

Millimeter-wave Techniques: Theory, Algorithms and Methods

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Chapter 1

INTRODUCTION

In this chapter, we will give a brief overview of the recent progress on mirror system design for QO launcher in high-power gyrotron and outline the future trend in this dynamic area.

1.1 High-power Gyrotron Review

Gyrotrons (electron cyclotron resonance masers) are a class of microwave sources which are capable of efficiently producing large amounts of power (> 1 MW pulse, 100 kW CW) by utilizing the interaction between an electron beam and fast-wave electromagnetic fields, at millimeter and submillimeter wave frequencies, a frequency band in which few other sources (slow-wave RF tubes, lasers) are able to effectively operate. In general, their output power is of the order of hundreds of kW and their frequency can range from several hundred GHz to 170 GHz and above. A typical semantical plot of gyrotron manufactured by Communication and Power Industries (CPI) is shown in Figure 1.1. For many years, gyrotron development has been driven by the need for fusion plasma ECRH heating; however, other applications like high energy particle accelerators, millimeter wave radars and deep space communications are of increasing interest.

QO modes excited in the gyrotron cavity are difficult to transmit, mainly because of the high loss level in the transmission lines. Therefore, the mode conversion is necessary and the efficiency of the mode conversion as well as the maintenance of low level of diffraction losses are crucial for the implementation of powerful gyrotrons as radiation sources for electron-cyclotron-resonance heating of fusion plasmas. For such purpose, a QO mode converters (launcher) are used to transform electromagnetic waves of complex structure and polarization generated in gyrotron cavities into a linearly polarized, Fundamental Gaussian Beam

(FGB) suitable for transmission in highly overmoded corrugated waveguides. The conventional waveguide mode converters used outside earlier gyrotrons are in the form of rippled-wall or corrugated circular waveguides and serpentine structures. These are devices of high efficiency but large size and narrow bandwidth and thus are not appropriate for implementation in high-power gyrotrons. There are mainly two types of quasi-optical converters that are used instead, namely the Vlasov converter and its modification, the Denisov converter. The Vlasov converter represents a smooth surface circular waveguide, with a cut (stepped, slant or helical one, as shown on figure 2) acting as a launcher, combined with a parabolic reflector that forms the Gaussian wave beam. In general, Vlasov converters are of moderate efficiency (typically 80 %), require large mirrors for higher order modes and are somewhat impractical to be embedded in gyrotrons. The Denisov converter consists of a pre-bunching section (instead of the smooth surface circular waveguide in Vlasov converters) with a helical-cut aperture and up to 4 reflectors (cylindrical, quasi-parabolic and phase correcting ones). The pre-bunching (or field pre-shaping) section represents a rippled-wall (dimpled) waveguide of relatively short length; its role is to transform the gyrotron cavity mode into a bundle of modes that form a Gaussian intensity profile at the aperture, prior to the reflectors. Denisov converters have higher conversion efficiency (about 95 % and more) and smaller mirror size that makes them suitable to built-in in gyrotrons.

The research behavior [1]-[12] at the microwave research laboratory in Electrical and Computer Engineering (ECE), the University of Wisconsin, Madison (UW-Madison) mainly focuses on the mirror system design part (see Figure 1.2) of gyrotron project in the application of Electron Cyclotron Heating (ECH) for plasma fusion. We collaborate with researchers from CPI and Calabazas Creek Research (CCR) to provide beam-shaping mirror system for gyrotrons used in the the DIII-D tokamak at General Atomics (GA). The DIII-D national fusion facility has a microwave ECH system, which is comprised of several gyrotrons operating at 110 GHz for heating the fusion plasma at different positions. The gyrotrons are designed by CPI in the U.S. and Gycom Ltd in Russia. Based on the pioneer work by Rong Cao and Michael P. Perkins, we have developed rigorous theory foundation, new efficient algorithms and novel methods to dramatically improve the beam-shaping mirror system design in high-power gyrotron application.

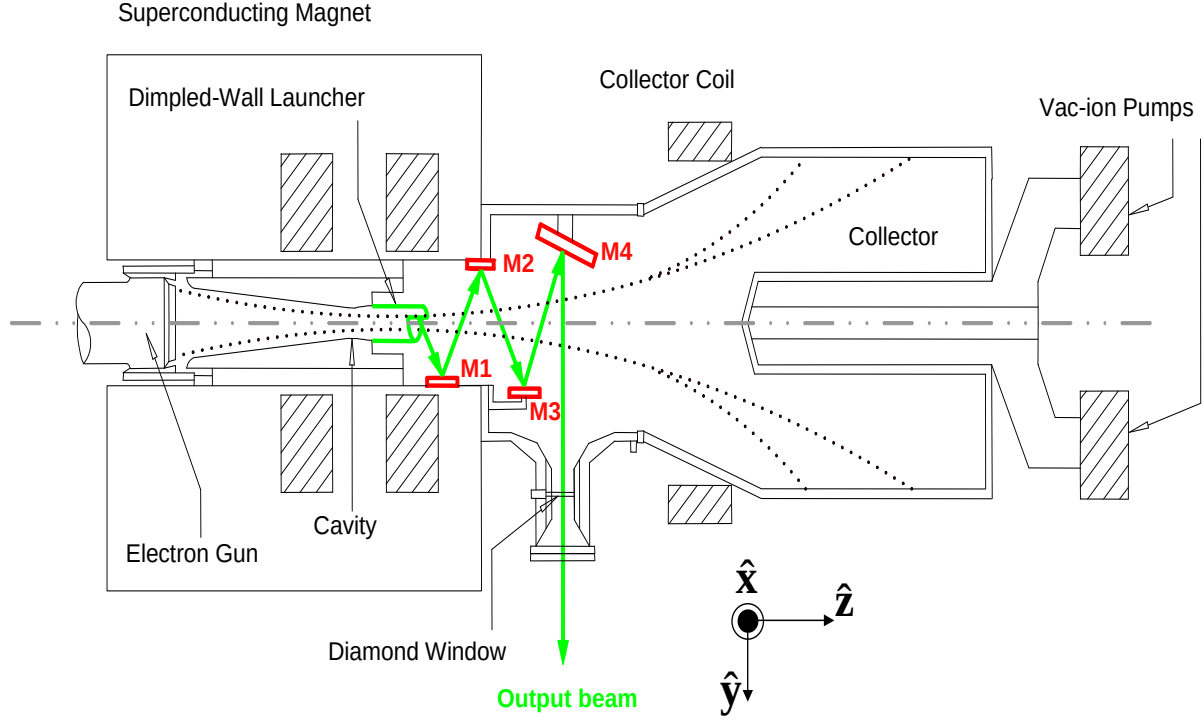


Figure 1.1: The illustration of the goal of a sub-THz beam-shaping 4-mirror system (M_1 , M_2 , M_3 and M_4) in the QO gyrotron: to shape the measured input beam from the dimpled-wall QO launcher into the target FGB to be injected into the corrugated waveguide through the diamond window. Our research mainly focuses on the beam-shaping mirror system, whose detailed schematic setup is also given in Figure 1.2.

1.2 The History of Beam-shaping Mirror Design

From Figure 1.1 and 1.2, it can be seen that the goal of the sub-THz beam-shaping mirror system in the high-power gyrotron is to shape the sub-THz input beam from a dimpled-wall QO launcher into the target Fundamental Gaussian Beam (FGB) to be injected into the corrugated waveguide through the diamond window, as shown in Figure 1.2. The design of a high-quality sub-THz beam-shaping mirror system for the QO gyrotron has been considered as a hard problem for a long time [1], [3], [4], [6]. For example, in the 1.5 MW, 110 GHz gyrotron design at MIT for electron cyclotron resonance plasma heating at DIII-D where

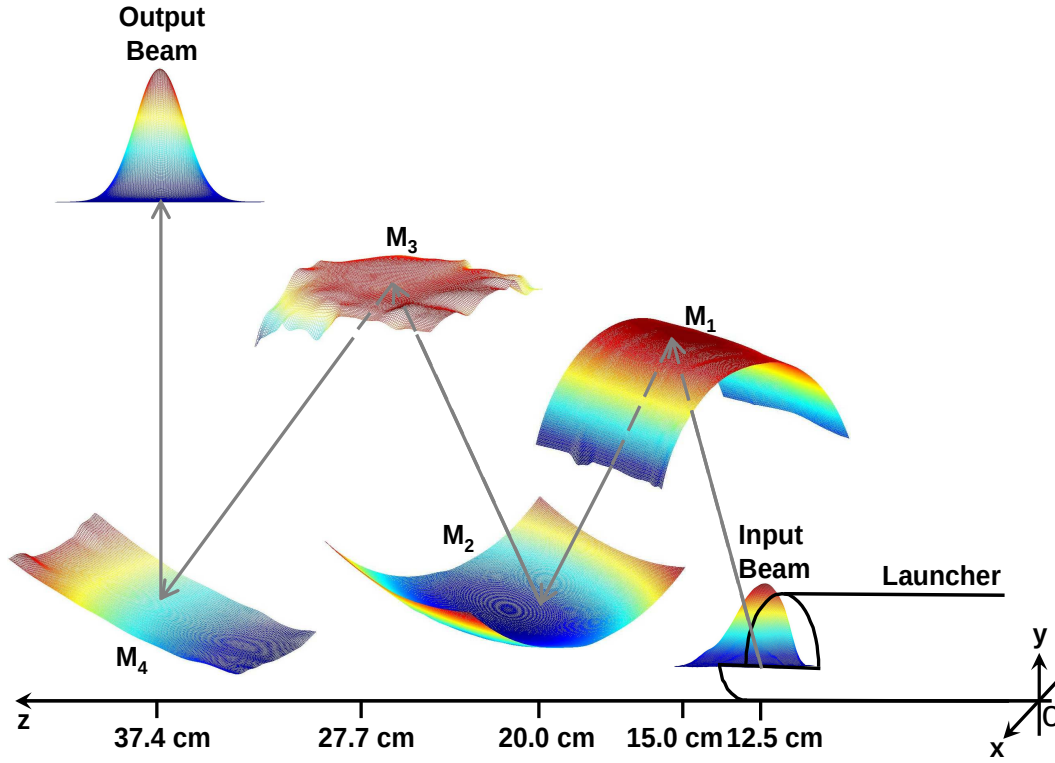


Figure 1.2: The diagram of the beam-shaping mirror system consisting of 4 pieces of PEC mirrors to shape the input beam from the QO launcher into the desired FGB output beam. The approximate z coordinates of the 4 pieces of PEC mirrors have been marked on z axis.

a $TE_{22,6}$ mode cavity is utilized, the internal mode converter of the $TE_{22,6}$ mode to a Gaussian beam consists of an irregular waveguide launcher and four QO mirrors. Rong Cao [1], [4] *et. al.* and Michael P. Perkins *et. al.* [3], [6] used the iterative Geometrical Optics Phase Correction (GOPC) method to correct the 4 pieces of Perfect Electric Conductor (PEC) mirrors adaptively and achieved good designed output beams with $> 99.9\%$ efficiency. Also, Rong *et. al.* [1], [4] have done the illuminating work on the multi-mode mirror system design and obtained an average coupling coefficient of greater than 99.8% between the designed output beam and the target FGB. Researchers in other groups have also made great progress on the beam-shaping mirror system design, either using the iterative phase correction method or the other new scheme.

Chapter 2

SUB-THZ BEAM-SHAPING MIRROR SYSTEM DESIGN

In this chapter, we will outline Iterative Phase Correction Design Procedure for the beam-shaping mirror system design. The detailed explanation of each step of the design procedure will be shown in later chapters.

2.1 The Iterative Design Procedure

Without loss of generality, a sub-THz beam-shaping 2-mirror system has been used to illustrate the design procedure, which has been shown in Figure 2.1, with different steps summarized as: 1) retrieve the phase of the input beam from the measured magnitude patterns on 2 $\sim$ 4 reference planes using the Iterative Phase Retrieval (IPR) technique; 2) compute the forward/backward beam propagations (denoted as $\mathcal{P}^f/\mathcal{P}^b$ in Figure 1.1) onto the PEC mirror surfaces through the *spatial* TI-FFT algorithm; 3) Compute the beam scatterings (denoted as $\mathcal{S}^f/\mathcal{S}^b$ in Figure 1.1) by the PEC mirror surfaces through the *spectral* TI-FFT algorithm; and 4) adaptively correct the PEC mirror surfaces through the iterative GOPC method.

2.1.1 The IPR Phase Retrieval

In the sub-THz regime, it is difficult to measure the phase of the beam accurately; while the measurement of the magnitude pattern is relatively easier [1]-[6]. So it is necessary to retrieve the phase out of the given magnitude patterns. The scheme to do the phase retrieval will be given in Chapter 3.

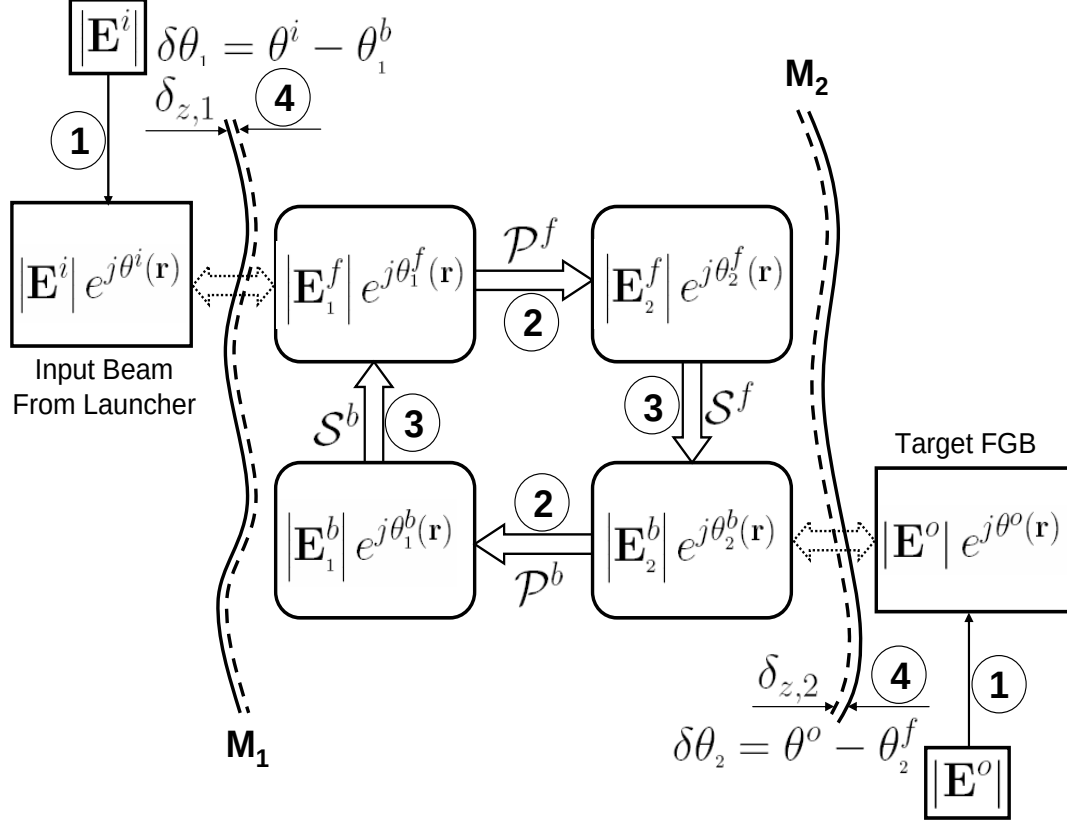


Figure 2.1: Design procedure of a sub-THz beam-shaping 2-mirror system for QO gyrotron application: 1) phase retrieval method; 2) forward/backward beam propagations ($\mathcal{P}^f/\mathcal{P}^b$); 3) beam scatterings ($\mathcal{S}^f/\mathcal{S}^b$) by PEC mirror surfaces; and 4) GOPC method for the PEC mirror surface corrections. The solid and dashed curves denotes the positions of PEC mirror surfaces before and after surface corrections respectively.

2.1.2 The Image Theorem Approximation

The computation of the scattered electromagnetic field requires the knowledge of the equivalent surface current on the PEC surface. The most simple way is to approximate the equivalent surface current using the image theorem like the PO approximation. However, it is well-known that the image theorem only works well for smooth PEC surface; so it is helpful to have some formula to describe the behavior of the image theorem, i.e., in which case the image theorem is valid. We have derive such formula in Chapter 4.

2.1.3 Beam Propagation and Scattering between PEC Mirror Surfaces

From Fig. 2.1, Step 3) and Step 4) require the computation of electromagnetic wave propagation and scattering between M_1 and M_2 and from incident (desired FGB) onto M_1 (M_2). The old algorithm [1], [3], [4], [6] used to do such computation is the direct integration method, which has a computational complexity of $\mathcal{O}(N^4)$. To make the computation fast, we have developed the TI-FFT algorithm for “quasi-planar” and “quasi-cylindrical” surfaces., which will be presented in Chapter 5.

2.1.4 The Phase Correction

The phase correction using PEC mirror surface requires the 2D phase unwrapping of the phase difference between the incident and reflected beams. The old method used to do the 2D phase unwrapping is various path-following algorithm and has a slow computational speed. What’s more, the old method used to correct the PEC surface iteratively is the GOPC method, which is only approximate and has a slow convergency rate. To solve the above problems, we have proposed a fast algorithm based on FFT to unwrap the 2D phase and the PGPC method to correct the PEC surface, which will be explained in details in Chapter 7.

2.1.5 Multi-mode optimization

For multi-mode beam-shaping mirror system design, the optimization scheme is also important to obtain the best performance. Chapter 8 will discuss such optimization scheme based on the minimum total r-norm phase difference.

2.2 Example Pseudocodes for Single-mode Single-frequency Design

The pseudocodes for the design procedure of 4-mirror single-mode single-frequency system are shown below,

(SINGLE-MODE SINGLE-FREQUENCY PROGRAM BEGINS)

Make initial guesses for all 4 mirror surfaces (M_1 , M_2 , M_3 and M_4) and choose the total iteration number N_ℓ (usually $N_\ell = 10 \sim 20$ is required);

Obtain the phase θ^i of the TE_{22,6}/110 GHz input beam through the IPR technique and denote the input beam as $\mathbf{E}^i = \mathbf{E}^i e^{j\theta^i}$;

For $\ell = 1$ to N_ℓ (iteration loop begins)

Forward-propagate the input beam $\mathbf{E}^i$ onto M_1 to obtain $(|\mathbf{E}_1^{f,(\ell)}|, \theta_1^{f,(\ell)})$;

Forward-propagate $\mathbf{E}_1^{f,(\ell)}$ onto M_2 to obtain $(|\mathbf{E}_2^{f,(\ell)}|, \theta_2^{f,(\ell)})$;

Forward-propagate $\mathbf{E}_2^{f,(\ell)}$ onto M_3 to obtain $(|\mathbf{E}_3^{f,(\ell)}|, \theta_3^{f,(\ell)})$;

Forward-propagate $\mathbf{E}_3^{f,(\ell)}$ onto M_4 to obtain $(|\mathbf{E}_4^{f,(\ell)}|, \theta_4^{f,(\ell)})$;

Back-propagate the target FGB onto M_4 to obtain $(|\mathbf{E}_4^{b,(\ell)}|, \theta_4^{b,(\ell)})$;

Correct M_4 surface using the GOPC method through $(\delta\theta_4 = \theta_4^{f,(\ell)} - \theta_4^{b,(\ell)})$;

Back-propagate the target FGB onto M_3 and update $(|\mathbf{E}_3^{b,(\ell)}|, \theta_3^{b,(\ell)})$;

Correct M_3 surface using the GOPC method through $(\delta\theta_3 = \theta_3^{f,(\ell)} - \theta_3^{b,(\ell)})$;

Back-propagate the target FGB onto M_2 and update $(|\mathbf{E}_2^{b,(\ell)}|, \theta_2^{b,(\ell)})$;

Correct M_2 surface using the GOPC method through $(\delta\theta_2 = \theta_2^{f,(\ell)} - \theta_2^{b,(\ell)})$;

Back-propagate the target FGB onto M_1 and update $(|\mathbf{E}_1^{b,(\ell)}|, \theta_1^{b,(\ell)})$;

Correct M_1 surface using the GOPC method through $(\delta\theta_1 = \theta_1^{f,(\ell)} - \theta_1^{b,(\ell)})$;

Forward-propagate $\mathbf{E}^i$ (input) onto diamond window to obtain $\mathbf{E}^{o,(\ell)}$ (output);

Exit iteration loop when the criterion $\chi^{(\ell)}$ defined in (2.1) is met or $\ell = N_\ell$.

$\ell = \ell + 1$ (iteration loop continues)

(SINGLE-MODE SINGLE-FREQUENCY PROGRAM ENDS)

The convergence criterion defined on the diamond window for the ℓ^{th} iteration of the sub-THz beam-shaping mirror system design is given as [1]-[6],

$$\chi^{(\ell)} \equiv \left| \frac{\iint dS' \mathbf{E}^{o,(\ell)} [\mathbf{E}^{o,(\ell-1)}]^*}{\sqrt{\iint dS' |\mathbf{E}^{o,(\ell)}|^2} \sqrt{\iint dS' |\mathbf{E}^{o,(\ell-1)}|^2}} \right|_{\text{(on diamond window)}} \quad (2.1)$$

Pseudocodes for the multi-mode multi-frequency design is similar and will be shown in Chapter 9: Computer Simulation.

Chapter 3

THE IPR PHASE RETRIEVAL

As has been said in Section 2.1.1, the phase of the initial incident beam has to be known during the beam-shaping mirror system design. Here we will review the conventional IPR technique and we will propose the more rigorous IPR technique based on the probe compensation later in Chapter 10.2. The conventional IPR technique [13] can be summarized as below,

- 1) measure the magnitude patterns of the electric fields on several reference planes (denoted as $|\mathbf{E}_p|$, $p = 1, 2, 3 \dots$) that are far away from each other to ensure the convergence of the IPR technique;
- 2) assign some initial phase $\theta_1^{f(1)}$ (usually set $\theta_1^{f(1)} = 0$) for the electric field on the 1st plane and express the new electric field as $\tilde{\mathbf{E}}_1 = |\mathbf{E}_1| e^{j\theta_1^{f(1)}}$;
- 3) iteration loop begins ($\ell = 1$): forward-propagate $\tilde{\mathbf{E}}_1$ onto the 2nd reference plane to obtain the electric field $(|\tilde{\mathbf{E}}_2|, \theta_2^{f(\ell)})$ through the Fast Fourier Transform/Inverse Fast Fourier Transform (FFT/IFFT);
- 4) update the electric field on the 2nd reference plane by using the new phase $\theta_2^{f(\ell)}$ and replacing the magnitude $|\tilde{\mathbf{E}}_2|$ with the measurement data $|\mathbf{E}_2|$, which is expressed as $\tilde{\mathbf{E}}_2 = |\mathbf{E}_2| e^{j\theta_2^{f(\ell)}}$;
- 5) continue to forward-propagate $\tilde{\mathbf{E}}_2$ onto other reference planes and update the phases, similar to step 3) and step 4) given above;
- 6) back-propagate the electric field onto all reference planes and update the phases;
- 7) iteration loop ends when the program converges;
- 8) iteration loop continues: $\ell = \ell + 1$ and go to step 3).

The above procedure has also been illustrated in Figure 3.1 for 2 reference planes. Usually, 2 $\sim$ 4 reference planes is enough for the IPR technique to converge. However, it has been shown that the IPR technique works better for

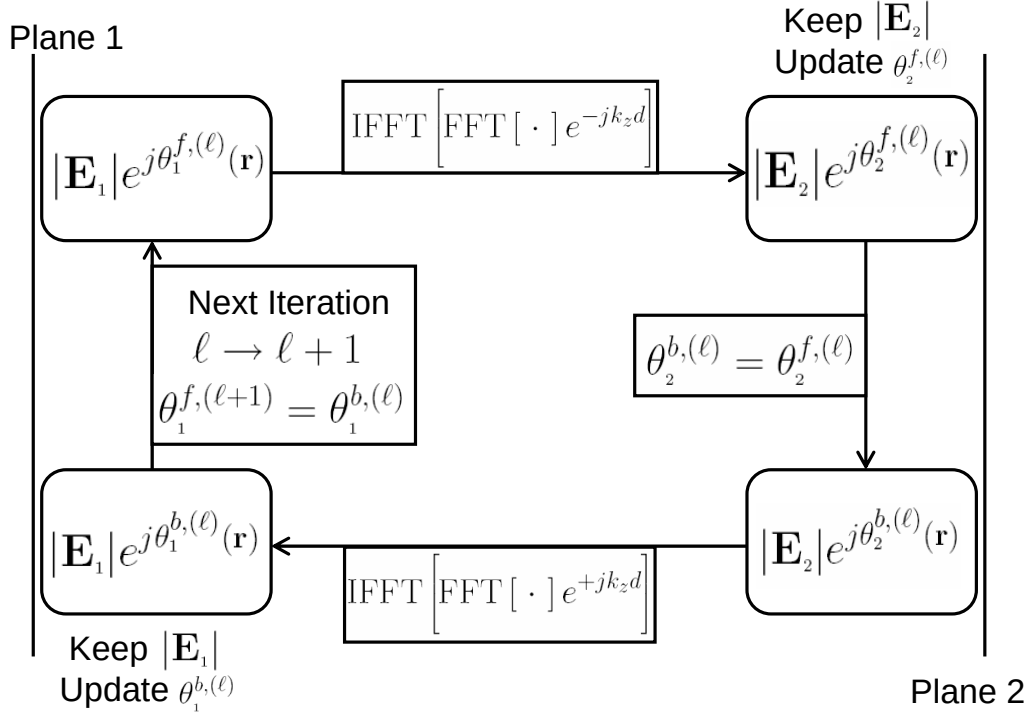


Figure 3.1: IPR technique for 2 reference planes (Step 1 in Figure 2.1): keep the magnitude on each plane and update the phase after each forward/backward beam propagation through the FFT/IFFT. d is the separation distance between the 2 reference planes.

≥ 3 reference planes. As an example, in a test of a beam pattern consisting of 5 FBGs with different offsets and tilting angles, the IPR technique converges after 20 iterations for 3 reference planes; while it doesn't converge until 500 iterations for 2 reference planes [3]. Finally, it should be pointed out that the IPR technique also works for cylindrical reference surfaces but it only works well for beams without significant side lobes.

Chapter 4

THE VALIDITY OF IMAGE THEOREMS

The image theorems have been frequently used as approximate solutions for Love's equivalence theorem in diffraction and radiation phenomena [14], [15], [16]. There are also many scattering problems, where the induction theorem and Physical Optics (PO) approximation have been extensively adopted in evaluations of Radar Cross Section (RCS) [17], [18], [19], [20], [21]. Recently, image theorems also find their applications in mirror system designs for high-power quasi-optical gyrotrons [1]-[6], [10]. However, it is known that image theorems only work well for smooth surfaces with large radii (small curvatures), where equivalent surface currents $\mathbf{M}_s$ ($\mathbf{J}_s$) can be approximated as the product of incident fields $\mathbf{E}^i$ ($\mathbf{H}^i$) and the unit surface normal $\hat{\mathbf{n}}^+$: $\mathbf{M}_s = 2\mathbf{E}^i \times \hat{\mathbf{n}}^+$ ($\mathbf{J}_s = 2\hat{\mathbf{n}}^+ \times \mathbf{H}^i$) on fictitious surfaces [14], [16], [22], [23]. What's more, although exact solutions of scattering problems for cylindrical and spherical geometries have been found [22], [24], [25], no rigorous theoretical criterions have been reported so far to evaluate the validity of image theorems. So, it is helpful to derive such criterions in their closed-form expressions in order to use image theorems efficiently.

The derivation of theoretical criterions of image theorems can be done through the following steps: 1) specify some arbitrary initial incident fields in forms of TE and TM cylindrical (spherical) modal expansions on the initial cylindrical (spherical) surfaces [25], [26]; 2) apply image theorems (induction theorem or PO approximation) to get $\mathbf{M}_s$ or $\mathbf{J}_s$; 3) calculate the forward-propagating (back-scattered) fields exactly on the initial cylindrical (spherical) surfaces through either the vector potential method or the dyadic Green's function method; And 4) compare cylindrical (spherical) modal expansion coefficients of the calculated forward-propagating (back-scattered) fields with those of the initial incident fields

specified on the initial cylindrical (spherical) surfaces, which leads to the theoretical criterions of image theorems.

4.1 Image Theorems in the Cylindrical Geometry

The scheme used to illustrate image theorems for the cylindrical geometry is shown in Fig. 4.1, where the incident field $\mathbf{E}^i$ propagates onto cylindrical surface S' , after which $\mathbf{E}^i$ may continue to propagate to $\mathbf{E}^+$ or it could be back-scattered to $\mathbf{E}^-$, depending on whether surface S' serves as a Perfect Electric Conductor (PEC) scatter or as a fictitious surface where the Love's equivalence theorem applies.

4.1.1 The Vector Potential method

The cylindrical modal expansion

The cylindrical modal expansion of vector potential $\mathbf{F}(\mathbf{r})$ for the magnetic current $\mathbf{M}_s(\mathbf{r}')$ on an arbitrary surface can be derived through the expansion of the scalar Green's function $g(\mathbf{r} - \mathbf{r}')$ in the cylindrical coordinate,

$$\begin{aligned}\mathbf{F}(\mathbf{r}) &= \epsilon_0 \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') g(\mathbf{r} - \mathbf{r}') \\ &= \frac{-j\epsilon_0}{8\pi} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') \int_{-\infty}^{\infty} dk_z H_0^{(2)}(\Lambda[\rho - \rho']) e^{-jk_z(z-z')}\end{aligned}\quad (4.1)$$

where, ϵ_0 is permittivity of free space and $H_0^{(2)}(\cdot)$ is Hankel function of the second kind of order 0. Λ and the scalar Green's function $g(\cdot)$ are defined as,

$$\Lambda = \sqrt{k^2 - k_z^2}; \quad g(\cdot) = \frac{e^{-jk|\cdot|}}{4\pi|\cdot|}. \quad (4.2)$$

According to the cylindrical addition theorem [22], [26],

$$H_0^{(2)}(\Lambda[\rho - \rho']) = \sum_{m=-\infty}^{\infty} J_m(\Lambda\rho') H_m^{(2)}(\Lambda\rho) e^{im(\phi' - \phi)} \quad (4.3)$$

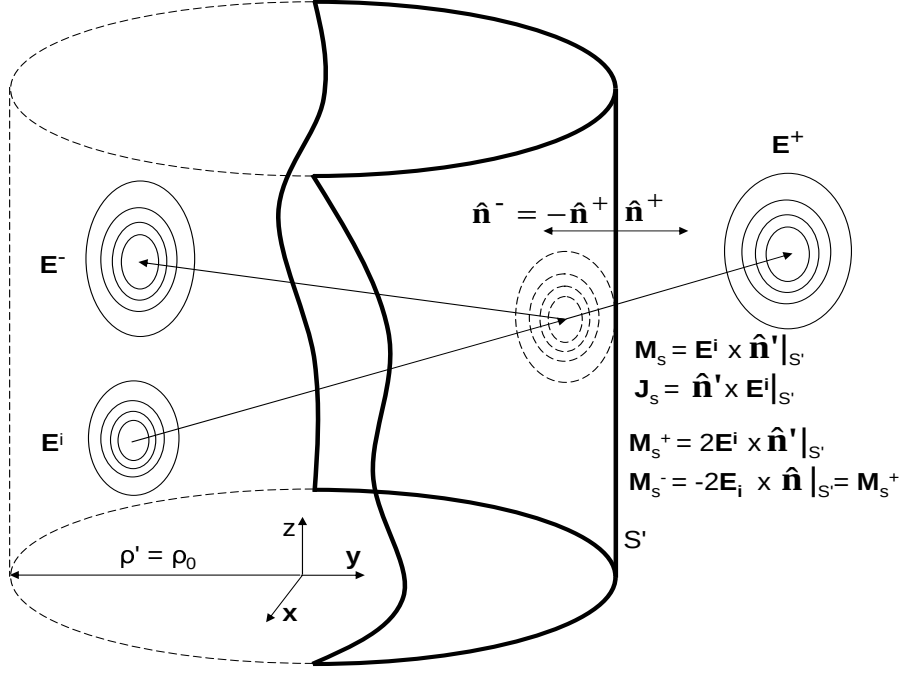


Figure 4.1: The cylindrical geometry: the incident field $\mathbf{E}^i$ propagates onto cylindrical surface S' , then it may forward-propagates to $\mathbf{E}^+$ or it could be back-scattered to $\mathbf{E}^-$, depending on whether surface S' serves as a PEC scatter or as a fictitious surface where the equivalence theorem applies. $\hat{\mathbf{n}}^+$ and $\hat{\mathbf{n}}^-$ are the outward and inward surface normal on cylindrical surface S' respectively. $\mathbf{M}_s$ and $\mathbf{J}_s$ are equivalent surface currents for Love's equivalence theorem. $\mathbf{M}_s^+$ is the image approximation of Love's theorem and $\mathbf{M}_s^-$ is the image approximation for the induction theorem.

where, $J_m(\cdot)$ is Bessel function of the first kind of integral order m . Substituting (4.3) into (4.1), the cylindrical modal expansion of $\mathbf{F}(\mathbf{r})$ on an arbitrary surface is obtained,

$$\begin{aligned} \mathbf{F}(\mathbf{r}) &= \text{IFT} \left\{ \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s} H_m^{(2)}(\Lambda \rho) \right\} \\ \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s} &= \frac{-j\epsilon_0}{4} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') J_m(\Lambda \rho') e^{im\phi'} e^{jk_z z'} \end{aligned} \quad (4.4)$$

with the Inverse Fourier Transform (IFT) defined as,

$$\text{IFT} \{ \cdot \} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \{ \cdot \} e^{-im\phi} e^{-jk_z z}. \quad (4.5)$$

The validity of (4.4) can be further confirmed by the Near-Field Far-Field (NF-FF) transform. The Far-Field $\mathbf{F}(\mathbf{r})|_{\text{FF}}$ can be obtained from (4.4) by letting $r \equiv |\mathbf{r}| \rightarrow \infty$,

$$\mathbf{F}(\mathbf{r})|_{\text{FF}} = \text{IFT} \left\{ \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s} H_m^{(2)}(\Lambda \rho) \right\} \Big|_{r \rightarrow \infty} \quad (4.6)$$

Also, for $r \rightarrow \infty$, the following relation holds [22],

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \left\{ \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s} H_m^{(2)}(\Lambda \rho) e^{-jk_z z} \right\} \Big|_{r \rightarrow \infty} \\ = \frac{1}{\pi} \frac{e^{-jkR}}{R} j^{m+1} \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s} \end{aligned} \quad (4.7)$$

Substituting (4.7) into (4.6),

$$\mathbf{F}(\mathbf{r})|_{\text{FF}} = \frac{1}{\pi} \frac{e^{-jkR}}{R} \sum_{-\infty}^{\infty} j^{m+1} \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s} e^{-im\phi}. \quad (4.8)$$

$\mathbf{F}(\mathbf{r})|_{\text{FF}}$ in (4.8) is the NF-FF transform in the cylindrical coordinate [27], [28], which can also be obtained from (4.1) by letting $r \rightarrow \infty$,

$$\mathbf{F}(\mathbf{r})|_{\text{FF}} = \epsilon_0 \frac{e^{-jkr}}{4\pi r} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') e^{j\Lambda \rho' \cos(\phi - \phi')} e^{jk_z z'} \quad (4.9)$$

Now, express $e^{j\Lambda \rho' \cos(\phi - \phi')}$ in a Fourier series [29],

$$e^{j\Lambda \rho' \cos(\phi - \phi')} = \sum_{m=-\infty}^{\infty} j^m J_m(\Lambda \rho') e^{-jm(\phi - \phi')}. \quad (4.10)$$

The substitution of (4.10) into (4.9) also gives (4.8).

