry and isotropy requirements from [87], since it not only led to an isotropic macroscopic description, but also ensure pressure term independence from velocity terms of the Navier-Stokes description. Furthermore, a skewed rectangular 9-speed lattice with independent a and b axis lengths was proposed[1], whose weights exactly recover those of the well-known D2Q9 lattice for $a = b$, over which the argument that valid weights '[...] *can always be found by choosing a large enough set of* (lattice) *velocity vectors* [...]' [47].

One driving application for increased velocity count lattices is thermal flows. On reference [3] an *unnamed* 2D, hexagonal (triangular), 13-velocity lattice having velocity magnitudes of 0, 1, and 2 lattice units [17] was employed for adiabatic sound propagation and heat transfer Couette flow, whose results were shown to be in agreement with corresponding analytical solutions.

Some 'nDmV' lattices, with n being the Euclidean space dimension and m the lattice velocity count, namely, 1D5V, 2D16V, and 3D40V, were introduced in [18] for shock wave front structure and shear wave flow application cases. The 2D16V lattice, for instance, was said to be comprised of four *sublattices*—a term that appeared in subsequent references—with each sublattice having 4 discrete velocities of same magnitude and forming adjacent right angles, which led to possibly *multiple sublattices per lattice energy level* $\varepsilon \equiv 2e = c^2$, with c being the microscopic (lattice) velocity magnitude, and e the corresponding specific kinetic energy, as was the case with the $\varepsilon = 1^2 + 2^2 = 5$ energy level of a square lattice, represented by the 8 discrete velocities obtained from permutations of $(\pm\{1,2\}, \pm\{2,1\})$ in lattice units, which were grouped in two distinct sublattices. This is in contrast to later works in which energy levels are treated as single groups.

Late 1990's to mid-2000's:

Most likely borrowing from mesh-based continuous mechanics numerical methods, a study [41] has proposed a LB algorithm for non-uniform mesh grids, by decoupling spatial and momentum space discretizations in the LB scheme. The underlying momentum space discretization was the well-known D2Q9 lattice, referred to in the study as '9-bit BGK model in 2D space' and other semantically equivalent sentences, in which BGK stands for kinetic theory's Bhatnagar-Gross-Krook collision model for the continuous Boltzmann equation [12, 50, 38].

Nine years after the debut of LB methods, a study [40] showed that they could be directly derived from the continuous Boltzmann equation with linearized collision operator under the BGK approximation [38], while lattice stencils from requirements of matching continuous and discrete velocity moments up to a desired order—a decisive publication, not only in making LB methods theory independent from its LGA historical predecessor, but also to pave the way towards later methods for lattice weights determination for the lattice velocity set based on some discrete-to-continuous equivalences [76, 65]. The lattices in [40] were verbosely referred to as 'd-dimensional b-bit g lattice model', with d being the Euclidean space dimension, b the lattice velocity count, and g a geometry term, such as 'triangular,' etc.

[1] That lattice was named 'face-centered rectangular' by the author.

A review article by Shen and Doolen [17] published a decade after McNamara and Zanetti's premiere LB publication [54] and seven years after Qian's paper introducing the now-prevailing 'DdQb' lattice naming scheme [72], would still refer to LB lattices either with LGA-style or verbose nomenclatures, and to overall LB schemes based on its collision term treatment, such as 'lattice BGK (LBGK)' models.

Higher-order lattices were proposed in [62] for two- and three-dimensional Euclidean spaces. They were referred to as 'octagonal grid (17-bit),' and as '3D "octagonal" 53-bit' models, respectively, and were isotropic up to the sixth-order. Since octagons are not space-filling, plane-tiling geometries, the proposed lattices were not of the Bravais type, meaning they impose a decoupling between the spatial and the momentum space discretizations, as with the non-uniform mesh [41], and the method has to resort to interpolations, which was later shown to cause spurious numerical diffusion [76, p. 429].

