

Unifying causal model of rate-independent linear damping for effectively reducing seismic response in low-frequency structures

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Summary

Rate-independent linear damping (RILD) demonstrates similar performance to that of linear viscous damping for the same loss factor when incorporated in a structure to control seismic response displacement. Nevertheless, the damping force generated by the RILD is relatively low in frequency ranges higher than the natural frequency of the primary structure. This leads to efficient displacement control with low damping force and floor response acceleration when RILD is integrated with low-frequency structures. However, the noncausality of RILD hinders its practical applications, and thus, causal models are widely studied to mimic the RILD behavior.

This paper proposes a causal model of RILD using Maxwell elements whose damping force is generated according to the fractional-order derivative of displacement. The proposed model, further represented by a fractional-order damping function, is found to be a unifying model that includes existing causal RILD models from literature, thereby providing further insights to better understand the nature of RILD. Furthermore, several methods were examined to physically realize the proposed model.

KEYWORDS:

Rate-independent linear damping, Maxwell model, fractional-order derivative, low-frequency structure.

1 | INTRODUCTION

Rate-independent linear damping (RILD) is found to aid low-frequency structures by effectively reducing excessive seismic response displacements without increasing floor response accelerations when subjected to strong ground motions^{1,2}. This paper proposes a fractional-order causal approximation model of RILD that leads to a novel theory to unify causal RILD models reported in literature.

The concept of RILD is introduced based on the energy losses of several solid materials per strain cycle proportional to the square of strain amplitude and independent of the strain rate over a considerable frequency range. RILD is also referred to as structural damping, linear hysteretic damping, or ideal hysteretic damping^{3,4,5,6}.

Crandall⁶ examined the impulse response of a structure incorporated with RILD and elucidated that the RILD is non-causal. Owing to its non-causality, it is impossible to achieve its exact realization using physical devices, which hinders its practical applications to utilize its benefit in low-frequency structures.

Researchers developed approximation methods of RILD to overcome the non-causality issue. The first successful causal model that exhibits approximated rate-independent dissipation behavior was proposed by Biot⁷. Caughey⁸ stated that the Biot model could be constructed by parallelly arranging an infinite number of Maxwell elements, which was considered to be a special

case of Maxwell-Wiechert model⁹. Makris¹⁰ proposed a causal linear hysteretic model that exhibited accurate rate-independent dissipation behavior and showed that the model approached Biot model with an increase in the excitation frequency.

Recently, Muscolino *et al.*¹¹ used the Laguerre polynomial approximation method to predict the dynamic response of different RILD models under deterministic and random excitations. Spanos and Tsavachidis¹² introduced time-domain methods to minimize computational costs in the dynamic analyses of structures that incorporated the Biot model. Muravskii^{13,14} studied several nonlinear hysteretic models that led to rate-independent damping and then constructed a causal linear model that exhibited nearly rate-independent dynamic stiffness by considering the entire structural system. Nakamura¹⁵ proposed a causal model with an approximately constant loss modulus over piecewise frequency ranges. Genta and Amanti¹⁶ proposed a tuned Maxwell-Wiechert (TMW) model that could mimic the loss modulus of RILD over a required frequency range; however, limited number of Maxwell elements were used. For nonlinear dynamic analyses of rate-independently damped systems, Huang *et al.*¹⁷ presented a causal model to mimic RILD, which reduced to a TMW model in a linear case.

Keivan *et al.*^{18,19} proposed semi-active control methods to mimic the behavior of RILD using a first-order causal digital filter, which was further passively realized using a mechanical system comprising a negative stiffness element coupled in parallel with a Maxwell element². In this study, to improve the approximation of the dissipation behavior of RILD, a novel model is proposed by generalizing the first-order causal filter to a fractional-order causal filter. Further, several methods are suggested to physically realize the proposed model.

Although a few attempts have been made to mimic the RILD using fractional-order models^{20,21}, to the best of our knowledge, relationships between a fractional-order model and existing causal models for RILD have not been discussed. The novelty of this study lies in the insights inferred regarding the relationships between the proposed model and existing causal models that lead to a novel theory to unify the causal RILD models.

The remainder of this paper is organized as follows. In Section 2, several well-known causal RILD approximation models are reviewed. The frequency- and time-domain representations of the proposed model are presented in Section 3, followed by the discussions on the relations between the proposed model and existing causal RILD models in Section 4. In Section 6, a time-domain method is developed for the dynamic analysis of structure systems incorporated with the proposed model, and several numerical examples (linear and nonlinear types) are used to verify the effectiveness of the proposed method. Further, the conclusions are presented.

2 | MATHEMATICAL MODELS FOR RILD

This section presents the mathematical models of non-causal and causal RILD, followed by the derivations of their dynamic stiffnesses, damping functions, damping kernel functions, and resistive force functions.

2.1 | Complex-stiffness model to represent ideal RILD

Consider a complex stiffness consisting of a linear main spring and an RILD element coupled in parallel, as shown in Fig. 1. Its transfer function from deformation to the resistive force, also referred to as the dynamic stiffness, is:

$$\mathcal{H}_1(i\omega) = \frac{F(i\omega)}{X(i\omega)} = k_0 [1 + \eta \mathcal{Z}_1(i\omega)] \quad (1)$$

where k_0 and η denote spring stiffness and loss factor, respectively, and $\mathcal{Z}_1(i\omega)$ denotes normalized dynamic stiffness of an RILD element, also known as a damping function. The damping function can be expressed as follows,

$$\mathcal{Z}_1(i\omega) = i \operatorname{sgn}(\omega) \quad (2)$$

where $i = \sqrt{-1}$ and $\operatorname{sgn}(\cdot)$ denotes the imaginary unit and the signum function, respectively. Applying an inverse Fourier transform to Eq.(2) gives²²

$$z_1(t) = \mathcal{F}^{-1}[\mathcal{Z}_1(i\omega)] = -\frac{1}{\pi t} \quad (3)$$

where $\mathcal{F}^{-1}(\cdot)$ denotes the inverse Fourier transform. The aforementioned equation implies that this model is non-causal because it yields non-zero response force prior to the application of an impulse deformation, *i.e.*, $z_1(t < 0) \neq 0$ ²³.