Finally, the vector potential $\mathbf{A}(\mathbf{r})$ for the electric current $\mathbf{J}_s(\mathbf{r}')$ can be obtained through duality as,

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \text{IFT} \left\{ \mathbf{g}_m^{k_z}(\mathbf{J}_s) H_m^{(2)}(\Lambda \rho) \right\} \\ \mathbf{g}_{m, \text{TE}}^{k_z, \mathbf{J}_s} &= \frac{-j\mu_0}{4} \iint_{S'} dS' \mathbf{J}_s(\mathbf{r}') J_m(\Lambda \rho') e^{im\phi'} e^{jk_z z'}. \end{aligned} \quad (4.11)$$

The back-scattered and forward-propagating waves

In Fig. 4.1, $\mathbf{M}_s^-$ is the image approximation of the Love's equivalence theorem for the forward-propagating wave and $\mathbf{M}_s^+$ is the induction theorem approximation for the back-scattered wave. It is not difficult to show that they are equal,

$$\mathbf{M}_s^+ = 2(-\hat{\mathbf{n}}^+) \times \mathbf{E}_s^i = 2\hat{\mathbf{n}}^- \times \mathbf{E}^i = \mathbf{M}_s^- . \quad (4.12)$$

where $\hat{\mathbf{n}}^+$ ($\hat{\mathbf{n}}^-$) are the outward (inward) unit surface normal on cylindrical surface S' . The total field calculated from the equivalent current $\mathbf{M}_s^+(\mathbf{r}')$ or $\mathbf{M}_s^-(\mathbf{r}')$ is thus the combination of the forward-propagating and back-scattered fields. Mathematically, (4.4) can be separated into two parts as,

$$\begin{aligned} J_m(\Lambda\rho') &= \frac{1}{2} \left\{ H_m^{(1)}(\Lambda\rho') + H_m^{(2)}(\Lambda\rho') \right\} \\ \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s^+} &= \frac{-j\epsilon_0}{8} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') H_m^{(1)}(\Lambda\rho') e^{im\phi'} e^{jk_z z'} \\ \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s^-} &= \frac{-j\epsilon_0}{8} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') H_m^{(2)}(\Lambda\rho') e^{im\phi'} e^{jk_z z'} . \end{aligned} \quad (4.13)$$

Now, suppose the initial incident field $\mathbf{E}(\mathbf{r}')$ on the initial cylindrical surface S' with radius of ρ_0 is given as the combination of TE and TM modes [22], [24], [25],

$$\begin{aligned} \mathbf{E}(\rho_0) &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \left\{ a_{m,o}^{k_z,e} \mathbf{M}_{m,o}^{k_z,e+}(\mathbf{r}') + b_{m,o}^{k_z,e} \mathbf{N}_{m,o}^{k_z,e+}(\mathbf{r}') \right\} \\ \psi_{m,o}^{k_z,e+}(\mathbf{r}') &= H_m^{(2)}(\Lambda\rho_0) e^{-jk_z z'} e^{-jm\phi'} \\ \mathbf{L}_{m,o}^{k_z,e+}(\mathbf{r}') &= \nabla \psi_{m,o}^{k_z,e+}(\mathbf{r}') \\ \mathbf{M}_{m,o}^{k_z,e+}(\mathbf{r}') &= \nabla \times \left\{ \mathbf{a}_z \psi_{m,o}^{k_z,e+}(\mathbf{r}') \right\} \\ \mathbf{N}_{m,o}^{k_z,e+}(\mathbf{r}') &= \frac{1}{k} \nabla \times \mathbf{M}_{m,o}^{k_z,e+}(\mathbf{r}') . \end{aligned} \quad (4.14)$$

Under the image theorem approximation, from (4.4) and (4.12), the approximate field $\tilde{\mathbf{E}}(\mathbf{r})$ can be obtained as,

$$\tilde{\mathbf{E}}(\mathbf{r}) = -\frac{1}{2\pi\epsilon_0} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \left\{ \mathbf{L}_{m,o}^{k_z,e+}(\mathbf{r}) \times \mathbf{f}_{m, \text{TE}}^{k_z, \mathbf{M}_s} \right\} \quad (4.15)$$

The field $\tilde{\mathbf{E}}(\rho_0)$ on the initial cylindrical surface S' is evaluated through (4.15) and the result is [10],

$$\tilde{\mathbf{E}}(\rho_0) = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} dk_z \left\{ \tilde{a}_{m,o}^{k_z,e} \mathbf{M}_{n,o}^{k_z,e+}(\mathbf{r}_0) + \tilde{b}_{m,o}^{k_z,e} \mathbf{N}_{m,o}^{k_z,e+}(\mathbf{r}_0) \right\} \quad (4.16)$$

$$\begin{aligned} \xi_{\text{TE}}^{\mathbf{M}_s} &\equiv \frac{\tilde{a}_{m,o}^{k_z,e}}{a_{m,o}^{k_z,e}} = j\pi\rho_0 J_m(\Lambda\rho_0) \frac{\partial H_m^{(2)}(\Lambda\rho)}{\partial \rho} \Big|_{\rho=\rho_0} \\ \xi_{\text{TM}}^{\mathbf{M}_s} &\equiv \frac{\tilde{b}_{m,o}^{k_z,e}}{b_{m,o}^{k_z,e}} = -j\pi\rho_0 H_m^{(2)}(\Lambda\rho_0) \frac{\partial J_m(\Lambda\rho)}{\partial \rho} \Big|_{\rho=\rho_0} \end{aligned}$$

Comparing the calculated approximate field $\tilde{\mathbf{E}}(\rho_0)$ in (4.16) and the initial incident field $\mathbf{E}(\rho_0)$ in (4.14), it is not difficult to show that the discrepancy $\xi_{\text{TE,TM}}(\mathbf{M}_s)$ in (4.16) is due to the approximation of the image theorem. Similar to (4.13), $\xi_{\text{TE,TM}}(\mathbf{M}_s)$ can also be separated into $\xi_{\text{TE,TM}}^+(\mathbf{M}_s)$ and $\xi_{\text{TE,TM}}^-(\mathbf{M}_s)$ as,

$$\begin{aligned} \xi_{\text{TE}}^{\mathbf{M}_s,\pm} &= \frac{j\pi\rho_0}{2} H_m^{(1),(2)}(\Lambda\rho) \frac{\partial H_m^{(2)}(\Lambda\rho_0)}{\partial \rho} \Big|_{\rho=\rho_0} \\ \xi_{\text{TM}}^{\mathbf{M}_s,\pm} &= -\frac{j\pi\rho_0}{2} H_m^{(2)}(\Lambda\rho_0) \frac{\partial H_m^{(1),(2)}(\Lambda\rho)}{\partial \rho} \Big|_{\rho=\rho_0} . \end{aligned} \quad (4.17)$$

Finally, it can be shown that the following relations hold for $\mathbf{M}_s$ and $\mathbf{J}_s$ image approximations,

$$\xi_{\text{TE}}^{\mathbf{J}_s} = \xi_{\text{TM}}^{\mathbf{M}_s}, \quad \xi_{\text{TM}}^{\mathbf{J}_s} = \xi_{\text{TE}}^{\mathbf{M}_s} \quad (4.18)$$

$$\xi_{\text{TE}}^{\mathbf{M}_s,\pm} = \xi_{\text{TM}}^{\mathbf{J}_s,\pm} = [\xi_{\text{TE}}^{\mathbf{J}_s,\pm}]^* = [\xi_{\text{TM}}^{\mathbf{M}_s,\pm}]^* .$$

4.1.2 The Dyadic Green's Function Method

The magnetic dyadic Green's function in the cylindrical coordinate [24], [25] is given as,

$$\begin{aligned} \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') &= -\frac{\mathbf{a}_\rho \mathbf{a}_\rho}{k^2} \delta(\rho - \rho') + \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{j8\pi\Lambda^2} \\ &\times \left\{ [\mathbf{M}_{m,o}^{k_z,e}(\mathbf{r}')]^* \mathbf{M}_{m,o}^{k_z,e+}(\mathbf{r}) + [\mathbf{N}_{m,o}^{k_z,e}(\mathbf{r}')]^* \mathbf{N}_{m,o}^{k_z,e+}(\mathbf{r}) \right\} dk_z \end{aligned} \quad (4.19)$$

where $\mathbf{M}_{m,o}^{k_z,e}$ ($\mathbf{N}_{m,o}^{k_z,e}$) is obtained by replacing $H_m^{(2)}$ with J_m in $\mathbf{M}_{m,o}^{k_z,e+}$ ($\mathbf{N}_{m,o}^{k_z,e+}$). The approximate field $\tilde{\mathbf{E}}(\mathbf{r})$ is thus obtained from (4.12) as,

$$\tilde{\mathbf{E}}(\mathbf{r}) = -\nabla \times \int_{S'} dS' \mathbf{M}_s^+(\mathbf{r}') \cdot \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') \quad (4.20)$$

Substituting (4.19) into (4.20) and applying orthogonal properties of cylindrical modal functions on the cylindrical surface, the following expression is obtained,

$$\begin{aligned} \tilde{\mathbf{E}}(\rho_0) &= \frac{jk}{4\pi\Lambda^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \frac{a_{m,o}^{k_z,e} \mathbf{M}_{m,o}^{k_z,e+}(\mathbf{r})}{b_{m,o}^{k_z,e} \mathbf{N}_{m,o}^{k_z,e+}(\mathbf{r})} \\ &\times \int_{S'} dS' \frac{[\mathbf{N}_{m,o}^{k_z,e}(\mathbf{r}')]^* \times \mathbf{M}_{m,o}^{k_z,e+}(\mathbf{r}')}{[\mathbf{M}_{m,o}^{k_z,e}(\mathbf{r}')]^* \times \mathbf{N}_{m,o}^{k_z,e+}(\mathbf{r}')} \cdot \mathbf{a}_{\rho'} . \end{aligned} \quad (4.21)$$

The evaluation of (4.21) also gives the same results as in (4.16) for $\mathbf{M}_s$ image approximation, after the separation of back-scattered and forward-propagating waves in (4.13). Similar arguments hold for $\mathbf{J}_s$ image approximation.

4.1.3 Image Theorem Criteria in the Cylindrical Geometry

$\xi_{\text{TE,TM}}^{\mathbf{M}_s, \mathbf{J}_s+}$ in (4.17) and (4.18) can be regarded as criteria of image theorems. For $\rho_0 \rightarrow \infty$, it is not difficult to see that $\xi_{\text{TE,TM}}^{\mathbf{M}_s+}$ for $\mathbf{M}_s$ approximation and $\xi_{\text{TE,TM}}^{\mathbf{J}_s+}$ for $\mathbf{J}_s$ approximation give the same accuracy by noting that,

$$\begin{aligned} H_m^{(2)}(\Lambda\rho_0) &= H_m^{(1)*}(\Lambda\rho_0) \sim \sqrt{\frac{2j}{\pi\Lambda\rho_0}} j^m e^{-j\Lambda\rho_0}, \quad \Lambda\rho_0 \rightarrow \infty \\ \xi_{\text{TE,TM}}^{\mathbf{M}_s, \mathbf{J}_s+} \big|_{\rho_0 \rightarrow \infty} &= 1 \end{aligned} \quad (4.22)$$

The combination of (4.16) and (4.22) implies the fact that $\tilde{\mathbf{E}}(\rho_0) \rightarrow \mathbf{E}(\rho_0)$ for $\rho_0 \rightarrow \infty$ for narrow-band fields.

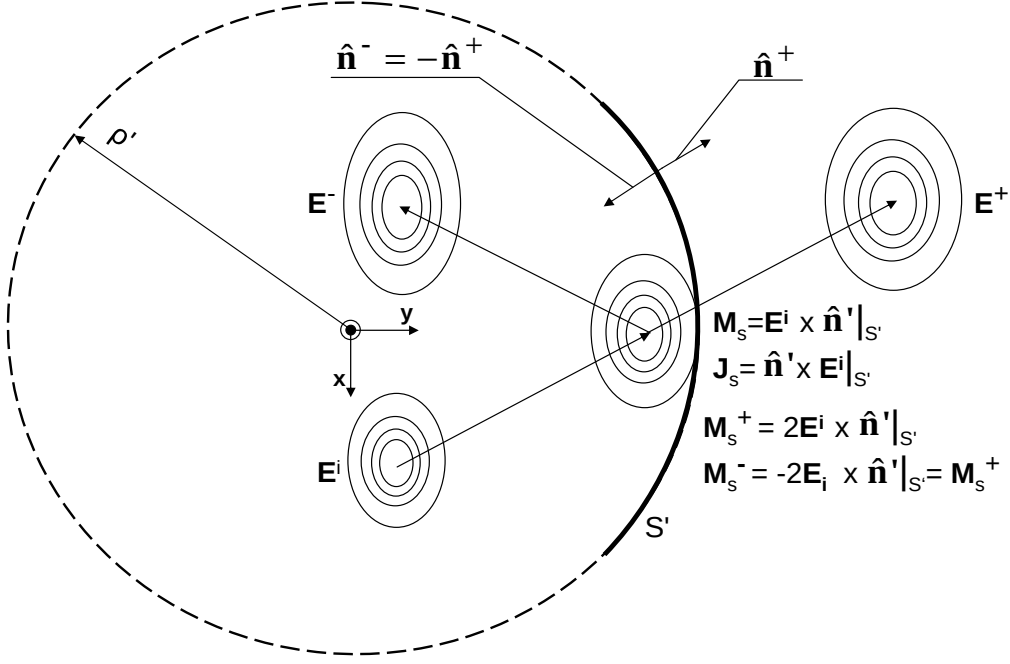


Figure 4.2: The spherical geometry: similar to the cylindrical geometry in Fig. 4.1, the incident field $\mathbf{E}^i$ propagates onto spherical surface S' , then it may forward-propagates to $\mathbf{E}^+$ or it could be back-scattered to $\mathbf{E}^-$, depending on whether surface S' serves as a PEC scatter or as a fictitious surface where the equivalence theorem applies. $\hat{\mathbf{n}}^+$ and $\hat{\mathbf{n}}^-$ are the outward and inward surface normal on spherical surface S' respectively. $\mathbf{M}_s$ and $\mathbf{J}_s$ are equivalent surface currents for Love's equivalence theorem. $\mathbf{M}_s^+$ is the image approximation of Love's theorem and $\mathbf{M}_s^-$ is the image approximation for the induction theorem.

4.2 Image Theorems in the Spherical Geometry

Fig. 4.2 shows the scheme in the spherical geometry with similar explanations as in the cylindrical geometry (Fig. 4.1).

4.2.1 The Vector Potential method

The spherical modal expansion

In the spherical coordinate, the vector potential $\mathbf{F}(\mathbf{r})$ for $\mathbf{M}_s(\mathbf{r}')$ is given as

[22], [26],

$$\begin{aligned}\mathbf{F}(\mathbf{r}) &= \epsilon_0 \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') g(\mathbf{r} - \mathbf{r}') \\ &= \frac{-jk\epsilon_0}{4\pi} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') h_0^{(2)}(k[\rho - \rho'])\end{aligned}\quad (4.23)$$

where, $h_0^{(2)}$ is spherical Hankel function of the second kind of order 0. According to spherical addition theorem [22], [26],

$$\begin{aligned}h_0^{(2)}(k[\rho - \rho']) &= \sum_{n=0}^{\infty} (2n+1) j_n(kr') h_n^{(2)}(kr) \\ &\times \sum_{m=0}^n (2 - \delta_m^0) \frac{(n-m)!}{(n+m)!} P_n^m(\theta') P_n^m(\theta) \cos m(\phi - \phi')\end{aligned}\quad (4.24)$$

where, j_n is the spherical bessel function of the first kind of integral order n and δ_m^0 is the Kronecker delta function ($\delta_m^0 = 1$ for $m=0$ and $\delta_m^0 = 0$ for $m \neq 0$). Substituting (4.24) into (4.23), the modal expansion of $\mathbf{F}(r)$ is obtained as,

$$\begin{aligned}\mathbf{F}(\mathbf{r}) &= \sum_{n=0}^{\infty} \sum_{m=0}^n \mathbf{f}_{\text{TE}}^{\mathbf{M}_s}(n, m) h_n^{(2)}(kr) P_n^m(\theta) \frac{\cos m\phi}{\sin m\phi} \\ \mathbf{f}_{n, \text{TE}}^{m, \mathbf{M}_s} &= \chi \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') j_n(kr') P_n^m(\theta') \frac{\cos m\phi'}{\sin m\phi'} \\ \chi &= (2 - \delta_m^0) \frac{-jk\epsilon_0}{4\pi} \frac{(2n+1)(n-m)!}{(n+m)!}.\end{aligned}\quad (4.25)$$

The NF-FF transform of (4.29) in the spherical coordinate is given as [27],

$$\mathbf{F}(\mathbf{r}) \Big|_{r \rightarrow \infty} = \frac{je^{-jkr}}{kr} \sum_{n=0}^{\infty} \sum_{m=0}^n j^n \mathbf{f}_{\text{TE}}^{\mathbf{M}_s}(n, m) P_n^m(\theta) \frac{\cos m\phi}{\sin m\phi}\quad (4.26)$$

Similar to the cylindrical geometry, (4.26) can also be derived from (4.23) by letting $r \rightarrow \infty$,

$$\mathbf{F}(\mathbf{r}) \Big|_{r \rightarrow \infty} = \epsilon_0 \frac{e^{-jkr}}{4\pi r} \iint_{S'} dS' \mathbf{M}_s(r') e^{jkr' \cos \gamma'}\quad (4.27)$$

where γ' is the angle between $\mathbf{k}$ and $\mathbf{r}'$. Also, $e^{jkr' \cos \gamma'}$ can be expressed in the plane wave expansion [29],

$$e^{jkr' \cos \gamma'} = \sum_{n=0}^{\infty} \sum_{m=0}^n j^n (2 - \delta_m^0) \frac{(2n+1)(n-m)!}{(n+m)!} \frac{\cos m\phi}{\sin m\phi} \\ \times j_n(kr') P_n^m(\cos \theta') P_n^m(\cos \theta) \frac{\cos m\phi'}{\sin m\phi'}. \quad (4.28)$$

The substitution of (4.28) into (4.27) also gives (4.26).

Similar to the cylindrical geometry, duality relations can be used to obtain $\mathbf{A}(\mathbf{r})$ for the $\mathbf{J}_s$ approximation as,

$$\mathbf{A}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=0}^n \mathbf{g}_{\text{TE}}^{M_s}(n, m) h_n^{(2)}(kr) P_n^m(\theta) \frac{\cos m\phi}{\sin m\phi} \\ \mathbf{g}_{n, \text{TE}}^{m, M_s} = \chi \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') j_n(kr') P_n^m(\theta') \frac{\cos m\phi'}{\sin m\phi'} \\ \chi = (2 - \delta_m^0) \frac{-jk\mu_0}{4\pi} \frac{(2n+1)(n-m)!}{(n+m)!}. \quad (4.29)$$

The back-scattered and forward-propagating waves

Similar to the cylindrical geometry, we can separate (4.29) into back-scattered and forward-propagating waves as,

$$j_m(\Lambda\rho') = \frac{1}{2} \left\{ h_n^{(1)}(\Lambda\rho') + h_n^{(2)}(\Lambda\rho') \right\} \quad (4.30)$$

$$\mathbf{f}_{n, \text{TE}}^{m, M_s+} = \frac{\chi}{2} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') h_n^{(1)}(kr') P_n^m(\theta') \frac{\cos m\phi'}{\sin m\phi'} \\ \mathbf{f}_{n, \text{TE}}^{m, M_s-} = \frac{\chi}{2} \iint_{S'} dS' \mathbf{M}_s(\mathbf{r}') h_n^{(2)}(kr') P_n^m(\theta') \frac{\cos m\phi'}{\sin m\phi'}$$

Suppose the initial incident electric field $\mathbf{E}(\mathbf{r}')$ on the initial spherical surface S' with radius of r_0 (in Fig. 4.2) is given as,

$$\mathbf{E}(r_0) = \sum_{n=0}^{\infty} \sum_{m=0}^n c_{n,o}^{m,e} \mathbf{M}_{n,o}^{m,e+}(\mathbf{r}') + d_{n,o}^{m,e} \mathbf{N}_{n,o}^{m,e+}(\mathbf{r}')$$

Table 4.1: Summary of $\xi_{\text{TE, TM}}^{+,-}(\mathbf{m}_s, \mathbf{j}_s)$ and $\zeta_{\text{TE, TM}}^{+,-}(\mathbf{m}_s, \mathbf{j}_s)$ for the cylindrical and spherical cases

TE/TM modes and $\mathbf{M}_s/\mathbf{J}_s$	The relations	$\xi_{\text{TE, TM}}^+(\mathbf{M}_s, \mathbf{J}_s) / \zeta_{\text{TE, TM}}^+(\mathbf{M}_s, \mathbf{J}_s)$	$\rho_0, r_0 \rightarrow \infty$
TE & $\mathbf{M}_s$ / TM & $\mathbf{J}_s$			
Cylinder: back-scattered wave	$\xi_{\text{TE}}^-(\mathbf{M}_s) = \xi_{\text{TM}}^-(\mathbf{J}_s)$	$j \frac{\pi \rho_0}{2} H_m^{(2)}(\Lambda \rho_0) \frac{\partial H_m^{(2)}(\Lambda \rho)}{\partial \rho} \Big _{\rho=\rho_0}$	$j(-1)^m e^{-j2\Lambda \rho_0}$
Cylinder: forward-propagating wave	$\xi_{\text{TE}}^+(\mathbf{M}_s) = \xi_{\text{TM}}^+(\mathbf{J}_s)$	$j \frac{\pi \rho_0}{2} H_m^{(1)}(\Lambda \rho_0) \frac{\partial H_m^{(2)}(\Lambda \rho)}{\partial \rho} \Big _{\rho=\rho_0}$	1
Sphere: back-scattered wave	$\zeta_{\text{TE}}^-(\mathbf{M}_s) = \zeta_{\text{TM}}^-(\mathbf{J}_s)$	$-jkr_0 h_n^{(2)}(kr_0) \frac{\partial [kr h_n^{(2)}(kr)]}{\partial kr} \Big _{r=r_0}$	$(-1)^n e^{-j2kr_0}$
Sphere: forward-propagating wave	$\zeta_{\text{TE}}^+(\mathbf{M}_s) = \zeta_{\text{TM}}^+(\mathbf{J}_s)$	$-jkr_0 h_n^{(2)}(kr_0) \frac{\partial [kr h_n^{(1)}(kr)]}{\partial kr} \Big _{r=r_0}$	1
TM & $\mathbf{M}_s$ / TE & $\mathbf{J}_s$			
Cylinder: back-scattered wave	$\xi_{\text{TM}}^-(\mathbf{M}_s) = \xi_{\text{TE}}^-(\mathbf{J}_s)$	$-j \frac{\pi \rho_0}{2} H_m^{(2)}(\Lambda \rho_0) \frac{\partial H_m^{(2)}(\Lambda \rho)}{\partial \rho} \Big _{\rho=\rho_0}$	$-j(-1)^m e^{-j2\Lambda \rho_0}$
Cylinder: forward-propagating wave	$\xi_{\text{TM}}^+(\mathbf{M}_s) = \xi_{\text{TE}}^+(\mathbf{J}_s)$	$[\xi_{\text{TE}}^+(\mathbf{M}_s)]^* / [\xi_{\text{TM}}^+(\mathbf{J}_s)]^*$	1
Sphere: back-scattered wave	$\zeta_{\text{TM}}^-(\mathbf{M}_s) = \zeta_{\text{TE}}^-(\mathbf{J}_s)$	$jkr_0 h_n^{(2)}(kr_0) \frac{\partial [kr h_n^{(2)}(kr)]}{\partial kr} \Big _{r=r_0}$	$-(-1)^n e^{-j2kr_0}$
Sphere: forward-propagating wave	$\zeta_{\text{TM}}^+(\mathbf{M}_s) = \zeta_{\text{TE}}^+(\mathbf{J}_s)$	$[\zeta_{\text{TE}}^+(\mathbf{M}_s)]^* / [\zeta_{\text{TM}}^+(\mathbf{J}_s)]^*$	1

$$\begin{aligned}
\psi_{n,o}^{m,e+}(\mathbf{r}') &= h_m^{(2)}(kr') P_n^m(\cos \theta') \frac{\cos(m\phi')}{\sin(m\phi')} \\
\mathbf{L}_{n,o}^{m,e+}(\mathbf{r}') &= \nabla \psi_{n,o}^{m,e+}(\mathbf{r}') \\
\mathbf{M}_{n,o}^{m,e+}(\mathbf{r}') &= \nabla \times \{ \mathbf{a}_r r \psi_{n,o}^{m,e+}(\mathbf{r}') \} \\
\mathbf{N}_{n,o}^{m,e+}(\mathbf{r}') &= \frac{1}{k} \nabla \times \mathbf{M}_{n,o}^{m,e+}(\mathbf{r}') .
\end{aligned} \tag{4.31}$$

From (4.29) and noting that $\mathbf{M}_s^+(\mathbf{r}') = 2\mathbf{E}(\mathbf{r}') \times \mathbf{a}_r$, on spherical surface S' in Fig. 4.2,

$$\tilde{\mathbf{E}}(r_0) = -\frac{1}{\epsilon_0} \sum_{n=0}^{\infty} \sum_{m=0}^n \{ \mathbf{L}_{n,o}^{m,e+}(\mathbf{r}) \times \mathbf{f}_{n,\text{TE}}^{m,\mathbf{M}_s} \} \tag{4.32}$$

The approximate field $\tilde{\mathbf{E}}(\mathbf{r}_0)$ on the initial spherical surface S' is obtained from (4.29) through the image approximation [10],

$$\tilde{\mathbf{E}}(\mathbf{r}_0) = \sum_{n=0}^{\infty} \sum_{m=0}^n \tilde{c}_{n,o}^{m,e} \mathbf{M}_{n,o}^{m,e}(\mathbf{r}_0) + \tilde{d}_{n,o}^{m,e} \mathbf{N}_{n,o}^{m,e+}(\mathbf{r}_0) \tag{4.33}$$

$$\begin{aligned}
\zeta_{\text{TE}}^{\mathbf{M}_s} &= \frac{\tilde{c}_{n,o}^{m,e}}{c_{n,o}^{m,e}} = -j2kr_0 h_n^{(2)}(kr_0) \frac{\partial[kr j_n(kr)]}{\partial kr} \Big|_{r=r_0} \\
\zeta_{\text{TM}}^{\mathbf{M}_s} &= \frac{\tilde{d}_{n,o}^{m,e}}{d_{n,o}^{m,e}} = j2kr_0 j_n(kr_0) \frac{\partial[kr h_n^{(2)}(kr)]}{\partial kr} \Big|_{r=r_0}
\end{aligned}$$

and,

$$\begin{aligned}
\zeta_{\text{TE}}^{\mathbf{M}_s, \pm} &= -j2kr_0 h_n^{(2)}(kr_0) \frac{\partial[kr h_n^{(1),(2)}(kr)]}{\partial kr} \Big|_{r=r_0} \\
\zeta_{\text{TM}}^{\mathbf{M}_s, \pm} &= j2kr_0 h_n^{(1),(2)}(kr_0) \frac{\partial[kr h_n^{(2)}(kr)]}{\partial kr} \Big|_{r=r_0} .
\end{aligned} \tag{4.34}$$

Similar expressions exist for $\mathbf{J}_s$ image approximation,

$$\zeta_{\text{TE}}^{\mathbf{J}_s} = \zeta_{\text{TM}}^{\mathbf{M}_s}, \quad \zeta_{\text{TM}}^{\mathbf{J}_s} = \zeta_{\text{TE}}^{\mathbf{M}_s} \tag{4.35}$$

$$\zeta_{\text{TE}}^{\mathbf{M}_s, \pm} = \zeta_{\text{TM}}^{\mathbf{J}_s, \pm} = [\zeta_{\text{TE}}^{\mathbf{J}_s, \pm}]^* = [\zeta_{\text{TM}}^{\mathbf{M}_s, \pm}]^* .$$

4.2.2 The Dyadic Green's Function Method

The magnetic dyadic Green's function in the spherical coordinate is,

$$\begin{aligned} \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') &= -\frac{\mathbf{a}_r \mathbf{a}_r}{k^2} \delta(\mathbf{r} - \mathbf{r}') - \sum_{n=-\infty}^{\infty} \frac{j\pi}{2kn(n+1)} \\ &\times \sum_{m=0}^n \frac{1}{Q_{nm}} \left\{ \mathbf{M}_{n,o}^{m,e}(\mathbf{r}') \mathbf{M}_{n,o}^{m,e+}(\mathbf{r}) + \mathbf{N}_{n,o}^{m,e}(\mathbf{r}') \mathbf{N}_{n,o}^{m,e+}(\mathbf{r}) \right\} \\ \text{and,} \quad Q_{nm} &= \frac{2\pi^2(n+m)!}{(2-\delta_m^0)(2n+1)(n-m)!} \end{aligned} \quad (4.36)$$

where $\mathbf{M}_{n,o}^{m,e}$ ($\mathbf{N}_{n,o}^{m,e}$) is obtained by replacing $h_n^{(2)}$ with j_n in $\mathbf{M}_{n,o}^{m,e+}$ ($\mathbf{N}_{n,o}^{m,e+}$).

The approximate field $\tilde{\mathbf{E}}(\mathbf{r})$ for $\mathbf{M}_s^+(\mathbf{r}')$ is given as,

$$\tilde{\mathbf{E}}(\mathbf{r}) = -\nabla \times \iint_{S'} dS' \mathbf{M}_s^+(\mathbf{r}') \cdot \bar{\mathbf{G}}_m(\mathbf{r}, \mathbf{r}') \quad (4.37)$$

Substituting (4.36) into (4.37) and using the orthogonal properties of spherical modal functions, the approximate field $\tilde{\mathbf{E}}(r_0)$ on initial spherical surface S' is obtained as,

$$\begin{aligned} \tilde{\mathbf{E}}(r_0) &= \sum_{n=-\infty}^{\infty} \sum_{m=0}^n \frac{j\pi}{n(n+1)Q_{nm}} \frac{c_{n,o}^{m,e} \mathbf{M}_{n,o}^{m,e+}(\mathbf{r})}{d_{n,o}^{m,e} \mathbf{N}_{n,o}^{m,e+}(\mathbf{r})} \\ &\times \iint_{S'} dS' \frac{[\mathbf{N}_{n,o}^{m,e}(\mathbf{r}')]^* \times \mathbf{M}_{n,o}^{m,e+}(\mathbf{r}')}{[\mathbf{M}_{n,o}^{m,e}(\mathbf{r}')^* \times \mathbf{N}_{n,o}^{m,e+}(\mathbf{r}')] \cdot \mathbf{a}_{r'}} \cdot \mathbf{a}_{r'} \cdot \end{aligned} \quad (4.38)$$

The evaluation of (4.38) also gives (4.33) and (4.34).

4.2.3 The Image Theorem Criteria in the Spherical Geometry

Similar to the cylindrical geometry, $\zeta_{\text{TE,TM}}^{\mathbf{M}_s, \mathbf{J}_s^+}$ in (4.34) and (4.35) can be considered as theoretical criteria of the image theorems for narrow-band fields in the spherical geometry. The large argument asymptotic behaviors of $\zeta_{\text{TE,TM}}^{\mathbf{M}_s, \mathbf{J}_s^+}$ for $r_0 \rightarrow \infty$ can be obtained by noting that,

$$\begin{aligned} h_n^{(2)}(kr_0) &= [h_n^{(1)}(kr_0)]^* \sim \frac{1}{kr_0} j^{(n+1)} e^{-jkr_0}, \quad kr_0 \rightarrow \infty \\ \zeta_{\text{TE,TM}}^{\mathbf{M}_s, \mathbf{J}_s^+} \Big|_{r_0 \rightarrow \infty} &= 1. \end{aligned} \quad (4.39)$$

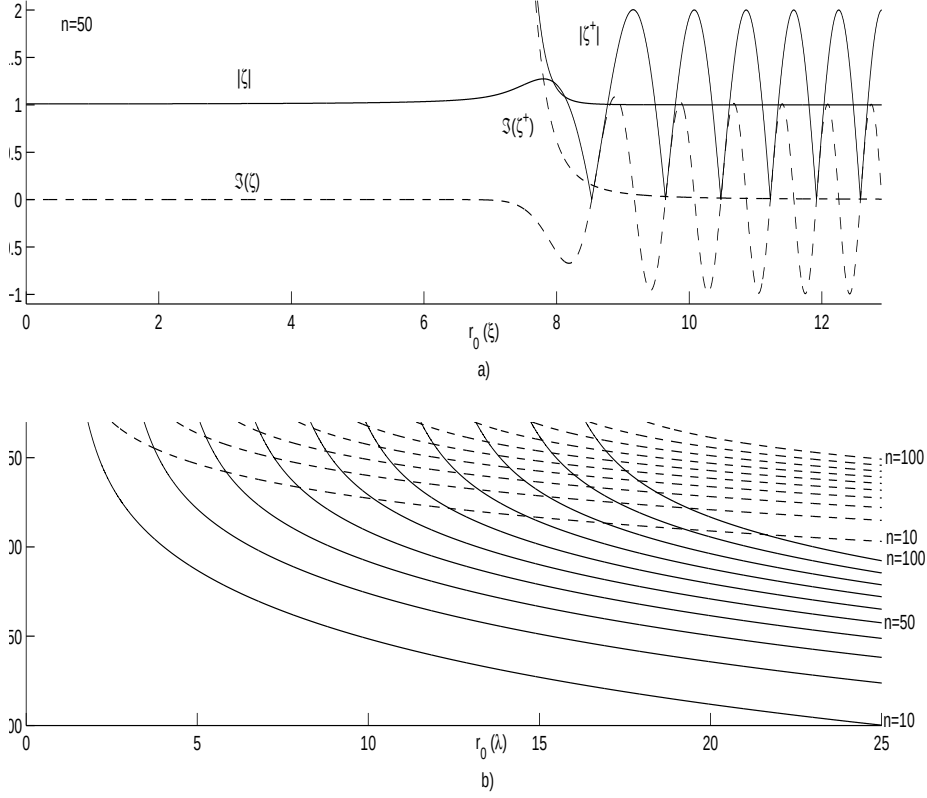


Figure 4.3: The spherical geometry - plots of $\zeta_{TE}^{M_s}$ and $\zeta_{TE}^{M_s+}$: a) shows oscillations of $\zeta_{TE}^{M_s}$ and the asymptotic behavior of $\zeta_{TE}^{M_s+}$ ($m=50$) for both magnitudes (solid lines) and imaginary parts (dashed lines); b) shows $20 \log_{10}(|\zeta_{TE}^{M_s+}| - 1)$ and $20 \log_{10}\{\Im[\zeta_{TE}^+(\mathbf{M}_s)]\}$ Vs. r_0 , from 0 to 25λ for $n=10, 20, \dots, 90, 100$.

4.3 Results and Discussion

TABLE 4.1 summarizes the properties of $\zeta_{TE, TM}^{M_s, J_s \pm}$ and $\zeta_{TE, TM}^{M_s, J_s \pm}$, for the back-scattered and forward-propagating waves respectively. For $\rho_0, r_0 \rightarrow \infty$, both $\zeta_{TE, TM}^{M_s, J_s} = \zeta_{TE, TM}^{M_s, J_s+} + \zeta_{TE, TM}^{M_s, J_s-}$ and $\zeta_{TE, TM}^{M_s, J_s} = \zeta_{TE, TM}^{M_s, J_s+} + \zeta_{TE, TM}^{M_s, J_s-}$ show fast oscillations, which can be seen from TABLE 4.1 or plot a) of Fig. 4.3 (spherical geometry only, cylindrical geometry is similar and thus not shown). Mathematically, the oscillations only appear as modal expansion coefficients and disappear after the implementation of the sum and integration in (4.16) or the double sums in (4.33). Physically, the oscillations are due to back-scattered fields, which approach 0 for

$\rho_0, r_0 \rightarrow \infty$. For example, consider $\zeta_{\text{TE}}^{\text{M}_s^-}$ in (4.16),

$$\begin{aligned} \tilde{\mathbf{E}}^-(r_0, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^n \left\{ \zeta_{\text{TE}}^{\text{M}_s^-} c_{n,o}^{m,e} \mathbf{M}_{n,o}^{m,e+}(\mathbf{r}_0) \right. \\ \left. + \zeta_{\text{TE}}^{\text{M}_s^-} b_{n,o}^{m,e} \mathbf{N}_{n,o}^{m,e+}(\mathbf{r}_0) \right\} \end{aligned} \quad (4.40)$$

Changing the variable $\phi' = \phi - \pi$ and letting $r_0 \rightarrow \infty$, from TABLE 4.1, (4.40) reduces to,

$$\begin{aligned} \tilde{\mathbf{E}}^-(r_0, \phi') \Big|_{\rho_0 \rightarrow \infty} = \sum_{n=0}^{\infty} \sum_{m=0}^n e^{-j2kr_0} \\ \times \left\{ c_{n,o}^{m,e} \mathbf{M}_{n,o}^{m,e+}(\mathbf{r}_0) - b_{n,o}^{m,e} \mathbf{N}_{n,o}^{m,e+}(\mathbf{r}_0) \right\}. \end{aligned} \quad (4.41)$$

Now, the back-scattered field $\tilde{\mathbf{E}}^-(r_0, \phi') \Big|_{\rho_0 \rightarrow \infty} \rightarrow 0$ due to the fast variation phase term $e^{-j2\Lambda\rho_0}$, which means that the oscillation in $\zeta_{\text{TE}}^{\text{M}_s^-}$ doesn't appear in the actual field evaluation for $r_0 \rightarrow \infty$.