Other lattice namings of the early- and mid-2000's include verbose, spelled out ones [25, 51]; a '$D_d Q_b$' variant of Qian's 'DdQb' scheme [60]; a 'groupI' to 'groupIV' regular 2D polygon variant [85, 84]; a 'b (dD)' short designation for an otherwise verbose one [20]; an explicit lattice units velocity list, such as '$\{0, \pm 1, \pm 3\}$', in [20]; and the 'dodecahedron' and 'icosahedron' ones that were shown to be stable for supersonic thermal flows [86, 84].

2.3 The Year of 2006

The year of 2006 is seemingly a landmark for multi-velocity, higher-order LB schemes—and incidentally for lattice naming schemes—as evidenced by the appearance of three key publications, namely those of Shan, X., Yuan, X.-F., and Chen, H., [76], of Philippi, P. C., Hegele, L. A., dos Santos, L. O. E. and Surmas, R., [65], and of Chikatamarla, S. S. and Karlin, I. V., [20].

Shan and Coauthors:

A systematic discretization framework for the Boltzmann equation was proposed by Shan and coauthors in [76]. From kinetic theory [38, 50], the authors pointed out that successive Chapman-Enskog approximations of the Boltzmann equation obtain the (i) Euler, (ii) Navier-Stokes, (iii) Burnett, and (iv) higher-order macroscopic equations—meaning progressively higher-order moments of the continuous Boltzmann equation express progressively higher-order macroscopic thermohydrodynamic descriptions. Moreover, the authors demonstrated that projecting the Boltzmann equation onto order-N truncated tensorial Hermite polynomial expansion bases [34], lead to discrete LB models of corresponding order-N moments, since resulting Hermite expansion coefficients correspond to the velocity moments up to the chosen order.

In this discretization framework, the lattice is viewed as a Hermite expansion *quadrature*, and the naming convention was defined in terms of three parameters, namely, an Euclidean space dimension D, a quadrature velocity count d, and an algebraic degree of precision n encoded in an '$E_{D,n}^{d}$' naming scheme—an order-N Hermite expansion requires a quadrature degree $n \geqslant 2N$. Citing Qian and coauthors' lattices [68], they established the following comparisons, which were off only by a scaling factor: $D2Q9 \propto E_{2,5}^{9}$, $D3Q15 \propto E_{3,5}^{15}$, and $D3Q19 \propto E_{3,5}^{19}$.

Additionally, they established that Gauss-Hermite quadratures of the Boltzmann equation yield LB models with *minimum velocity count* for a given degree of precision and Euclidean spacial dimension, without, however, the ability to predefine (choose) the discrete velocity abscisae, which apart from special cases fails to produce a space-filling, Bravais lattice—recalling that for LB methods, this means lower memory requirements but decoupled spatial and momentum 'meshes' that require interpolations, thus introducing artifacts such as spurious numerical diffusion.

In the Appendix of reference [76], the authors include a brief overview on deriving quadratures on predefined Cartesian abscissae, which is the main requirement for space-filling, Bravais lattices for non-interpolating, exact advection LB schemes. The brief overview, however, is of scalar nature, while a tensorial treatment is needed for full clarity. Results for the space-filling $E_{2,7}^{17}$ and $E_{3,7}^{39}$ quadratures were listed among the ones obtained with Gauss-Hermite quadratures.

Philippi and Coauthors:

Tackling the aspects associated in deriving space-filling, Bravais lattices aiming at sufficiently high orders as to approach thermal hydrodynamic transport problems, Philippi and coauthors [65] have proposed a new *Method of Prescribed Abscissas*, MPA, for obtaining lattice weight values and scaling factor from predefined lattice arrangements.

Departing from the continuous Boltzmann equation, the derivation of discrete velocity sets, i.e., the lattice vectors, and corresponding weights, was considered as a quadrature problem aiming at (i) matching discrete equilibrium mass distribution function with its continuous counterpart and at (ii) warranting even-ranked velocity tensor isotropy, which, in turn, translates into isotropic fluid transport properties.