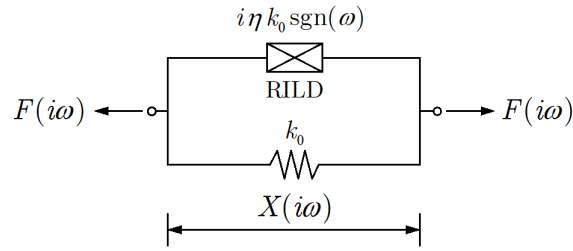


FIGURE 1 Complex-stiffness model

A damping kernel function can be obtained using inverse Fourier transform of the impedance function. Because the impedance function is a transfer function from velocity to damping force, it is $\mathcal{Z}_1(i\omega)/i\omega$. Thus, the damping kernel function of the non-causal model is derived as follows:

$$q_1(t) = \mathcal{F}^{-1} \left[\frac{\mathcal{Z}_1(i\omega)}{i\omega} \right] = -\frac{1}{\pi} \ln |t| \quad (4)$$

Notably, the following relationship holds:

$$\frac{d}{dt} q_1(t) = z_1(t) \quad (5)$$

The resistive force generated by this non-causal model from a deformation $x(t)$ can be expressed as follows:

$$f_1(t) = k_0 x(t) * [\delta(t) + \eta z_1(t)] = k_0 [x(t) + \eta \hat{x}(t)] \quad (6)$$

where the asterisk $*$ indicates the convolution integral, $\delta(t)$ denotes the Dirac's delta function, which is defined as an inverse Fourier transform of unity, and hat $\hat{\cdot}$ indicates the Hilbert transform, which can be expressed as follows^{23,24},

$$\hat{x}(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau \quad (7)$$

Furthermore, this shows that this model is non-causal because its resistive force depends on the past and future input deformations.

The difficulty in conducting numerical analysis on a nonlinear system incorporated with a non-causal model demands the development of causal models.

2.2 | Biot model

Biot⁷ proposed the first causal model that obtains an approximated rate-independent dissipation behavior. Caughey⁸ stated that the Biot model was physically represented using an infinite number of Maxwell elements coupled in parallel. Such a representation is considered to be a particular case of the Maxwell-Wiechert model shown in Fig. 2 with the number of Maxwell elements tending to infinity. In Fig. 2, c_j and k_j denote the damping coefficient and stiffness of the j -th Maxwell element, respectively, and v_j denotes the deformation of the j -th damping element.

For the Maxwell-Wiechert model shown in Fig. 2, the dynamic stiffness is represented as follows:

$$\mathcal{H}(i\omega) = k_0 + \sum_{j=1}^n k_j \frac{i\omega}{i\omega + r_j} \quad (8)$$

where r_j denotes the ratio of the spring stiffness and damping coefficient of the j -th Maxwell element, i.e., $r_j = k_j/c_j$, which is also known as relaxation frequency or the inverse of relaxation time.

It can be proven that as $n \rightarrow \infty$, the Maxwell-Wiechert model reduces to Biot model. Thus, it is assumed that the incremental relaxation frequencies of neighboring Maxwell elements are uniformly distributed, i.e., $\Delta r = r_{j+1} - r_j$ ($j = 1, 2, \dots, n$). The stiffness of the j -th Maxwell element is determined such that

$$k_j = k_1 \frac{\Delta r}{r_j} \quad (9)$$

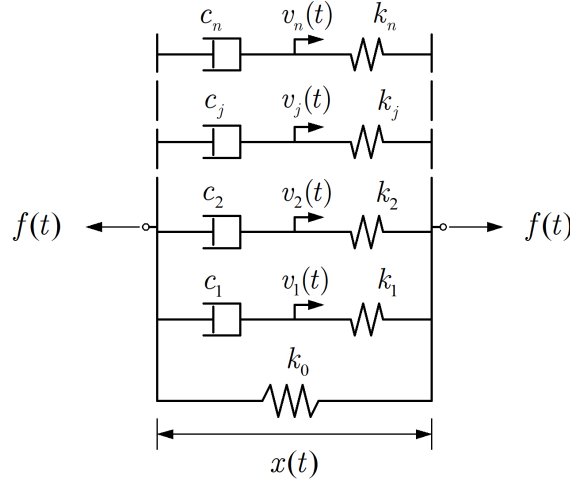


FIGURE 2 Maxwell-Wiechert model

where $k_1 = 2\eta k_0/\pi$. Subsequently, the dynamic stiffness of the designed Maxwell-Wiechert model can be expressed as follows:

$$\mathcal{H}(i\omega) = k_0 \left(1 + \frac{2\eta}{\pi} \sum_{j=1}^n \frac{i\omega}{i\omega + r_j} \frac{\Delta r}{r_j} \right) \quad (10)$$

provided that $r_1 = \varepsilon$ and $\Delta r = r_a/\sqrt{n}$, where r_a is an arbitrary positive parameter with a dimension of circular frequency. As $n \rightarrow \infty$, we have $r_n = \varepsilon + (n-1)\Delta r \rightarrow \infty$ and $\Delta r = dr$; therefore, the aforementioned equation can be expressed as follows:

$$\mathcal{H}(i\omega) = k_0 \left(1 + \frac{2\eta}{\pi} \int_{\varepsilon}^{\infty} \frac{i\omega}{i\omega + r} \frac{dr}{r} \right) \quad (11)$$

The dynamic stiffness of the Biot model can be obtained by integrating Eq.(11) in terms of its real and imaginary parts, as follows:

$$\mathcal{H}_B(i\omega) = k_0 \left\{ 1 + \frac{2\eta}{\pi} \left[\ln \sqrt{1 + \left(\frac{\omega}{\varepsilon} \right)^2} + i \arctan \left(\frac{\omega}{\varepsilon} \right) \right] \right\} \quad (12)$$

where $\arctan(\cdot)$ denotes the inverse tangent function. By introducing the principal branch of the complex logarithm function $\ln(1 + i\omega/\varepsilon)$, we can revise Eq.(12) in a compact form as

$$\mathcal{H}_B(i\omega) = k_0 \left[1 + \frac{2\eta}{\pi} \ln \left(1 + \frac{i\omega}{\varepsilon} \right) \right] \quad (13)$$

Corresponding to Eq.(1), the dynamic stiffness of the Biot model can be reformulated as follows:

$$\mathcal{H}_B(i\omega) = k_0 [1 + \eta \mathcal{Z}_B(i\omega)] \quad (14)$$

where the damping function of the Biot model can be obtained from Eqs.(11)-(13) as follows,

$$\begin{aligned} \mathcal{Z}_B(i\omega) &= \frac{2}{\pi} \int_{\varepsilon}^{\infty} \frac{i\omega}{i\omega + r} \frac{dr}{r} \\ &= \frac{2}{\pi} \left[\ln \sqrt{1 + \left(\frac{\omega}{\varepsilon} \right)^2} + i \arctan \left(\frac{\omega}{\varepsilon} \right) \right] \\ &= \frac{2}{\pi} \ln \left(1 + \frac{i\omega}{\varepsilon} \right) \end{aligned} \quad (15)$$

Furthermore, from the first equality in Eqs.(15), the damping kernel function of the Biot model can be obtained as follows:

$$q_B(t) = \mathcal{F}^{-1} \left[\frac{\mathcal{Z}_B(\omega)}{i\omega} \right] = -\frac{2}{\pi} \text{Ei}(-\varepsilon t) u(t) \quad (16)$$

where $u(t)$ denotes the Heaviside's unit step function (*i.e.*, $u(t) = 0$, if $t \leq 0$; otherwise, $u(t) = 1$); $\text{Ei}(-\varepsilon t)$ denotes the exponential integral, which can be expressed as follows²⁵,

$$\text{Ei}(-\varepsilon t) = \int_{\varepsilon}^{\infty} \frac{-1}{e^{rt}} \frac{dr}{r} = \ln |\varepsilon t| + \gamma_0 + \sum_{n=1}^{\infty} \frac{(-1)^n (\varepsilon t)^n}{n n!} \quad (17)$$

where $\gamma_0 \doteq 0.577$ is the Euler constant.