Due to the above reasons, $\xi_{\text{TE,TM}}^{\text{M}_s, \text{J}_s^+}$ and $\zeta_{\text{TE,TM}}^{\text{M}_s, \text{J}_s^\pm}$ are theoretical criterions of interests to evaluate the validity of image theorems.

4.3.1 The Cylindrical Geometry

As can be seen from TABLE 4.1, in the cylindrical geometry, $\xi_{\text{TE}, \text{TM}}^{\text{M}_s, \text{J}_s^+}$ are functions of three parameters - m , Λ and ρ_0 . As an example, in Fig. 4.4, $\xi_{\text{TE}}^{\text{M}_s^+}$ is plotted Vs. ρ_0 for different m and Λ . What's more, Fig. 4.5 shows the threshold radius ρ_{th} (above which image theorems meet the required accuracy) for $\xi_{\text{TE}}^{\text{M}_s^+}$, with respect to Λ for an accuracy of -30 dB and $m=0, 10, \dots, 90, 100$. Solid lines are for $20 \log_{10}(|\xi_{\text{TE}}^{\text{M}_s^+}| - 1)$ and dashed lines are for $20 \log_{10}\{\Im[\xi_{\text{TE}}^{\text{M}_s^+}]\}$. Both Fig. 4.4 and Fig. 4.5 show that a smaller m and a larger $\Lambda\rho_0$ will give a higher accuracy. For example, to achieve an accuracy of -30 dB, on one hand, in Fig. 4.5, for $m=50$ and $\Lambda = 0.9k$ ($\frac{k-\Lambda}{k} = 0.1$), $\rho_{\text{th}} \sim 9.5\lambda$ is required for the magnitude $|\xi_{\text{TE}}^{\text{M}_s^+}|$ and $\rho_{\text{th}} \sim 13\lambda$ is required for the imaginary part $\Im[\xi_{\text{TE}}^{\text{M}_s^+}]$. On the other hand, to achieve the same accuracy, for $m=100$ and $\Lambda = 0.5k$ ($\frac{k-\Lambda}{k} = 0.5$), $\rho_{\text{th}} \sim 37\lambda$ and $\rho_{\text{th}} \sim 32.5\lambda$ are necessary for $|\xi_{\text{TE}}^{\text{M}_s^+}|$ and $\Im[\xi_{\text{TE}}^{\text{M}_s^+}(\mathbf{M}_s)]$ respectively. It is clear that the imaginary part $\Im[\xi_{\text{TE}}^{\text{M}_s^+}]$ dominates the accuracy and image theorems work well for narrow-band fields and cylindrical surfaces of large ρ_0 .

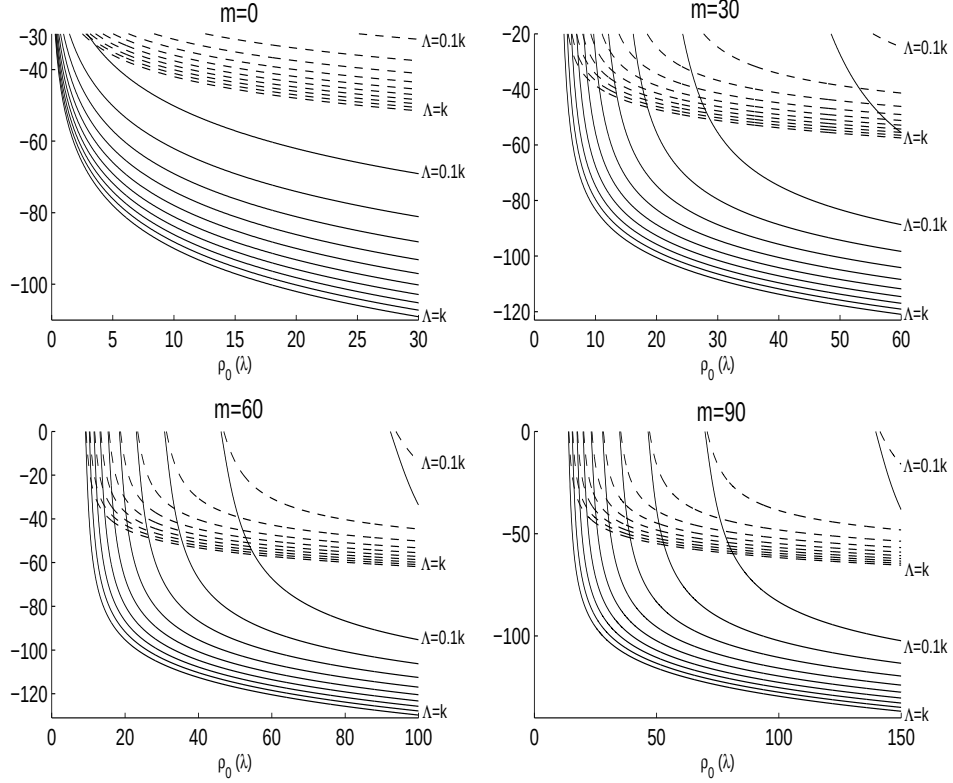


Figure 4.4: The cylindrical geometry - plots of $\xi_{\text{TE}}^{\text{M}_s^+}$ Vs. ρ_0 for $m=0, 30, 60, 90$ and $\Lambda = 0.1k$ to k (in $0.1k$ increment, from top to bottom): solid lines are for magnitudes $20 \log_{10}(|\xi_{\text{TE}}^{\text{M}_s^+}| - 1)$, and dashed lines are for imaginary parts $20 \log_{10}\{\Im[\xi_{\text{TE}}^{\text{M}_s^+}]\}$. The plots show that the image theorem works well for a small m and a large $\Lambda\rho_0$.

4.3.2 The Spherical Geometry

From TABLE 4.1, in the spherical geometry, $\zeta_{\text{TE}, \text{TM}}^{\text{M}_s, \text{J}_s^+}$ has only two parameters - n and r_0 , as shown in b) of Fig. 4.3 for $\zeta_{\text{TE}}^{\text{M}_s^+}$. It is also helpful to plot the corresponding threshold radius r_{th} with respect to n , for both $20 \log_{10}(|\zeta_{\text{TE}}^{\text{M}_s^+}| - 1)$ and $20 \log_{10}\{\Im[\zeta_{\text{TE}}^{\text{M}_s^+}]\}$, with different accuracies ranging from -60 dB to -30 dB (in 3 dB increment), as in Fig. 4.6. It can be seen from both Fig. 4.3 and Fig. 4.6 that, in order to achieve an accuracy of -30 dB for $|\zeta_{\text{TE}}^{\text{M}_s^+}|$ (with respect to 1), $r_{\text{th}} \sim 8\lambda$ and $r_{\text{th}} \sim 16\lambda$ for $n = 50$ and $n = 100$ respectively. However, for the imaginary part $\Im[\zeta_{\text{TE}}^{\text{M}_s^+}]$, $r_{\text{th}} \sim 9.5\lambda$ and $r_{\text{th}} \sim 18\lambda$ are required for $n = 50$ and $n = 100$ respectively, which again implies that the imaginary part $\zeta_{\text{TE}}^{\text{M}_s^+}$

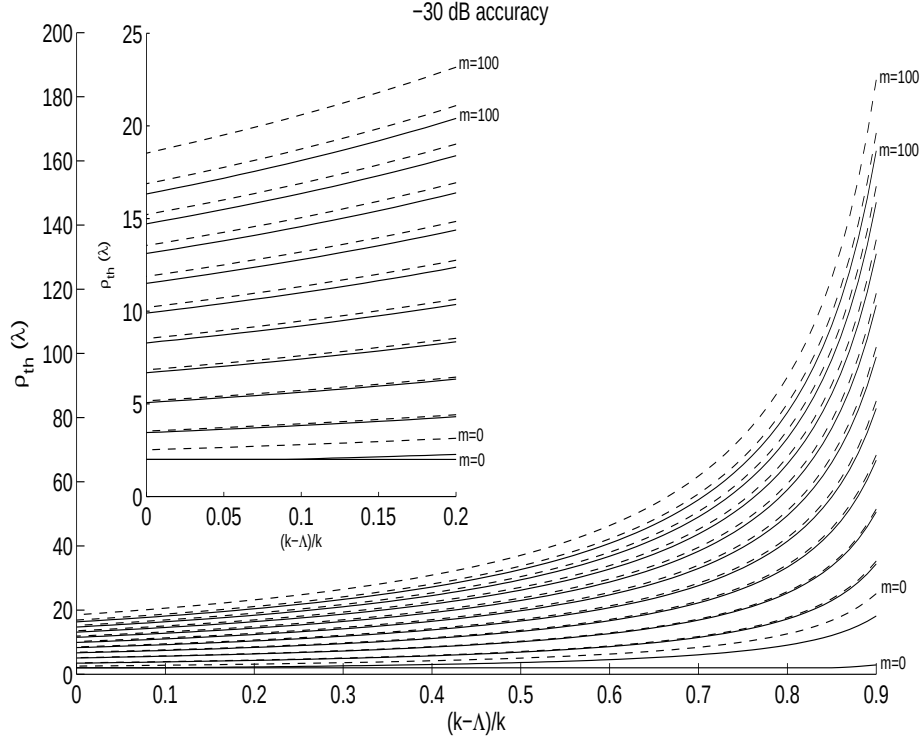


Figure 4.5: The cylindrical geometry - threshold radii ρ_{th} Vs. $\Lambda = 0.1k$ to k , corresponding to $(k - \Lambda)/k = 0$ to 0.2 , for an accuracy of -30 dB. Solid lines are for magnitudes $20 \log_{10}(|\xi_{TE}^{M_s+}| - 1)$ and dashed lines are for imaginary parts $20 \log_{10}\{\Im[\xi_{TE}^{M_s+}]\}$. The inset plot is also shown to make the display clearer for $\Lambda = 0.8k$ to k , corresponding to $(k - \Lambda)/k = 0$ to 0.2 . It can be seen that imaginary parts $\xi_{TE}^{M_s+}$ require larger threshold radii ρ_{th} for the same accuracy.

dominates the accuracy of image theorems.

4.4 Summary of the Image Theorem Approximation

The theoretical criteria for image theorems (both $\mathbf{M}_s$ and $\mathbf{J}_s$ approximations) have been derived through two equivalent methods - the vector potential method and the dyadic Green's function method, for both TE and TM modes, in cylindrical and spherical geometries. The criteria are related to each other within the same geometry and have the same accuracy for surfaces of large radii. The

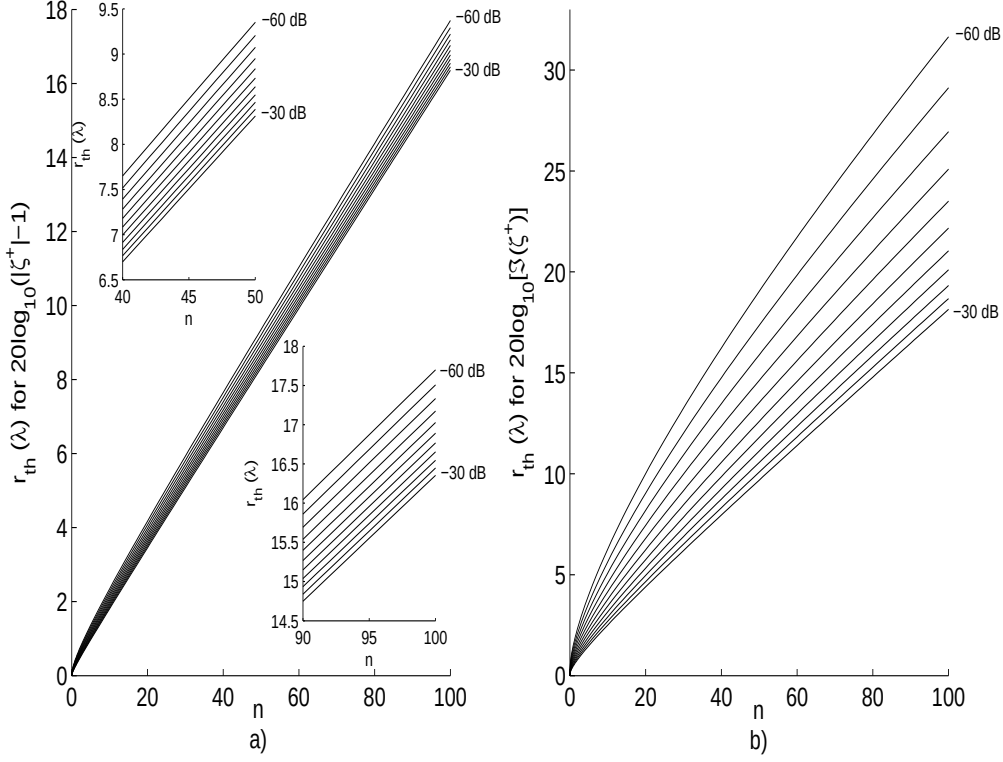


Figure 4.6: The spherical geometry - threshold radii r_{th} Vs. $n=0$ to 100, for different accuracies, from -60 dB to -30 dB (in 10 dB increment, from bottom to top): a) the magnitudes $20 \log_{10}(|\zeta_{TE}^{M_s+}| - 1)$, and b) the imaginary parts $20 \log_{10}[\Im(\zeta_{TE}^{M_s+})]$. The inset plots in a) are used to make the display clearer. Similar to the cylindrical geometry, imaginary parts $\zeta_{TE}^{M_s+}$ require larger threshold radii r_{th} for the same accuracy.

threshold radii ρ_{th} and r_{th} for a given accuracy depend on properties of the incident fields (m , Λ for the cylindrical geometry and n for the spherical geometry), and radii ρ_0 and r_0 of cylindrical and spherical surfaces. Generally, narrow-band fields and large radii will give high accuracies. In the cylindrical geometry, for an accuracy of -30 dB, $\Lambda = 0.9k$ and $m = 50$, typical values of ρ_{th} are about 9.5λ for the magnitude and 13λ for the imaginary part. While in the spherical geometry, for the same accuracy and $n = 50$, r_{th} are about 9.5λ for the magnitude and 18λ for the imaginary part. The theoretical criteria are valuable in the validity evaluation of image theorems for many problems, including RCS computation, mirror system designs and scattering phenomena in both cylindrical and spherical geometries.

Chapter 5

THE PLANAR TI-FFT ALGORITHM

In this chapter, we will introduce the Taylor Interpolation scheme using FFT algorithm (TI-FFT) that will be used to speed the numerical computation of electromagnetic wave propagation and scattering in the half-space scenario. There are two types of TI-FFT algorithm, i.e., the spatial TI-FFT for computation of electromagnetic wave on the quasi-planar surface; and the spectral TI-FFT for computation of the 2-Dimensional (2D) Fourier spectrum of the electromagnetic wave. For each type of TI-FFT algorithm, there are also two numerical implementation methods, i.e., the spatial domain division method and the spectral domain division method. Based on the Fourier spectrum of the scalar Green's function, detailed closed-form expressions of the 2D Fourier spectra of various diffraction formulas (i.e., Kirchhoff, Stratton-Chu, Kottler, and Franz formulas) have been found and can be efficiently evaluated through the planar TI-FFT algorithm for the quasi-planar geometry. The planar TI-FFT algorithm can be optimized for the computational complexity. The optimized order of Taylor series only depends on the algorithm's computational accuracy γ_{TI} : $\mathcal{N}_o^{\text{opt}} \sim -\ln \gamma_{\text{TI}}$ and the optimized spatial slicing spacing between two adjacent spatial reference planes only depends on the characteristic wavelength λ_c of the electromagnetic wave: $\delta_z^{\text{opt}} \sim \frac{1}{17}\lambda_c$, from which the optimized number of spatial reference planes $\mathcal{N}_r^{\text{opt}}$ can be found. The optimized computational complexity is then given as $[\mathcal{N}_r^{\text{opt}} \times \mathcal{N}_o^{\text{opt}}] \mathcal{O}(N \log_2 N)$ for an $N = \mathcal{N}_x \times \mathcal{N}_y$ computational grid, comparable to the computational complexity of the FMM. Also, the planar TI-FFT algorithm allows a large sampling spacing only limited by the Nyquist sampling rate. In addition, the algorithm is free of singularities. The algorithm works particularly well for the narrow-band beam and the quasi-planar geometry.

5.1 Basic Concepts

5.1.1 Electromagnetic wave in the spectral domain

The planar TI-FFT algorithm is a frequency-domain FFT-based algorithm which requires the computation of the 2D Fourier spectrum of the electromagnetic wave. In this section, the spectral property of the electromagnetic wave is discussed.

5.1.2 The Maxwell's equations and the scalar Green's function

The frequency-domain representation of the Maxwell's equations in the homogeneous medium is given as

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E}, \quad \nabla \times \mathbf{E} = -\mathbf{J}_m - j\omega\mu\mathbf{H}, \quad (5.1.2.1)$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon}\rho, \quad \nabla \cdot \mathbf{H} = \frac{1}{\mu}\rho_m, \quad (5.1.2.2)$$

where, $\mathbf{J}$ and ρ are the electric current density (volume) and electric charge density (volume) respectively. The magnetic current density (volume) $\mathbf{J}_m$ and the magnetic charge density (volume) ρ_m are fictitious and complimentary to $\mathbf{J}$ and ρ . If the vector potentials $\mathbf{A} \equiv \mu\nabla \times \mathbf{H}$ (corresponding to $\mathbf{J}$) and $\mathbf{A}_m \equiv -\epsilon\nabla \times \mathbf{E}_m$ (corresponding to $\mathbf{J}_m$) are used in the Maxwell's equations in (5.1.2.1)-(5.1.2.2), the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ are obtained as

$$\mathbf{E} = -\nabla\phi - j\omega\mathbf{A} - \frac{1}{\epsilon}\nabla \times \mathbf{A}_m, \quad (5.1.2.3)$$

$$\mathbf{H} = -\nabla\phi_m - j\omega\mathbf{A}_m + \frac{1}{\mu}\nabla \times \mathbf{A}, \quad (5.1.2.4)$$

with the scalar and vector potentials satisfying the Helmholtz wave equations

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu\mathbf{J}, \quad \nabla^2 \mathbf{A}_m + k^2 \mathbf{A}_m = -\epsilon\mathbf{J}_m, \quad (5.1.2.5)$$

$$\nabla^2 \phi + k^2 \phi = -\frac{\rho}{\epsilon}, \quad \nabla^2 \phi_m + k^2 \phi_m = -\frac{\rho_m}{\mu}, \quad (5.1.2.6)$$

and the Lorenz conditions

$$\nabla \cdot \mathbf{A} + j\omega\epsilon\mu\phi = 0, \quad \nabla \cdot \mathbf{A}_m + j\omega\epsilon\mu\phi_m = 0, \quad (5.1.2.7)$$

where the explicit solutions for the Helmholtz equations in (5.1.2.5) and (5.1.2.6) are convolutions of the source terms with the scalar Green's function given as

$$\mathbf{A}(\mathbf{r}) = \iiint_V \mu \mathbf{J}(\mathbf{r}') G(\mathbf{R}) dV', \quad \mathbf{A}_m(\mathbf{r}) = \iiint_V \epsilon \mathbf{J}_m(\mathbf{r}') G(\mathbf{R}) dV' \quad (5.1.2.8)$$

$$\phi(\mathbf{r}) = \iiint_V \frac{1}{\epsilon} \rho(\mathbf{r}') G(\mathbf{R}) dV', \quad \phi_m(\mathbf{r}) = \iiint_V \frac{1}{\mu} \rho_m(\mathbf{r}') G(\mathbf{R}) dV' \quad (5.1.2.9)$$

where the scalar Green's function $G(\mathbf{R})$ is the solution for a 3-Dimensional (3D) point source,

$$\nabla^2 G(\mathbf{R}) + k^2 G(\mathbf{R}) = -\delta^{(3)}(\mathbf{R}), \quad (5.1.2.10)$$

$$G(\mathbf{R}) = \frac{e^{-jk|\mathbf{R}|}}{4\pi|\mathbf{R}|}, \quad \mathbf{R} \equiv \mathbf{r} - \mathbf{r}'. \quad (5.1.2.11)$$

Substitute the Lorenz condition (5.1.2.7), the vector potentials (5.1.2.8) and the scalar potentials (5.1.2.9) into (6.1.1.4) and (6.1.1.5), the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ are obtained as

$$\begin{aligned} \mathbf{E} = \frac{-j}{\omega\epsilon} \iiint_V \left[k^2 \mathbf{J}(\mathbf{r}') G(\mathbf{R}) + \left(\mathbf{J}(\mathbf{r}') \cdot \nabla' \right) \nabla' G(\mathbf{R}) \right. \\ \left. - j\omega\epsilon \mathbf{J}_m(\mathbf{r}') \times \nabla' G(\mathbf{R}) \right] dV'. \end{aligned} \quad (5.1.2.12)$$

$$\begin{aligned} \mathbf{H} = \frac{-j}{\omega\mu} \iiint_V \left[k^2 \mathbf{J}_m(\mathbf{r}') G(\mathbf{R}) + \left(\mathbf{J}_m(\mathbf{r}') \cdot \nabla' \right) \nabla' G(\mathbf{R}) \right. \\ \left. + j\omega\mu \mathbf{J}(\mathbf{r}') \times \nabla' G(\mathbf{R}) \right] dV'. \end{aligned} \quad (5.1.2.13)$$

The electric field $\mathbf{E}$ in (5.1.2.12) and the magnetic field $\mathbf{H}$ in (5.1.2.13) can also be expressed in the form of the dyadic Green's functions of the electric type ($\bar{\bar{G}}_e$) and the magnetic type ($\bar{\bar{G}}_m$) [25, 37],

$$\mathbf{E} = \iiint_V \left[(-j\omega\mu)\bar{\bar{G}}_e(\mathbf{R}) \cdot \mathbf{J}(\mathbf{r}') - \bar{\bar{G}}_m(\mathbf{R}) \cdot \mathbf{J}_m(\mathbf{r}') \right] dV', \quad (5.1.2.14)$$

$$\mathbf{H} = \iiint_V \left[(-j\omega\epsilon)\bar{\bar{G}}_e(\mathbf{R}) \cdot \mathbf{J}_m(\mathbf{r}') + \bar{\bar{G}}_m(\mathbf{R}) \cdot \mathbf{J}(\mathbf{r}') \right] dV', \quad (5.1.2.15)$$

The dyadic Green's functions of the electric type ($\bar{\bar{G}}_e$) and the magnetic type ($\bar{\bar{G}}_m$) are given as

$$\bar{\bar{G}}_e(\mathbf{R}) = \left(\bar{\bar{I}} + \frac{1}{k^2} \nabla \nabla \right) G(\mathbf{R}). \quad (5.1.2.16)$$

$$\bar{\bar{G}}_m(\mathbf{R}) = \nabla G(\mathbf{R}) \times \bar{\bar{I}}. \quad (5.1.2.17)$$

It is clear that (5.2.1.55) can be obtained from (5.2.1.54) using the duality relations below,

$$\mathbf{E} \rightarrow \mathbf{H}, \quad \mathbf{H} \rightarrow -\mathbf{E}, \quad \mathbf{J} \rightarrow \mathbf{J}_m, \quad \mathbf{J}_m \rightarrow -\mathbf{J}, \quad (5.1.2.18)$$

$$\bar{\bar{G}}_e(\mathbf{R}) \rightarrow \bar{\bar{G}}_m(\mathbf{R}), \quad \bar{\bar{G}}_m(\mathbf{R}) \rightarrow -\bar{\bar{G}}_e(\mathbf{R}), \quad \epsilon \rightarrow \mu, \quad \mu \rightarrow \epsilon.$$

5.1.3 The spectral property of the scalar Green's function

Now apply the 2D Fourier transform on the scalar Green's function $G(\mathbf{R})$ in (5.1.2.11),

$$\text{FT}_{2D} \left[\nabla^2 G(\mathbf{R}) + k^2 G(\mathbf{R}) \right] = \text{FT}_{2D} \left[-\delta^{(3)}(\mathbf{R}) \right], \quad (5.1.3.19)$$

It is easy to show that (5.1.3.19) reduces to the 1-Dimensional (1D) problem for coordinate z , which is

$$\left[\frac{\partial^2}{\partial z^2} + k_z^2 \right] \mathcal{G}(k_x, k_y, \mathbf{r}') = \left[-\frac{1}{2\pi} e^{j(k_x x' + k_y y')} \right] \delta(z - z'), \quad (5.1.3.20)$$

with

$$\mathcal{G}(k_x, k_y, \mathbf{r}') \equiv \text{FT}_{2D} \left[G(\mathbf{R}) \right], \quad (5.1.3.21)$$

$$k_z = \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2}, & k_x^2 + k_y^2 < k^2 \\ -j\sqrt{k_x^2 + k_y^2 - k^2}, & k_x^2 + k_y^2 \geq k^2 \end{cases}, \quad (5.1.3.22)$$

where the 2D Fourier transform has been defined as

$$\text{FT}_{2\text{D}} \left[\cdot \right] = \frac{1}{2\pi} \int_{x=-\infty}^{\infty} \left\{ e^{jk_x x} \int_{y=-\infty}^{\infty} \left[\cdot \right] e^{jk_y y} dy \right\} dx, \quad (5.1.3.23)$$

In derivation of (5.1.3.20), the following property of the Fourier transform [30] has been used,

$$\text{FT}_{2\text{D}} \left[\frac{\partial G(\mathbf{R})}{\partial v} \right] = -jk_v \mathcal{G}(k_x, k_y, \mathbf{r}'), \quad v = x, y. \quad (5.1.3.24)$$

The solution of the 2D Fourier spectrum of the scalar Green's function in (5.1.3.20) is given as

$$\mathcal{G}(k_x, k_y, \mathbf{r}') = \left[\frac{-j}{4\pi k_z} e^{j(k_x x' + k_y y')} \right] e^{-jk_z |z - z'|}, \quad (5.1.3.25)$$

$$\mathcal{G}_{<}^>(k_x, k_y, \mathbf{r}') = \left[\frac{-j}{4\pi k_z} e^{j(k_x x' + k_y y')} \right] e^{\mp jk_z (z - z')}, \quad \begin{array}{l} > \text{ for half-space } z > z' \\ < \text{ for half-space } z < z' \end{array} \quad (5.1.3.26)$$

The correctness of $\mathcal{G}(k_x, k_y, \mathbf{r}')$ in (5.1.3.25) can be verified through the line integral of (5.1.3.20), from the minus side of z' (z'_-) to the positive side of z' (z'_+),

$$\begin{aligned} \text{LHS} &= \int_{z=z'_-}^{z'_+} \left[\frac{\partial^2}{\partial z^2} + k_z^2 \right] \mathcal{G}(k_x, k_y, \mathbf{r}') dz = \int_{z'_-}^{z'_+} \frac{\partial^2}{\partial z^2} \mathcal{G}(k_x, k_y, \mathbf{r}') dz \\ &= -\frac{1}{2\pi} e^{j(k_x x' + k_y y')} = \left[-\frac{1}{2\pi} e^{j(k_x x' + k_y y')} \right] \int_{z=z'_-}^{z'_+} \delta(z - z') dz = \text{RHS}. \end{aligned} \quad (5.1.3.27)$$

where LHS and RHS stand for the Left Hand Side and Right Hand Side of (5.1.3.20) respectively. The 2D Fourier spectrum of the scalar Green's function in (5.1.3.25) can also be derived in the cylindrical coordinate, which has been given in Appendix A. Because only half-space $z > z'$ is of interest, only $\mathcal{G}^>(k_x, k_y, \mathbf{r}')$ will be used in the rest of this article.

The 2D Fourier spectra of the derivatives (order n) of the scalar Green's function can be obtained from the property of the Fourier transform [30],

$$\text{FT}_{2\text{D}} \left[\frac{\partial^{(n)} G(\mathbf{R})}{\partial v^{(n)}} \right] = (-jk_v)^n \mathcal{G}^>(k_x, k_y, \mathbf{r}'), \quad v = x, y, z. \quad (5.1.3.28)$$

Particularly, for the first-order and second-order derivatives,

$$\text{FT}_{2\text{D}} \left[\frac{\partial G(\mathbf{R})}{\partial v} \right] = \left[\frac{-k_v}{4\pi k_z} e^{j(k_x x' + k_y y')} \right] e^{-jk_z(z-z')}, \quad v = x, y, z. \quad (5.1.3.29)$$

$$\text{FT}_{2\text{D}} \left[\frac{\partial^2 G(\mathbf{R})}{\partial v^2} \right] = \left[\frac{jk_v^2}{4\pi k_z} e^{j(k_x x' + k_y y')} \right] e^{-jk_z(z-z')}, \quad v = x, y, z. \quad (5.1.3.30)$$

Similarly, the 2D Fourier spectra of the following expressions can be obtained for half-space $z > z'$,

$$\text{FT}_{2\text{D}} \left[\nabla G(\mathbf{R}) \right] = -j\mathbf{k} \mathcal{G}^>(k_x, k_y, \mathbf{r}'). \quad (5.1.3.31)$$

$$\text{FT}_{2\text{D}} \left[\nabla^2 G(\mathbf{R}) \right] = -k^2 \mathcal{G}^>(k_x, k_y, \mathbf{r}'). \quad (5.1.3.32)$$

$$\text{FT}_{2\text{D}} \left[\nabla \nabla G(\mathbf{R}) \right] = -\mathbf{k} \mathbf{k} \mathcal{G}^>(k_x, k_y, \mathbf{r}'). \quad (5.1.3.33)$$

$$\text{FT}_{2\text{D}} \left[\overline{\overline{G}}_e(\mathbf{R}) \right] = \mathcal{G}^>(k_x, k_y, \mathbf{r}') \left[\overline{\overline{I}} - \frac{\mathbf{k} \mathbf{k}}{k^2} \right]. \quad (5.1.3.34)$$

$$\text{FT}_{2\text{D}} \left[\overline{\overline{G}}_m(\mathbf{R}) \right] = -j \mathcal{G}^>(k_x, k_y, \mathbf{r}') \left[\mathbf{k} \times \overline{\overline{I}} \right]. \quad (5.1.3.35)$$

Table 5.1 summarizes the 2D Fourier spectra for different Green's function related expressions mentioned above for half-space $z > z'$, together with the corresponding far-fields, which will be discussed in Section 5.1.4.

Table 5.1: The 2D Fourier spectra and far-fields of the scalar Green's function related expressions for half-space $z > z'$

The expressions	The far-fields, only $\frac{1}{ \mathbf{r} }$ term ($\mathbf{r} \rightarrow \infty$)	The 2D Fourier spectra
$G(\mathbf{R}) = \frac{e^{-jk \mathbf{R} }}{4\pi \mathbf{R} }$ $\frac{\partial^{(n)}G(\mathbf{R})}{\partial v^{(n)}}, (v = x, y, z)$ $\nabla G(\mathbf{R})$ $\nabla^2 G(\mathbf{R})$ $\nabla \nabla G(\mathbf{R})$ $\bar{\bar{G}}_e(\mathbf{R})$ $\bar{\bar{G}}_m(\mathbf{R})$	$G(\mathbf{r}) = \frac{e^{-jk \mathbf{r} }}{4\pi \mathbf{r} }$ $(-jk_v)^n G(\mathbf{r})$ $-j\mathbf{k}G(\mathbf{r})$ $-k^2 G(\mathbf{r})$ $-\mathbf{k}\mathbf{k}G(\mathbf{r})$ $G(\mathbf{r}) \left[\bar{\bar{I}} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right]$ $-jG(\mathbf{r}) \left[\mathbf{k} \times \bar{\bar{I}} \right]$	$\mathcal{G}^>(k_x, k_y, \mathbf{r}') = \frac{-je^{-jk_z z}}{4\pi k_z} e^{j\mathbf{k} \cdot \mathbf{r}'}$ $(-jk_v)^n \mathcal{G}^>(k_x, k_y, \mathbf{r}')$ $-j\mathbf{k}\mathcal{G}^>(k_x, k_y, \mathbf{r}')$ $-k^2 \mathcal{G}^>(k_x, k_y, \mathbf{r}')$ $-\mathbf{k}\mathbf{k}\mathcal{G}^>(k_x, k_y, \mathbf{r}')$ $\mathcal{G}^>(k_x, k_y, \mathbf{r}') \left[\bar{\bar{I}} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right]$ $-j\mathcal{G}^>(k_x, k_y, \mathbf{r}') \left[\mathbf{k} \times \bar{\bar{I}} \right]$

5.1.4 The far-fields

The far-field of the Green's function can be found by letting $\mathbf{R} \rightarrow \infty$,

$$G(\mathbf{r}) = \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}, \quad \mathbf{R} = \mathbf{r} \text{ in the far-field limit.} \quad (5.1.4.36)$$

Similarly, the first-order derivative of the Green's function in the far-field limit can be obtained as

$$\begin{aligned} \frac{\partial G(\mathbf{r})}{\partial v} &= \frac{v}{|\mathbf{r}|} \left(-jk - \frac{1}{|\mathbf{r}|^2} \right) \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|} \\ &\simeq -j \left(\frac{v}{|\mathbf{r}|} k \right) \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|} = -jk_v \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}, \quad v = x, y, z \end{aligned} \quad (5.1.4.37)$$

where only $\frac{1}{|\mathbf{r}|}$ term is kept and the other terms ($\frac{1}{|\mathbf{r}|^2}, \frac{1}{|\mathbf{r}|^3}, \dots$) are ignored.

In derivation of (5.1.4.37), the following relation has been used in the far-field limit,

$$\frac{k_v}{k} = \frac{v}{|\mathbf{r}|}, \quad v = x, y, z. \quad (5.1.4.38)$$

Following the similar procedure given in (5.1.4.37), the far-fields of derivatives (order n) of the scalar Green's function are obtained as

$$\frac{\partial^{(n)} G(\mathbf{r})}{\partial v^{(n)}} = (-jk_v)^n \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}, \quad v = x, y, z. \quad (5.1.4.39)$$

The far-fields for other Green's function related expressions are summarized in Table 5.1, together with the corresponding 2D Fourier spectra given in Section 5.1.3. It is not difficult to see that the far-fields and the 2D Fourier spectra are closely related to each other, which means that when one quantity is known, the other one can be easily obtained through Table 5.1.

5.1.5 Huygens-Fresnel principle and the diffraction formulas

Consider the problem of finding the electromagnetic fields ($\mathbf{E}$, $\mathbf{H}$) at observation point P in volume V (see Fig. 5.1) generated by the electric and magnetic current densities ($\mathbf{J}$, $\mathbf{J}_m$) in a bounded volume V' enclosed by surface S . According to the Huygens-Fresnel principle, it is equivalent to represent the current densities ($\mathbf{J}$, $\mathbf{J}_m$) in volume V' by surface currents $\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}$ and $\mathbf{J}_{ms} = \mathbf{E} \times \hat{\mathbf{n}}$ on surface S , which are referred as the secondary sources for the electromagnetic fields ($\mathbf{E}$, $\mathbf{H}$) in volume V .