The Method of Prescribed Abscissas, MPA, yields *implicit* equations for lattice weights and lattice scale factor in the form of polynomial tensor products, which are generally excessively numerous, especially for higher-order cases. They have to be selected (reduced) and converted either into a non-linear system of equations. The apparent lack of literature guidance in tensor component equation selection criteria and solution approach has motivated works [5, 24].

In their prescribed abscissas quadrature discussion, authors state that [65, p. 6]:

> *"[...] Nth-order approximation to the [Maxwell-Boltzmann] equilibrium distribution is required when Nth-order macroscopic equilibrium moments are to be correctly described in [lattice-Boltzmann methods,]"*

which is homologous to many Shan and coauthor's statements in [76]. Observations like these, allied to the new and consistent methods of deriving higher-order LB stencils by Gauss-Hermite or Prescribed Abscissas quadratures, allowed for the immediate and subsequent appearances of lattices in 2D- and in 3D-Euclidean spaces with increased velocity counts, many of which requiring changes or adaptations in the naming scheme, as the sequence will show.

Immediate examples [65] include (i) *two* forms of bidimensional, 17-velocity ones, named D2Q17 and D2V17 for distinction; (ii) a D2Q21 one; as well as (iii) *two* forms of bidimensional, 25-velocity ones, named D2V25(W1) and D2V25(W6), containing the energy levels $\varepsilon \in \{0, 2, 4, 8, 9, 16, 18\}$, and $\varepsilon \in \{0, 1, 2, 4, 8, 9, 16\}$, respectively—thus without the energy levels (hence, weights) labeled '1' and '6' for '(W1)' and '(W6)', respectively—and (iv) a fourth-order ($N = 4$) Hermite D2V37 lattice, suitable for thermal flow LB simulations.

Chikatamarla and Karlin:

Seeking to systematically derive *stable* and *Galilean invariant* LB models, Chikatamarla and Karlin [20] set about the problem of LB stencil construction from a discrete form of Boltzmann's H-theorem—in which the $(-H)$ quantity represents a sort of generalized thermodynamic entropy for non-equilibrium states in the Boltzmann Gas Limit (BGL) that increases according to the second law of thermodynamics until equilibrium is reached [38]. By maximizing the entropy, i.e., by minimizing H in the discrete description,

$$H = \sum_{i=1}^{N} f_i \ln\left(\frac{f_i}{W_i}\right), \tag{1}$$

with appropriately chosen weights W_i under a set of macroscopic constraints of mass and energy conservation as well as constitutive relations for higher-order macroscopic tensors, authors arrived at *explicit* expressions for the weights W_i and for the stencil reference temperature for several one-dimensional lattice models having from 3 to 5 velocities. Methods obtained by this systematic were named 'entropic' LB methods (ELBM).

Remarkably, due to a pattern in Gauss-Hermite quadrature [6], Chikatamarla and Karlin proposed a straightforward way of obtaining higher-dimension lattice stencils [20]:

> *"[...] discrete velocities [$\mathbf{c}_i$] in the D-dimensional case are tensor products of D copies of the one-dimensional velocities, whereas the corresponding weights [W_i] are algebraic products of the corresponding weights in one dimension."*

As the immediate examples of [65] evidence, as soon as one allows for including several energy levels (but not necessarily all, nor in monotonic order) in a Bravais lattice, velocity count no longer uniquely identifies lattices, and thus, *velocity count-based lattice naming schemes are bound to be ambiguous and require additional information as to uniquely identify the lattice.* Usually, the additional information is laying out *all* velocity vectors, whether by energy level listings, a quiver-like lattice picture, or tabulated lattice velocities (plus weights, and either scaling factor or reference temperature)—all of which seem, to varying degrees, excessively wordy and lengthy. Moreover, the very need for providing additional information as to uniquely identify an already named lattice seems to defeat the purpose of naming it, at least in part.