Therefore, the resistive force generated by the Biot model can be obtained as follows:

$$f_B(t) = k_0[x(t) + \eta \dot{x}(t) * q_B(t)] \quad (18)$$

Substituting Eq.(16) into the above equation gives

$$f_B(t) = k_0 \left\{ x(t) - \frac{2\eta}{\pi} \dot{x}(t) * [\text{Ei}(-\varepsilon t)u(t)] \right\} \quad (19)$$

that coincides with the time-domain representation of the Biot model reported by Caughey⁸.

2.3 | Makris model

An alternative explanation for the non-causality of the complex-stiffness model is that the real and imaginary parts of its dynamic stiffness do not relate with each other by Hilbert transform. Such relations, referred to as Kramers–Kronig relation, are the sufficient and required conditions for a strictly proper transfer function to ensure causality²⁶.

By modifying the real part of Eq.(2) to the Hilbert transform of its imaginary part to satisfy the causality requirement, Makris¹⁰ constructed a causal model having a dynamic stiffness as follows:

$$\mathcal{H}_M(i\omega) = k_0 [1 + \eta \mathcal{Z}_M(i\omega)] \quad (20)$$

where the damping function of this model is given as follows:

$$\mathcal{Z}_M(i\omega) = \frac{2}{\pi} \ln \left| \frac{\omega}{\varepsilon} \right| + i \operatorname{sgn} \left(\frac{\omega}{\varepsilon} \right) \quad (21)$$

Similar to the Biot model, by introducing the principal branch of the complex logarithm function $\ln(i\omega/\varepsilon)$, Eq.(21) can be expressed in a compact form as follows:

$$\mathcal{Z}_M(i\omega) = \frac{2}{\pi} \ln \left(\frac{i\omega}{\varepsilon} \right) \quad (22)$$

Notably, the last term in Eq.(15) approaches Eq.(22) when $\omega \gg \varepsilon$, *i.e.*, the Makris and Biot models behave similarly in high-frequency regions. In other words, the Biot model can be used as an approximation of the Makris model¹⁰ in a high-frequency region. Applying an inverse Fourier transform to Eq.(21) as a generalized function²⁷ provides

$$z_M(t) = \mathcal{F}^{-1} [\mathcal{Z}_M(i\omega)] = -\frac{2u(t)}{\pi t} \quad (23)$$

The damping kernel function of this model is¹⁰,

$$q_M(t) = \mathcal{F}^{-1} \left[\frac{\mathcal{Z}_M(i\omega)}{i\omega} \right] = -\frac{2}{\pi} (\ln |\varepsilon t| + \gamma_0) u(t) \quad (24)$$

Further, Eq.(16) reduces to Eq.(24) as εt tends to zero (such that the last term in Eq.(17) is eliminated). The resistive force provided by the Makris model from a deformation $x(t)$ can be obtained as follows:

$$f_M(t) = k_0 [x(t) + \eta \dot{x}(t) * q_M(t)] \quad (25)$$

Substituting Eq.(24) in the above equation gives

$$f_M(t) = k_0 \left\{ \left(1 - \frac{2\eta}{\pi} \gamma_0 \right) x(t) - \frac{2\eta}{\pi} \dot{x}(t) * [\ln |\varepsilon t| u(t)] \right\} \quad (26)$$

that includes a negative constant term in the displacement-dependent part. This suggests that a negative stiffness element can be used in the physical approximation of RILD^{28,29}.

2.4 | Tuned Maxwell-Wiechert model

For numerical implementation, a Prony series with a finite number of exponential relaxation functions was used by Makris and Zhang³⁰ to approximate the behavior of the Biot model. Such a series is also known as the relaxation modulus or damping kernel function of a Maxwell-Wiechert model, as shown in Fig. 2. Furthermore, a Prony series was used by Spanos and Tsavachidis¹² to develop a recursive procedure for dynamic analysis of structural systems incorporated with the Biot model. Genta and Amati¹⁶ proposed the concept of a TMW model such that its loss moduli can fit those of the non-causal RILD over a required frequency range. A similar concept was later used by Reggio *et al.*³¹ for system identification applications and by Huang *et al.*¹⁷ in the nonlinear analysis of rate-independent damped structures.

The dynamic stiffness of a TMW model is given as follows:

$$\mathcal{H}_T(i\omega) = k_0 [1 + \eta \mathcal{Z}_T(i\omega)] \quad (27)$$

where the damping function is given as follows:

$$\mathcal{Z}_T(i\omega) = \sum_{j=1}^n \phi_j \frac{i\omega}{i\omega + r_j} \quad (28)$$

where ϕ_j is an adjusting parameter with respect to a discrete frequency point $\omega = r_j$, or physically represents the normalized stiffness of the j -th Maxwell element with a relaxation frequency of r_j . The damping kernel function of this model is represented as follows:

$$q_T(t) = \mathcal{F}^{-1} \left[\frac{\mathcal{Z}_T(i\omega)}{i\omega} \right] = \sum_{j=1}^n \phi_j e^{-r_j t} u(t) \quad (29)$$

By incorporating appropriate terms, Eq.(29) can be used to approximate any physical relaxation process. A regression analysis can be conducted to determine the parameters ϕ_j and r_j ($j = 1, 2, \dots, n$) by incorporating a target function, and the frequency- and time-domain methods³² can be availed for this purpose.

Using Eq.(29), the resistive force generated by the TMW model can be expressed as follows:

$$f_T(t) = k_0 \left[x(t) + \eta \dot{x}(t) * \sum_{j=1}^n \phi_j e^{-r_j t} u(t) \right] \quad (30)$$

The above model is believed to be suitable for numerical implementation because several commercial software provide entrances for the parameters ϕ_j and r_j ($j = 1, 2, \dots, n$).

3 | PROPOSED MODEL FOR RILD

3.1 | Fractional-order causal model

Recently, an alternative approach is developed to approximate ideal RILD (Eq.(2)) that uses the first-order all-pass filter¹⁸ defined as follows:

$$\mathcal{Z}_1(i\omega) = \frac{i\omega - \varepsilon}{i\omega + \varepsilon} \quad (31)$$

that produces unity amplitude along the frequency and an equal phase lead of $\pi/2$ rad at $\omega = \varepsilon$ similar to the ideal RILD. However, the above filter demonstrates strong rate-dependency in loss modulus, which may compromise its capability to simulate the energy dissipation behavior of the RILD over a broad frequency range.