Various equivalent diffraction formulas (i.e., the Kirchhoff, Stratton-Chu, Kottler and Franz formulas [31, 32, 33, 34, 35, 36]) are available to calculate the electromagnetic fields ($\mathbf{E}$, $\mathbf{H}$) for a closed surface (S). The explicit forms of electric field $\mathbf{E}$ for different diffraction formulas are given as

$$\mathbf{E}(\mathbf{r})|_{\text{Kirchhoff}} = \oint_S \left[\mathbf{E}(\mathbf{r}') \frac{\partial G(\mathbf{R})}{\partial n'} - \frac{\partial \mathbf{E}(\mathbf{r}')}{\partial n'} G(\mathbf{R}) \right] dS'. \quad (5.1.5.40)$$

$$\begin{aligned} \mathbf{E}(\mathbf{r})|_{\text{Stratton-Chu}} = \oint_S \left[\frac{\omega\mu}{j} \mathbf{J}_s(\mathbf{r}') G(\mathbf{R}) + \nabla' G(\mathbf{R}) \times \mathbf{J}_{ms}(\mathbf{r}') \right. \\ \left. + \nabla' G(\mathbf{R}) \frac{\rho_s(\mathbf{r}')}{\epsilon} \right] dS'. \end{aligned} \quad (5.1.5.41)$$

$$\mathbf{E}(\mathbf{r})|_{\text{Kottler}} = \frac{-j}{\omega\epsilon} \oint_S \left[k^2 \mathbf{J}_s(\mathbf{r}') G(\mathbf{R}) + \left(\mathbf{J}_s(\mathbf{r}') \cdot \nabla' \right) \nabla' G(\mathbf{R}) \right] dS' \quad (5.1.5.42)$$

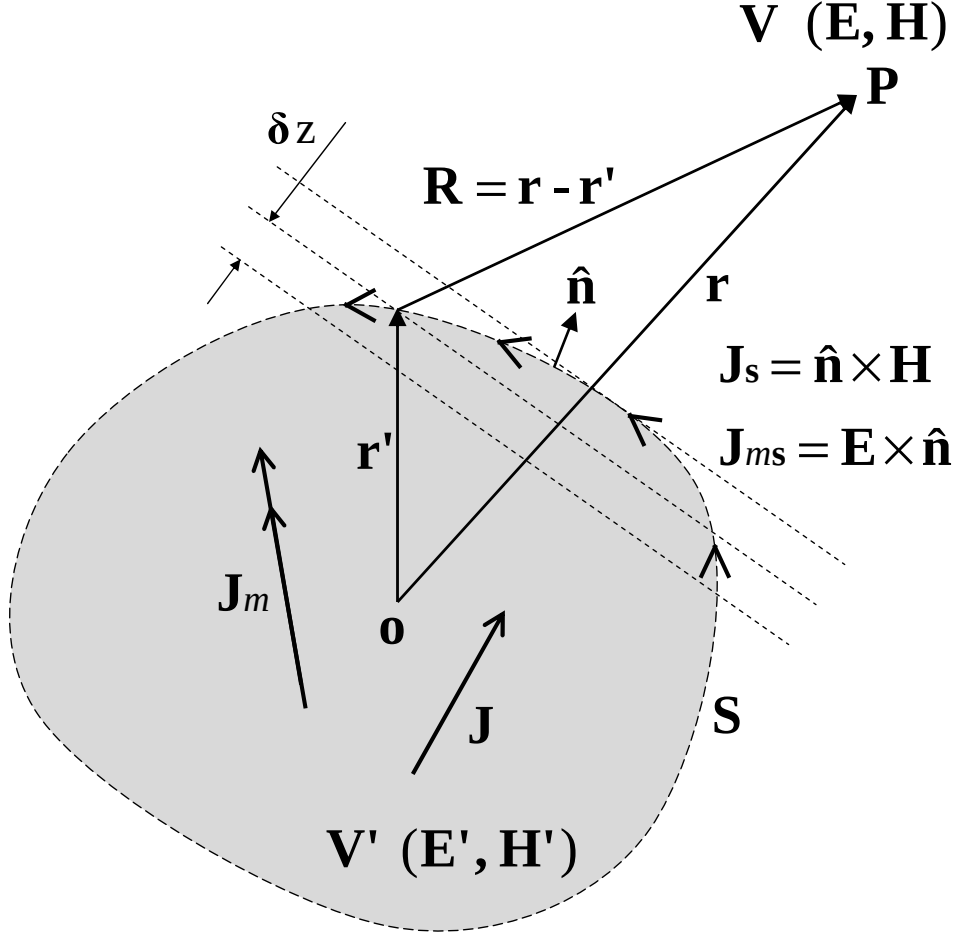


Figure 5.1: Closed surface diffraction problem: the electric and magnetic current density ($\mathbf{J}, \mathbf{J}_m$) are given in a bounded volume V' enclosed by surface S , where the Huygens-Fresnel principle applies ($\mathbf{J}_s = \hat{\mathbf{n}} \times \mathbf{H}$ and $\mathbf{J}_{ms} = \mathbf{E} \times \hat{\mathbf{n}}$ are the equivalent surface currents). P is the observation point in volume V where the electromagnetic fields ($\mathbf{E}, \mathbf{H}$) need to be evaluated. $\hat{\mathbf{n}}$ is the surface normal to S , pointing towards volume V . The dotted lines are spatial reference planes that are used in the planar TI-FFT algorithm (the number of the spatial reference planes depends on individual problem). δ_z is the the spatial slicing spacing between two adjacent spatial reference planes.

$$-j\omega\epsilon\mathbf{J}_{ms}(\mathbf{r}') \times \nabla' G(\mathbf{R}) \Big] dS'.$$

$$\mathbf{E}(\mathbf{r})|_{\text{Franz}} = \frac{-j}{\omega\epsilon} \nabla \times \oint_S \left[\nabla G(\mathbf{R}) \times \mathbf{J}_s(\mathbf{r}') + \frac{\omega\epsilon}{j} G(\mathbf{R}) \mathbf{J}_{ms}(\mathbf{r}') \right] dS' \quad (5.1.5.43)$$

where, $\frac{\partial}{\partial n'} \equiv \hat{\mathbf{n}} \cdot \nabla'$ and ∇' is the gradient operator on the source coordinate $\mathbf{r}'$. The electric surface charge has been defined as $\rho_s \equiv \epsilon \hat{\mathbf{n}} \cdot \mathbf{E}(\mathbf{r}')$.

It has been shown that the above diffraction formulas are equivalent for a close surface S [31, 32, 33, 34, 35, 36],

$$\mathbf{E}(\mathbf{r})|_{\text{Kirchhoff}} = \mathbf{E}(\mathbf{r})|_{\text{Stratton-Chu}} = \mathbf{E}(\mathbf{r})|_{\text{Kottler}} = \mathbf{E}(\mathbf{r})|_{\text{Franz}}. \quad (5.1.5.44)$$

The magnetic field $\mathbf{H}$ can be obtained from the electric field $\mathbf{E}$ in (5.1.5.40) -(5.1.5.43) using the duality relations given in (5.2.1.58),

$$\mathbf{H}(\mathbf{r})|_{\text{Kirchhoff}} = \oint_S \left[\mathbf{H}(\mathbf{r}') \frac{\partial G(\mathbf{R})}{\partial n'} - \frac{\partial \mathbf{H}(\mathbf{r}')}{\partial n'} G(\mathbf{R}) \right] dS'. \quad (5.1.5.45)$$

$$\begin{aligned} \mathbf{H}(\mathbf{r})|_{\text{Stratton-Chu}} = \oint_S \left[\frac{\omega\epsilon}{j} \mathbf{J}_{ms}(\mathbf{r}') G(\mathbf{R}) - \nabla' G(\mathbf{R}) \times \mathbf{J}_s(\mathbf{r}') \right. \\ \left. + \nabla' G(\mathbf{R}) \frac{\rho_{ms}(\mathbf{r}')}{\mu} \right] dS'. \end{aligned} \quad (5.1.5.46)$$

$$\begin{aligned} \mathbf{H}(\mathbf{r})|_{\text{Kottler}} = \frac{-j}{\omega\mu} \oint_S \left[k^2 \mathbf{J}_{ms}(\mathbf{r}') G(\mathbf{R}) + \left(\mathbf{J}_{ms}(\mathbf{r}') \cdot \nabla' \right) \nabla' G(\mathbf{R}) \right. \\ \left. + j\omega\mu \mathbf{J}_s(\mathbf{r}') \times \nabla' G(\mathbf{R}) \right] dS'. \end{aligned} \quad (5.1.5.47)$$

$$\mathbf{H}(\mathbf{r})|_{\text{Franz}} = \frac{-j}{\omega\mu} \nabla \times \oint_S \left[\nabla G(\mathbf{R}) \times \mathbf{J}_{ms}(\mathbf{r}') + j\omega\mu G(\mathbf{R}) \mathbf{J}_s(\mathbf{r}') \right] dS' \quad (5.1.5.48)$$

where the magnetic surface charge is defined as $\rho_{ms} \equiv \mu \hat{\mathbf{n}} \cdot \mathbf{H}(\mathbf{r}')$.

Note that the Kottler formula in (5.1.5.42) and (5.1.5.47) reduces to those in (5.1.2.12) and (5.1.2.13) after the surface currents ($\mathbf{J}_s, \mathbf{J}_{ms}$) are replaced with the volume current densities ($\mathbf{J}, \mathbf{J}_m$).

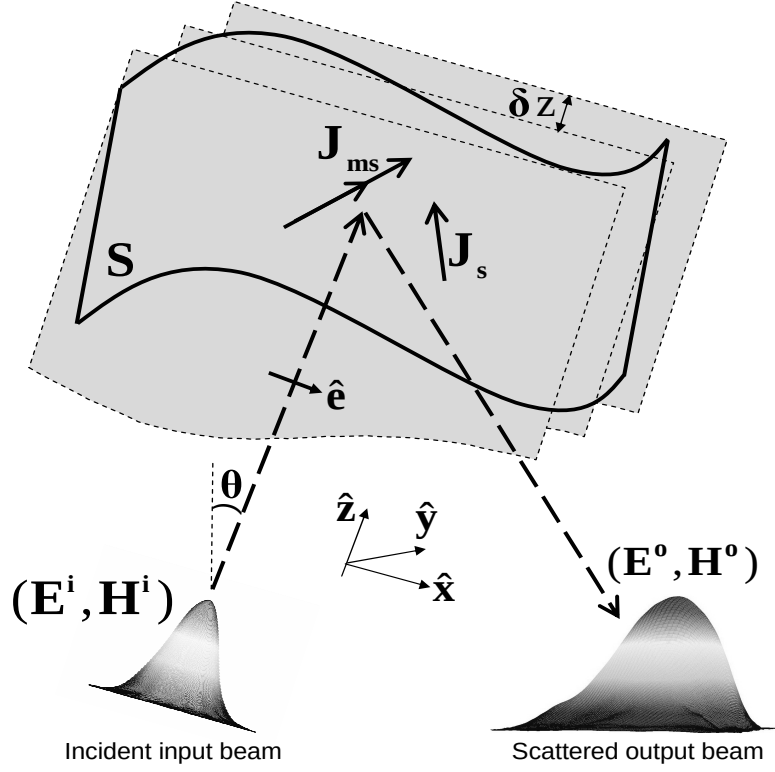


Figure 5.2: Illustration of electromagnetic wave scattering for the quasi-planar geometry: The equivalent electric and magnetic surface currents are denoted as $(\mathbf{J}_s, \mathbf{J}_{ms})$. Two equivalent principles are available: 1) in the induction equivalent, the equivalent magnetic surface current is given as $\mathbf{J}_{ms} = \hat{\mathbf{n}} \times \mathbf{E}^i$ with PEC boundary condition (or approximately, $\mathbf{J}_{ms} = \mathbf{J}_{ms}^{\text{ind}} = 2\hat{\mathbf{n}} \times \mathbf{E}^i$ with free space boundary, when surface S is smooth enough where the image theorem can be used); and 2) in the physical equivalent, the rigorous equivalent electric surface current $\mathbf{J}_s$ has to be obtained using the EFIE or the MFIE method (through MoM); or approximately, $\mathbf{J}_s = \mathbf{J}_s^{\text{PO}} = 2\hat{\mathbf{n}} \times \mathbf{H}^i$ can be used for a smooth enough surface (S). δz is the spatial slicing spacing between two adjacent spatial reference planes used in the planar TI-FFT algorithm.

5.1.6 Electromagnetic wave scattering

The electromagnetic wave scattering problem has been illustrated in Fig. 5.2 for the quasi-planar geometry (surface S). The calculation of the scattered electromagnetic output fields ($\mathbf{E}^\circ$, $\mathbf{H}^\circ$) from the PEC surface (S) requires the knowledge of the equivalent electric and magnetic surface currents ($\mathbf{J}_s$, $\mathbf{J}_{ms}$).

Two equivalent principles are available to obtain ($\mathbf{J}_s$, $\mathbf{J}_{ms}$), 1) the induction equivalent; and 2) the physical equivalent [16], [22] which will be explained in details in Section 5.1.6 and Section 5.1.6 respectively.

The induction equivalent There are two boundary conditions for the induction equivalent, i.e., the PEC boundary condition and the free space boundary condition (approximate solution),

$$\mathbf{J}_{ms}(\mathbf{r}_s) = \hat{\mathbf{n}} \times \mathbf{E}^i(\mathbf{r}_s), \quad \text{PEC boundary,} \quad (5.1.6.49)$$

$$\mathbf{J}_{ms}(\mathbf{r}_s) \sim \mathbf{J}_{ms}^{\text{ind}}(\mathbf{r}_s) = 2\hat{\mathbf{n}} \times \mathbf{E}^i(\mathbf{r}_s), \quad \text{free space (approximation)} \quad (5.1.6.50)$$

where $\mathbf{r}_s$ denotes the coordinate on PEC surface S . The free space boundary condition comes from the approximation of the image theorem for a smooth surface (S). Note that the equivalent electric surface current $\mathbf{J}_s = 0$ in the induction equivalent.

The physical equivalent The rigorous solution of the equivalent electric surface current $\mathbf{J}_s$ has to be obtained using Electric Field Integral Equation (EFIE) or the Magnetic Field Integral Equation (MFIE) method through the Method of Moment (MoM),

$$\text{EFIE: } \frac{j}{\omega\epsilon} \left[\iint_S \left[\nabla \left(\nabla' \cdot \mathbf{J}_s(\mathbf{r}') G(\mathbf{R}_s) \right) + k^2 \mathbf{J}_s(\mathbf{r}') G(\mathbf{R}_s) \right] dS' \right]_{//} = \mathbf{E}_{//}^i(\mathbf{r}_s) \quad (5.1.6.51)$$

$$\text{MFIE: } \mathbf{J}_s(\mathbf{r}_s) - \lim_{\mathbf{r} \rightarrow S} \left[\hat{\mathbf{n}} \times \iint_S \left[\mathbf{J}_s(\mathbf{r}') \times \nabla' G(\mathbf{R}) \right] dS' \right] = \hat{\mathbf{n}} \times \mathbf{H}^i(\mathbf{r}_s) \quad (5.1.6.52)$$

where “//” denotes the tangential component of the electric field on surface S and $\mathbf{R}_s \equiv \mathbf{r}_s - \mathbf{r}'$. Particularly, when surface S is smooth enough, the PO approximation can be used,

$$\mathbf{J}_s(\mathbf{r}_s) \sim \mathbf{J}_s^{\text{PO}}(\mathbf{r}_s) = 2\hat{\mathbf{n}} \times \mathbf{H}^i(\mathbf{r}_s). \quad (5.1.6.53)$$

Note that $\mathbf{J}_{ms} = 0$ in the physical equivalent.

The scattered electromagnetic wave After the equivalent electric and magnetic surface currents ($\mathbf{J}_s, \mathbf{J}_{ms}$) are obtained according to (5.1.6.49)-(5.1.6.53), the scattered electromagnetic field ($\mathbf{E}, \mathbf{H}$) can be obtained through (5.1.2.12)-(5.2.1.55), with the surface currents ($\mathbf{J}_s, \mathbf{J}_{ms}$) replacing the volume current densities ($\mathbf{J}, \mathbf{J}_m$), or equivalently through Kottler formula (5.1.5.42) and (5.1.5.47).

The scattered electromagnetic beams can be evaluated as, A) obtain the Fourier spectra (or cylindrical modal expansion coefficients) through the *spectral* TI-FFT algorithm [8], [11]; and B) obtain the scattered beams through the *spatial* TI-FFT algorithm, which has been discussed in Section 5.2.3. Because the electric field $\mathbf{E}$ is related to the vector potential $\mathbf{F}$ through $\mathbf{E} = -\frac{1}{\epsilon_0} \nabla \times \mathbf{F}$, only the vector potential $\mathbf{F}$ is discussed here.

5.2 Behavior of Electromagnetic Wave in Spectral Domain

5.2.1 The 2D Fourier spectra of the 3D spatial convolutions

The electric field $\mathbf{E}$ in (5.1.2.12) and the magnetic field $\mathbf{H}$ in (5.1.2.13) can also be expressed in the form of the dyadic Green's functions of the electric type ($\bar{\bar{G}}_e$) and the magnetic type ($\bar{\bar{G}}_m$) [25, 37],

$$\mathbf{E} = \iiint_V \left[(-j\omega\mu) \bar{\bar{G}}_e(\mathbf{R}) \cdot \mathbf{J}(\mathbf{r}') - \bar{\bar{G}}_m(\mathbf{R}) \cdot \mathbf{J}_m(\mathbf{r}') \right] dV', \quad (5.2.1.54)$$

$$\mathbf{H} = \iiint_V \left[(-j\omega\epsilon) \bar{\bar{G}}_e(\mathbf{R}) \cdot \mathbf{J}_m(\mathbf{r}') + \bar{\bar{G}}_m(\mathbf{R}) \cdot \mathbf{J}(\mathbf{r}') \right] dV', \quad (5.2.1.55)$$

The dyadic Green's functions of the electric type ($\bar{\bar{G}}_e$) and the magnetic type ($\bar{\bar{G}}_m$) are given as

$$\bar{\bar{G}}_e(\mathbf{R}) = \left(\bar{\bar{I}} + \frac{1}{k^2} \nabla \nabla \right) G(\mathbf{R}). \quad (5.2.1.56)$$

$$\bar{\bar{G}}_m(\mathbf{R}) = \nabla G(\mathbf{R}) \times \bar{\bar{I}}. \quad (5.2.1.57)$$

It is clear that (5.2.1.55) can be obtained from (5.2.1.54) using the duality relations below,

$$\mathbf{E} \rightarrow \mathbf{H}, \quad \mathbf{H} \rightarrow -\mathbf{E}, \quad \mathbf{J} \rightarrow \mathbf{J}_m, \quad \mathbf{J}_m \rightarrow -\mathbf{J}, \quad (5.2.1.58)$$

$$\overline{\overline{G}}_e(\mathbf{R}) \rightarrow \overline{\overline{G}}_e(\mathbf{R}), \quad \overline{\overline{G}}_m(\mathbf{R}) \rightarrow \overline{\overline{G}}_m(\mathbf{R}), \quad \epsilon \rightarrow \mu, \quad \mu \rightarrow \epsilon.$$

The computation of the beam propagation onto the PEC mirror surface through the direct integration method is a time-consuming process [4], [6]. Fortunately, for “quasi-planar”/“quasi-cylindrical” PEC mirror surfaces, Shaolin *et al.* [5], [8], [7], [11] have developed the efficient TI-FFT algorithm to speed up the computation. It has been shown that the TI-FFT algorithm has a computational complexity of $\mathcal{O}(N^2 \log_2 N^2)$ for an $N \times N$ computational grid, instead of $\mathcal{O}(N^4)$ for the direct integration method and the algorithm allows for a low sampling rate according to the Nyquist sampling theorem. There are 2 types of TI-FFT algorithm: the *spatial* TI-FFT and the *spectral* TI-FFT. The *spatial* TI-FFT is for computations of electromagnetic beam propagations onto “quasi-planar”/“quasi-cylindrical” surfaces; while the *spectral* TI-FFT is for computations of Fourier spectra of the scattered beams due to the PEC mirror surfaces, which will be discussed in Section 5.2.3.

It is not difficult to see that, the diffraction formulas in (5.1.5.40)-(5.1.5.48) can be expressed as the sum of the 3D spatial convolutions of some source terms with the Green’s function related expressions. According to the property of the Fourier transform, the 2D spatial convolution corresponds to the product of the 2D Fourier transforms in the spectral domain [30]. So it is interesting to investigate the 2D Fourier transform of the 3D spatial convolution. Consider the 3D spatial convolution of some source term $\mathbf{s}(\mathbf{r})$ with the scalar Green’s function $G(\mathbf{r})$,

$$\mathbf{s}(\mathbf{r}) \overset{3D}{\otimes} G(\mathbf{r}) = \oint\!\!\!\oint_S \mathbf{s}(\mathbf{r}') G(\mathbf{R}) dS', \quad (5.2.1.59)$$

Now, apply the 2D Fourier transform on (5.2.1.59) and express the scalar Green’s function $G(\mathbf{R})$ in the spectral domain,

$$\mathcal{S}(k_x, k_y) \equiv \text{FT}_{2D} \left[\mathbf{s}(\mathbf{r}) \overset{3D}{\otimes} G(\mathbf{r}) \right], \quad (5.2.1.60)$$

$$= \frac{1}{2\pi} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \left\{ e^{jk_x x} e^{jk_y y} \oint\!\!\!\oint_S \left[\mathbf{s}(\mathbf{r}') \left(\frac{1}{2\pi} \int_{k'_x=-\infty}^{\infty} \int_{k'_y=-\infty}^{\infty} \right. \right. \right.$$

$$\times \frac{-je^{-jk'_z(z-z')}}{4\pi k'_z} e^{-jk'_x(x-x')} e^{-jk'_y(y-y')} dk'_x dk'_y \Big] dS' \Big\} dx dy,$$

First, do the integral over (x, y) , (5.2.1.60) reduces to

$$\begin{aligned} \mathcal{S}(k_x, k_y) = \oint_S \Big\{ \mathbf{s}(\mathbf{r}') \int_{k'_x=-\infty}^{\infty} \int_{k'_y=-\infty}^{\infty} \left[\frac{-je^{-jk'_z(z-z')}}{4\pi k'_z} \right. \quad (5.2.1.61) \\ \left. \times e^{jk'_x x'} e^{jk'_y y'} \delta(k'_x - k_x) \delta(k'_y - k_y) dk'_x dk'_y \right] \Big\} dS', \end{aligned}$$

Next, do the integral over (k'_x, k'_y) and (5.2.1.61) reduces to

$$\mathcal{S}(k_x, k_y) = \frac{-je^{-jk_z z}}{4\pi k_z} \oint_S \mathbf{s}(\mathbf{r}') e^{j\mathbf{k} \cdot \mathbf{r}'} dS' = \frac{-je^{-jk_z z}}{4\pi k_z} \mathbf{L}(\mathbf{s}), \quad (5.2.1.62)$$

where $\mathbf{L}$ in (5.2.1.62) is the radiation vector [16] of the source term $\mathbf{s}(\mathbf{r}')$ defined as

$$\mathbf{L}(\mathbf{s}) = \oint_S \mathbf{s}(\mathbf{r}') e^{j\mathbf{k} \cdot \mathbf{r}'} dS', \quad (5.2.1.63)$$

It is not difficult to see that the radiation vector $\mathbf{L}$ in (5.2.1.63) reduces to the regular 2D Fourier spectrum when surface S is a plane located at $z' = 0$.

$$\mathbf{L}(\mathbf{s})|_{z'=0} = \int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} \mathbf{s}(x', y') e^{jk_x x'} e^{jk_y y'} dx' dy' = \text{FT}_{2D} \left[\mathbf{s}(x, y) \right] \quad (5.2.1.64)$$

where the dummy primed (x', y') have been replaced with (x, y) . Substitute (5.2.1.63) into (5.2.1.62),

$$\mathcal{S}(k_x, k_y) = \frac{-je^{-jk_z z}}{4\pi k_z} \mathbf{L}(\mathbf{s}) = \mathcal{G}(k_x, k_y, 0) \mathbf{L}(\mathbf{s}). \quad (5.2.1.65)$$

It is not difficult to see that the radiation vector $\mathbf{L}$ in (5.2.1.59) is closely related to the far-field by letting $\mathbf{r} \rightarrow \infty$,

$$\mathbf{s}(\mathbf{r}) \bigotimes_{\mathbf{r} \rightarrow \infty}^{\text{3D}} G(\mathbf{r}) \Big| = \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|} \oint_S \mathbf{s}(\mathbf{r}') e^{j\mathbf{k} \cdot \mathbf{r}'} dS' = \frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|} \mathbf{L}(\mathbf{s}). \quad (5.2.1.66)$$

From (5.2.1.66), if $\frac{e^{-jk|\mathbf{r}|}}{4\pi|\mathbf{r}|}$ can be ignored, the radiation vector $\mathbf{L}$ is exactly the far-field, which means that when the far-field is obtained, the radiation vector and the 2D Fourier spectrum of the 3D convolution are also obtained, from (5.2.1.66) and (5.2.1.65) respectively.

5.2.2 The 2D Fourier spectra of the diffraction formulas

The derivation of the 2D Fourier spectra of the diffraction formulas in (5.1.5.40) - (5.1.5.48) is based on (5.2.1.65), which is discussed in this section.

Separation of the source terms and Green's function related expressions In order to use (5.2.1.65), the source terms have to be separated from the Green's function related expressions. For example, consider the Kirchhoff formula in (5.1.5.40), which can be rewritten as

$$\mathbf{E}(\mathbf{r})|_{\text{Kirchhoff}} = \oint_S \left\{ \sum_{v'=x',y',z'} \left[\frac{\partial G(\mathbf{R})}{\partial v'} \left(n_{v'}(\mathbf{r}') \mathbf{E}(\mathbf{r}') \right) \right] - \sum_{v'=x',y',z'} \left[G(\mathbf{R}) \left(\frac{\partial \mathbf{E}(\mathbf{r}')}{\partial v'} n_{v'}(\mathbf{r}') \right) \right] \right\} dS', \quad (5.2.2.67)$$

Now the source terms are,

$$\mathbf{s}_{v'}^1(\mathbf{r}') = n_{v'}(\mathbf{r}') \mathbf{E}(\mathbf{r}') \quad \text{and} \quad \mathbf{s}_{v'}^2(\mathbf{r}') = -\frac{\partial \mathbf{E}(\mathbf{r}')}{\partial v'} n_{v'}(\mathbf{r}'), \quad v' = x', y', z', \quad (5.2.2.68)$$

The corresponding Green's function related expressions are,

$$G_{v'}^1(\mathbf{R}) = \frac{\partial G(\mathbf{R})}{\partial v'} = -\frac{\partial G(\mathbf{R})}{\partial v} \quad \text{and} \quad G_{v'}^2(\mathbf{R}) = G(\mathbf{R}), \quad v = x, y, z, \quad (5.2.2.69)$$

Obtain the radiation vectors of the source terms The radiation vectors of $\mathbf{s}_{v'}^1(\mathbf{r}')$ and $\mathbf{s}_{v'}^2(\mathbf{r}')$ can be obtained through (5.2.1.63), which are given as

$$\mathbf{L}(\mathbf{s}_{v'}^1) = \oint_S n_{v'}(\mathbf{r}') \mathbf{E}(\mathbf{r}') e^{j\mathbf{k} \cdot \mathbf{r}'} dS', \quad (5.2.2.70)$$

$$\mathbf{L}(\mathbf{s}_{v'}^2) = -\oint_S \frac{\partial \mathbf{E}(\mathbf{r}')}{\partial v'} n_{v'}(\mathbf{r}') e^{j\mathbf{k} \cdot \mathbf{r}'} dS', \quad (5.2.2.71)$$

Obtain 2D Fourier spectra of the Green's function related expressions

The 2D Fourier spectra of $G_{v'}^1(\mathbf{r})$ and $G_{v'}^2(\mathbf{r})$ can be obtained from Table 5.1 by setting $\mathbf{r}' = 0$,

$$\mathcal{G}_{v'}^1(k_x, k_y) = \text{FT}_{2D} \left[G_{v'}^1(\mathbf{r}) \right] = jk_v \mathcal{G}^>(k_x, k_y, 0) = \frac{k_v}{4\pi k_z} e^{-jk_z z}, \quad (5.2.2.72)$$

$$\mathcal{G}_{v'}^2(k_x, k_y) = \text{FT}_{2D} \left[G_{v'}^2(\mathbf{r}) \right] = \mathcal{G}^>(k_x, k_y, 0) = \frac{-j}{4\pi k_z} e^{-jk_z z}, \quad (5.2.2.73)$$

The closed-form expressions of 2D Fourier spectra Substitute (5.2.2.70) -(5.2.2.73) into (5.2.1.65) and do the sum for all source terms, the 2D Fourier spectrum of the Kirchhoff formula $\mathcal{F}|_{\text{Kirchhoff}}$ is obtained,

$$\begin{aligned} \mathcal{F}|_{\text{Kirchhoff}} &= \sum_{v'=x',y',z'} \left[\mathbf{L}(\mathbf{s}_{v'}^1) \mathcal{G}_{v'}^1(k_x, k_y) + \mathbf{L}(\mathbf{s}_{v'}^2) \mathcal{G}_{v'}^2(k_x, k_y) \right] \quad (5.2.2.74) \\ &= \mathcal{G}^>(k_x, k_y, 0) \sum_{v'=x',y',z'} \left[jk_{v'} \mathbf{L} \left(n_{v'}(\mathbf{r}') \mathbf{E}(\mathbf{r}') \right) - \mathbf{L} \left(n_{v'}(\mathbf{r}') \frac{\partial \mathbf{E}(\mathbf{r}')}{\partial v'} \right) \right]. \end{aligned}$$

The 2D Fourier spectra for the electric field $\mathbf{E}$ of other diffraction formulas and for the magnetic fields $\mathbf{H}$ can be obtained similarly. For example, the 2D Fourier spectrum of the electric field $\mathbf{E}$ of the Kottler formula $\mathcal{F}|_{\text{Kottler}}$ is given as

$$\begin{aligned} \mathcal{F}|_{\text{Kottler}} &= \frac{-j}{\omega\epsilon} \mathcal{G}^>(k_x, k_y, 0) \left\{ k^2 \mathbf{L} \left(\mathbf{J}_s(\mathbf{r}') \right) \right. \quad (5.2.2.75) \\ &\quad \left. - \mathbf{k} \sum_{v'=x',y',z'} \left[k_{v'} \mathbf{L} \left(\mathbf{J}_{sv'}(\mathbf{r}') \right) \right] + \omega\epsilon \mathbf{L} \left(\mathbf{J}_{ms}(\mathbf{r}') \right) \times \mathbf{k} \right\}. \end{aligned}$$

The 2D Fourier spectrum of the electric field $\mathbf{E}$ of the Stratton-Chu and Franz formulas are given in Appendix B.

5.2.3 Two special cases

In this section, the 2D Fourier spectra of the diffraction formulas are applied on two special cases: 1) electromagnetic wave propagation from a plane; and 2) electromagnetic wave scattering from a flat PEC plate.

Electromagnetic wave propagation from a plane Assume that $\mathbf{E}(z=0)$ is given on a plane located at $z=0$, where the surface normal is given as $(n_x, n_y, n_z) = (0, 0, 1)$. The 2D Fourier spectrum of the Kirchhoff formula in (5.2.2.74) reduces to

$$\mathcal{F}|_{\text{Kirchhoff}} = j4\pi k_z \mathcal{G}^>(k_x, k_y, 0) \text{FT}_{2\text{D}} \left[\mathbf{E}(z=0) \right], \quad (5.2.3.76)$$

Substitute the explicit form of $\mathcal{G}^>(k_x, k_y, 0)$ into (5.2.3.76),

$$\mathcal{F}|_{\text{Kirchhoff}} = \text{FT}_{2\text{D}} \left[\mathbf{E}(z=0) \right] e^{-jk_z z}. \quad (5.2.3.77)$$

Eqn. (5.2.3.77) is well-known as the Plane Wave Spectrum (PWS) representation of the electromagnetic wave [38, 39]. According to the Maxwell's equation (5.1.2.2),

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \nabla \cdot \left\{ \text{IFT}_{2\text{D}} \left[\text{FT}_{2\text{D}} \left[\mathbf{E}(\mathbf{r}) \right] \right] \right\} = 0 \quad (5.2.3.78)$$

$$\rightarrow \sum_{v=x,y,z} \left[jk_v \text{FT}_{2\text{D}} \left[E_v(\mathbf{r}) \right] \right] = 0, \quad (5.2.3.79)$$

It can be seen from (5.2.3.79) that the following relation holds,

$$\text{FT}_{2\text{D}} \left[E_z(\mathbf{r}) \right] = -\frac{1}{k_z} \left[k_x \text{FT}_{2\text{D}} \left[E_x(\mathbf{r}) \right] + k_y \text{FT}_{2\text{D}} \left[E_y(\mathbf{r}) \right] \right] \quad (5.2.3.80)$$

Electromagnetic wave scattering from a flat PEC plate Assume that the incident field $\mathbf{E}^i(z=0)$ approaches a flat PEC plate located at $z=0$ from the positive side (or propagates in the $-\hat{\mathbf{z}}$ direction). The surface normal is given as $(n_x, n_y, n_z) = (0, 0, 1)$. Now, consider the induction equivalent,

$$\mathbf{J}_{ms}^{\text{ind}}(z=0) = 2 \hat{\mathbf{z}} \times \mathbf{E}^i(z=0) \quad \text{and} \quad \mathbf{J}_s(z=0) = 0, \quad (5.2.3.81)$$

The 2D Fourier spectrum of the Kottler formula is available from (5.2.2.75),

$$\begin{aligned} \mathcal{F}|_{\text{Kottler}} &= -j2\pi \mathcal{G}^>(k_x, k_y, 0) \text{FT}_{2\text{D}} \left[\mathbf{J}_{ms}(z=0) \right] \times \mathbf{k} \\ &= -j4\pi \mathcal{G}^>(k_x, k_y, 0) \left[\hat{\mathbf{z}} \times \text{FT}_{2\text{D}} \left[\mathbf{E}^i(z=0) \right] \times \mathbf{k} \right], \end{aligned} \quad (5.2.3.82)$$

Expressing $\mathcal{G}^>(k_x, k_y, 0)$ in its explicit form, (5.2.3.82) becomes,

$$\begin{aligned} \mathcal{F}|_{\text{Kottler}} &= e^{-jk_z z} \left\{ -\text{FT}_{2\text{D}} \left[E_x^i(z=0) \right] \hat{\mathbf{x}} \right. \\ &\quad \left. -\text{FT}_{2\text{D}} \left[E_y^i(z=0) \right] \hat{\mathbf{y}} + \text{FT}_{2\text{D}} \left[E_z^i(z=0) \right] \hat{\mathbf{z}} \right\}, \end{aligned} \quad (5.2.3.83)$$

where, the PWS property in (5.2.3.80) has been used to derive (5.2.3.83) for the incident field that propagates in $-\hat{\mathbf{z}}$ direction,

$$\begin{aligned} \text{FT}_{2\text{D}} \left[\begin{array}{c} \mathbf{E}_z^i(z=0) \end{array} \right] &= \frac{1}{k_z} \left[k_x \text{FT}_{2\text{D}} \left[\begin{array}{c} \mathbf{E}_x^i(z=0) \end{array} \right] \right. \\ &\quad \left. + k_y \text{FT}_{2\text{D}} \left[\begin{array}{c} \mathbf{E}_y^i(z=0) \end{array} \right] \right]. \end{aligned} \quad (5.2.3.84)$$

From (5.2.3.83), it is clear that the signs of tangential components (x- and y- components here) of the scattered field (or reflected field here) have been reversed after scattering (reflection), which confirms the PEC boundary condition. What's more, the z-component remains unchanged after scattering (reflection). The total field after scattering (reflection) is thus twice the values of the z-component before scattering (reflection),

$$\mathbf{E}_z^{\text{total}}(z=0) = 2 \mathbf{E}_z^i(z=0). \quad (5.2.3.85)$$

5.3 The planar TI-FFT algorithm

As has been shown in Section 5.2.2, the evaluation of the 2D Fourier spectra of the diffraction formulas requires the calculation of the radiation vector $\mathbf{L}$, which can be done through the spectral TI-FFT algorithm (introduced in Section 5.3.1). After the 2D Fourier spectra have been obtained, the electromagnetic fields ($\mathbf{E}$, $\mathbf{H}$) on a plane can be evaluated through the IFT; for a quasi-planar surface, the electromagnetic fields ($\mathbf{E}$, $\mathbf{H}$) can be evaluated through the spatial TI-FFT algorithm (introduced in Section 5.3.2).