2.4 Higher-order lattice proliferation

Late 2000's:

The onset of systematic techniques for LB stencils fabrication in 2006 [20, 65, 76], allied to the expansion of LB simulation applications and domains, contributed to the proliferation of LB stencils over the following years. This further highlighted the ambiguities in velocity count-based naming schemes, as well as prompted the appearance of further lattice naming variety.

The following 'DdVb' lattices were derived by Ortiz [61], using prescribed abscissas quadrature [65]: second-order hexagonally regular (Bravais) D2V7; third-order irregular, i.e., not space-filling, D2V12; fourth-order irregular D2V19, D2V20, D2V21, and regular D2V37; fifth-order irregular D2V28a, D2V28b, and regular D2V53a and D2V53b; sixth-order regular D2V81. In three-dimensional Euclidean space, the following: second-order irregular D3V13, third-order irregular D3V27, and fourth-order irregular D3V52 and D3V53.

The D2V37, D2V53 and D2V81 lattices appear on [66]. Analytically derived exact weights and scale factors for the D2V17 and D2V37 lattices are reported in [78].

A '59-velocity model in three dimensions' is said to be of third order Hermite expansion, with sixth-order tensor isotropy in [15]; however, no weights, velocity list nor scaling factors of such lattice are given. One-dimensional D1Q3, D1Q4, D1Q5, and D1Q6, as well as two-dimensional D2Q12 and D2Q21 lattices are presented in [46], with finite Knudsen number applications in view. Reference [73] studies various two- and three-dimensional lattices of up to 51 velocities, while referring to a 'DdQq' notation as being 'standard'.

On patent [75], authors designate Cartesian, space-filling models with 21, 37, 39, and 103 velocities, as '2D-1', '2D-2', '3D-1', and '3D-2', respectively.

Discussion on a plurality of lattices and lattice operations, such as stretching, extending (product), pruning, and superimposing, takes place in [22]; noteworthy ones are the D1Q$(1+2k)$ lattices for $k \in \{1, \dots, 4\}$, the higher-order ones also referred to '1D seven/nine velocity set'; the 'ZOT', i.e., '$\{0, \pm 1, \pm 3\}$', or 'zero-one-three' 1-D lattice; the 'D1Q5-ZOT lattice', defined as "*the shortest integer-valued discrete velocity set in the family of five-velocity sets*" [20]; the 'D2Q25-ZOT' lattice [21], of eight-order isotropy, obtained by extending the 'D1Q5-ZOT' through a tensor product with itself, so that

$$D2Q25\text{-ZOT} \equiv (D1Q5\text{-ZOT}) \otimes (D1Q5\text{-ZOT}); \quad (2)$$

the 'D3Q125-ZOT lattice', also of eight-order isotropy, obtained by extending either the 'D1Q5-ZOT' or the 'D2Q25-ZOT' lattice through a tensor product between them, so that

$$D3Q125\text{-ZOT} \equiv (D2Q25\text{-ZOT}) \otimes (D1Q5\text{-ZOT}); \quad (3)$$

and also it's pruned version 'D3Q41-ZOT', obtained by pruning, or, symmetrically removing velocity subsets.

It is worth noting that pruning operations deal with discrete velocity groups of *same magnitude*; hence, energy levels—so that pruning remove entire energy levels from a departure lattice configuration.

Several lattice stencils are given in [82] *mostly* in the 'DdVb' naming scheme from [65], but also including an 'n' suffix, as in D1V9n, D2V28n, and D3V53n, as to indicate the lattice is not space-filling, and also in the '$\{0, \pm a, \pm b\}$' format, in which b can be an explicit multiple of a, as in $\{0, \pm a, \pm 3a\}$, naming schemes. Bravais lattices are given up to D1V15, D2V53, and D3V107 in one-, two-, and three- Euclidean spacial dimensions. Appendix tables list full velocity sets as the 'DdVb[n]' naming scheme, in which the 'n' suffix is optional, is not uniquely determined.