Here, the first-order causal filter is generalized and extended to a fractional-order filter defined in the following form

$$\mathcal{Z}_\alpha(i\omega) = \beta_\alpha \frac{(i\omega)^\alpha - \varepsilon^\alpha}{(i\omega)^\alpha + \varepsilon^\alpha} \quad (32)$$

where the real-valued parameter α is defined within the range $[0, 1]$. The power of a complex value $(i\omega)^\alpha$ is defined over the principal branch as follows:

$$(i\omega)^\alpha = |\omega|^\alpha \exp[i \alpha \operatorname{sgn}(\omega) \pi/2] \quad (33)$$

and β_α is a real-valued adjusting coefficient. For example, assuming the amplitude of the filter be adjusted to unity (*i.e.* the same as ideal RILD defined in Eq.(2)) at $\omega = \varepsilon$, we obtain

$$\beta_\alpha = \cot \left(\frac{\alpha\pi}{4} \right) \quad (34)$$

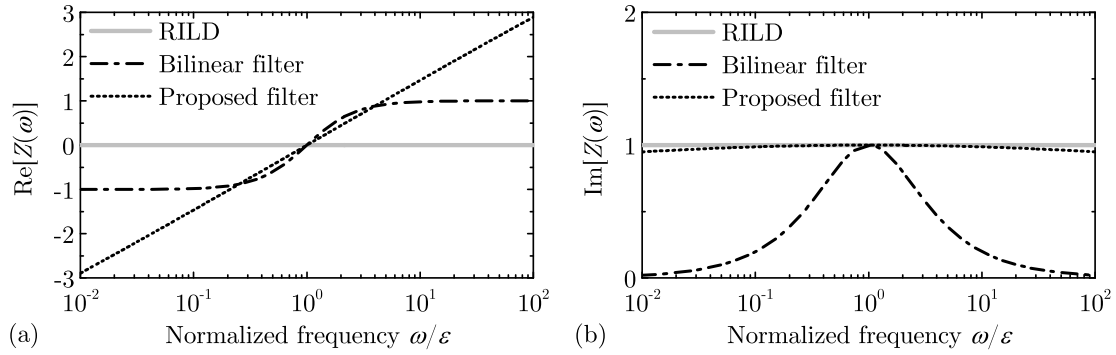


FIGURE 3 Comparison between the damping functions of different elements: (a) real and (b) imaginary parts.

where $\cot(\cdot)$ indicates a cotangent function. A novel fractional-order filter is then proposed as follows:

$$\mathcal{Z}_\alpha(i\omega) = \cot\left(\frac{\alpha\pi}{4}\right) \frac{(i\omega)^\alpha - \epsilon^\alpha}{(i\omega)^\alpha + \epsilon^\alpha} \quad (35)$$

which reduces to Eq.(31) as α tends to unity.

The proposed filter can provide a less rate-dependent loss modulus over a considerable frequency range when compared with the first-order causal filter. For example, Fig. 3 compares the real and imaginary parts of the proposed filter (e.g., $\alpha = 0.1$), first-order causal filter, and ideal RILD. It is observed that the proposed filter (e.g., $\alpha = 0.1$) provides an improved approximation of the RILD in terms of loss modulus while yielding similar storage moduli to those of a first-order causal filter over a frequency range near $\omega = \epsilon$.

Corresponding to the complex-stiffness model given in Eq.(1), the proposed model for RILD is constructed with a dynamic stiffness as follows:

$$\mathcal{H}_\alpha(i\omega) = k_0[1 + \eta \mathcal{Z}_\alpha(i\omega)] \quad (36)$$

From the aforementioned equation, we can observe that the fractional-order filter $\mathcal{Z}_\alpha(i\omega)$ (Eq.(35)) is a damping function.

3.2 | Causality of the proposed model

Here, the causality of the proposed model is investigated first. Accordingly, assuming $0 < \alpha < 1$, the damping function of the proposed model is reformulated as follows:

$$\mathcal{Z}_\alpha(i\omega) = \beta_\alpha [2i\omega\mathcal{G}(i\omega) - 1] \quad (37)$$

where an auxiliary function $\mathcal{G}(i\omega) = (i\omega)^{\alpha-1}/[(i\omega)^\alpha + \epsilon^\alpha]$ is introduced. The damping kernel function of the proposed model can be obtained from an inverse Fourier transform of the impedance function as follows:

$$q_\alpha(t) = \mathcal{F}^{-1} \left[\frac{\mathcal{Z}_\alpha(i\omega)}{i\omega} \right] = \beta_\alpha [2g(t) - u(t)] \quad (38)$$

where $g(t)$ denotes an inverse Fourier transform of $\mathcal{G}(i\omega)$, i.e.,

$$g(t) = \mathcal{F}^{-1}[\mathcal{G}(i\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(i\omega)^{\alpha-1}}{(i\omega)^\alpha + \epsilon^\alpha} e^{i\omega t} d\omega \quad (39)$$

Further, for $0 < \alpha < 1$, the denominator of the integrand of the above equation

$$(i\omega)^\alpha + \epsilon^\alpha = |\omega|^\alpha [\cos(\alpha\pi/2) + i \operatorname{sgn}(\omega) \sin(\alpha\pi/2)] + \epsilon^\alpha$$

comprises positive real and imaginary parts, i.e., $(i\omega)^\alpha + \epsilon^\alpha \neq 0$ for any ω when $0 < \alpha < 1$. Therefore, the integrand $\mathcal{G}(i\omega)$ has no pole, thereby eliminating the residual terms when the Cauchy integration theorem is applied to evaluate the integral in Eq.(39).

In the case of $t < 0$, $g(t < 0)$ is verified to be eliminated because of Jordan's lemma³³, acquiring $q_\alpha(t < 0) \equiv 0$. Thus, the proposed model is causal and can be realized using physical systems.

3.3 | Time-domain representation

For $t > 0$, the time-domain representation of the proposed model is derived as follows: Assuming $s = i\omega$, we can reformulate Eq.(39) as follows:

$$g(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{s^{\alpha-1}}{s^{\alpha} + \varepsilon^{\alpha}} e^{st} ds = \mathcal{L}^{-1} \left[\frac{s^{\alpha-1}}{s^{\alpha} + \varepsilon^{\alpha}} \right] \quad (40)$$

where $\mathcal{L}^{-1}(\cdot)$ indicates the inverse Laplace transform. The Mittag-Leffler relaxation function is defined as Eq.(41)

$$E_{\alpha}(-t^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{\alpha n}}{\Gamma(\alpha n + 1)} \quad (41)$$

where $\Gamma(\cdot)$ denotes the Euler's Gamma function.