5.3.1 The spectral TI-FFT algorithm

There are two methods to compute the radiation vector $\mathbf{L}$ defined in (5.2.1.63), 1) the spatial domain division method, in which $e^{jk_z z'}$ of (5.2.1.63) can be expressed as a Taylor series on the spatial reference planes, with respect to the source coordinate z' ; the source field surface then divided into a number of spatial surface subdomains, enabling the computation of the radiation vector $\mathbf{L}$ through the superposition of the spatial surface subdomains using the FFT; and 2) the spectral domain division method, in which $e^{jk_z z'}$ of (5.2.1.63) can be expressed as a Taylor series on the spectral reference planes, with respect to the axial wave vector k_z ; the spherical spectral surface ($k_x^2 + k_y^2 + k_z^2 = k^2$) is then divided

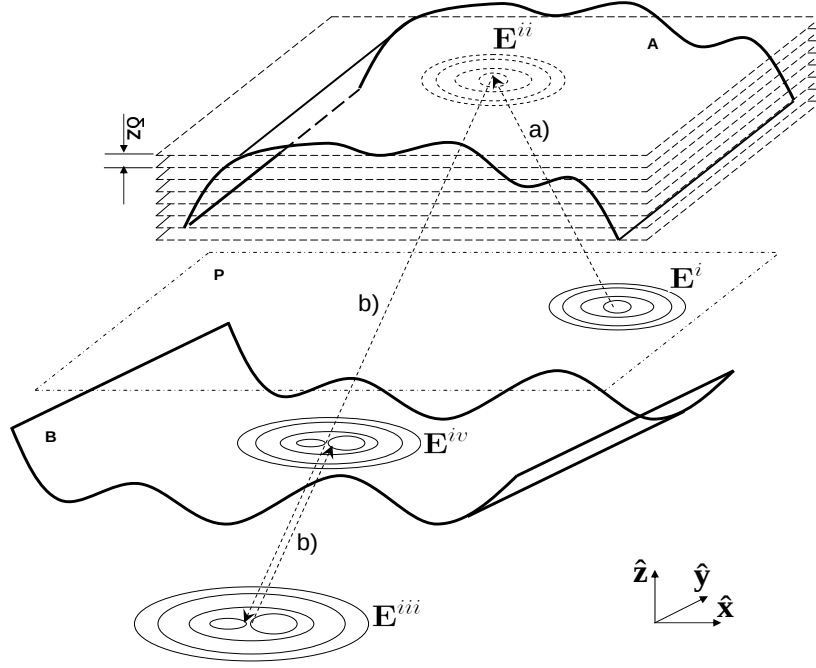


Figure 5.3: Field propagation between two smooth PEC surfaces: the top surface (A) serves as the scattering surface and the bottom surface (B) serves as the observation surface. Two steps are involved in the computation, a) the propagation of the electromagnetic field from the initial plane (P) onto surface A is obtained through the *spatial* TI-FFT: $\mathbf{E}^i \rightarrow \mathbf{E}^{ii}$; and b) the scattering of the electromagnetic field from surface A onto surface B is found through the combination of the *spatial* and *spectral* TI-FFTs: $\mathbf{E}^{ii} \rightarrow \mathbf{E}^{iii}$ (FF) $\rightarrow \mathbf{E}^{iv}$. δz is the slicing interval in the algorithm.

into a number of spectral surface subdomains, enabling the computation of the radiation vector $\mathbf{L}$ using the whole source field surface through the FFT. It will be shown that the optimized computational complexity is the same for both the spatial and spectral domain division methods.

Spatial domain division method The radiation vector $\mathbf{L}$ in (5.2.1.63) can be rewritten as

$$\mathbf{L}(\mathbf{s}) = e^{-j \begin{bmatrix} k - k_z \end{bmatrix} z_{\min}} \oint_S \begin{bmatrix} e^{jk_x x} e^{jk_y y} \end{bmatrix} \quad (5.3.1.86)$$

$$\begin{aligned}
& \times \left[\mathbf{s}(\mathbf{r}) e^{j[k_z z(x,y)]} e^{-j[k - k_z][z(x,y) - z_{\min}]} \right] dS \\
& = e^{-j\Delta k_z z_{\min}} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \tilde{\mathbf{s}}(\mathbf{r}) e^{jk_x x} e^{jk_y y} e^{-j\Delta k_z \Delta z(x,y)} dx dy,
\end{aligned}$$

where the dummy primed variables (x', y', z') and dS' have been changed to the unprimed variables (x, y, z) and dS . The following quantities have also been defined,

$$\tilde{\mathbf{s}}(\mathbf{r}) \equiv \frac{\mathbf{s}(\mathbf{r})}{\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}} e^{j[k_z z(x,y)]}, \quad \Delta k_z \equiv k - k_z, \quad \Delta z(x, y) \equiv z(x, y) - z_{\min}. \quad (5.3.1.87)$$

Now, express $e^{-j\Delta k_z \Delta z(x,y)}$ into a Taylor series on the spatial reference plane located at $z = z_r$,

$$e^{-j\Delta k_z \Delta z(x,y)} = e^{-j\Delta k_z \Delta z_r} \sum_{n=0}^{\mathcal{N}_o} \left[\frac{1}{n!} (-j\Delta k_z)^n \left(z(x, y) - z_r \right)^n \right], \quad (5.3.1.88)$$

where $\Delta z_r = z_r - z_{\min}$ and $\mathcal{N}_o$ is the order of Taylor series.

Substitute (5.3.1.88) into (5.3.1.86), the planar TI-FFT algorithm (spatial domain division method) for the radiation vector $\mathbf{L}$ is obtained as

$$\begin{aligned}
\mathbf{L}(\mathbf{s}) = 2\pi \sum_{r=1}^{\mathcal{N}_r} \left\{ e^{-j\Delta k_z z_r} \sum_{n=0}^{\mathcal{N}_o} \left[\frac{1}{n!} (-j\Delta k_z)^n \right. \right. \\
\left. \left. \times \text{FT}_{2D} \left[\tilde{\mathbf{s}}(\mathbf{r}') \left(z(x, y) - z_r \right)^n \right] \right] \right\}, \quad (5.3.1.89)
\end{aligned}$$

where $\mathcal{N}_r$ is the number of spatial reference planes required in the computation, which depends on the spatial slicing spacing $\delta_z \equiv \max \left[z(x, y) - z_r \right] = z_{r+1} - z_r$ and the characteristic surface variation Δz_c (within which the electromagnetic field is of interest): $\mathcal{N}_r \propto \Delta z_c / \delta_z$. The readers should note that the actual maximum interpolation distance is $\frac{\delta_z}{2}$, which is located at the middle of two adjacent spatial reference planes, but δ_z is used in this article to simplify the notation. Apparently, to achieve the desired computational accuracy (denoted as γ_{TI}), the choice of the spatial slicing spacing δ_z between two adjacent spatial reference planes depends on $\Delta k_{z,c}$ defined as

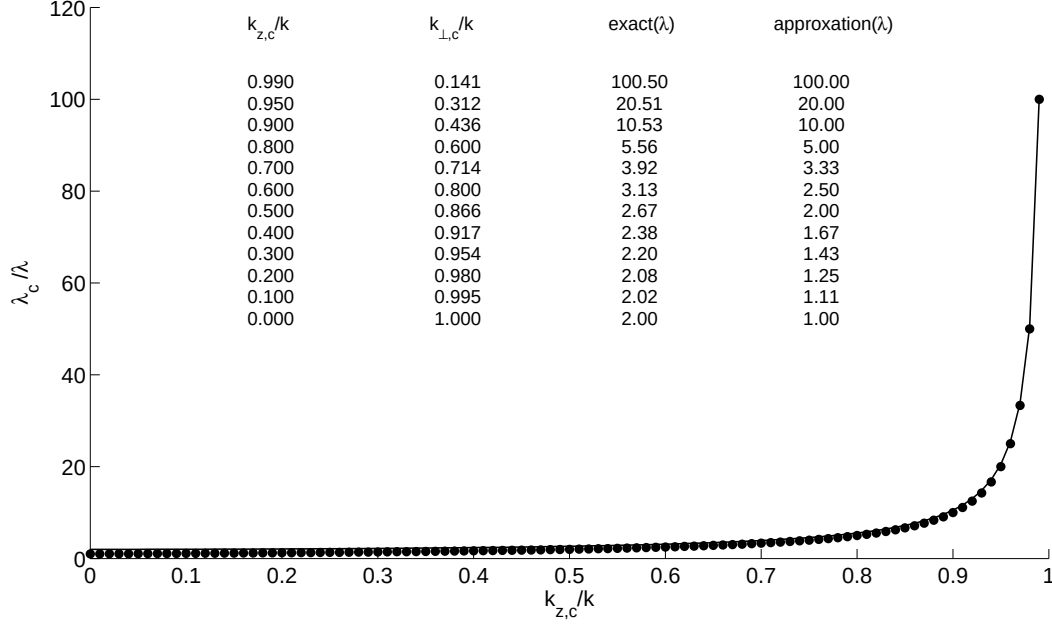


Figure 5.4: The plots of the characteristic wavelength λ_c for different $k_{z,c}$. The exact value (line) is given in (5.3.1.91) and the approximation (dots) is given in (5.3.1.92). The plots show that $\lambda_c \gg \lambda$ for a narrow-band beam. The maximum deviation of the approximation from the exact value is 1λ , which occurs at $k_{z,c} = 0$ ($k_{\perp,c} = k$).

$$\Delta k_{z,c} \equiv k - k_{z,c} = k - \sqrt{k^2 - k_{\perp,c}^2} = k\alpha, \quad \alpha = 1 - \sqrt{1 - \left(\frac{k_{\perp,c}}{k}\right)^2} \quad (5.3.1.90)$$

where, $k_{\perp,c}$ is the characteristic bandwidth defined on x-y plane and the 2D Fourier spectrum outside the characteristic bandwidth $k_{\perp,c}$ is negligible. For this reason, $k_{\perp,c}$ can also be regarded as the characteristic bandwidth of the electromagnetic wave. It is clear that the smaller the bandwidth $k_{\perp,c}$, the larger the δ_z could be, which also means a smaller $\mathcal{N}_r$. So, a narrow-band beam and a small surface variation Δz_c (quasi-planar geometry) are in favor of the planar TI-FFT algorithm. Note that the computational grid subdomains divided by the spatial reference planes should satisfy the position requirement ($z \in [z_r, z_r + \delta_z]$), but they don't need to be continuous in the spatial domain.

In view of the importance of the δ_z , it is interesting to define the characteristic wave length λ_c for a narrow-band beam, from (5.3.1.90),

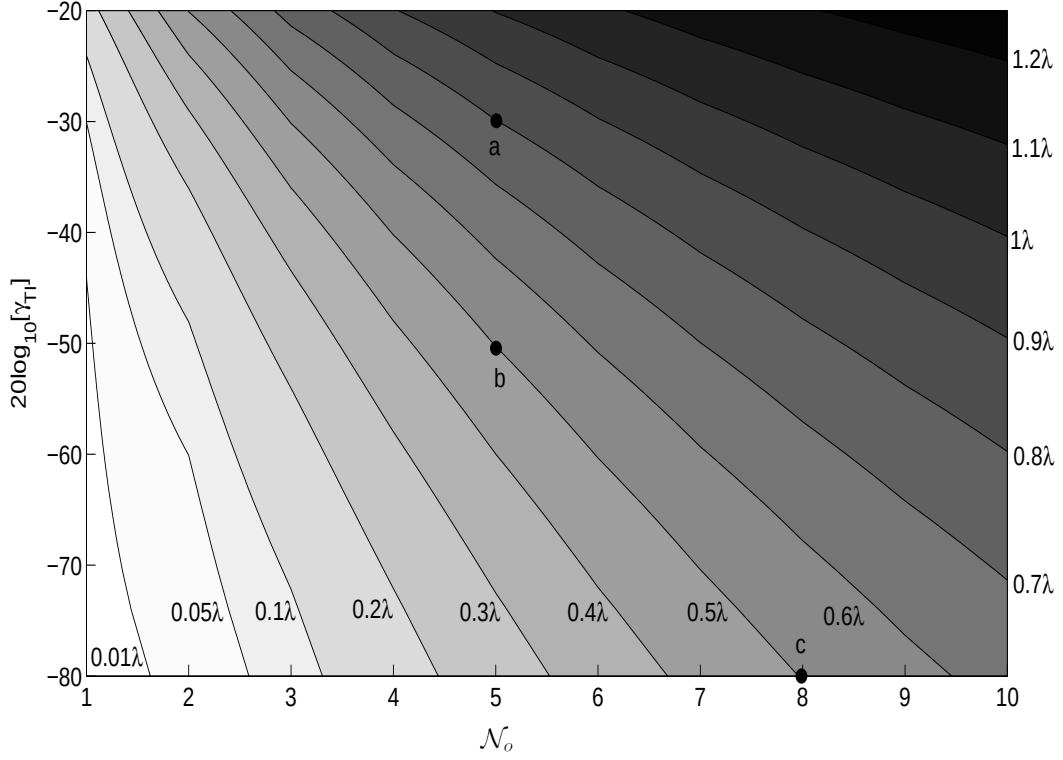


Figure 5.5: Contour plot of δ_z given in (5.3.1.93) for different order of Taylor series $\mathcal{N}_o$ and different computational accuracy γ_{TI} of the planar TI-FFT algorithm, for $k_{z,c} = 0.9k$ ($k_{\perp,c} = 0.436k$, see Fig. 5.4). Some typical values are marked as black filled circles: black filled circle **a** denotes ($\gamma_{\text{TI}} = -30$ dB, $\mathcal{N}_o = 5$, $\delta_z = 0.8\lambda$), black filled circle **b** denotes ($\gamma_{\text{TI}} = -50$ dB, $\mathcal{N}_o = 5$, $\delta_z = 0.5\lambda$) and black filled circle **c** denotes ($\gamma_{\text{TI}} = -80$ dB, $\mathcal{N}_o = 8$, $\delta_z = 0.5\lambda$).

$$\lambda_c \equiv \frac{2\pi}{\Delta k_{z,c}} = \frac{2\pi}{k - \sqrt{k^2 - k_{\perp,c}^2}} = \frac{\lambda}{\alpha}, \quad (5.3.1.91)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$\lambda_c \sim 2 \left(\frac{k}{k_{\perp,c}} \right)^2 \lambda. \quad (5.3.1.92)$$

Fig. 5.4 plots the exact value (line) and approximation (dots) of the characteristic wavelength λ_c for different $k_{z,c}$, together with some selected points

given in the table, from which it can be seen that $\lambda_c > 100\lambda$ for $k_{z,c} > 0.99k$ ($k_{\perp,c} < 0.141k$) and the maximum deviation of the approximate from the exact value is 1λ , which occurs at $k_{z,c} = 0$.

It can be seen from (5.3.1.89) that, for a given computational accuracy γ_{TI} , the spatial slicing spacing δ_z should satisfy the following relation,

$$\gamma_{\text{TI}} \sim \mathcal{O} \left[(\Delta k_{z,c} \delta_z)^{\mathcal{N}_o} \right] \rightarrow \delta_z \sim \frac{1}{k\alpha} \left(\frac{1}{\gamma_{\text{TI}}} \right)^{-\frac{1}{\mathcal{N}_o}} = \frac{\lambda_c}{2\pi\alpha} \left(\frac{1}{\gamma_{\text{TI}}} \right)^{-\frac{1}{\mathcal{N}_o}}, \quad (5.3.1.93)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$\delta_z \sim \frac{1}{\pi} \left(\frac{k}{k_{\perp,c}} \right)^2 \left(\frac{1}{\gamma_{\text{TI}}} \right)^{-\frac{1}{\mathcal{N}_o}} \lambda. \quad (5.3.1.94)$$

Fig. 5.5 shows the contour plot of δ_z with respect to different order of Taylor series $\mathcal{N}_o$ and different computational accuracy γ_{TI} of the planar TI-FFT algorithm, for a narrow-band beam with $k_{z,c} = 0.9k$ ($k_{\perp,c} = 0.436k$), from which it can be seen that the spatial slicing spacing $\delta_z \in 0.3\lambda \sim 0.8\lambda$ and the order of Taylor series $\mathcal{N}_r \in 5 \sim 10$ will give the computational accuracy $\gamma_{\text{TI}} \in -30 \text{ dB} \sim -80 \text{ dB}$.

Now consider a quasi-planar surface with a characteristic surface variation of $\Delta z_c = \mathcal{N}_z \lambda$, from (5.3.1.93) the number of spatial reference planes $\mathcal{N}_r$ is given as

$$\mathcal{N}_r = \frac{\Delta z_c}{\delta_z} \sim 2\pi\alpha \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_z, \quad (5.3.1.95)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$\mathcal{N}_r \sim \pi \left(\frac{k_{\perp,c}}{k} \right)^2 \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_z. \quad (5.3.1.96)$$

Now the number of FFT operations N_{FFT} and the computational complexity CPU are obtained as

$$N_{\text{FFT}} = \mathcal{N}_o \times \mathcal{N}_r = 2\pi\alpha \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_o \mathcal{N}_z, \quad (5.3.1.97)$$

$$\text{CPU} = N_{\text{FFT}} \mathcal{O} [N \log_2 N] = 2\pi\alpha \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_o \mathcal{N}_z \mathcal{O} [N \log_2 N], \quad (5.3.1.98)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$N_{\text{FFT}} \sim \pi \left(\frac{k_{\perp,c}}{k} \right)^2 \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_o \mathcal{N}_z. \quad (5.3.1.99)$$

$$\text{CPU} \sim \pi \left(\frac{k_{\perp,c}}{k} \right)^2 \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_o \mathcal{N}_z \mathcal{O} [N \log_2 N]. \quad (5.3.1.100)$$

where an $N = \mathcal{N}_x \times \mathcal{N}_y$ computational grid has been assumed. Obviously, the computational complexity CPU has a square law dependence on the characteristic bandwidth $k_{\perp,c}$ of the electromagnetic wave and have a linear dependence on the surface variation ($\Delta z_c = \mathcal{N}_z \lambda$). The computational complexity CPU also has an inverse $\mathcal{N}_o^{\text{th}}$ -root dependence on the computational accuracy γ_{TI} . So the characteristic bandwidth $k_{\perp,c}$ has the most significant effect on the computational complexity of the planar TI-FFT.

It can be seen from (5.3.1.97) and (5.3.1.98) that the number of FFT operations $\mathcal{N}_{\text{FFT}}$ and the computational complexity CPU depend on the order of Taylor series $\mathcal{N}_o$. As an example, Fig. 5.6 shows the number of FFT operations $\mathcal{N}_{\text{FFT}}$ (per λ surface variation Δz_c) for the electromagnetic wave with $k_{z,c} = 0.9k$ ($k_{\perp,c} = 0.436$), at different computational accuracies $\gamma_{\text{TI}} = (-20, -40, -60, -80)$ dB, from which the optimized order of Taylor series $\mathcal{N}_o^{\text{opt}}$ can be found as $\mathcal{N}_o^{\text{opt}} = (2, 5, 7, 9)$ respectively. Mathematically, $\mathcal{N}_o^{\text{opt}}$ can also be obtained through finding the minimum value of N_{FFT} in (5.3.1.97) by assuming that $\mathcal{N}_o$ is a continuous variable,

$$\left. \frac{\partial N_{\text{FFT}}}{\partial \mathcal{N}_o} \right|_{\mathcal{N}_o^{\text{opt}}} = 0 \rightarrow \left. \frac{\partial [\ln [\mathcal{N}_o] - \ln [\gamma_{\text{TI}}] \frac{1}{\mathcal{N}_o}]}{\partial \mathcal{N}_o} \right|_{\mathcal{N}_o^{\text{opt}}} = 0, \quad (5.3.1.101)$$

$$\mathcal{N}_o^{\text{opt}} \sim \text{round} \left[\ln \left[\frac{1}{\gamma_{\text{TI}}} \right] \right] = \text{round} \left[-0.1151 \gamma_{\text{TI}} (\text{dB}) \right]. \quad (5.3.1.102)$$

where “round” means to round the value to its nearest integer (actually, to achieve a higher computational accuracy γ_{TI} , the upper-bound could be used but

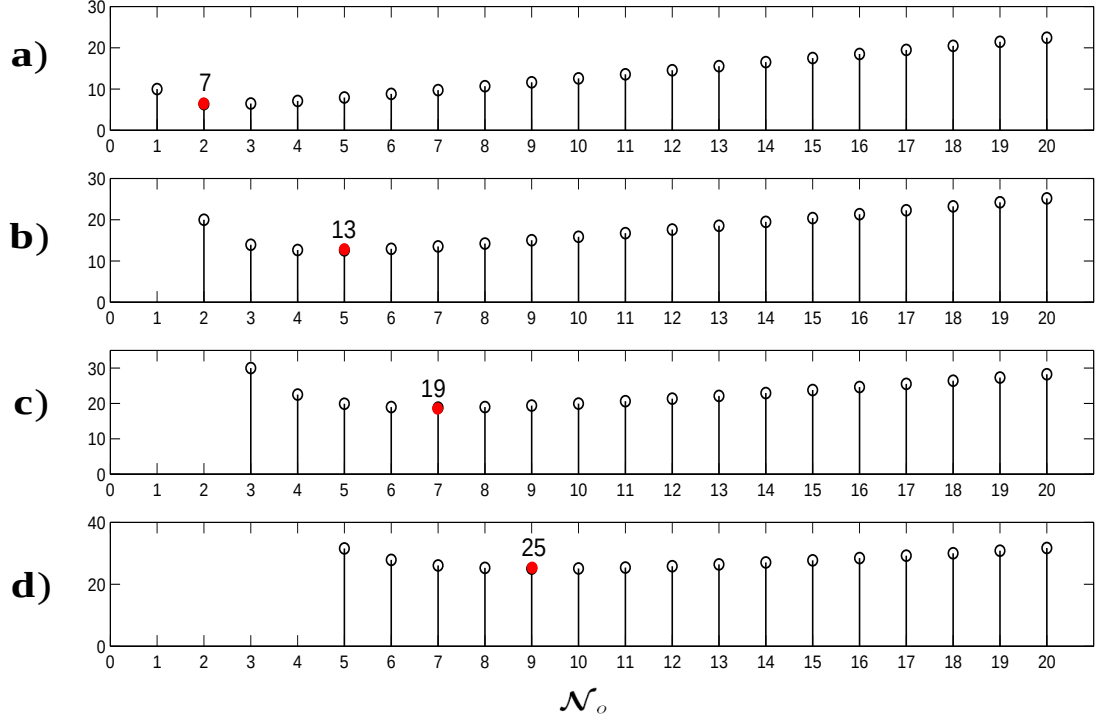


Figure 5.6: The Values of expression $(1/\gamma_{\text{TI}})^{1/N_o}$ in (5.3.1.97) for different accuracies $\gamma_{\text{TI}} = (-20, -40, -60, -80)$ dB, in a), b), c) and d) respectively. The corresponding optimized order of Taylor series (red circles) is found as $N_o^{\text{opt}} = (2, 5, 7, 9)$, which confirms the mathematical linear prediction in (5.3.1.102) and its plot in Fig. 5.7. The corresponding values of $(1/\gamma_{\text{TI}})^{1/N_o}$ are given as $N_{\text{FFT}}^{\text{opt}} \sim (7, 13, 19, 25)$ respectively.

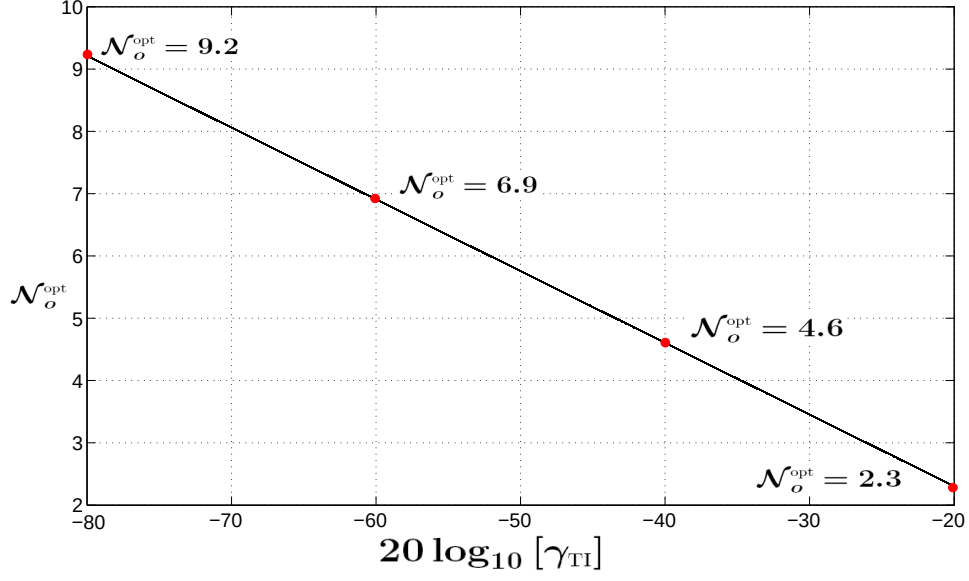


Figure 5.7: The linear dependence of the optimized order of Taylor series $\mathcal{N}_o^{\text{opt}}$ on the computational accuracy γ_{TI} (dB). It can be seen that $\mathcal{N}_o^{\text{opt}} \sim (2.3, 4.6, 6.9, 9.2)$ for different computational accuracy $\gamma_{\text{TI}} = (-20, -40, -60, -80)$ dB, which corresponds to the exact values given in Fig. 5.6 after $\mathcal{N}_o^{\text{opt}}$ is rounded to the nearest integer, which is $\mathcal{N}_o^{\text{opt}} = (2, 5, 7, 9)$.

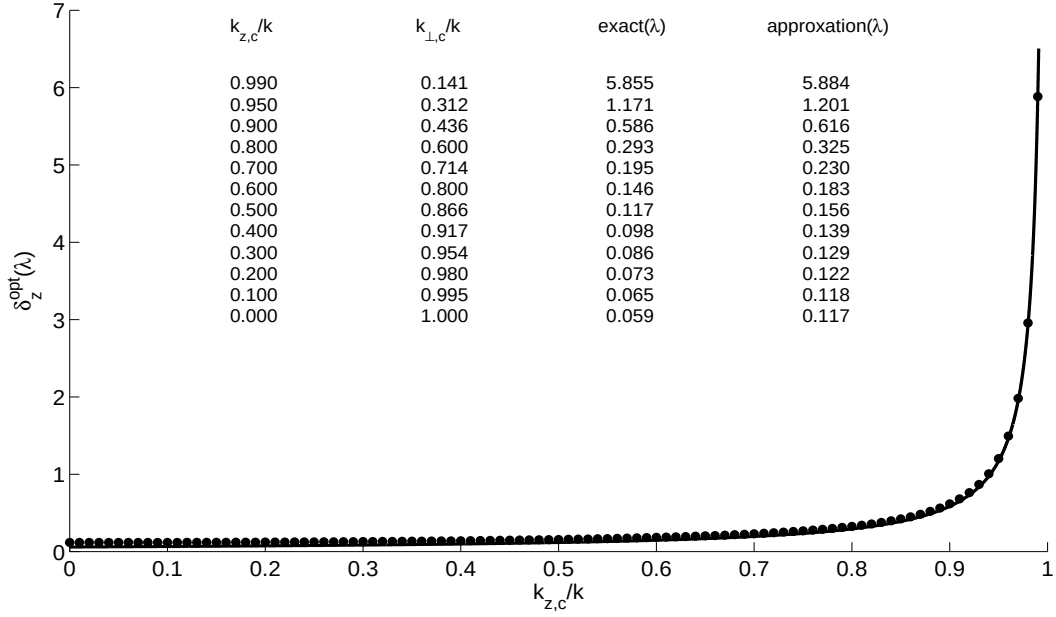


Figure 5.8: Plots of the exact value (line) of δ_z^{opt} given in (5.3.1.103) and approximation (dots) given in (5.3.1.104) for different $k_{z,c}$ ($k_{\perp,c}$).

the computational complexity is a little higher). Fig. 5.7 also shows the linear dependence of the optimized order of Taylor series $\mathcal{N}_o^{\text{opt}}$ on the computational accuracy γ_{TI} (dB), which has been shown in (5.3.1.102).

The optimized spatial slicing spacing δ_z^{opt} can be obtained from (5.3.1.93), which is

$$\delta_z^{\text{opt}} \sim \frac{\lambda}{2\pi\alpha} \left(\frac{1}{\gamma_{\text{TI}}} \right)^{-\frac{1}{\ln\left[\frac{1}{\gamma_{\text{TI}}}\right]}} = \frac{\lambda}{2\pi e\alpha} \sim \frac{1}{17}\lambda_c, \quad (5.3.1.103)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$\delta_z^{\text{opt}} \sim \frac{1}{e\pi} \left(\frac{k}{k_{\perp,c}} \right)^2 \lambda, \quad (5.3.1.104)$$

where $e \sim 2.718$ is the natural logarithmic base. It is interesting to note that the optimized spatial slicing spacing δ_z doesn't depend on the computational accuracy γ_{TI} and strongly depends on the characteristic bandwidth $k_{\perp,c}$ (inverse square law). Fig. 5.8 shows δ_z^{opt} for different $k_{z,c}$ ($k_{\perp,c}$), from which it can be seen that $\delta_z^{\text{opt}} > 0.5\lambda$ for $k_{z,c} > 0.9k$ ($k_{\perp,c} < 0.436k$).

The optimized number of spatial reference planes $\mathcal{N}_r^{\text{opt}}$ is given as

$$\mathcal{N}_r^{\text{opt}} = \frac{\Delta z_c}{\delta_z} = 2\pi e\alpha \mathcal{N}_z \sim 17\alpha \mathcal{N}_z, \quad (5.3.1.105)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$\mathcal{N}_r^{\text{opt}} \sim \pi e \left(\frac{k_{\perp,c}}{k} \right)^2 \mathcal{N}_z. \quad (5.3.1.106)$$

Substitute (5.3.1.102) into (5.3.1.97) and (5.3.1.98), the optimized number of FFT operations $N_{\text{FFT}}^{\text{opt}}$ and the optimized computational complexity CPU^{opt} can also be obtained,

$$N_{\text{FFT}}^{\text{opt}} = 2\pi e\alpha \ln \left[\frac{1}{\gamma_{\text{TI}}} \right] \mathcal{N}_z, \quad (5.3.1.107)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$N_{\text{FFT}}^{\text{opt}} \sim \pi e \left(\frac{k_{\perp,c}}{k} \right)^2 \ln \left[\frac{1}{\gamma_{\text{TI}}} \right] \mathcal{N}_z. \quad (5.3.1.108)$$

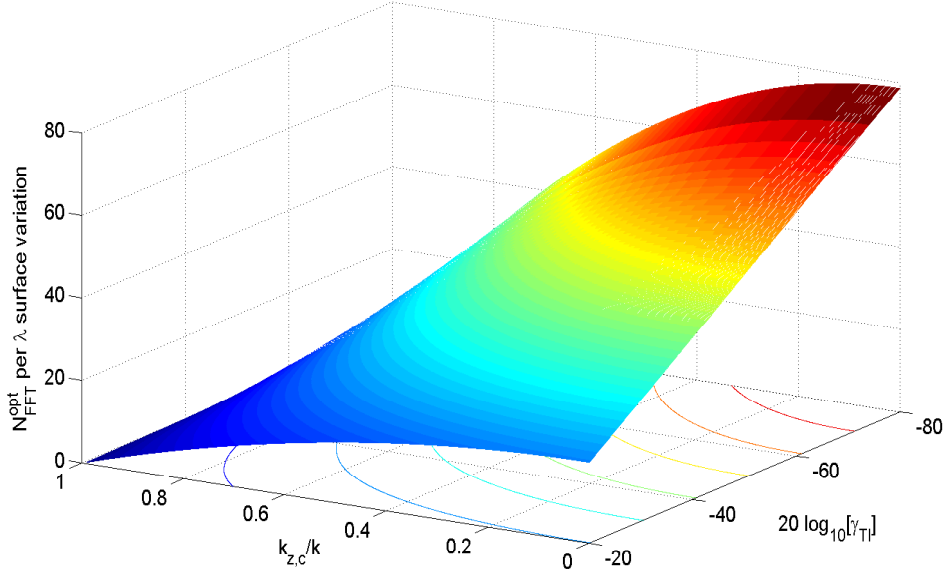


Figure 5.9: The dependence of the optimized number of FFT operations $N_{\text{FFT}}^{\text{opt}}$ (per λ surface variation Δz_c) on $k_{z,c}$ and the computational accuracy γ_{TI} .

The optimized computational complexity CPU^{opt} is given as

$$\text{CPU}^{\text{opt}} \sim N_{\text{FFT}}^{\text{opt}} \mathcal{O} \left[N \log_2 N \right]. \quad (5.3.1.109)$$

Fig. 5.9 shows the optimized number of FFT operations $N_{\text{FFT}}^{\text{opt}}$ (per λ surface variation Δz_c), with respect to the $k_{z,c}$ ($k_{\perp,c}$) and the computational accuracy γ_{TI} .

To have a concrete picture of the optimized quantities given above, consider a quasi-planar surface with a characteristic surface variation $\Delta z_c = 1\lambda$ ($N_z = 10$) and a narrow-band beam with $k_{z,c} = 0.9k$ ($k_{\perp,c} = 0.436k$). To obtain the computational accuracy $\gamma_{\text{TI}} = -80$ dB, the following optimized quantities are obtained from (5.3.1.102)-(5.3.1.109),

$$\mathcal{N}_o^{\text{opt}} \sim 0.1151 \times 80 \sim 9. \quad (5.3.1.110)$$

$$\delta_z^{\text{opt}} \sim \frac{1}{e\pi} \left(\frac{1}{0.436} \right)^2 \sim 0.6\lambda. \quad (5.3.1.111)$$

$$\mathcal{N}_r \sim \frac{1}{0.6} \sim 2. \quad (5.3.1.112)$$

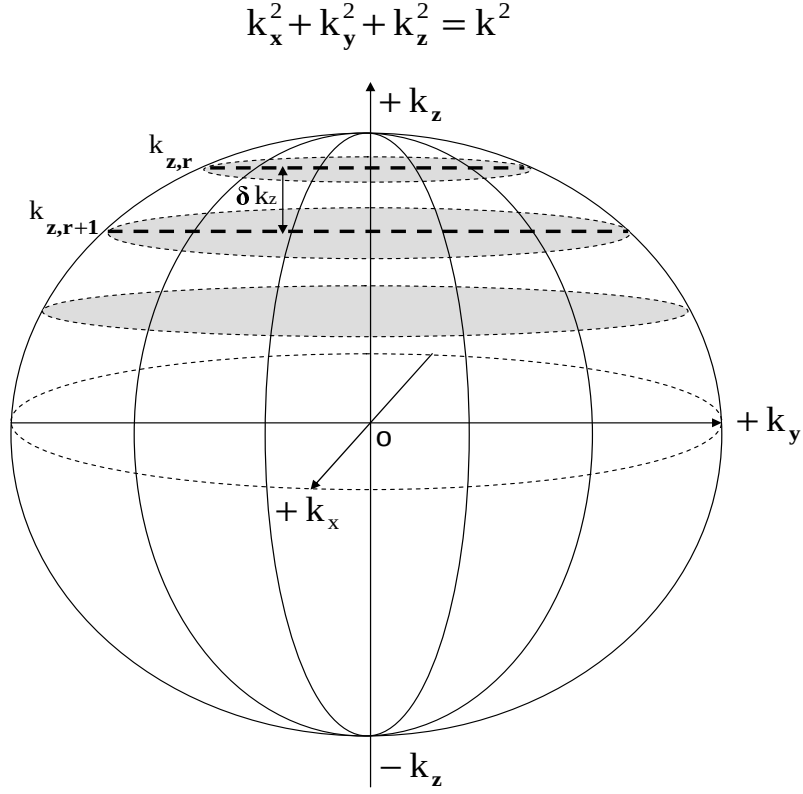


Figure 5.10: The spectral domain division method for both the *spectral* (Section 5.3.1) and *spatial* (Section 5.3.2) types of TI-FFT algorithm. Only $k_z > 0$ half sphere surface is used for half-space $z > z'$ computation in this article. $k_{z,r}$ and $k_{z,r+1}$ denote the r^{th} and $(r+1)^{\text{th}}$ spectral reference planes respectively. δk_z is the spectral slicing spacing.

$$N_{\text{FFT}}^{\text{opt}} \sim 9 \times 2 = 18 \quad \text{per } \lambda \text{ surface variation.} \quad (5.3.1.113)$$

$$\text{CPU}^{\text{opt}} \sim 18 \mathcal{O} \left[N \log_2 N \right] \quad \text{per } \lambda \text{ surface variation.} \quad (5.3.1.114)$$

Spectral domain division method The radiation vector $\mathbf{L}$ in (5.2.1.63) can also be expressed as

$$\mathbf{L}(\mathbf{s}) = e^{jk_z z_{\min}} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} dx dy \left[e^{jk_x x} e^{jk_y y} \frac{\mathbf{s}(\mathbf{r})}{\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}} e^{jk_z \Delta z(x,y)} \right], \quad (5.3.1.115)$$

The Taylor expansion of $\mathbf{L}$ in (5.3.1.115) over k_z is given as

$$\begin{aligned} \mathbf{L}(\mathbf{s}) = 2\pi e^{jk_z z_{\min}} \sum_{r=1}^{\mathcal{N}_r} \left\{ \sum_{n=0}^{\mathcal{N}_o} \left[\frac{1}{n!} \left(j [k_z - k_{z,r}] \right)^n \right. \right. \\ \left. \left. \times \text{FT}_{2D} \left[\tilde{\mathbf{s}}(\mathbf{r}) \left(\Delta z(x,y) \right)^n \right] \right] \right\}, \end{aligned} \quad (5.3.1.116)$$

where $\tilde{\mathbf{s}}(\mathbf{r})$ is given as

$$\tilde{\mathbf{s}}(\mathbf{r}) \equiv \frac{\mathbf{s}(\mathbf{r})}{\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}} e^{jk_{z,r} \Delta z(x,y)}, \quad (5.3.1.117)$$

$k_{z,r}$ denotes the spectral reference plane. From (5.3.1.116), for the given computational accuracy γ_{TI} , the spectral slicing spacing $\delta k_z \equiv \max [k_z - k_{z,r}] = k_{z,r+1} - k_{z,r}$ should satisfy the following relation,

$$\gamma_{\text{TI}} \sim \mathcal{O} \left[(\delta k_z \Delta z_c)^{\mathcal{N}_o} \right], \quad (5.3.1.118)$$

$$\rightarrow \delta k_z \sim \frac{1}{\Delta z_c} \left(\frac{1}{\gamma_{\text{TI}}} \right)^{-\frac{1}{\mathcal{N}_o}} \sim \frac{1}{\mathcal{N}_z \lambda} \left(\frac{1}{\gamma_{\text{TI}}} \right)^{-\frac{1}{\mathcal{N}_o}}, \quad (5.3.1.119)$$

The number of spectral reference planes $\mathcal{N}_r$ is given as

$$\mathcal{N}_r = \frac{\Delta k_{z,c}}{\delta k_z} \sim 2\pi\alpha \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_z, \quad (5.3.1.120)$$

The number of FFT operations is given as

$$\mathcal{N}_{\text{FFT}} \sim 2\pi\alpha \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_o \mathcal{N}_z, \quad (5.3.1.121)$$

For a narrow-band beam ($k_{\perp,c} \ll k$),

$$\mathcal{N}_r \sim \pi \left(\frac{k_{\perp,c}}{k} \right)^2 \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_z. \quad (5.3.1.122)$$

$$\mathcal{N}_{\text{FFT}} \sim \pi \left(\frac{k_{\perp,c}}{k} \right)^2 \left(\frac{1}{\gamma_{\text{TI}}} \right)^{\frac{1}{\mathcal{N}_o}} \mathcal{N}_o \mathcal{N}_z. \quad (5.3.1.123)$$

It is obvious that $\mathcal{N}_r$ in (5.3.1.120) and $\mathcal{N}_{\text{FFT}}$ in (5.3.1.121) are the same as those given in the spatial domain division method in (5.3.1.96) and (5.3.1.97) respectively, which also means that the spatial and spectral division methods have the same optimized quantities, i.e., $\mathcal{N}_r^{\text{opt}}$ in (5.3.1.105), $\mathcal{N}_o^{\text{opt}}$ in (5.3.1.102), δ_z^{opt} in (5.3.1.103), $N_{\text{FFT}}^{\text{opt}}$ in (5.3.1.107) and CPU^{opt} in (5.3.1.109). The readers are encouraged to see Section 5.3.1 for details.