From 2010 to 2020:

The following lattices are referenced in the following works: a D2Q36 in [2, p. 452]; a different D2V25, D2V33, D2V29-{l, r, rl}, D3V39, D2V45, D2V77, rectangular D2Q13R, hexagonal D2V19H (also referred to as 'GBL', after Grosfils, Boon and Lallemand), D2V55H, D2V85H, and D2V115H, the last three of fifth, sixth, and seventh order, respectively, and D3V107 in [43]; an unnamed one, described as 'a ninth-order accurate Gauss-Hermite quadrature formula in three dimensions' in [59] having 121 velocities in three-dimensional Euclidean space.

Spherical shell 'SLB$(N;K,L,M)$' lattices—of order N, K spherical shells, L shell latitudes, each K-L intersection circle with M uniformly distributed discrete velocities, so that models have $K \times L \times M$ velocities—with $1 \leqslant N \leqslant 7$, $K, L > N$, and $M > 2N$, i.e., SLB(1; 2, 2, 3), SLB(2; 3, 3, 5), SLB(3; 4, 4, 7), SLB(4; 5, 5, 9), and so on up to SLB(7; 8, 8, 15), and even an SLB(N; 20, 20, 17) are given in [4], the last ones having, respectively, 960 and 6800 velocities!

Rhombic D2Q9, rectangular D2R11, and orthorhombic Bravais D3R23 lattices are found in [42].

Mattila and coauthors [53] have shown that spurious currents emerge along liquid-vapor interface in multiphase simulations. They have shown that higher-order stencils, such as the fourth-order D2V37, yields more localized and isotropic spurious currents than lower order ones, such as third-order D2V17 and D2Q25-ZOT ones. Thus, multiphase flows became another application requiring multi-speed, higher-order LB methods [77]. In fact, shortly after, the group published [52] prescribed abscissas-derived D2V81 and D2V141 lattices, along with an equivalent, however far simpler, form of the prescribed abscissas method in their Section 2.

Reference [55] lists lattice velocities, weights and scaling factor for D2Q16, D2Q17 (reference [78]'s D2V17 and reference [74]'s $E_{2,7}^{17}$), D2Q37 (reference [65]'s D2V37), and for a D3Q121, originally unnamed on reference [59]. Situations like this—in which a given lattice is referred to by different names in different sources yet without ruling out ambiguities—illustrate the need for improved lattice naming schemes.

The 'DdQb^d' notation—as in D3Q5^3, D2Q7^2, D3Q7^3, and D3Q11^3—with a more explicit origin and relationship with a lower-dimensionality, entropic 'D1Qb' lattice of b velocities, through tensor product extension from it [20]—appears in [27, 28] in connection to compressible flow applications. In this scheme, if

$$(D1Qb)^d \equiv \underbrace{(D1Qb) \otimes (D1Qb) \ldots}_{(d \text{ times})}, \text{ then} \tag{4}$$

$$DdQb^d \equiv (D1Qb)^d. \tag{5}$$

It is worth noting that the D3Q7^3 and D3Q11^3 lattices have 343 and 1331 velocities, respectively.

Supersonic and hypersonic flow speeds are comparable to and greater than molecular thermal velocity scales, respectively; moreover, many supersonic flows have a well defined prevailing flow direction, especially when simple, slender objects move with high speeds through quiescent media. Changing from a rest to the object's reference frame causes molecule velocity populations to be shifted by the object's speed. References [29, 30] present the $D2Q7^2$ lattice, i.e., a D2Q49 one, in the (i) symmetric variety, rest reference frame, and (ii) shifted variety, comoving reference frame of $U_x = 1$ lattice units. Better yet, authors demonstrate that departing from a Galilean-invariant symmetric, rest reference frame lattice, reference-frame shiftings *do not change* the lattice weights, meaning

$$W_i(U,T) = W_i(0,T), \tag{6}$$

for arbitrary U, where W_i's are the lattice weights in terms of the lattice reference speed and temperature, and U is the reference frame speed shifting. Better still, the Galilean invariance property allow for the construction of higher-order lattices through tensor products of lower-order Galilean invariant ones, whether they are shifted or not.