Or, its alternative expression³⁴ is

$$E_{\alpha}(-t^{\alpha}) = \mathcal{L}^{-1} \left[\frac{s^{\alpha-1}}{s^{\alpha} + 1} \right] \quad (42)$$

As $g(t < 0) \equiv 0$, from Eqs.(40) and (42), the following equation can be obtained

$$g(t) = E_{\alpha}(-\varepsilon^{\alpha} t^{\alpha}) u(t) \quad (43)$$

Substituting Eq.(43) in Eq.(38) provides the damping kernel function of the proposed model as follows:

$$q_{\alpha}(t) = \beta_{\alpha} u(t) [2E_{\alpha}(-\varepsilon^{\alpha} t^{\alpha}) - 1] \quad (44)$$

Subsequently, the resistive force $f_{\alpha}(t)$ generated by the proposed model is obtained as follows:

$$f_{\alpha}(t) = k_0 [x(t) + \eta \dot{x}(t) * q_{\alpha}(t)] \quad (45)$$

Substituting Eq.(44) in the above equation gives

$$f_{\alpha}(t) = k_0 \{ (1 - \eta\beta_{\alpha})x(t) + 2\eta\beta_{\alpha}\dot{x}(t) * [E_{\alpha}(-\varepsilon^{\alpha} t^{\alpha})u(t)] \} \quad (46)$$

With $\alpha = 1$, we have $E_{\alpha}(-\varepsilon^{\alpha} t^{\alpha}) = e^{-\varepsilon t}$ and $\beta_{\alpha} = 1$, followed by

$$f_{\alpha}(t) = k_0 \{ (1 - \eta)x(t) + 2\eta\dot{x}(t) * [e^{-\varepsilon t}u(t)] \} \quad (47)$$

that represents the resistive force generated by Keivan model¹⁸. The negative constant term $-\eta$ in the displacement-dependent part corresponds to a negative stiffness element used in the physical realization of RILD as reported by Luo *et al.*².

4 | UNIFYING FRAMEWORK OF RILD

For comparison, Table 1 summarizes the damping functions and kernel functions of different models for RILD introduced previously. The relationships between the proposed and existing causal models listed in Table 1 are discussed in the following section.

4.1 | Relationship with Makris model

Here, it is proved that the proposed model can include the Makris model as a specific case of $\alpha = 0$. Assuming $\lambda = \omega/\varepsilon$, Eq.(35) can be reformulated as follows:

$$\mathcal{Z}_{\alpha}(i\lambda) = \frac{\cos(\alpha\pi/4)\{(i\lambda)^{\alpha} - 1\}}{\sin(\alpha\pi/4)\{(i\lambda)^{\alpha} + 1\}} \quad (48)$$

Because the denominator and numerator of the above equation approach 0 as α approaches 0, the l'Hôpital's rule is applied to obtain the limit

$$\lim_{\alpha \rightarrow 0} \mathcal{Z}_{\alpha}(i\lambda) = \lim_{\alpha \rightarrow 0} \frac{-(\pi/4) \sin(\alpha\pi/4)\{(i\lambda)^{\alpha} - 1\} + \cos(\alpha\pi/4)\{\ln(i\lambda) \cdot (i\lambda)^{\alpha}\}}{(\pi/4) \cos(\alpha\pi/4)\{(i\lambda)^{\alpha} + 1\} + \sin(\alpha\pi/4)\{\ln(i\lambda) \cdot (i\lambda)^{\alpha}\}} = \frac{2}{\pi} \ln(i\lambda) \quad (49)$$

As $\lambda = \omega/\varepsilon$, the aforementioned equation can be reformulated as follows

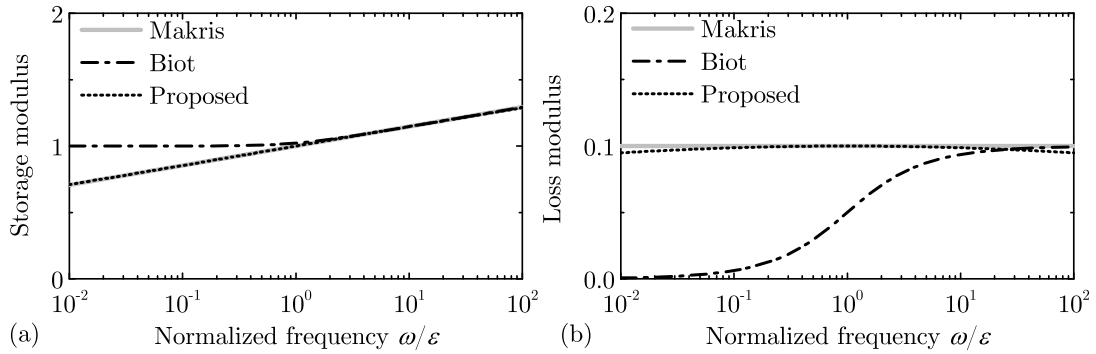
$$\lim_{\alpha \rightarrow 0} \mathcal{Z}_{\alpha}(i\omega) = \frac{2}{\pi} \ln \left(\frac{i\omega}{\varepsilon} \right) = \frac{2}{\pi} \ln \left| \frac{\omega}{\varepsilon} \right| + i \operatorname{sgn} \left(\frac{\omega}{\varepsilon} \right) \quad (50)$$

TABLE 1 Comparison of different RILD elements

Model	Damping function $\mathcal{Z}(i\omega)$	Damping kernel function $q(t)$	Causality
RILD	$i \operatorname{sgn}(\omega)$	$-\frac{1}{\pi} \ln t $	Non-Causal
Biot	$\frac{2}{\pi} \ln \left(1 + i \frac{\omega}{\varepsilon} \right)$	$-\frac{2u(t)}{\pi} \operatorname{Ei}(-\varepsilon t)$	Causal
Makris	$\frac{2}{\pi} \ln \left(\frac{i\omega}{\varepsilon} \right)$	$-\frac{2u(t)}{\pi} (\ln \varepsilon t + \gamma_0)$	Causal
TMW	$\sum_{j=1}^n \phi_j \frac{i\omega}{i\omega + r_j}$	$\sum_{j=1}^n u(t) \phi_j e^{-r_j t}$	Causal
Keivan	$\frac{i\omega - \varepsilon}{i\omega + \varepsilon}$	$u(t) [2e^{-\varepsilon t} - 1]$	Causal
Proposed	$\beta_\alpha \frac{(i\omega)^\alpha - \varepsilon^\alpha}{(i\omega)^\alpha + \varepsilon^\alpha}$	$\beta_\alpha u(t) [2E_\alpha(-\varepsilon^\alpha t^\alpha) - 1]$	Causal

that is similar to the damping function of the Makris model, as defined in Eqs.(21) and (22).

The similarity between the damping function of the two models in the frequency domain further suggests that the damping kernel function of the proposed model in the case of $\alpha = 0$ can be expressed as Eq.(24). Furthermore, because the dynamic stiffness and damping kernel function of the proposed model vary continuously with the value of α , it can be predicted that the proposed model with a small order of α (e.g., $\alpha = 0.1$) can provide a suitable approximation of the Makris model, at least over a required frequency range.