5.3.2 The spatial TI-FFT algorithm

After the radiation vector $\mathbf{L}$ of the source term has been found through (5.3.1.89) or (5.3.1.116), the 2D Fourier spectrum $\mathcal{F}$ of the electric field $\mathbf{E}(\mathbf{r})$ can be obtained through the 2D Fourier spectrum of Kottler formula (5.2.2.75). The electric field $\mathbf{E}$ on a plane is then computed through the IFT in (5.2.3.77). For a quasi-planar surface, the TI technique can also be used, which leads to the spatial TI-FFT algorithm. There are also two numerical implementations for the spatial TI-FFT, i.e., the spatial domain division method given in Section 5.3.2 and the spectral domain division method given in Section 5.3.2.

Spatial domain division method Consider the PWS expression for the electric field $\mathbf{E}(\mathbf{r})$,

$$\mathbf{E}(\mathbf{r}) = e^{-jkz(x,y)} \text{IFT}_{2\text{D}} \left[\left[\mathcal{F}(k_x, k_y) e^{jk_z z(x,y)} \right] e^{j\Delta k_z z(x,y)} \right], \quad (5.3.2.124)$$

Note that (5.3.2.124) is actually very similar to the spatial domain division method for the radiation vector $\mathbf{L}$ in (5.3.1.86) of Section 5.3.1. Following the similar procedure, the electric field $\mathbf{E}$ is finally obtained as

$$\mathbf{E}(\mathbf{r}) = e^{-jkz(x,y)} \sum_{r=1}^{\mathcal{N}_r} \left\{ \sum_{n=0}^{\mathcal{N}_o} \left[\frac{1}{n!} \left(-j [z(x,y) - z_r] \right)^n \right] \right\} \quad (5.3.2.125)$$

$$\times \text{IFT}_{2\text{D}} \left[\begin{array}{c} \tilde{\mathcal{F}}(k_x, k_y) (\Delta k_z)^n \\ \end{array} \right] \Bigg\},$$

where $\tilde{\mathcal{F}}(k_x, k_y)$ is given as

$$\tilde{\mathcal{F}}(k_x, k_y) = \mathcal{F}(k_x, k_y) e^{jk_z z(x,y)} e^{j\Delta k_z z_r}. \quad (5.3.2.126)$$

Obviously, the optimized quantities are the same as those in Section 5.3.1.

Spectral domain division method The PWS representation (5.3.2.124) can also be expressed as

$$\mathbf{E}(\mathbf{r}) = e^{-jk\Delta z(x,y)} \text{IFT}_{2\text{D}} \left[\begin{array}{c} \tilde{\mathcal{F}}(k_x, k_y) e^{jk_x x} e^{jk_y y} e^{j\Delta k_z \Delta z(x,y)} \\ \end{array} \right], \quad (5.3.2.127)$$

where $\tilde{\mathcal{F}}(k_x, k_y)$ is given as

$$\tilde{\mathcal{F}}(k_x, k_y) = \mathcal{F}(k_x, k_y) e^{jk_z z(x,y)} e^{-jk_z z_{\min}}, \quad (5.3.2.128)$$

Similarly, the Taylor expansion of $\mathbf{E}(\mathbf{r})$ for the spectral domain division method can be obtained as

$$\begin{aligned} \mathbf{E}(\mathbf{r}) = \sum_{r=1}^{\mathcal{N}_r} \left\{ e^{-jk_{z,r} \Delta z(x,y)} \sum_{n=0}^{\mathcal{N}_o} \left[\frac{1}{n!} \left(-j\Delta z(x,y) \right)^n \right. \right. \\ \left. \left. \times \text{IFT}_{2\text{D}} \left[\begin{array}{c} \tilde{\mathcal{F}}(k_x, k_y) \left(k_z - k_{z,r} \right)^n \\ \end{array} \right] \right] \right\}. \end{aligned} \quad (5.3.2.129)$$

Again, the optimized quantities are also the same as those in Section 5.3.1.

5.3.3 Pseudocodes for the planar TI-FFT algorithm

In this section, the pseudocodes are shown to describe the procedure of the planar TI-FFT algorithm (spatial domain division method) in the electromagnetic scattering problem shown in Fig. 5.2, where both the incident input beam and the scattered output beam are located on plane $z = 0$.

(*PROGRAM BEGINS*)

Calculate the 2D Fourier spectrum $\mathcal{F}^i$ of the incident input beam $\mathbf{E}^i$ according to (5.2.3.77);

(Algorithm preconditioning begins)

Estimate the characteristic bandwidth $k_{\perp,c}$ of the incident input beam and obtain the optimized spatial slicing spacing δ_z^{opt} according to (5.3.1.103);

Obtain the characteristic surface (S) variation Δz_c and calculate the optimized number of spatial reference planes N_r^{opt} according to (5.3.1.105);

Specify the desired computational accuracy γ_{TI} of the planar TI-FFT algorithm and obtain the optimized order of Taylor series N_o^{opt} according to (5.3.1.102);

(Algorithm preconditioning ends)

(The spatial TI-FFT algorithm begins)

For $r = 1$ **to** N_r (**iteration loop begins**)

Obtain the electric field on the r^{th} spatial reference plane, denoted as $\mathbf{E}_r^{(r)}$;

Obtain the electric field on the r^{th} spatial surface subdomain (denoted as $\mathbf{E}_{s,r}$) through the spatial TI-FFT algorithm (spatial domain division method), which is given in (5.3.2.125);

$r = r + 1$ (**iteration loop continues until** $r = N_r$)

(The spatial TI-FFT algorithm ends)

IF (surface S is smooth enough)

Approximate the equivalent surface current as $\mathbf{J}_{s,r} \sim 2\hat{\mathbf{n}} \times \mathbf{E}_{s,r}$ (physical equivalent);

ELSE

Obtain the equivalent surface current $\mathbf{J}_{s,r}$ using the EFIE or MFIE method (through MoM);

END

(The spectral TI-FFT algorithm begins)

For $r = 1$ **to** N_r (**iteration loop begins**)

Obtain the 2D Fourier spectrum $\mathcal{F}_r^\circ$ of the scattered output beam $\mathbf{E}^\circ$ for the r^{th} spatial surface subdomain, using $\mathbf{J}_{s,r}$;

Update the complete 2D Fourier spectrum $\mathcal{F}^\circ$ by summing the contribution from the 1st to the r^{th} subdomains as $\mathcal{F}^\circ = \mathcal{F}^\circ + \mathcal{F}_r^\circ$ (note that $\mathcal{F}_{r=0}^\circ = 0$), according to the spectral TI-FFT (spatial domain division method) in (5.3.1.89);

$r = r + 1$ (**iteration loop continues until $r = \mathcal{N}_r$**)

(The spectral TI-FFT algorithm ends)

Obtain the scattered output field on plane $z = 0$ through the IFT, $\mathbf{E}^\circ = \text{IFT}_{2D}[\mathcal{F}^\circ]$;

(PROGRAM ENDS)

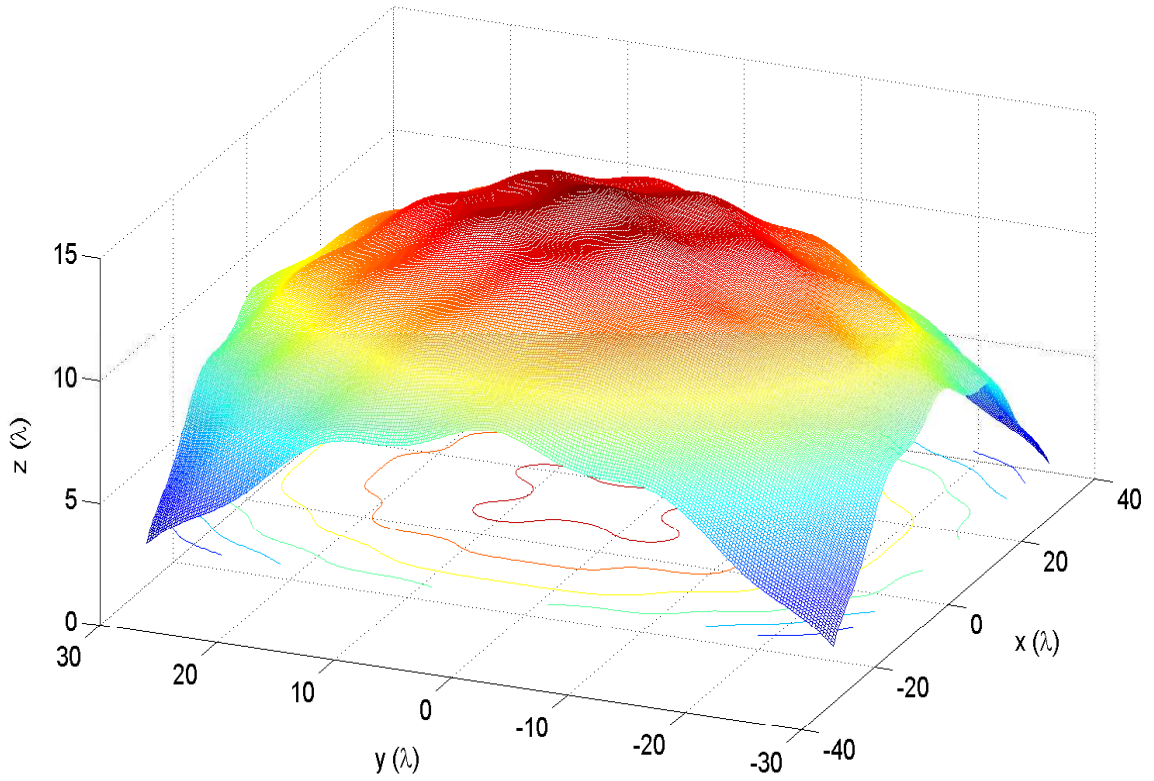


Figure 5.11: The “quasi-spherical” surface with a sine wave perturbation used in the numerical evaluation of the planar TI-FFT algorithm, see Section 5.3.5 for details. The mathematical form is given in (5.3.5.130).

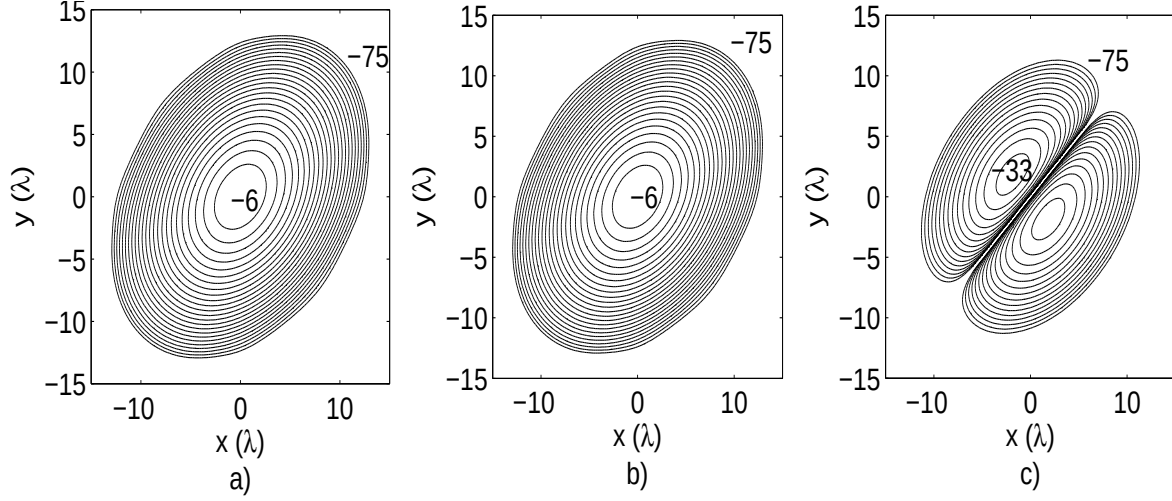


Figure 5.12: The input FGB magnitude patterns: a) x-component; b) y-component; and c) z-component. Contours are in 3 dB decrements.

5.3.4 Computational Results

To show the efficiency of the planar TI-FFT algorithm, the direct integration of the Kottler formula in (5.1.5.42) has been used to make comparison with the planar TI-FFT algorithm. Two examples are used for such purpose, both at 110 GHz with $\lambda \sim 2.7$ mm: 1) a 45° -tilted Fundamental Gaussian Beam (FGB) scattered by a PEC quasi-spherical surface with a sine wave perturbation; and 2) a beam-shaping 4-mirror system design for a single-mode ($\text{TE}_{22,6}/110$ GHz), 1.25 MW CPI gyrotron [4, 2].

5.3.5 The scattering of a 45° -tilted FGB

The incident FGB (110 GHz or $\lambda \sim 2.7$ mm) has an angle of 45° between direction $\hat{\mathbf{z}}$ and the beam propagation direction ($\hat{\mathbf{k}}_i = \frac{(\hat{\mathbf{x}}+\hat{\mathbf{y}})+\sqrt{2}\hat{\mathbf{z}}}{2}$). This 45° -tilted FGB is defined on a plane that is perpendicular to $\hat{\mathbf{k}}_i$, and it is linearly polarized at the direction $\hat{\mathbf{e}} = \frac{(\hat{\mathbf{x}}+\hat{\mathbf{y}})-\sqrt{2}\hat{\mathbf{z}}}{2}$, with symmetrical beam waist radius of $w = 1$ cm. The PEC “quasi-spherical” surface with a sine wave perturbation is given as

$$z = -59\lambda + \sqrt{(75\lambda)^2 - x^2 - y^2} + 10^{-3} \cos\left(2\pi \frac{x}{15\lambda}\right) \cos\left(2\pi \frac{y}{15\lambda}\right). \quad (5.3.5.130)$$

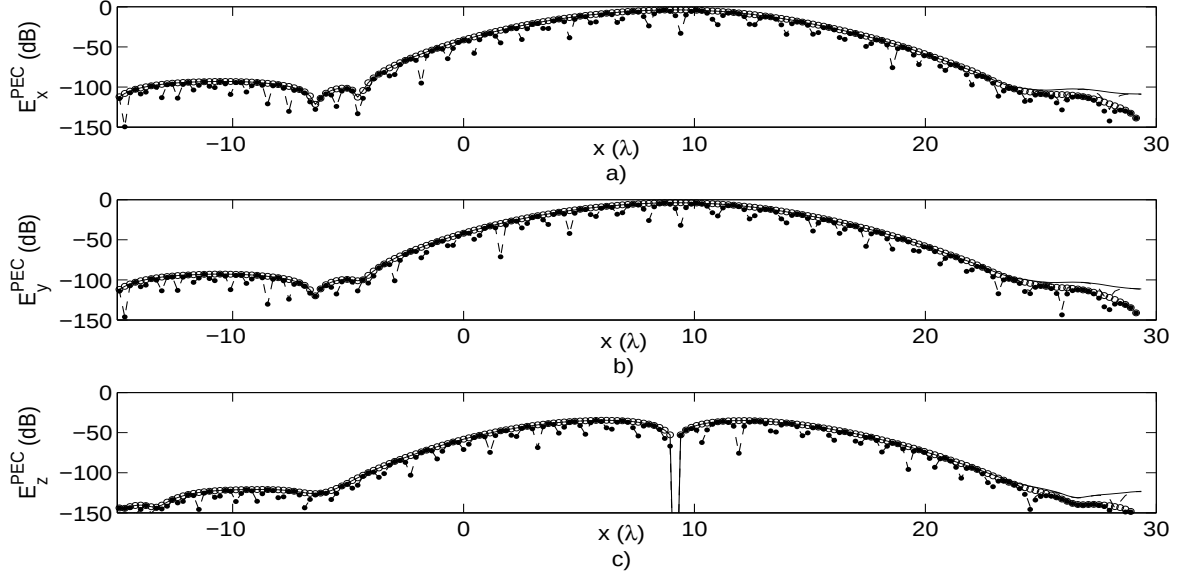


Figure 5.13: The comparison of the electric field on the quasi-spherical PEC surface: solid lines (TI-FFT) and circles (direct integration method) are magnitudes; dashed lines (TI-FFT) and dots (direct integration method) are real parts.

The quasi-spherical surface is shown in Fig. 5.11 and the x-, y- and z- components of the input FGB have been shown in Fig. 5.12.

In the numerical implementation of the planar TI-FFT algorithm, the computational accuracy $\gamma_{\text{TI}} = 0.0001$ (−80 dB) has been used and the following optimized quantities are obtained from (5.3.1.110)-(5.3.1.114),

$$\mathcal{N}_o^{\text{opt}} \sim 9, \quad \delta_z^{\text{opt}} \sim 0.6\lambda, \quad \mathcal{N}_r^{\text{opt}} \sim \frac{13}{0.6} \sim 22, \quad (5.3.5.131)$$

$$N_{\text{FFT}}^{\text{opt}} = \mathcal{N}_o^{\text{opt}} \times \mathcal{N}_r \sim 198, \quad \text{CPU}^{\text{opt}} \sim 198 \mathcal{O} \left[N \log_2 N \right]. \quad (5.3.5.132)$$

where the quasi-spherical surface (Fig. 5.11) has a characteristic surface variation $\Delta z_c \sim 13\lambda$.

The comparison of the electric field on the PEC quasi-spherical surface is given in Fig. 5.13 for both the magnitudes (solid lines are results from the planar TI-FFT algorithm and circles are for the direct integration method) and real parts (dashed lines are for the planar TI-FFT algorithm and dots are for the

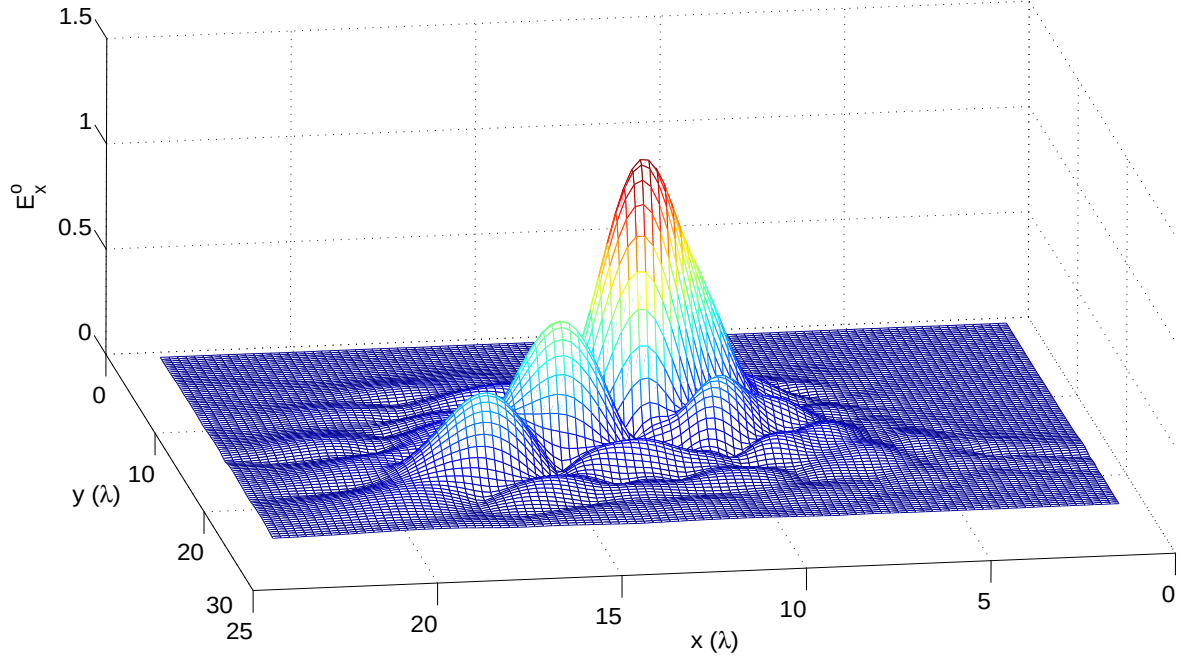


Figure 5.14: The x-component (magnitude) of the scattered output beam.

direct integration method), from which it can be seen that the planar TI-FFT algorithm has the desired -80 dB computational accuracy.

The scattered output field $\mathbf{E}^o$ are evaluated on plane $z = 0$, where the incident input field starts. The comparison of results between the planar TI-FFT algorithm and the direct integration method is given in Fig. 5.17 for both the magnitudes and real parts, which shows again that the planar TI-FFT algorithm has the desired -80 dB computational accuracy.

5.3.6 A beam-shaping 4-mirror system design for a single-mode, 1.25 MW CPI gyrotron

In this numerical simulation, a beam-shaping 4-mirror system has been designed for a single-mode ($\text{TE}_{22,6}/110$ GHz), 1.25 MW CPI gyrotron, to shape the measured input beam ($\text{TE}_{22,6}/110$ GHz) from a QO launcher into a target FGB, which is then injected into a corrugated waveguide through the diamond window. Fig. 1.2 shows the configuration of the beam-shaping 4-mirror system (the readers are encouraged to look at [1], [2] for details about the beam-shaping mirror system design for the QO gryotron application).

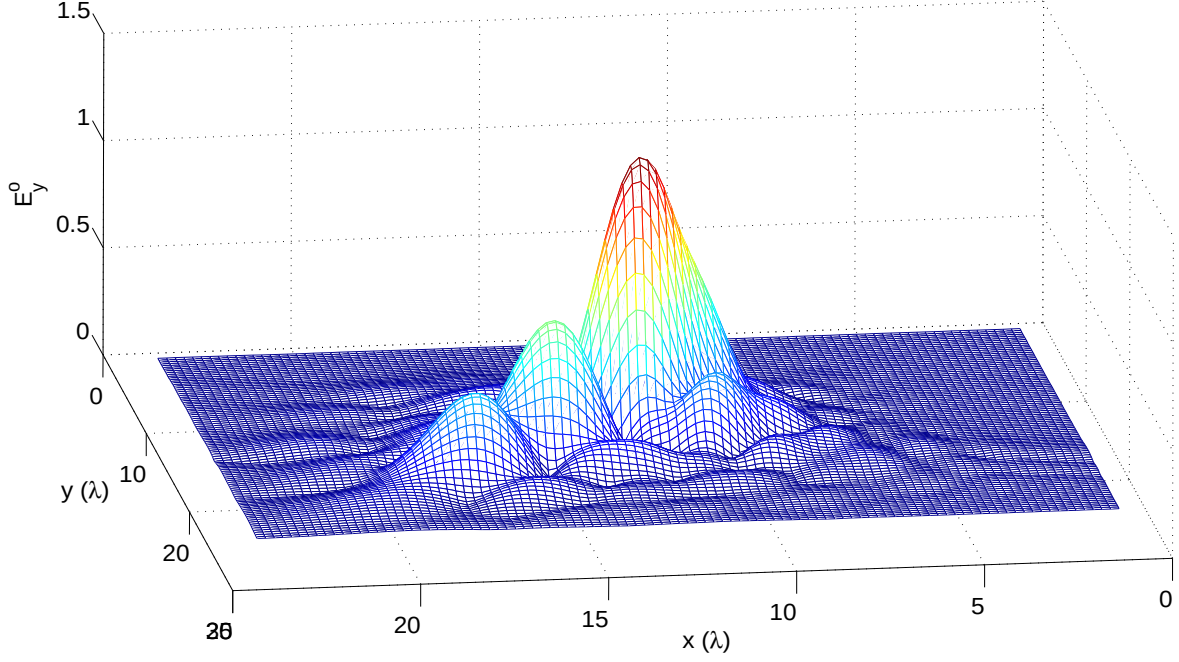


Figure 5.15: The y-component (magnitude) of the scattered output beam.

The measured input beam from the QO launcher contains x-component E_x^i and y-component E_y^i , which are shown in Fig. 5.18 (z-component $E_z^i \sim 0$ and not shown). Fig. 5.19 shows the computer-simulated output beam and its comparison with the target FGB. The comparison of results from the planar TI-FFT algorithm and the direct integration method has been shown in Fig. 5.20, with good agreement again.

5.3.7 The CPU Time and the Accuracy

The CPU time t_{TI} for the planar TI-FFT algorithm and t_{DI} for the direct integration method are summarized in Table 5.2 for example 1 (example 2 is similar), together with the coupling coefficient defined as

$$C_v \equiv \left| \frac{\iint E_{\text{TI},v}^{\circ} [E_{\text{DI},v}^{\circ}]^* dx dy}{\sqrt{\iint |E_{\text{TI},v}^{\circ}|^2 dx dy} \sqrt{\iint |E_{\text{DI},v}^{\circ}|^2 dx dy}} \right|_{z=0}, \quad v = x, y, z. \quad (5.3.7.133)$$

where, $E_{\text{TI},v}^{\circ}$ and $E_{\text{DI},v}^{\circ}$ denote the scattered output field components obtained through the planar TI-FFT algorithm and the direct integration method re-

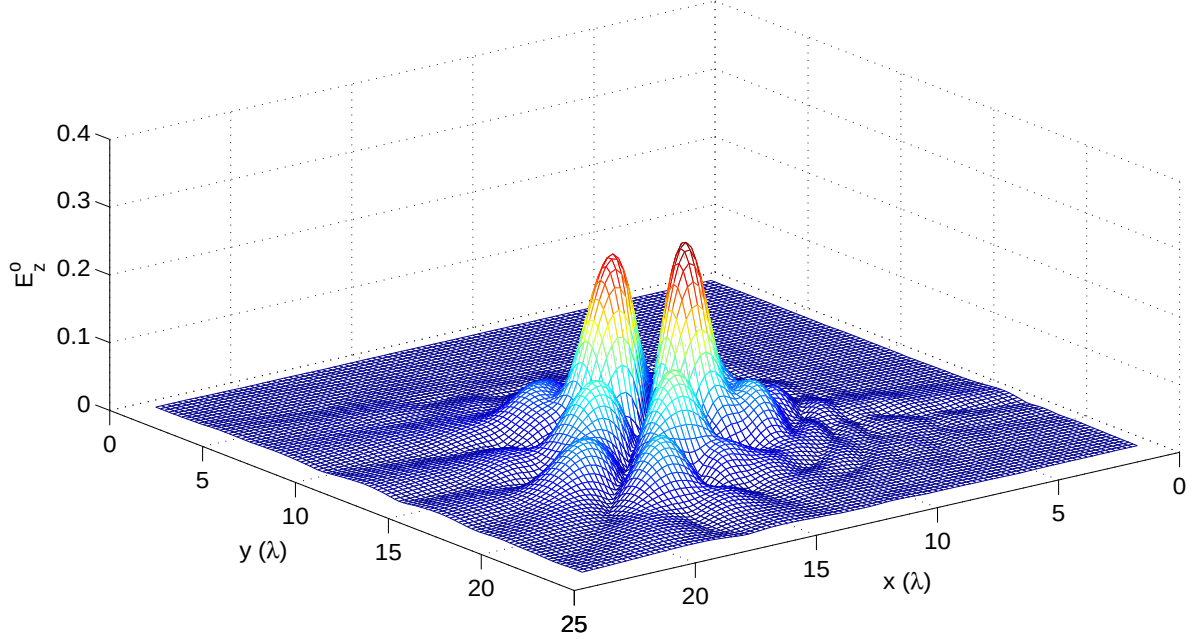


Figure 5.16: The z-component (magnitude) of the scattered output beam.

spectively. From Table 5.2, it can be seen that even though at large sampling spacings $\delta_x = \delta_y = 0.524\lambda$ ($\mathcal{N}_x = \mathcal{N}_y \sim 112$), the coupling coefficients are still well above 99.9%, which means that the characteristic bandwidth $k_{\perp,c}$ of the scattered electromagnetic beam is narrow. At this sampling rate, the direct integration using Simpson's 1/3 rule is not accurate enough [1]-[6]. Also note that the coupling coefficients $\mathcal{C}_v = 99.999\%$ ($v = x, y, z$) reach their maximum values at $\mathcal{N}_x = \mathcal{N}_y \sim 128$ ($\delta_x = \delta_y = 0.459\lambda$), after which the accuracies remain constant and thus the Nyquist rate can be estimated roughly as $\mathcal{N}_{\text{Nyquist}} \sim 128$. The reason for this phenomenon is, that after the sampling rate increases above the Nyquist rate, further increasing the sampling rate will not give more information or computational accuracy. It is clear that the planar TI-FFT allows a larger sampling spacing (or a smaller computational grid number: $N = \mathcal{N}_x \times \mathcal{N}_y$) to obtain a similar computational accuracy (coupling coefficient) due to the FFT, which means less CPU time.

The CPU times for the planar TI-FFT algorithm t_{TI} and for the direct integration method t_{DI} are shown in Fig. 6.5, together with their corresponding asymptotic computational complexities: $N \log_2 N$ for the planar TI-FFT algorithm and N^2 for the direct integration method. Fig. 6.6 shows the ratio $t_{\text{DI}}/t_{\text{TI}}$

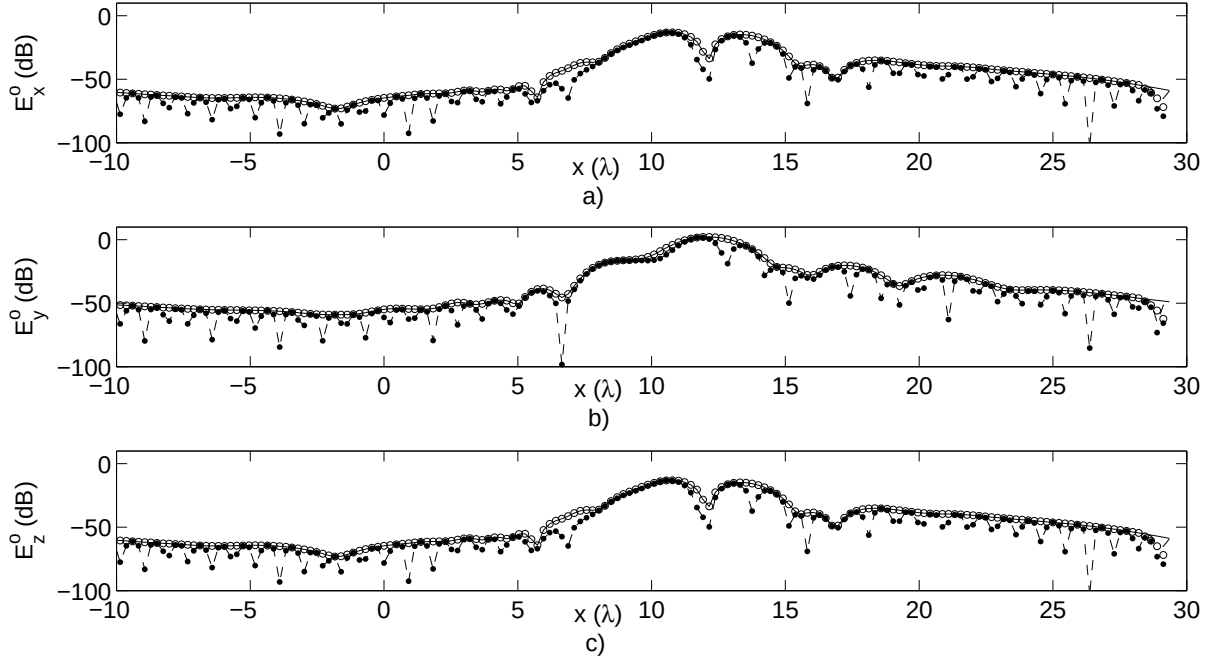


Figure 5.17: The comparison of the scattered output fields on plane $z = 0$: solid lines (TI-FFT) and circles (direct integration method) are magnitudes; dashed lines (TI-FFT) and dots (direct integration method) are real parts. The plots show that the planar TI-FFT algorithm has the desired -80 dB computational accuracy (γ_{TI}).

and its asymptotic value $N/\log_2 N$.

All work was done in Matlab 6.5 (release 13) on a Dell Pentium IV, 3.2 GHz PC with 2 GB RAM.

5.3.8 Discussion

In this section, the advantages and problems of the planar TI-FFT algorithm are discussed. Possible solutions and suggestions are presented for different problems.

Advantages of the planar TI-FFT algorithm It has been shown that the planar TI-FFT algorithm works in both the spatial and spectral domains, which leads to the spatial TI-FFT and the spectral TI-FFT respectively. Two methods are available to implement the spatial and spectral types of TI-FFT algorithm, i.e., the spatial domain division method and the spectral domain division method.