Therefore, let a symmetrical $D1Q7$ lattice, with velocity abscissas $V_7 = \{-3, -2, -1, 0, +1, +2, +3\}$, produce a unit U_x-shifted lattice with velocity abscissas $V_7' = \{-2, -1, 0, +1, +2, +3, +4\}$, then the velocity set of the unit U_x-shifted $D2Q7^2$ lattice is given by $V_{7x}' \otimes V_{7y}$.

Body-centered cubic, BCC, lattice arrangements arising from emphasizing spatial discretization over the momentum one in the discretization of the Boltzmann equation in [58] led to BCC lattices named 'RD3Q27'. A novel BCC lattice model named RD3Q67 is also proposed in [8]. Reference [49] adds capital roman numerals to 'DdQb' lattices depending on their underlying relaxation time scheme in Multiple Relaxation Time, MRT, models, thus yielding D2Q9-I to D2Q9-IV lattice model designations.

Velocity count-based lattice naming ambiguities arose in [63], in which the different yet homonymous 'D1Q5' lattices from references [70, 20]—one with $\{0, \pm1, \pm2\}$, and the other with $\{0, \pm1, \pm3\}$ velocity sets—were distinguished by an 'A' or 'B' suffix, making them D1Q5A and D1Q5B.

Finally, a space-filling regular Bravais D2V169 lattice, comprised of 169 velocity vectors, with corresponding weights and scaling factor appears in [24, p. 68].

3. Discussion

From the lattice naming systems survey of the previous Section, one finds that many lattice naming schemes have appeared over the years, with each new variety either introduced as to accommodate or reflect a new aspect brought in the research, as with [76, 21, 65, 4, 58], or to organize and distinguish multiple lattices within a publication, as with [72, 65, 61, 55, 49].

The present survey is unaware of any published effort in the direction of major standardizations across multiple lattice types and features, as well as of the existence of any concise *naming*[2] system that would allow for uniqueness by ruling out name ambiguity.

Lattice naming variety appears to be due to (i) the inherent decentralized nature of research, (ii) the inherent novelty and discovery associated to the practice of research, making future features, ideas, and demands unforeseeable to previous studies, as well as to (iii) the variety of lattice *types*.

On this last aspect, the history of the method has seen (a) space-filling, regular, Bravais types in linear, square, triangular (hexagonal), cubic, and projected hypercubic geometries; (b) irregular, non-space-filling ones; (c) those based on spherical coordinate systems; (d) those with shifted reference velocity frame; and (e) those more heavily focused on spatial space discretization rather than on momentum space. This facet alone may at best difficult efforts in creating concise, unambiguous lattice naming schemes of general scope.

With respect to the continuation and adoption of lattice naming systems, the Euclidean dimension, velocity counting based 'DdQb' template of 1990 due to Qian [71], and of 1991 of Qian and coauthors [72] seems to be the closest thing to a present-time de-facto standard for LB stencil naming, being thus acknowledged on research [73] and on review [2] papers, as well as textbooks [56, 48]. Nonetheless, reaching this current status hasn't been a quick process, as it seemingly took a considerable amount of years until the naming scheme became widely adopted in the LB literature, as evidenced by the lack of its usage in the 1998 review paper of Chen and coauthors, which refer to some of them as 'LBM models based on 21 and 25 velocities' [17, p. 357], and on the 2002 review paper by Succi and coauthors, which refer to Qian and coworkers' model as 'LBGK' [81, p. 1215] after the collision model.

[2] As opposed to velocity *listing*.

Moreover, other naming conventions such as the '$E^{d}_{D,n}$' one due to Shan and coworkers [76], the 'DdVb' one due to Philippi and coworkers [65], and the 'DdQb'-based variations such as '-ZOT' suffix and integer power velocity count ones due to Chikatamarla and Karlin [21] frequently reappear in many subsequent publications, but its adoption seem to be more or less confined to the proposing author's research groups, and to direct citations—so that one may perceive then to be in competition.