**FIGURE 4** Comparison between the normalized dynamic stiffnesses of different models for RILD ($\eta = 0.1$)

For example, assuming $\eta = 0.1$, Fig. 4 compares the normalized dynamic stiffness of the proposed model (e.g., $\alpha = 0.1$) with those of the Makris and Biot models. Fig. 4 shows that the differences between the dynamic stiffness of the proposed model with $\alpha = 0.1$ and that of Makris model are negligible over a considerable frequency range. Hence, the proposed model with a small order of α can be used to approximate the Makris model without significantly compromising the accuracy. In certain cases, such an approximation seems to be a useful requirement, which is further discussed in Section 5.

4.2 | Relationship with Biot model

Here, it is first shown that the proposed model is equivalent to a first-order approximation of the Biot model in terms of dynamic stiffness as the order α tends to unity (*i.e.*, Keivan model). Hence, Eq.(15) can be revised as follows:

$$\mathcal{Z}_B(i\omega) = \frac{2}{\pi} \left[\ln 2 - \ln \left(1 - \frac{i\omega - \varepsilon}{i\omega + \varepsilon} \right) \right] \quad (51)$$

Using the Mercator series to expand the frequency-dependent term (the second $\ln(\cdot)$ term) in the above equation, the following equation can be obtained.

$$\mathcal{Z}_B(i\omega) = \frac{2}{\pi} \left[\ln 2 + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{i\omega - \varepsilon}{i\omega + \varepsilon} \right)^n \right] \quad (52)$$

Thereafter, substituting Eq.(52) in Eq.(14) provides an alternative expression for the dynamic stiffness of the Biot model as follows:

$$\mathcal{H}_B(i\omega) = k_0 \left[1 + \frac{2\eta}{\pi} \ln 2 + \frac{2\eta}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{i\omega - \varepsilon}{i\omega + \varepsilon} \right)^n \right] \quad (53)$$

Assuming $\bar{k}_0 = k_0[1 + (2\eta \ln 2)/\pi]$ and $\bar{\eta} = 2\eta/(\pi + 2\eta \ln 2)$, Eq.(53) can be reformulated as follows:

$$\mathcal{H}_B(i\omega) = \mathcal{H}_B^{(N)}(i\omega) + \bar{\eta} \bar{k}_0 \sum_{n=N+1}^{\infty} \frac{1}{n} \left(\frac{i\omega - \varepsilon}{i\omega + \varepsilon} \right)^n \quad (54)$$

where $\mathcal{H}_B^{(N)}(i\omega)$ denotes the N th-order truncated approximation of $\mathcal{H}_B(i\omega)$, which can be expressed as follows:

$$\mathcal{H}_B^{(N)}(i\omega) = \bar{k}_0 \left[1 + \bar{\eta} \sum_{n=1}^N \frac{1}{n} \left(\frac{i\omega - \varepsilon}{i\omega + \varepsilon} \right)^n \right] \quad (55)$$

Furthermore, assuming $N = 1$, we have

$$\mathcal{H}_B^{(1)}(i\omega) = \bar{k}_0 \left(1 + \bar{\eta} \frac{i\omega - \varepsilon}{i\omega + \varepsilon} \right) \quad (56)$$

that has the same form as the dynamic stiffness of the proposed model expressed in Eq.(36) with $\alpha = 1$ (*i.e.*, the Keivan model). Thus, the Keivan model is equivalent to a first-order approximation of the Biot model.

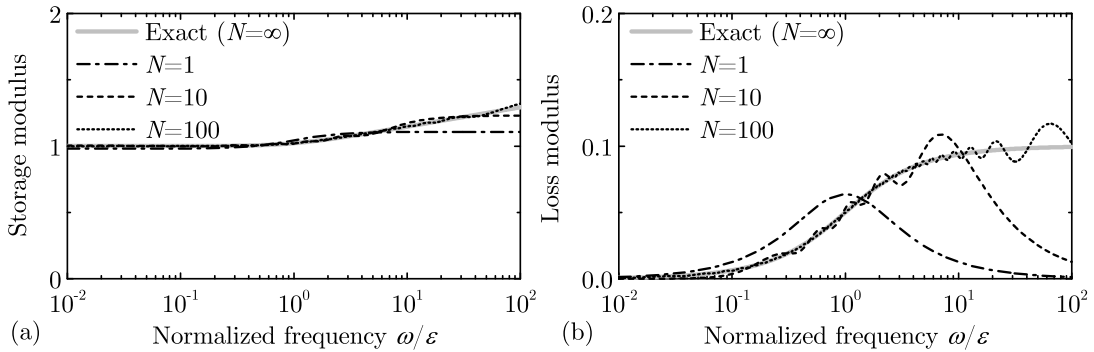


FIGURE 5 Approximation of the dynamic stiffness of Biot model using Mercator series ($\eta = 0.1$)

To further investigate the effect of N on the accuracy for approximating the dynamic stiffness of the Biot model, for example, assuming $\eta = 0.1$, Fig. 5 compares the N th-order truncated approximation of normalized dynamic stiffness of the Biot model with $N = 1, 10, 100$, and ∞ . Fig. 5 shows that a first-order approximation exhibits similar characteristics of dynamic stiffness of the Biot model in low-frequency region; to represent the high-frequency characteristics of the Biot model, a significantly higher-order approximation is required. Alternatively, the proposed model with a small order α (*e.g.*, $\alpha = 0.1$) can be used at the expense of distorted low-frequency characteristics.

As mentioned in Section 2, the dynamic stiffness of the Biot model approaches that of the Makris model when $\omega \gg \varepsilon$. Therefore, from the observed relation between the proposed and Makris models, the proposed model with a small order α (*e.g.*,

$\alpha = 0.1$) can be used to approximate the high-frequency characteristics of the Biot model. As shown in Fig. 4, the proposed and the Biot models exhibit a similar behavior in high frequency region.

4.3 | Equivalence to a modified TMW model

Here, it is shown that the proposed model is equivalent to a TMW model with its storage modulus modified by a rate-independent term. Accordingly, comparisons between the TMW and proposed models are quantified, and the method suggested by Huang *et al.*¹⁷ is used to design the order of the TMW model, ϕ_j , with respect to a given set of $r_j, j = 1, 2, \dots, n$, which are logarithmically spaced over a frequency range from 0.01ε to 100ε . Two different TMW models are designed with $n = 5$ and $n = 7$, respectively.

To approximate the non-causal model represented by Eq.(1) at a specified frequency $\omega = \varepsilon$, the dynamic stiffness of the TMW model is modified as follows:

$$\mathcal{H}'_T(i\omega) = k_0 [1 - \eta Z_{\text{Mod}}(\varepsilon) + \eta Z_T(i\omega)] \quad (57)$$

where $Z_{\text{Mod}}(\varepsilon)$ is a rate-independent modification term, *i.e.*,

$$Z_{\text{Mod}}(\varepsilon) = \mathcal{R} [Z_T(i\varepsilon)] = \sum_{j=1}^n \frac{\phi_j \varepsilon^2}{\varepsilon^2 + r_j^2} \quad (58)$$

In addition, a modification by adding a rate-independent term to the storage modulus does not affect the causality of the original model (*e.g.*, refer¹⁵). The modification can be physically interpreted by connecting the TMW model in parallel with a linear negative stiffness element whose absolute stiffness is $\eta k_0 Z_{\text{Mod}}(\varepsilon)$, as reported by Luo *et al.*².