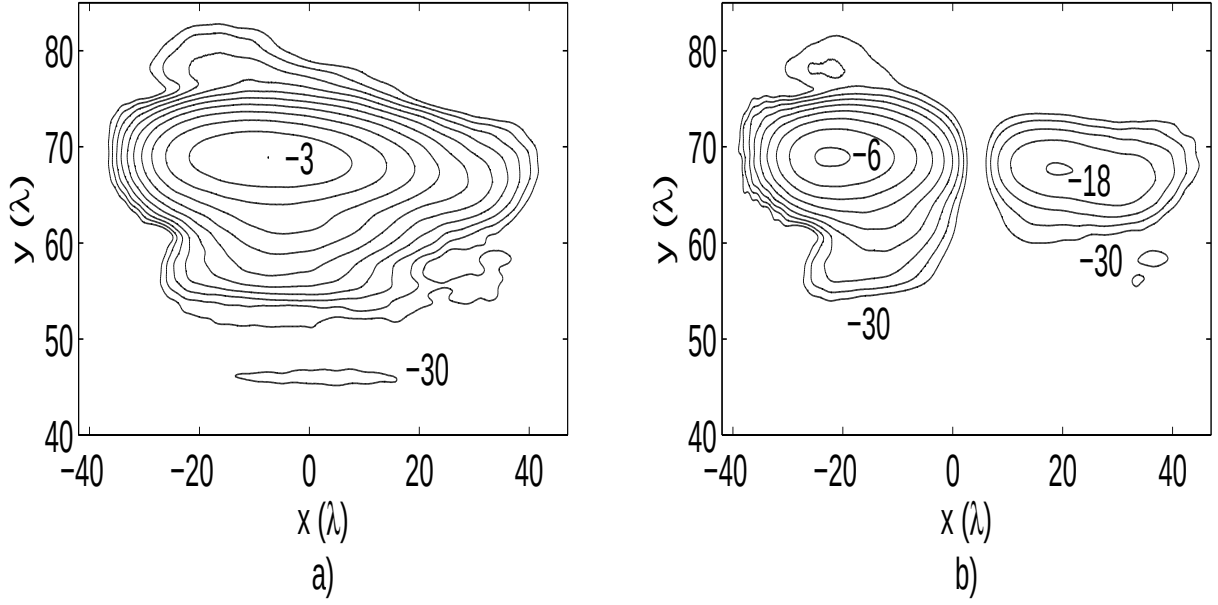


Figure 5.18: The measured input beam from QO launcher: a) $|E_x^i|$ 3 dB contour plots, from -3 dB to -30 dB; and b) $|E_y^i|$ 3 dB contour plots, from -6 dB to -30 dB. Values below -30 dB are close to the noise floor and thus not shown.

The optimized computational complexity of the planar TI-FFT algorithm is $\mathcal{O}(\mathcal{N}_r^{\text{opt}} \mathcal{N}_o^{\text{opt}} N \log_2 N)$, for an optimized order of Taylor series $\mathcal{N}_o^{\text{opt}} \sim -\ln(\zeta_{\text{TI}})$ and an optimized spatial slicing spacing between two adjacent spatial reference planes $\delta_z^{\text{opt}} \sim \frac{1}{17} \lambda_c$. Also, the planar TI-FFT algorithm allows a low sampling rate (required by the Nyquist rate $\mathcal{N}_{\text{Nyquist}}$) due to the use of an FFT, while the direct integration method requires a higher sampling rate - a sampling spacing smaller than $\lambda/5$ is suggested for Simpson's 1/3 rule integration [4, 6]. What's more, the planar TI-FFT avoids the singularity problem of the computation of the electromagnetic field on the source points, which has to be treated in the direct integration method and the FMM. The planar TI-FFT achieve this through the backward propagation of the electromagnetic wave onto the source field surface, through the *spatial* TI-FFT algorithm, and after the Fourier spectrum of the scattered electromagnetic field has been computed through the *spectral* TI-FFT algorithm.

Problems and possible solutions Although the planar TI-FFT algorithm has so many advantages given above, some problems do exist in the practical

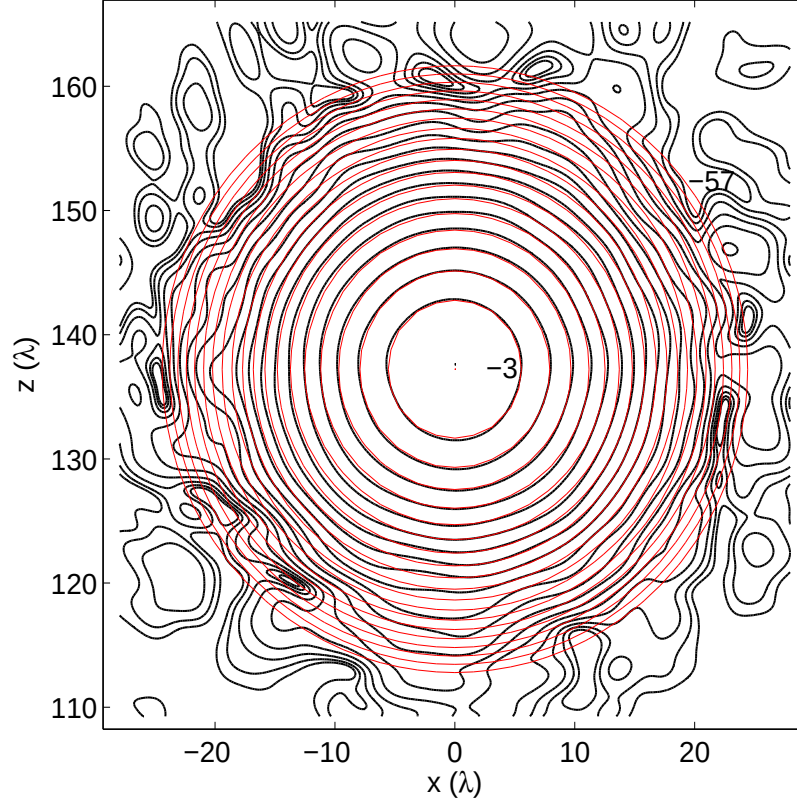


Figure 5.19: The computer-simulated output beam E_x^o (other components are negligible) is shown in black and the target FGB is shown in red, with 3 dB decrements, from -3 dB to -57 dB, from which it can be seen that the beam-shaping 4-mirror system has successfully shaped the input beam (Fig. 5.18) into the target FGB.

applications.

Complicate geometry As an example, consider surface S shown in Fig. 5.23, where the surface itself is not a quasi-planar surface and the direct implementation of the planar TI-FFT algorithm requires a large number of FFT operations, which can be seen from the spatial reference planes with a spatial slicing spacing δ_z . The problem can be solved by dividing surface S into two surface patches $\triangle S_1$ and $\triangle S_2$, which can be considered as quasi-planar surfaces and the planar TI-FFT can be used on them independently, with coordinate systems selected based on the spatial reference planes. At the extreme limit where surface patches $\triangle S_1$ and $\triangle S_2$ are planes, the number of FFT operations reduces to $\mathcal{N}_{\text{FFT}} = 2$.

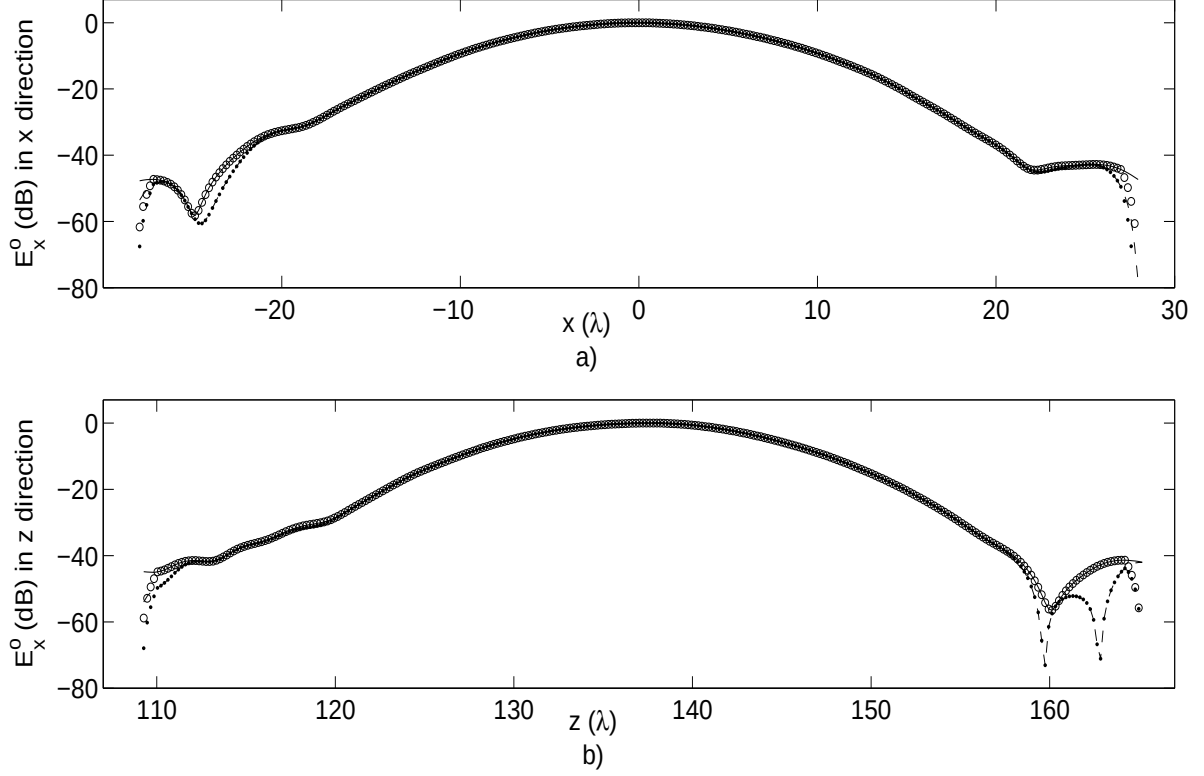


Figure 5.20: The scattered output beam E_x^0 (other components are negligible) from mirror 4 (M_4 in Fig. 1.2): a) plots in $\hat{x}$ direction; and b) plots in $\hat{y}$ direction. Solid lines and circles are for magnitudes, obtained from the planar TI-FFT algorithm and the direct integration method respectively; dashed lines and dots are for real parts, obtained from the planar TI-FFT algorithm and the direct integration method respectively.

Observation points not on the computational grid It is well-known that the FFT requires an even grid spacing (but δ_x and δ_y need not to be equal), which raises the question of how to calculate the electric field at points that are not exactly on the computational grid, e.g., the red filled circles in Fig. 5.24. One solution for this problem is to zero-pad the computational grid in the spectral domain, which corresponds to the interpolation of the computational grid in the spatial domain, as shown in Fig. 5.25. In the above example, it has been assumed that the observation points are evenly distributed and the interpolation results are exact provided that the sampling rate is above the Nyquist rate [30]. For complicate observation point configurations (e.g., unevenly distributed points),

Table 5.2: The CPU times and the coupling coefficients of $\mathbf{E}^\circ$ for different grid sizes: example 1

$N = \mathcal{N}_x \times \mathcal{N}_y$	$\delta_x = \delta_y \text{ (}\lambda\text{)}$	$t_{\text{TI}}(\text{sec.})$	$t_{\text{DI}}(\text{sec.})$	$t_{\text{DI}} / t_{\text{TI}}$	$\mathcal{C}_x(\%)$	$\mathcal{C}_y(\%)$	$\mathcal{C}_z(\%)$
64×64	0.917	4.0	12.4	3.1	75.534	75.337	73.976
80×80	0.734	6.2	29.6	4.8	94.756	94.483	91.249
112×112	0.524	10.9	115.0	10.6	99.987	99.969	99.902
128×128	0.459	14.0	196.6	14.0	99.999	99.999	99.999
144×144	0.408	16.6	310.2	18.7	99.999	99.999	99.999
162×162	0.362	21.2	474.6	22.4	99.999	99.999	99.999
176×176	0.334	25.2	696.0	27.6	99.999	99.999	99.999
192×192	0.306	30.3	1000.0	33.0	99.999	99.999	99.999
224×224	0.262	42.0	1861.2	44.3	99.999	99.999	99.999
256×256	0.229	50.1	3855.0	76.9	99.999	99.999	99.999
320×320	0.183	86.3	9311.4	107.9	99.999	99.999	99.999
512×512	0.115	247.1	64591.6	261.4	99.999	99.999	99.999
576×576	0.102	318.7	103463.0	324.6	99.999	99.999	99.999
1024×1024	0.057	1098.4	1033466.0	940.9	99.999	99.999	99.999

the approximate techniques like the Gauss's forward/backward interpolations can be used.

The translation in spatial domain In the real situation, the source field surface and the observation surface are separate far away from each other (see Fig. 5.26). It is not practical nor necessary to use a large computational grid that covers both the source field surface and the observation surface. This kind of problem can be solved by using two computational grids, one for the source field surface and the other for the observation surface, with the same grid spacings (δ_x, δ_y) . Then the translation of the observation coordinate system in the spatial domain, which is denoted as (x_0, y_0) , corresponds to the phase shift in the spectral domain. Suppose the electric field in the source coordinate system is expressed as $\mathbf{E}(x' - x_0, y' - y_0)$, according to the property of the Fourier transform [30], the electric field $\mathbf{E}(x, y)$ in the observation coordinate system is given as

$$\mathbf{E}(x, y) = \text{IFT}_{2\text{D}} \left[\text{FT}_{2\text{D}} \left[\mathbf{E}(x' - x_0, y' - y_0) \right] e^{-jk_x x_0} e^{-jk_y y_0} \right]. \quad (5.3.8.134)$$

Computational redundancy In the numerical implementation of the planar TI-FFT algorithm, the spatial domain or the spectral domain are divided into many small subdomains where the FFT can be used to interpolate the electromagnetic field (see Fig. 5.3). However, the FFT operation is done on the whole spatial or spectral domain even though the interpolation is only necessary on the relatively small subdomain, which causes the computational redundancy in the planar TI-FFT algorithm. Fortunately, the computational redundancy is small for a quasi-planar surface and a narrow-band beam. The improvement to the Taylor-FFT algorithm is to use the Hermite interpolation scheme to incorporate all the reference planes to do the interpolation on the whole surface.

5.4 Summary of the Planar TI-FFT Algorithm

In this article, the optimized planar TI-FFT algorithm for the computation of electromagnetic wave propagation, diffraction and scattering has been introduced for the case of the narrow-band beam and the quasi-planar geometry. Two types of TI-FFT algorithm are available, i.e., the spatial TI-FFT and the spectral TI-FFT. The former is for computation of electromagnetic wave on the quasi-planar surface and the latter is for computation of the 2D Fourier spectrum of the electromagnetic wave. There are also two numerical methods to implement the two types of TI-FFT algorithm, i.e., the spatial domain division method and the spectral domain division method. The optimized order of Taylor series used in the planar TI-FFT algorithm is found to be closely related to the algorithm's computational accuracy γ_{TI} , which is given as $\mathcal{N}_o^{\text{opt}} \sim -\ln \gamma_{\text{TI}}$ and the optimized spatial slicing spacing between two adjacent spatial reference planes only depends on the characteristic wavelength λ_c of the electromagnetic wave, which is $\delta_z^{\text{opt}} \sim \frac{1}{17}\lambda_c$. The optimized computational complexity is given as $\mathcal{O}(\mathcal{N}_r^{\text{opt}} \mathcal{N}_o^{\text{opt}} N \log_2 N)$ for an $N = \mathcal{N}_x \times \mathcal{N}_y$ computational grid, which is comparable to the computational complexity of the FMM. The planar TI-FFT algorithm allows a low sampling rate required by the Nyquist sampling theorem. Also, the algorithm doesn't have the problem singularities. The planar TI-FFT algorithm has applications in near-field and far-field computations, beam-shaping mirror system designs, diffraction and scattering phenomena, millimeter wave propagation, and microwave imaging in the half-space scenario.

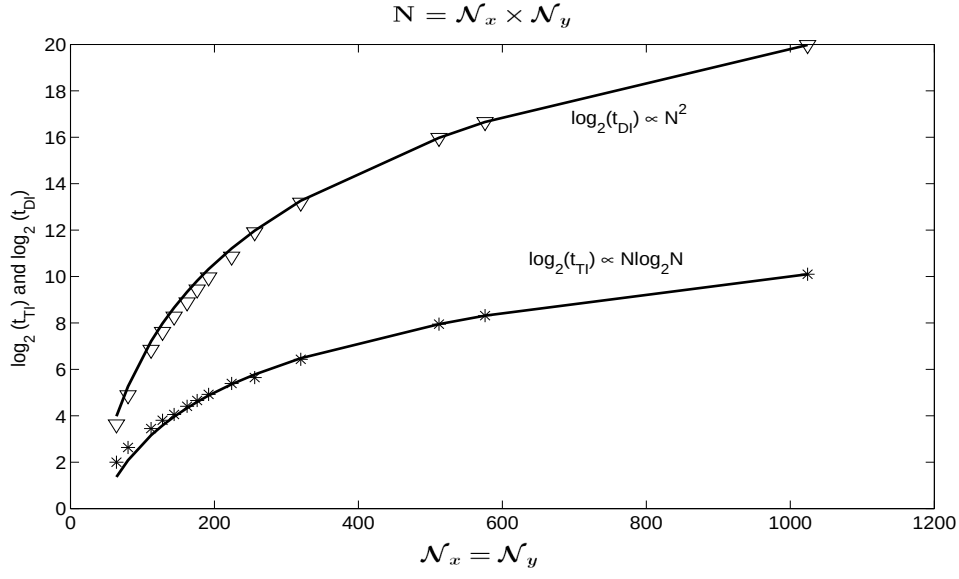


Figure 5.21: The actual CPU times t_{TI} and t_{DI} are shown as symbols, together with their corresponding asymptotic behaviors (lines): $O(N \log_2 N)$ for the planar TI-FFT algorithm and $O(N^2)$ for the direct integration method.

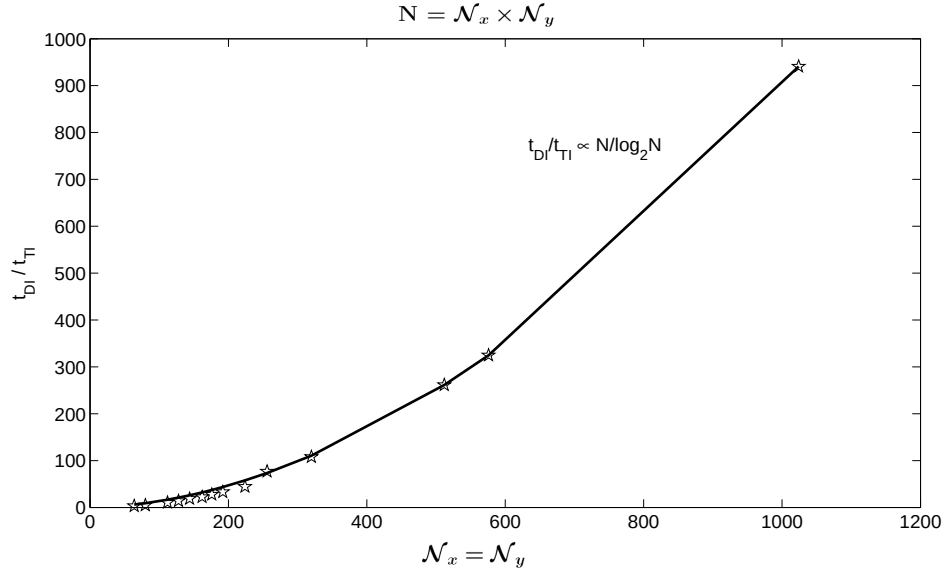


Figure 5.22: Ratio of $t_{\text{DI}}/t_{\text{TI}}$ (symbols) with their corresponding asymptotic values (line): $N / \log_2 N$.

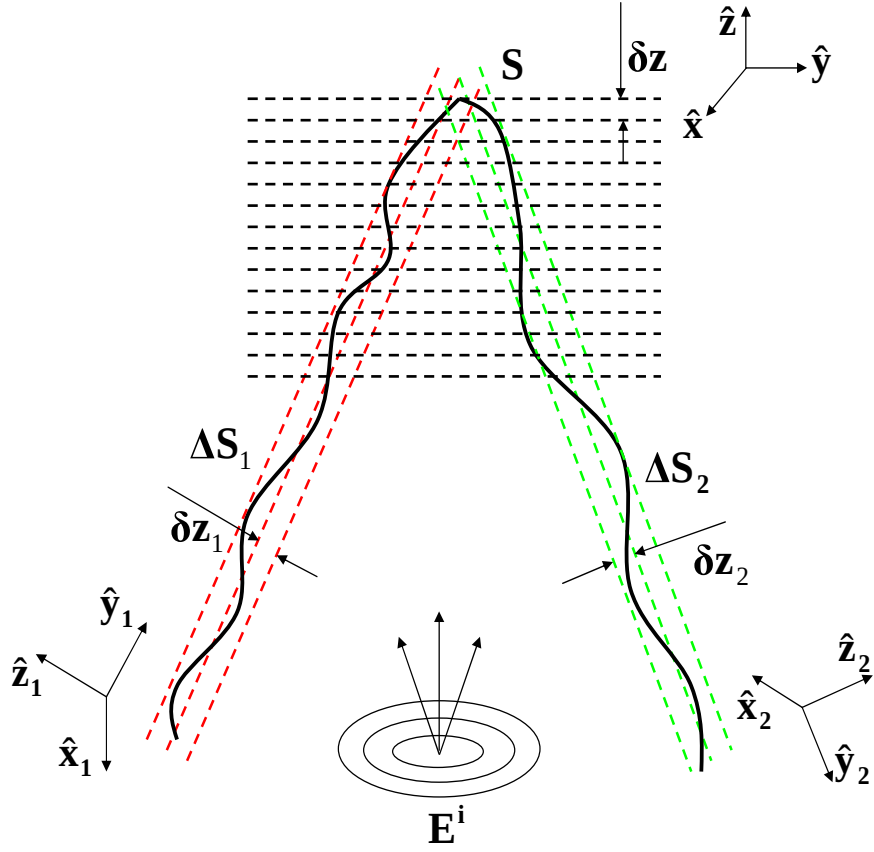


Figure 5.23: An example of complicate surface S that can be divided into two quasi-planar surface patches ΔS_1 and ΔS_2 . The computations of each surface patch is done in its corresponding coordinate system whose z -coordinate is perpendicular to the slicing spatial reference planes.

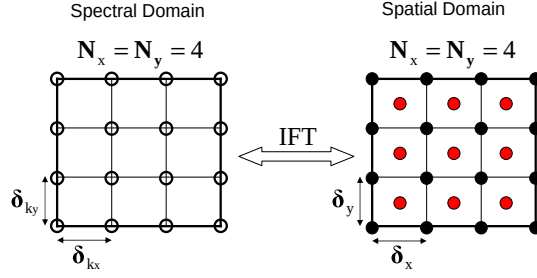


Figure 5.24: The problem of computation of electromagnetic field on the observation points that are not on the computational grid (4×4), which are denoted as red filled circles in the spatial domain (assume that they are evenly distributed). $(\delta_{k_x}, \delta_{k_y})$ are grid spacings in the spectral domain. (δ_x, δ_y) are grid spacings in the space domain.

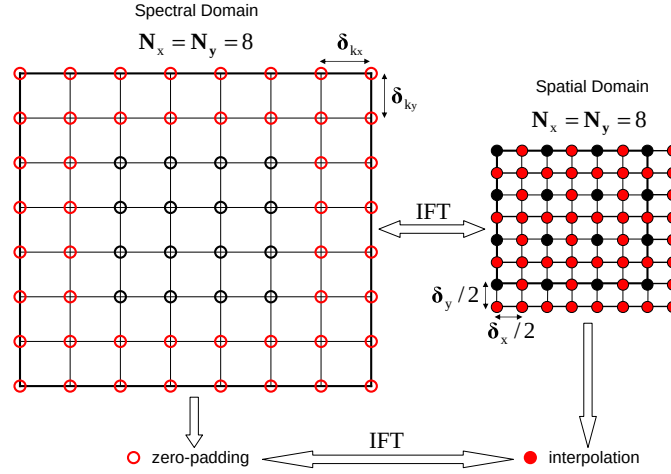


Figure 5.25: The zero-padding in the spectral domain ($4 \times 4 \rightarrow 8 \times 8$) corresponding to the interpolation in the spatial domain ($4 \times 4 \rightarrow 8 \times 8$). $(\delta_{k_x}, \delta_{k_y})$ are still the same after zero-padding. But grid spacings in the spatial domain become $(\delta_x/2, \delta_y/2)$ after interpolation.

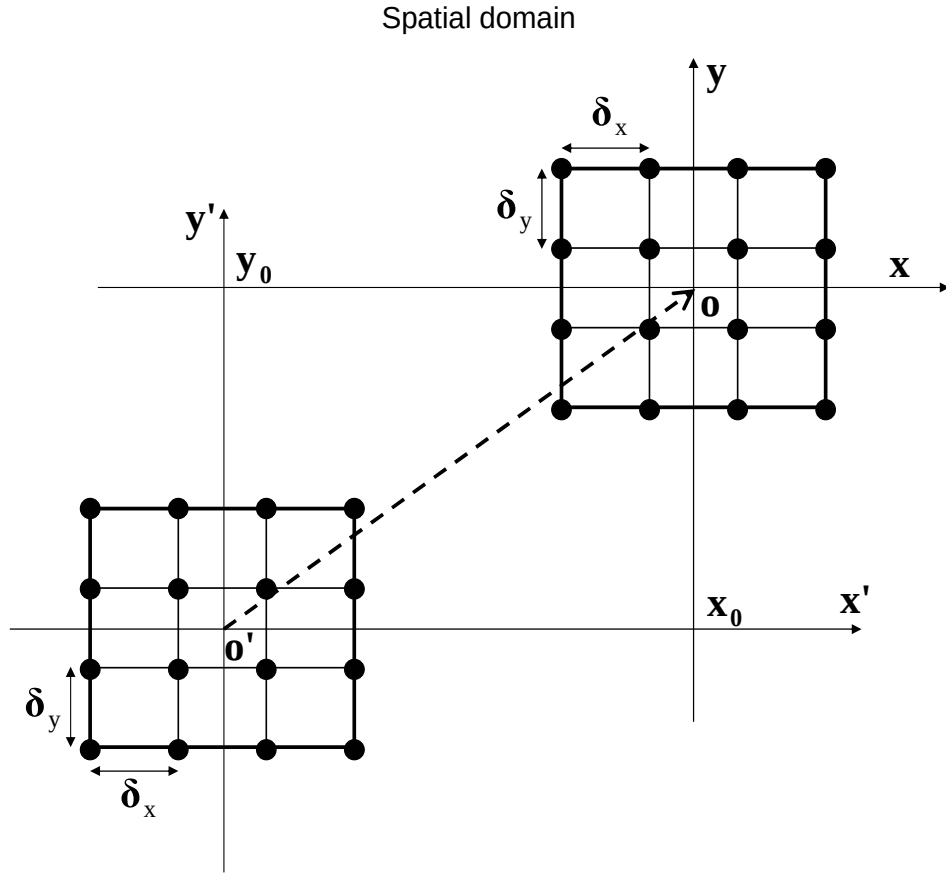


Figure 5.26: The translation of the source coordinate system $\mathbf{o}'(0,0)$ to the observation coordinate system $\mathbf{o}(x_0, y_0)$ in the spatial domain. Both the source and observation coordinate systems should have the same grid spacings (δ_x, δ_y) .

Chapter 6

THE CYLINDRICAL TI-FFT ALGORITHM

The planar Taylor Interpolation through FFT (TI-FFT) algorithm introduced in Chapter 5 and also in our publications [5], [7] has been shown to be efficient in the computation of narrow-band beam propagation and scattering for the quasi-planar geometry. However, cylinder-like geometry is not uncommon in the electromagnetic engineering, e.g., the input mirror system design [2] for the high-power gyrotron application. In such case, the planar TI-FFT algorithm is not efficient and we have developed the cylindrical TI-FFT to solve the problem.

In this chapter, we will introduce the cylindrical Taylor Interpolation through FFT (TI-FFT) algorithm for computation of the near-field and far-field in the quasi-cylindrical geometry. The modal expansion coefficient of the vector potentials $\mathbf{F}$ and $\mathbf{A}$ within the context of the cylindrical harmonics (TE and TM modes) can be expressed in the closed-form expression through the cylindrical addition theorem. For the quasi-cylindrical geometry, the modal expansion coefficient can be evaluated through FFT with the help of the Taylor Interpolation (TI) technique. The near-field on any arbitrary cylindrical surface can be obtained through the Inverse Fourier Transform (IFT). The far-field can be obtained through the Near-Field Far-Field (NF-FF) transform. The cylindrical TI-FFT algorithm has the advantages of $\mathcal{O}(N \log_2 N)$ computational complexity for $N = N_\phi \times N_z$ computational grid, small sampling rate (large sampling spacing) and no singularity problem.

For the cylindrical geometry, the computation is efficient because the electromagnetic field that is expressed in the cylindrical harmonics can be numerically implemented through the FFT. For the quasi-cylindrical geometry, the FFT can still be used, with the help of the Taylor Interpolation (TI) technique. Fig. 5.3

shows the scheme used to illustrate the cylindrical TI-FFT algorithm and the time dependence $e^{i\omega t}$ ($i \equiv \sqrt{-1}$) is used in this article.

6.1 The Near-field and the Far-field

In this section, the near-field and the far-field for surface currents ($\mathbf{M}_s, \mathbf{J}_s$) are presented within the context of the cylindrical harmonics.

6.1.1 The Near-field

It can be shown [9], [10], [11] that the vector potential ($\mathbf{F}, \mathbf{A}$) due to surface currents ($\mathbf{M}_s, \mathbf{J}_s$) for the scattering phenomenon in the region $\rho > \rho'$ can be expressed as

$$\begin{pmatrix} \mathbf{F}(\mathbf{r}) \\ \mathbf{A}(\mathbf{r}) \end{pmatrix} = \text{IFT} \left\{ \begin{pmatrix} \mathbf{f}_m^h \\ \mathbf{g}_m^h \end{pmatrix} H_m^{(2)}(\Lambda\rho) \right\}, \quad (6.1.1.1)$$

$$\begin{pmatrix} \mathbf{f}_m^h \\ \mathbf{g}_m^h \end{pmatrix} = \frac{1}{i4} \iint_S dS' \begin{pmatrix} \epsilon \mathbf{M}_s(\mathbf{r}') \\ \mu \mathbf{J}_s(\mathbf{r}') \end{pmatrix} H_m^{(1)}(\Lambda\rho) e^{im\phi'} e^{ihz'}, \quad (6.1.1.2)$$

where $H_m^{(1)}(\cdot)$ and $H_m^{(2)}(\cdot)$ are Hankel functions of the first kind and the second kind of integer order m respectively. The Inverse Fourier Transform (IFT) has been defined as,

$$\text{IFT} \left\{ \cdot \right\} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dh \left\{ \cdot \right\} e^{-im\phi} e^{-ihz}. \quad (6.1.1.3)$$

The electromagnetic field ($\mathbf{E}, \mathbf{H}$) is given as

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{\epsilon} \nabla \times \mathbf{F}(\mathbf{r}) - i\omega \mathbf{A}(\mathbf{r}) + \frac{1}{i\omega\epsilon\mu} \nabla' \left[\nabla' \cdot \mathbf{A}(\mathbf{r}) \right]. \quad (6.1.1.4)$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu} \nabla \times \mathbf{A}(\mathbf{r}) - i\omega \mathbf{F}(\mathbf{r}) + \frac{1}{i\omega\epsilon\mu} \nabla' \left[\nabla' \cdot \mathbf{F}(\mathbf{r}) \right]. \quad (6.1.1.5)$$

6.1.2 The cylindrical harmonics

The cylindrical TE and TM modes are obtained when the magnetic (electric) surface current has only $\hat{\mathbf{z}}$ -component, i.e., $\mathbf{M}_s = \hat{\mathbf{z}} M_{s,z}$ ($\mathbf{J}_s = \hat{\mathbf{z}} J_{s,z}$). From (6.1.1.1)-(6.1.1.5),

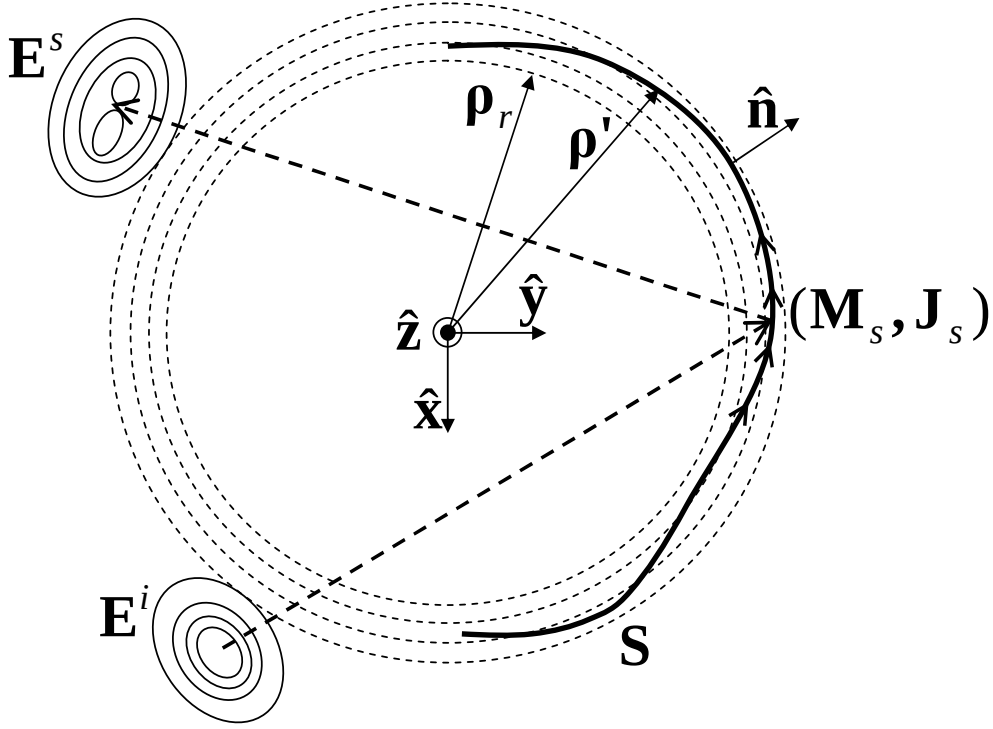


Figure 6.1: The scattering of the narrow-band beam: the incident field $\mathbf{E}^i$ propagates onto PEC surface S and is back-scattered to $\mathbf{E}^s$. The induced surface currents $(\mathbf{M}_s, \mathbf{J}_s)$ can be obtained through the Method of Moment (MoM) or Physical Optics (PO) approximation if PEC surface S is smooth enough. ρ' is the source coordinate and ρ_r is the radius of the reference cylindrical surface. $\hat{\mathbf{n}}$ is the surface normal to S .

$$\mathbf{M}_m^h(\mathbf{r}) = \left[\hat{\rho} \frac{m}{i\rho} H_m^{(2)}(\Lambda\rho) - \hat{\phi} \Lambda \frac{\partial H_m^{(2)}(\Lambda\rho)}{\partial(\Lambda\rho)} \right] e^{-im\phi} e^{-ihz}. \quad (6.1.2.6)$$

$$\mathbf{N}_m^h(\mathbf{r}) = \left[\hat{\rho} \frac{h\Lambda}{ik} \frac{\partial H_m^{(2)}(\Lambda\rho)}{\partial(\Lambda\rho)} - \hat{\phi} \frac{mh}{k\rho} H_m^{(2)}(\Lambda\rho) + \hat{\mathbf{z}} \frac{\Lambda^2}{k} H_m^{(2)}(\Lambda\rho) \right] e^{-im\phi} e^{-ihz}. \quad (6.1.2.7)$$

The electromagnetic field $(\mathbf{E}, \mathbf{H})$ can be expressed as the combination of the TE and TM modes,

$$\mathbf{E}(\rho) = \sum_m \left\{ \int_{-\infty}^{\infty} \left[a_m^h \mathbf{M}_m^h(\rho) + b_m^h \mathbf{N}_m^h(\rho) \right] dh \right\}, \quad (6.1.2.8)$$

$$\mathbf{H}(\rho) = \frac{i}{\eta} \sum_m \left\{ \int_{-\infty}^{\infty} \left[a_m^h \mathbf{N}_m^h(\rho) + b_m^h \mathbf{M}_m^h(\rho) \right] dh \right\}, \quad (6.1.2.9)$$

$$a_m^h = -\frac{1}{2\pi\epsilon} f_{m,z}^h, \quad b_m^h = -\frac{iv}{2\pi} g_{m,z}^h, \quad (6.1.2.10)$$

where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ and $v = \frac{1}{\sqrt{\mu\epsilon}}$ is the electromagnetic wave velocity in the homogeneous medium.

6.1.3 The Far-field

The far-field can be obtained through the Near-Field Far-Field (NF-FF) transform [28],

$$\mathbf{E}(\mathbf{R}) = -\frac{2k \sin \theta e^{-ikR}}{R} \sum_m i^m e^{-im\phi} \left[\hat{\phi} a_m^h + \hat{\theta} i b_m^h \right], \quad (6.1.3.11)$$

$$\mathbf{H}(\mathbf{R}) = -\frac{2k \sin \theta e^{-ikR}}{\eta R} \sum_m i^m e^{-im\phi} \left[\hat{\phi} i b_m^h - \hat{\theta} a_m^h \right], \quad (6.1.3.12)$$

where $\mathbf{R}$ is the coordinate in the far-field and $R = |\mathbf{R}|$.