It is worth noting that all 'mainstream' lattice naming schemes—whether '$E^{d}_{D,n}$', 'DdQb' and its variations—are able to describe space-filling, Bravais lattices. Yet, all such lattice naming schemes are velocity count based and therefore suffer from ambiguity, as, for instance, the sole 'D2Q25' (or 'D2V25', or 'D2Q5^2') information *can* mean many different lattice configurations, having completely different envelope shapes, conception strategy, set of populated energy levels, and optional shiftings, since it only specifies a set of 25 discrete velocities in two Euclidean space dimensions, and, in the case of the '$E^{d}_{D,n}$' scheme, the resulting order of approximation.

Aiming at space-filling, regular, Bravais types in one- to three-dimensional Euclidean spaces with complete, fully-populated energy levels, the authors conjecture [57] that a scheme with (i) energy-level-based primitives, (ii) that allows for operations such as (tensor product) extensions and shiftings; is able to produce relatively *concise* and *unambiguous* lattice names, while being sufficiently *generic* within its category.

4. Citations by Year

The following are the citations indexed by year in chronological order: **1949**: [34], **1954**: [12], **1973**: [37], **1976**: [36], **1986**: [33, 87], **1987**: [32, 80], **1988**: [54], **1989**: [44, 45], **1990**: [9, 11, 14, 71, 83], **1991**: [7, 13, 16, 23, 26, 31, 35, 47, 72, 79], **1992**: [10, 68], **1993**: [3, 67, 69], **1994**: [18], **1996**: [41], **1997**: [40, 39], **1998**: [17, 62, 70], **2001**: [25], **2002**: [81], **2003**: [6, 50, 60, 85], **2005**: [51], **2006**: [20, 19, 65, 76, 86], **2007**: [61, 66, 78, 84], **2008**: [15, 21, 46, 59, 73, 75], **2009**: [22, 82], **2010**: [2, 1, 43, 74], **2011**: [38, 56], **2012**: [4], **2013**: [42, 53], **2014**: [52, 55, 77], **2015**: [27], **2016**: [5, 28, 29, 58, 64], **2017**: [49, 63], **2018**: [8, 48], **2019**: [24], **2020**: [30, 57].

5. Conclusions

A survey of lattice naming systems for lattice-Boltzmann (LB) methods, from its Lattice-Gas Automata (LGA) historic predecessor to the present time has been performed, which correspond to the period of years of our Lord from 1973 to 2020.

From the survey, key findings include: (i) the appearance (and discontinuance) of many lattice naming schemes over the years, (ii) an apparent lack of published efforts solely geared towards major lattice name standardizations, (iii) the existence of a great diversity of lattice types, (iv) the prominence of Qian's (and coworkers)' [71, 72] velocity-count based 'DdQb' naming scheme—such as D2Q9—being the closest thing to a de-facto standard in the LB literature; (v) the existence of other, seemingly competing, velocity count naming standards; and (vi) the ambiguity of velocity-count based lattice naming schemes, plainly evident in [55, 63].

From the survey and from the diversity of lattice types, it becomes somewhat clear that (a) probably there will be no generic and concise 'one-size-fits-all' naming scheme for all surveyed models, let alone, published ones, and (b) a concise, unambiguous naming scheme, at least for the more regular lattice types is in order, as to enable the necessary distinctions between models of same dimensionality and velocity count. An upcoming work from the authors [57] is to make a proposition.

Conflict of Interest

The author declares that there is no conflict of interest in this work.

CRediT Author Statement

CN: Conceptualization, Methodology, Investigation, Data Curation, Writing - Original Draft, Writing - Review & Editing, Supervision, Project administration. **FNA:** Investigation, Data Curation, Writing - Review & Editing.

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To YHWH God the Father, the Son, and the Spirit, be the glory!

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