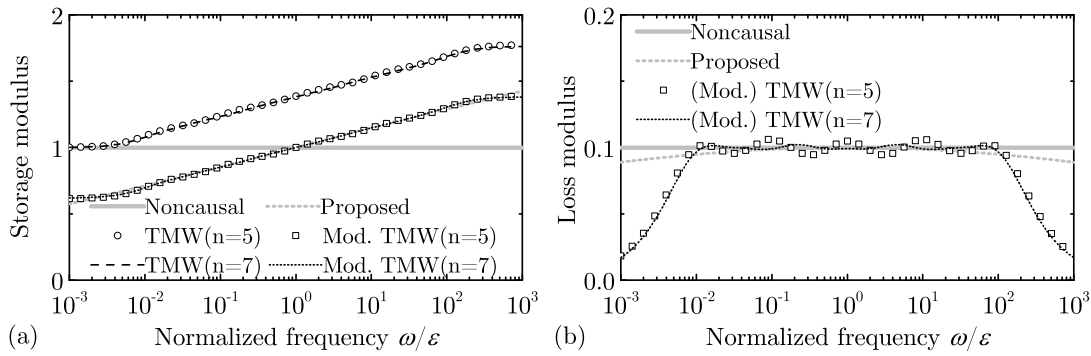


FIGURE 6 Comparison between the TMW and proposed models ($\eta = 0.1$)

Alternatively, Eq.(57) can be reformulated as follows:

$$\mathcal{H}'_T(i\omega) = k'_0 [1 + \eta' Z_T(i\omega)] \quad (59)$$

where $k'_0 = k_0 [1 - \eta Z_{\text{Mod}}(\varepsilon)]$ and $\eta' = \eta / [1 - \eta Z_{\text{Mod}}(\varepsilon)]$. The aforementioned equation has the same form as the dynamic stiffness of the TMW model expressed in Eq.(27). This suggests that the modified model can be equivalently realized by designing a TMW model with k'_0 and η' , instead of k'_0 and η . In other words, the modified model can be equally achieved without supplementing a physical negative stiffness element.

To further demonstrate the relationship between the TMW and proposed models, for example, assuming $\eta = 0.1$, Fig. 6 denotes the normalized dynamic stiffnesses of the TMW models and those of the non-causal and proposed models (*e.g.*, $\alpha = 0.1$) for comparison. It is shown that the accuracy of the TMW model for approximating the loss modulus of non-causal model is improved as the number n increases. Moreover, modifying the storage modulus of the TMW model to fit the non-causal model at $\omega = \varepsilon$ leads to a suitable fit to the proposed model over the required frequency range.

The modified TMW model can be equivalently represented by the proposed model to mimic the behavior of the non-causal RILD. This may be preferable in certain applications because substantially reduced empirical parameters are used for simulations in the proposed model. The proposed model requires one additional parameter α , whereas seven adjusting parameters ϕ_j ($j =$

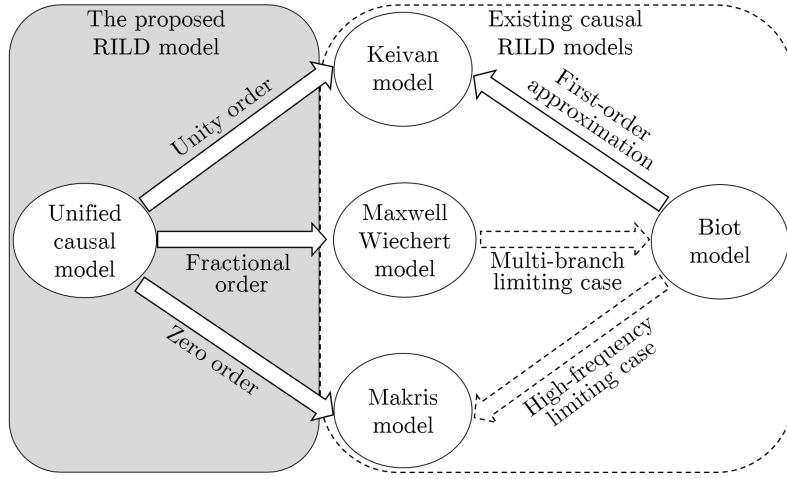


FIGURE 7 New framework of causal models for RILD

1, 2, ..., 7) are used for the (modified) TMW model to equalize with the proposed model in this example. This implies the advantage of the fractional-order operator over integer-order operators used in a constitutive law^{35,36}.

Each Maxwell element can be realized by connecting a viscous damper in series with a spring and then connecting with each other in parallel to construct a TMW model. Accordingly, the modified TMW model is considered as a viable option to physically realize the proposed model. Further discussions on the physical realization of the proposed model are presented in Section 5.

In summary, the relationships between different causal models of RILD are shown in Fig. 7. It has been proven that the proposed model can include the Keivan and Makris models as its particular cases with $\alpha = 1$ and $\alpha = 0$, respectively. From the comparison in Fig. 6, it is suggested that the proposed model with a fractional order (*e.g.*, $\alpha = 0.1$) is equivalent to a modified Maxwell-Wiechert model, which can approximate the Biot model for practical applications (*e.g.*, see references^{12,30}). Hence, the proposed model can be considered as a unifying causal model for RILD. Furthermore, in Subsection 4.2, Keivan model is shown to be equivalent to a first-order approximation of the Biot model. These new relationships (indicated by solid arrows in Fig. 7) along with the established relationships (indicated by dashed arrows in Fig. 7) between different models contribute to the development of a novel framework for causal models of RILD to better understand its nature.

5 | PHYSICAL REALIZATION OF RILD

The RILD cannot be realized strictly using real-life devices because of its non-causality. However, physically realizing approximated causal models that can utilize RILD in practical applications can be beneficial. A mechanical system was developed to physically realize the first-order causal filter by parallelly connecting a Maxwell element with a negative stiffness element². This section discusses the further expansion of this method to examine the feasibility of the realization using fractional-order model.

5.1 | Mechanical representation of the proposed model

Here, for physical realization of the proposed model, a mechanical representation is developed based on our previous study². Hence, a conventional Maxwell element is generalized into a fractional-order element comprising a linear spring arranged in series with a fractional-order damping element, as shown in Fig. 8, where k_α and k_N denote the stiffnesses of the fractional-order Maxwell and negative stiffness elements, respectively, and $v(t)$ denotes the deformation of fractional-order damping element.

Here, the dynamic stiffness of the mechanical representation in Fig. 8 is identical to that of the proposed model, as shown in Eq. (57). The equilibrium equation of motion of the fractional-order Maxwell element can be expressed as follows:

$$k_\alpha \{x(t) - v(t)\} = \frac{k_\alpha}{\epsilon^\alpha} D_{0+}^\alpha v(t) \quad (60)$$

where $D_{0+}^\alpha v(t)$ denotes an α -order derivative of $v(t)$ with respect to t .

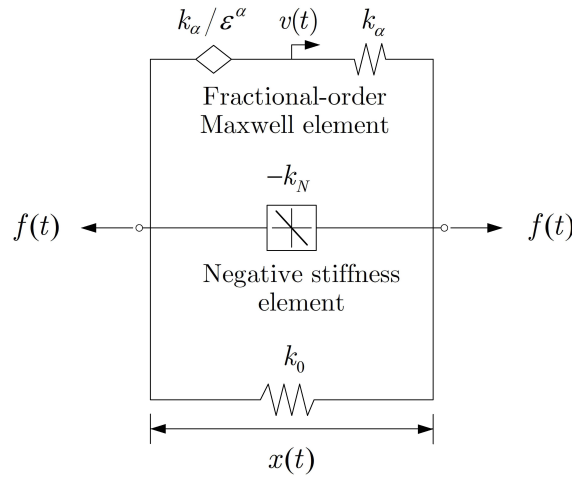


FIGURE 8 Mechanical representation of the proposed model

This can be deduced to

$$D_{0+}^\alpha v(t) + \varepsilon^\alpha v(t) = \varepsilon^\alpha x(t) \quad (61)$$

Here, the definition reported by Riemann-Liouville³⁷ of the fractional derivative is used and defined as follows:

$$D_{0+}^\alpha v(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(\tau)}{(t-\tau)^\alpha} d\tau \quad (62)$$

Using the property of fractional derivative³⁷, we have

$$\mathcal{F}[D_{0+}^\alpha v(t)] = (i\omega)^\alpha \mathcal{F}[v(t)] = (i\omega)^\alpha V(i\omega) \quad (63)$$

Thus, applying a Fourier transform to Eq.(61) provides

$$V(i\omega) = \frac{\varepsilon^\alpha}{(i\omega)^\alpha + \varepsilon^\alpha} X(i\omega) \quad (64)$$

The resistive force provided by the mechanical model in Fig. 8 can be represented as follows:

$$f(t) = (k_0 - k_N)x(t) + k_\alpha[x(t) - v(t)] \quad (65)$$

Applying a Fourier transform to Eq.(65) gives

$$F(i\omega) = (k_0 - k_N)X(i\omega) + k_\alpha[X(i\omega) - V(i\omega)] \quad (66)$$

With $k_N = \eta\beta_\alpha k_0$ and $k_\alpha = 2\eta\beta_\alpha k_0$, the dynamic stiffness can be obtained by substituting Eq.(64) in Eq.(66) as

$$\mathcal{H}(i\omega) = k_0 \left[1 + \eta\beta_\alpha \frac{(i\omega)^\alpha - \varepsilon^\alpha}{(i\omega)^\alpha + \varepsilon^\alpha} \right] \quad (67)$$

that coincides with the dynamic stiffness of the proposed model expressed in Eq.(36). Consequently, the proposed fractional-order filter is verified to be represented using a fractional-order Maxwell element in parallel with a negative stiffness element (hereafter referred to as fractional-Maxwell-negative-stiffness (FMNS) model).

5.2 | Feasibility of physical realizations

Considering the physical realization of the FMNS model, the negative stiffness element can be created by equivalently reducing the horizontal stiffness of the primary structure (or of isolators) or physically realized using passive negative stiffness devices³⁸, as discussed in our previous study². Considering the fractional-order Maxwell element, numerous engineering solid materials^{35,36} and certain damping devices^{39,40} have been reported to be accurately simulated using fractional-order damping elements.

Therefore, such real-life damping devices can be developed, which can physically realize the fractional-order element to be used in the FMNS model. Further challenges are encountered for developing real-life devices with desired damping characteristics.

Excluding the aforementioned method, at present, two types of alternative methods are believed to be promising in physical realization of the FMNS model: (i) to develop a passive system consisting of multiple existing damping devices, which is equivalent to the FMNS model discussed in Subsection 4.3. A modified TMW model can be a viable option); or (ii) to employ semi-active devices¹⁸ that can generate the desired damping force yielded by the FMNS model. An efficient time-domain calculation technique is required to ensure the real-time control based on FMNS model⁴¹.

As the order of α tends to unity, the fractional-order Maxwell element reduces to a conventional Maxwell element, and the FMNS model reduces to a Maxwell-negative-stiffness (MNS) model. The MNS model consisting of a Maxwell element and a negative stiffness element is considered to be the simplest realizable case of the FMNS (or modified TMW) model. The application of the MNS model used for seismic performance enhancement of base-isolated structures has been discussed in our past study².

As the order of α tends to zero, the fractional-order Maxwell element can be used to mechanically represent the damping (frequency-dependent) part of the Makris model in the theoretical form. However, an exact and physical realization of such an element is impossible in practical engineering applications because the required stiffnesses of negative stiffness element and Maxwell spring are infinite. Such elements with infinite stiffnesses can be difficult to be used in numerical simulation problems. However, the FMNS model with a small order α can be used for approximation, such that the parameters can be designed within an appropriate range. This suggests that the proposed model with an additional adjusting parameter α is a viable option for practical applications.

6 | CONCLUSIONS

Increasing the damping in a structure can effectively reduce excessive displacements induced by low-frequency components of ground motion, which may result in excessive damping force and floor response accelerations induced by high-frequency components. Accordingly, using the RILD can improve the performance in reducing damping force and floor response accelerations when incorporating a high amount of damping because its damping force does not increase with increase in the excitation frequency. The RILD performs selective damping to effectively reduce the displacement without increasing floor response accelerations when incorporated into a low-frequency structure.

In this study, a novel causal RILD model is proposed for low-frequency structures. The proposed model is obtained by modifying the order of Keivan model (first-order) to fractional-order. The proposed model improves the approximation in energy dissipation behavior in a considerably wide frequency range compared with the first-order model. The causality of the proposed model is proved, and a time-domain representation is derived using the Mittag-Leffler relaxation function.

Comparisons between the proposed model with existing causal models aided in better understanding the relationships between different causal models in literature, leading to the development of a unifying framework of causal RILD models. It is elucidated that the proposed model can include the Makris and Keivan models as its particular cases with $\alpha = 0$ and $\alpha = 1$, respectively. Further, comparing the proposed model with designed TMW models revealed that the proposed model with a fractional order is equivalent to a TMW model, which is used to approximate the Biot model for practical applications. Moreover, the Keivan model is proven to be equivalent to a first-order approximation of the Biot model.

Furthermore, a mechanical representation of the proposed model is developed to physically realize the proposed model in practical engineering applications.

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