6.2 The Cylindrical TI-FFT Algorithm

For the narrow-band beam and the quasi-cylindrical surface, both the electromagnetic field in (6.1.2.8)-(6.1.2.9) and the modal expansion coefficient in (6.1.1.2) can be expressed in the Taylor series, which facilitates the use of FFT. Due to the similarity, only TE mode will be considered in this article.

6.2.1 The Electromagnetic Field

Generally, the near-field $\mathbf{E}$ can be expressed in the Taylor series,

$$\mathbf{E}(\rho_r + \delta\rho) = \mathbf{E}(\rho_r) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\partial^{(n)} \mathbf{E}}{\partial \rho^{(n)}} \bigg|_{\rho_r} (\delta\rho)^n, \quad (6.2.1.13)$$

where, ρ_r is the reference cylindrical surface and the Taylor coefficient $\left. \frac{\partial^{(n)} \mathbf{E}}{\partial \rho^{(n)}} \right|_{\rho_r}$ can be expressed in the form of IFT. Take TE mode ($\mathbf{M}_m^h$) as an example, for $\hat{\phi}$ -component E_ϕ , from (6.1.2.6) and (6.1.2.8),

$$E_\phi(\rho) = \text{IFT} \left\{ \frac{\Lambda}{\epsilon} \frac{\partial H_m^{(2)}(\Lambda \rho)}{\partial (\Lambda \rho)} f_{m,z}^h \right\}, \quad (6.2.1.14)$$

Now, the Taylor coefficient for $E_\phi(\rho)$ are given as

$$\left. \frac{\partial^{(n)} \mathbf{E}}{\partial \rho^{(n)}} \right|_{\rho_r} = \text{IFT} \left\{ \frac{\Lambda^{n+1}}{\epsilon} \frac{\partial^{(n+1)} H_m^{(2)}(\Lambda \rho)}{\partial (\Lambda \rho)^{(n+1)}} \right|_{\rho_r} f_{m,z}^h \right\}. \quad (6.2.1.15)$$

Similar argument holds for other electromagnetic field components and TM mode.

6.2.2 The Modal Expansion Coefficient

Similarly, $H_m^{(1)}(\Lambda \rho')$ in the the modal expansion coefficients ($\mathbf{f}_m^h, \mathbf{g}_m^h$) in (6.1.1.2) can be expanded into the Taylor series,

$$H_m^{(1)} \left(\Lambda \left[\rho_r + \delta \rho' \right] \right) = H_m^{(1)}(\Lambda \rho_r) + \sum_{n=1}^{\infty} \frac{\Lambda^n}{n!} \frac{\partial^{(n)} H_m^{(1)}(\Lambda \rho)}{\partial (\Lambda \rho)^{(n)}} \bigg|_{\rho_r} (\delta \rho')^n \quad (6.2.2.16)$$

where $\delta \rho' = \rho' - \rho_r$. Now, the modal expansion coefficient in (6.1.1.2) is given as

$$\begin{pmatrix} \mathbf{f}_m^h \\ \mathbf{g}_m^h \end{pmatrix} = \sum_{\Delta S} \sum_{n=0}^{\infty} \text{FT} \left\{ \gamma_m(\Lambda \rho_r) \begin{pmatrix} \epsilon \tilde{\mathbf{M}}_s(\mathbf{r}') \\ \mu \tilde{\mathbf{J}}_s(\mathbf{r}') \end{pmatrix} (\delta \rho')^n \right\} \bigg|_{\Delta S} \quad (6.2.2.17)$$

where the Fourier Transform FT is defined similarly as IFT in (7.3.2.14) and ΔS is the small surface patch between two adjacent reference cylindrical surfaces; what's more, the following quantities have been defined,

$$\gamma_m(\Lambda \rho_r) = \frac{\pi}{i2} \frac{\Lambda^n \rho'}{n!} \frac{\partial^{(n)} H_m^{(1)}(\Lambda \rho)}{\partial (\Lambda \rho)^{(n)}} \bigg|_{\rho_r}, \quad \begin{pmatrix} \tilde{\mathbf{M}}_s(\mathbf{r}') \\ \tilde{\mathbf{J}}_s(\mathbf{r}') \end{pmatrix} = \frac{1}{\hat{\mathbf{n}} \cdot \hat{\rho}'} \begin{pmatrix} \mathbf{M}_s(\mathbf{r}') \\ \mathbf{J}_s(\mathbf{r}') \end{pmatrix}. \quad (6.2.2.18)$$

$\hat{\mathbf{n}}$ is the surface normal to S .

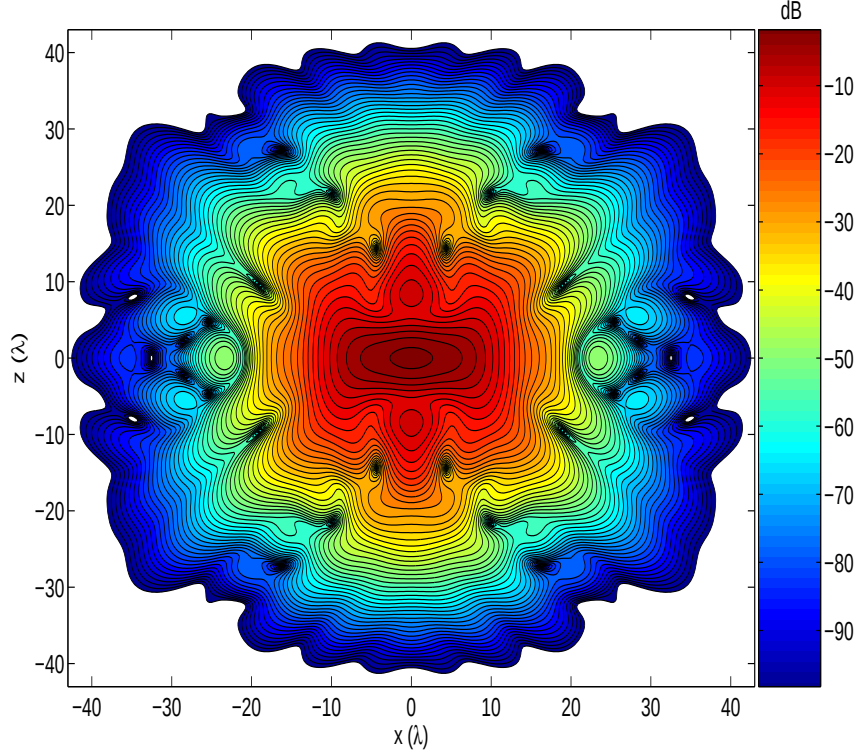


Figure 6.2: $20 \log_{10} |E_x^s|$.

6.3 Numerical Result

To show the efficiency of the cylindrical TI-FFT algorithm, the direct integration method [1], [3], [4], [6] has been used to make comparison with the cylindrical TI-FFT algorithm. The numerical example used for such purpose is a 110 GHz ($\lambda \sim 2.7$ mm) Fundamental Gaussian Beam (FGB) scattered by a PEC quasi-cylindrical surface with a cosine wave perturbation. The incident FGB is $\hat{\mathbf{x}}$ -polarized and propagates at $\hat{\mathbf{z}}$ direction, with symmetrical beam waist radii $w_x = w_y = 8\lambda$. The quasi-cylindrical PEC surface is given as

$$y(x, z) = \sqrt{(80\lambda)^2 - x^2} + 0.1\lambda \cos\left(2\pi \frac{x}{20\lambda}\right) \cos\left(2\pi \frac{z}{20\lambda}\right) \quad (6.3.0.19)$$

$$\rho(x, z) = x \cos \phi + y \sin \phi, \quad \phi = \arctan \left[\frac{y}{x} \right]$$

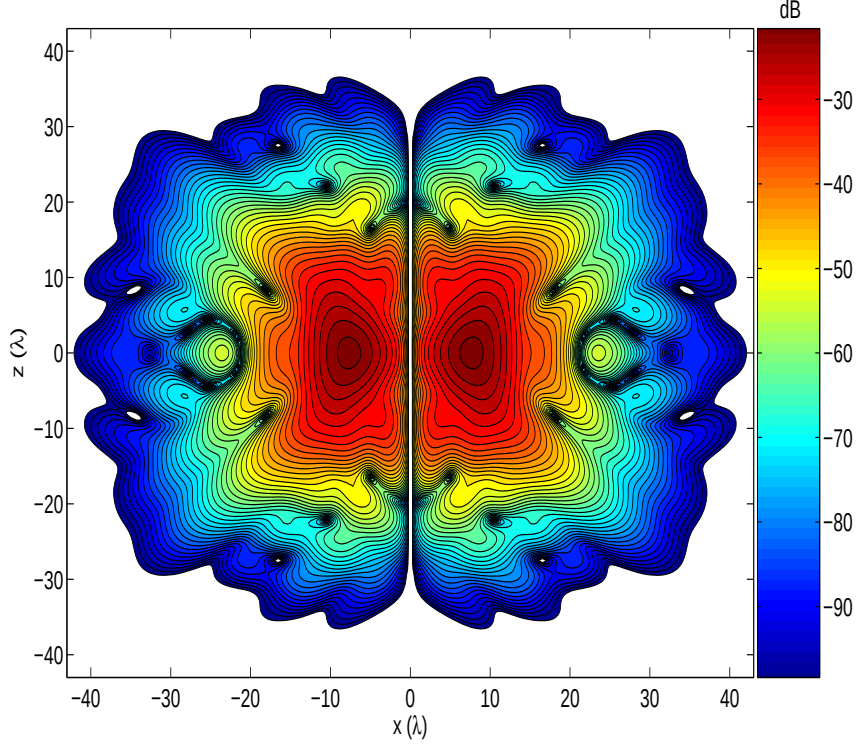


Figure 6.3: $20 \log_{10} |E_z^s|$.

The scattered field $\mathbf{E}^s$ is evaluated on plane $y = 0$ (where the incident FGB starts to propagate). Fig. 6.2 and Fig. 6.3 show the magnitude patterns of the x-component E_x^s and the z-component E_z^s of the scattered output field $\mathbf{E}^s$ (y-component E_y^s is small and not shown). The comparison of result obtained from the cylindrical TI-FFT algorithm and that from the direct integration method is given in Fig. 6.4, for both the magnitude and the real part.

The CPU time for the cylindrical TI-FFT algorithm t_{TI} and the CPU time for the direct integration method t_{DI} are shown in Fig. 6.5. The ratio $t_{\text{DI}}/t_{\text{TI}}$ is shown in Fig. 6.6, for different size of the computational grid ($N = N_\phi \times N_z$). All work was done in Matlab 7.0.1, on a 1.66 GHz PC, with Intel Core Duo and 512 MB RAM.

6.4 Summary of Cylindrical TI-FFT Algorithm

The cylindrical TI-FFT algorithm for the computation of the electromagnetic wave propagation and scattering has been introduced for the narrow-band beam and the quasi-geometry geometry. The cylindrical TI-FFT algorithm has the complexity of $\mathcal{O}(N \log_2 N)$ for $N = N_\phi \times N_z$ computational grid. The algorithm allows for a low sampling rate (limited the Nyquist sampling rate) and doesn't have the problem of singularity.

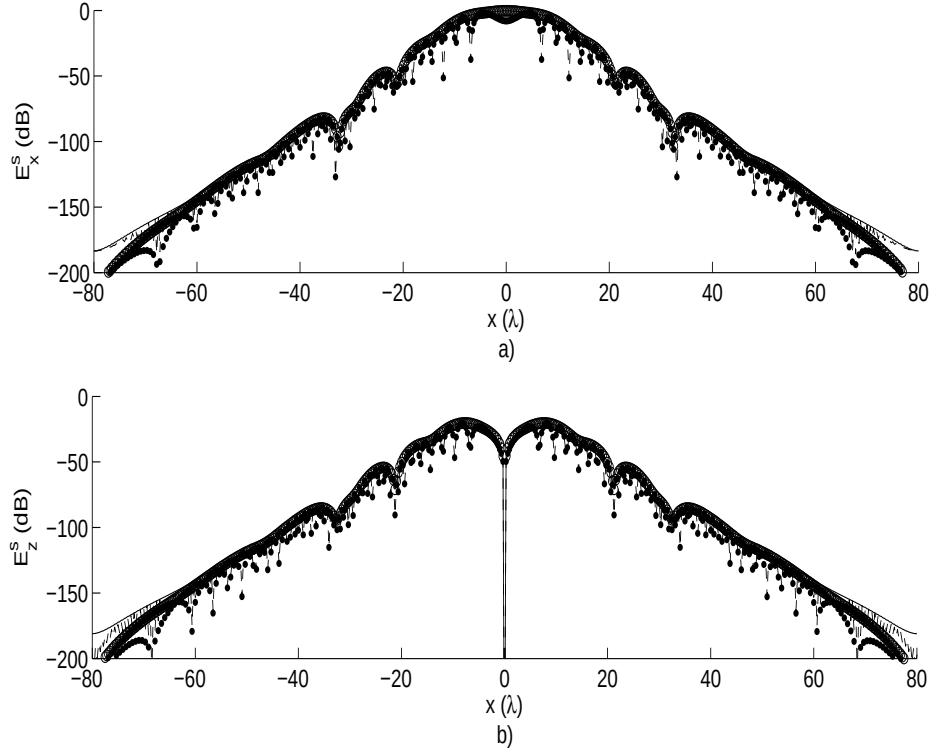


Figure 6.4: Comparison of the cylindrical TI-FFT algorithm with the direct integration method: a) E_x (dB); and b) E_z (dB). Plots are shown in $\hat{x}$ direction, across the maximum value point of $|E_x|$. Solid and dashed lines denote the magnitude and real part obtained from the direct integration method respectively; circles and dots denote the magnitude and real part obtained from the cylindrical TI-FFT algorithm respectively. E_y is small and not shown.

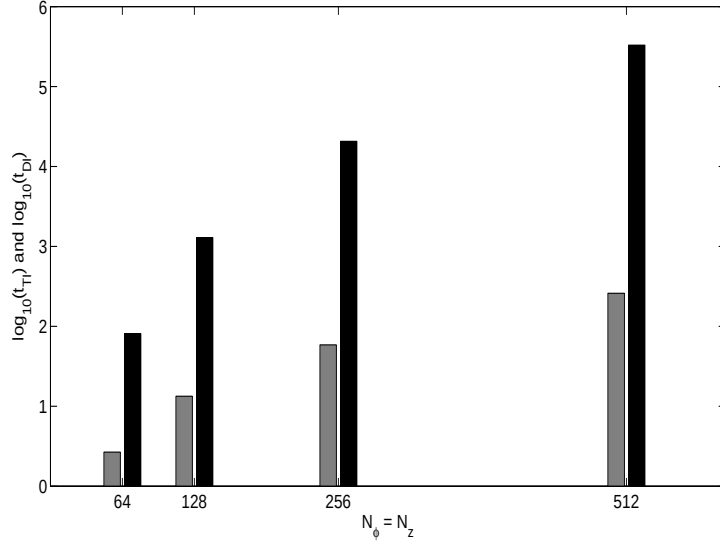


Figure 6.5: The logarithmic CPU time t_{TI} and t_{DI} .

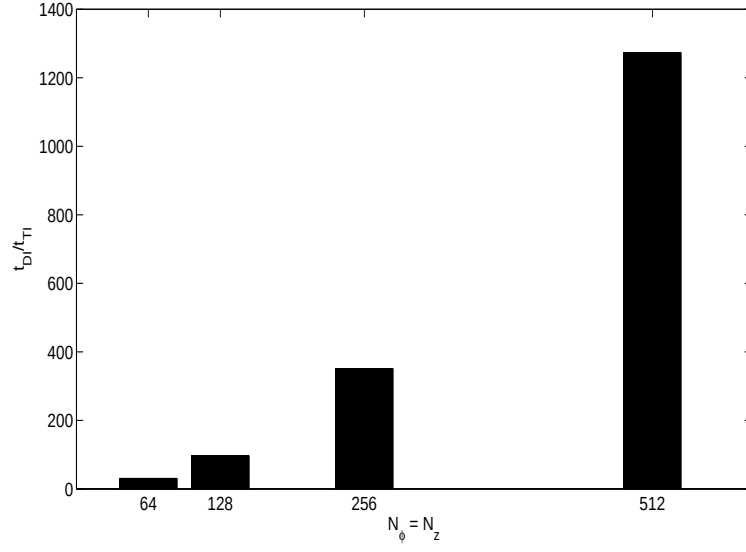


Figure 6.6: The CPU time ratio of $t_{\text{DI}}/t_{\text{TI}}$.

Chapter 7

THE PHASE CORRECTION

The Perfect Electric Conductor (PEC) mirror phase corrector plays an important role in the beam-shaping mirror system design for Quasi-Optical (QO) mode converter (launcher) in the sub-THz high-power gyrotron. In this Chapter, both the Geometry Optical (GO) method and the phase gradient method have been presented for the PEC mirror phase corrector design. The advantages and disadvantages are discussed for both methods. An efficient algorithm has been proposed for the phase gradient method.

7.1 2D Phase Unwrapping

The phase correction requires the knowledge of the unwrapped 2-Dimensional (2D) phases of the incident electric field and the reflected electric field (θ^i, θ^r). However, the phase obtained from the electric field $\mathbf{E}$ through $\check{\theta} = \arctan \left[\frac{\Im(\mathbf{E})}{\Re(\mathbf{E})} \right]$ ($\Re$ and $\Im$ denote the real part and the imaginary part respectively) is the wrapped 2D phase, which contains discontinuities of $2n\pi$ (n is an integer). So, in order to ensure the smoothness of the PEC mirror surface, the wrapped 2D phases ($\check{\theta}^i, \check{\theta}^r$) must be unwrapped through the 2D phase unwrapping methods [47, 48].

Mathematically, in the ideal situation where there is no residues in the wrapped 2D phase $\check{\theta}$, the discrete phase gradient $\nabla\theta = \nabla\check{\theta}$ (assuming that $\nabla\theta < \pi$) and the 2D phase unwrapping can be expressed as,

$$\theta = \int_C \nabla\check{\theta} \cdot d\mathbf{r} + \theta(\mathbf{r}_0) \quad (7.1.0.1)$$

where, θ denotes the 2D unwrapped phase along the integration path C and $\mathbf{r}_0$ denotes the starting point of the path integration. Note that the unwrapped

phase θ obtained through (7.1.0.1) should not depend on the integration path C . However, due to the residues in practice, the discrete phase gradient should be written as $\nabla\check{\theta} = (\nabla g + \nabla \times \mathbf{R})$ and the unwrapped phase θ is obtained as follows,

$$\theta = \int_C (\nabla g + \nabla \times \mathbf{R}) \cdot d\mathbf{r} + \theta(\mathbf{r}_0) \quad (7.1.0.2)$$

From (7.1.0.2), it can be seen that $\nabla \times \mathbf{R} \neq 0$ is caused by the existence of residues and the unwrapped phase θ depends on the integration path C . There are many 2D phase unwrapping algorithms to deal with the residues in the literatures [47], [48]. For example, the path following algorithm (e.g., “quality-guided” method and “mask-cut” method) gives faithful congruent unwrapped phase (with $2n\pi$ difference from the wrapped phase). However, path following algorithm is time-consuming and the unwrapped phase contains many discontinuities due to the existence of residues. Another commonly-used algorithm, the minimum norm method unwraps the wrapped phase by minimizing the r -norm phase difference between the gradients of the wrapped phase and the desired unwrapped phase [48],

$$Q = \sum_X \sum_Z \left[w_x \left| \frac{\partial \theta}{\partial x} - \frac{\partial \check{\theta}}{\partial x} \right|^r + w_z \left| \frac{\partial \theta}{\partial z} - \frac{\partial \check{\theta}}{\partial z} \right|^r \right] \quad (7.1.0.3)$$

where, w_x and w_z are weights for $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ directions respectively. When $r = 2$, it is called the Least Mean Square (LMS) method.

7.2 The GOPC Method

In the sub-THz QO regime, it is reasonable to assume that the intensity or magnitude of the electric field is locally constant and the local phase change can be evaluated through the GO method, as shown in Fig. 7.1. For fixed computational grid given on x-z plane (in favor of FFT operation), $\delta_{\hat{\mathbf{y}}}$ is preferred, which is rewritten as follows ($\cos \alpha^i = \frac{\mathbf{k}^i \cdot \hat{\mathbf{n}}}{k}$),

$$\delta_{\hat{\mathbf{y}}} = \frac{\delta \theta}{2 (\mathbf{k}^i \cdot \hat{\mathbf{n}}) (\hat{\mathbf{y}} \cdot \hat{\mathbf{n}})}, \quad \delta \theta = \theta^r - \theta^i \quad (7.2.0.4)$$

GO approximation:

$$\alpha^i = \alpha^r$$

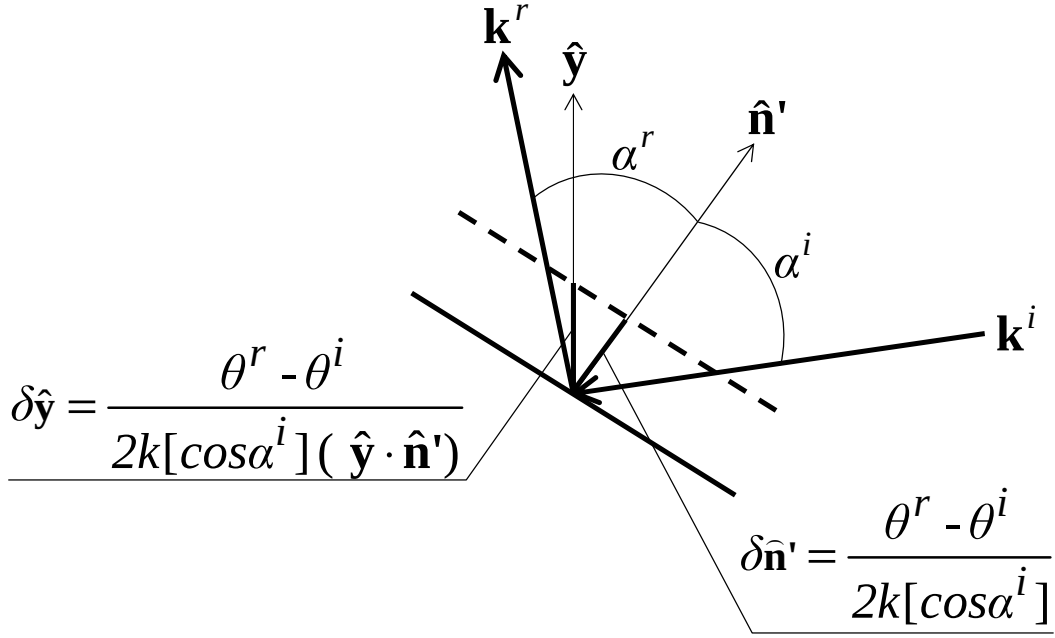


Figure 7.1: The PEC mirror surface correction in the sub-THz QO regime: $\mathbf{k}^i$ and $\mathbf{k}^r$ are wave vectors for the local incident beam (with incident angle α^i) and the local reflection beam (with reflected angle α^r). $\delta \hat{\mathbf{n}}$ is the PEC mirror surface correction in $\hat{\mathbf{n}}$ direction and $\delta \hat{\mathbf{y}}$ is the PEC mirror surface correction in $\hat{\mathbf{y}}$ direction.

There are two approaches to calculate the local wave vector $\mathbf{k}$ (incident wave vector $\mathbf{k}^i$ and reflected wave vector $\mathbf{k}^r$), i.e., 1) the Poynting vector approach; and 2) the phase gradient approach. The Poynting vector approach assumes that the local beam propagates in the direction given by the Poynting vector,

$$\mathbf{k} \propto \mathbf{E} \times (\mathbf{H})^* \propto \mathbf{E} \times \nabla \times (\mathbf{E})^* \quad (7.2.0.5)$$

The phase gradient approach approximates the local wave vector as the gradient of the phase,

$$\mathbf{k} \propto \nabla \theta \quad (7.2.0.6)$$

It is not difficult to show that the two approaches are equivalent in the far-field limit.

7.3 Phase Gradient Phase Correction Method

To overcome the shortcomings of the GOPC method, we have developed the PGPC method and the corresponding efficient algorithm to fasten the 2D phase unwrapping.

7.3.1 Basic Theory

Instead of (7.2.0.4), the expression of the PEC mirror surface correction $\delta_{\hat{\mathbf{y}}}$ in the phase gradient method is given as

$$\delta_{\hat{\mathbf{y}}} = \frac{\delta\theta}{\nabla(\delta\theta)} = \frac{\delta\theta}{\nabla\theta^r - \nabla\theta^i} \quad (7.3.1.7)$$

The phase gradient $\nabla\theta$ for the electric field $\mathbf{E} = |\mathbf{E}|e^{i\theta}$ can be found as

$$\begin{aligned} \nabla\mathbf{E} &= \nabla\{|\mathbf{E}|e^{i\theta}\} \\ &= \nabla\{|\mathbf{E}|\}e^{i\theta} + |\mathbf{E}|\nabla\{e^{i\theta}\} \\ &= \left[\nabla\{|\mathbf{E}|\} + i|\mathbf{E}|\nabla\theta \right] e^{i\theta} \end{aligned} \quad (7.3.1.8)$$

$$\begin{aligned} \rightarrow \nabla\theta &= \Im \left[\frac{\nabla\mathbf{E}}{\mathbf{E}} \right] = \Im \left[\nabla \ln \mathbf{E} \right] \\ \rightarrow \nabla\{|\mathbf{E}|\} &= \Re \left[\nabla\mathbf{E}e^{-i\theta} \right] \end{aligned} \quad (7.3.1.9)$$

From (7.3.1.8) and (7.3.1.9), the expression for $\nabla(\delta\theta)$ in (7.3.1.7) is obtained,

$$\nabla(\delta\theta) = \Im \left[\frac{\nabla\mathbf{E}^r}{\mathbf{E}^r} - \frac{\nabla\mathbf{E}^i}{\mathbf{E}^i} \right] = \Im \left[\nabla \ln \frac{\mathbf{E}^r}{\mathbf{E}^i} \right] \quad (7.3.1.10)$$

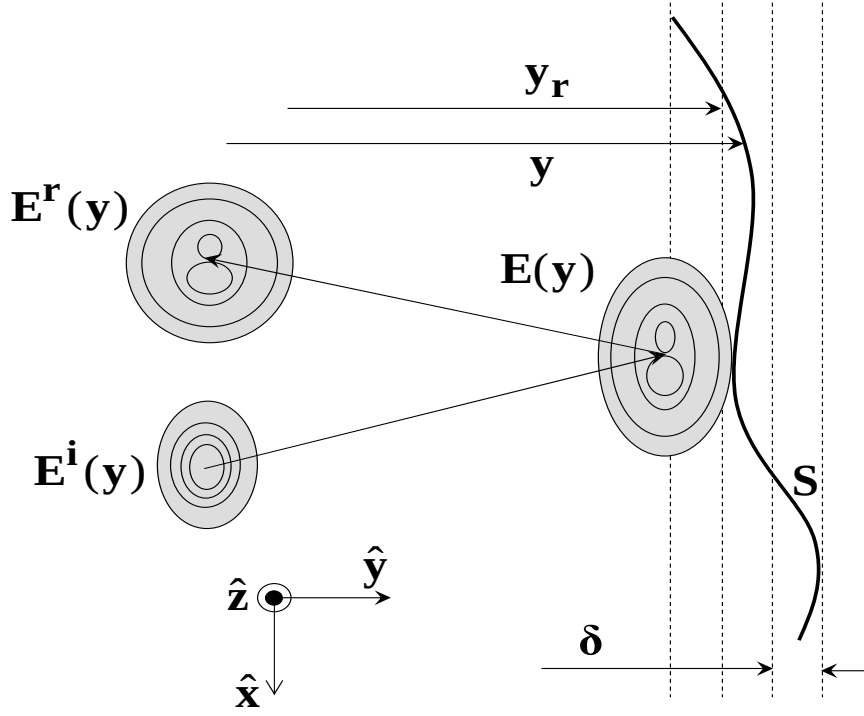


Figure 7.2: Illustration of the FFT-based efficient algorithm for the phase gradient method. $\mathbf{E}^i$ and $\mathbf{E}^r$ are the incident electric field and reflected electric field respectively. S is the PEC mirror phase corrector. y is the coordinate of the PEC mirror phase corrector and y_r is the coordinate of the slicing reference plane. δ is the spacing between two adjacent slicing reference planes.

7.3.2 An Efficient Algorithm for PGPC

By slicing the PEC mirror phase corrector into many subdomains, as shown in Fig. 7.2, the FFT can be used [7, 2] to compute the electric field $\mathbf{E}$ and its derivatives,

$$\mathbf{E}(y) = \text{IFT} \left\{ \mathbf{F}(k_x, k_z) e^{-ik_y y} \right\}, \quad \mathbf{F}(k_x, k_z) = \text{FT} \left\{ \mathbf{E}(y=0) \right\} \quad (7.3.2.11)$$

$$\frac{\partial \mathbf{E}(y)}{\partial v} = \text{IFT} \left\{ -ik_v \mathbf{F}(k_x, k_z) e^{-ik_y y} \right\}, \quad v = x, z \quad (7.3.2.12)$$

where, the Fourier Transform (FT) and the Inverse Fourier Transform (IFT) are defined as follows,

$$\text{FT} \{ \cdot \} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ik_x x} \int_{-\infty}^{\infty} \{ \cdot \} e^{ik_z z} dz \quad (7.3.2.13)$$

$$\text{IFT} \{ \cdot \} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_x e^{-ik_x x} \int_{-\infty}^{\infty} \{ \cdot \} e^{-ik_z z} dk_z \quad (7.3.2.14)$$

The wrapped phase difference $\delta\check{\theta} = (\check{\theta}^i - \check{\theta}^r)$ is obtained from (7.3.2.11). Due to similarity, only x-component $\hat{\mathbf{x}}\mathbf{E}_x$ is considered here,

$$\delta\check{\theta}_x = \arctan \left[\frac{\Im \left(\text{IFT} \{ \mathbf{F}_x^r(k_x, k_z) e^{-ik_y y} \} \right)}{\Re \left(\text{IFT} \{ \mathbf{F}_x^r(k_x, k_z) e^{-ik_y y} \} \right)} \right] - \arctan \left[\frac{\Im \left(\text{IFT} \{ \mathbf{F}_x^i(k_x, k_z) e^{-ik_y y} \} \right)}{\Re \left(\text{IFT} \{ \mathbf{F}_x^i(k_x, k_z) e^{-ik_y y} \} \right)} \right] \quad (7.3.2.15)$$

With the help of (7.3.2.11)-(7.3.2.14), the gradient of the phase difference $\nabla(\delta\theta_x)$ on the slicing reference plane y_r in Fig. 7.2 can be obtained from (7.3.1.10),

$$\nabla(\delta\theta_x) = \nabla(\delta\check{\theta}_x) = \Re \left[\frac{\text{IFT} \{ \mathbf{k} \mathbf{F}_x^i(k_x, k_z) e^{-ik_y y} \}}{\text{IFT} \{ \mathbf{F}_x^i(k_x, k_z) e^{-ik_y y} \}} - \frac{\text{IFT} \{ \mathbf{k} \mathbf{F}_x^r(k_x, k_z) e^{-ik_y y} \}}{\text{IFT} \{ \mathbf{F}_x^r(k_x, k_z) e^{-ik_y y} \}} \right] \quad (7.3.2.16)$$

To obtain the PEC mirror surface correction $\delta_{\hat{\mathbf{y}}}$ through (7.3.1.7), $\delta\check{\theta}_x$ has to be unwrapped. Here, an FFT-based phase unwrapping algorithm is presented for the r -norm minimum problem given in (7.1.0.3). Suppose that $\delta\theta_x$ can be expressed in the Fourier series,

$$\delta\theta_x = \text{IFT} \{ f(k_x, k_z) \} \quad (7.3.2.17)$$

Then,

$$\frac{\partial(\delta\theta_x)}{\partial v} = \text{IFT} \{ -ik_v f(k_x, k_z) \}, \quad v = x, z \quad (7.3.2.18)$$

To obtain $\delta\theta_x$, the Fourier coefficient $f(k_x, k_z)$ is chosen to minimize the cost function Q given in (7.1.0.3), with $w_x = w_z = 1$. For LMS method where $r = 2$, it can be shown that $f(k_x, k_z)$ takes the following form,

$$\frac{Q(f + \delta f) - Q(f)}{\delta f} = 0 \rightarrow f(k_x, k_z) = i \frac{k_x \text{FT} \left\{ \nabla \left(\delta\check{\theta}_x \right) \cdot \hat{\mathbf{x}} \right\} + k_z \text{FT} \left\{ \nabla \left(\delta\check{\theta}_x \right) \cdot \hat{\mathbf{z}} \right\}}{k_x^2 + k_z^2} \quad (7.3.2.19)$$

Now, the PEC mirror surface correction $\delta_{\hat{\mathbf{y}}}$ can be obtained from (7.3.1.7), with the help of (7.3.2.16)-(7.3.2.19).

7.4 Summary of the Phase Correction Techniques

In this Chapter, both the GO method and the phase gradient method have been presented for the PEC mirror phase corrector design. The FFT-based efficient algorithm has been proposed for the phase gradient method to speed up the design procedure.

Chapter 8

MULTI-MODE OPTIMIZATION

It has been shown in Chapter 7 that the single-mode surface correction to the PEC surface can be done through either the GOPC method or the PGPC method. For multi-mode surface correction, the phase correction can't be satisfied at the same time for all modes and the performance cost function has to be defined and minimized.

8.1 Least Mean Square (LMS) Optimization

Since we are concerned about the phase difference, let's define the r-norm performance cost function as follows,

$$P = \int_X \int_Z \sum_i \left[w_i |\nu_i \delta y - \delta \theta_i|^r \right] dx dz \quad (8.1.0.1)$$

where, δy is the multi-mode PEC surface correction; w_i is the weight for the i^{th} mode; θ_i is the phase difference of the i^{th} mode; and ν_i is given as

$$\nu_i = \begin{cases} 2k \cos \alpha^i & \text{GOPC method} \\ \nabla(\theta^i - \theta^r) \cdot \hat{\mathbf{y}} & \text{PGPC method} \end{cases} \quad (8.1.0.2)$$

Without loss of generality, let's take $r = 2$ (Least Mean Square phase difference) and minimize the performance cost function with respect to δy , from 8.1.0.1,

$$\begin{aligned}
\frac{\partial P}{\partial \delta y} &= \int_X \int_Z \sum_i \left[2w_i \nu_i (\nu_i \delta y - \delta \theta_i) \right] dx dz = 0 \\
&\Rightarrow \sum_i w_i \nu_i (\nu_i \delta y - \delta \theta_i) = 0 \\
&\Rightarrow \delta y = \frac{\sum_i w_i \nu_i^2 \delta \theta_i}{\sum_i w_i \nu_i^2}
\end{aligned} \tag{8.1.0.3}$$

8.2 Different Weighting Scheme

In this section, we will give different weighting scheme that is used in (8.1.0.2).

1) Single-mode single-frequency weighting

$$w_i = 1, \quad w_j = 0, \quad i \neq j \tag{8.2.0.4}$$

2) Even weighting

$$w_i = \frac{1}{2} \tag{8.2.0.5}$$

The even weighting scheme is the simplest one for multi-mode multi-frequency design.

3) Amplitude weighting

$$w_i = \frac{|\mathbf{E}_i|}{\sum_i |\mathbf{E}_i|} \tag{8.2.0.6}$$

This weighting scheme is based on the amplitude of the electric field distribution.

4) Power weighting

$$w_i = \frac{|\mathbf{E}_i|^2}{\sum_i |\mathbf{E}_i|^2} \tag{8.2.0.7}$$

This weighting scheme is based on the power of different modes.

Table 8.1: Typical tilting and offset values for four modes

	TE _{22,6}	TE _{23,6}	TE _{24,6}	TE _{25,6}
tilting at $\hat{\mathbf{x}}$ (deg.)	0.50	0.10	-0.25	-0.50
offset at $\hat{\mathbf{x}}$ (cm)	0.65	0.30	-0.05	-0.35
tilting at $\hat{\mathbf{z}}$ (deg.)	-0.05	-0.05	0.00	0.00
offset at $\hat{\mathbf{z}}$ (cm)	-0.10	0.10	0.35	0.30

8.3 Tilting and Offset of the Output Diamond Window

It has been shown that in multi-mode design practice, the tilting and shift of the output FGB position (refer to 8.1) is desirable and necessary [1]. This is because, different modes have different frequencies (wavelength) and they “see” different sizes of the mirrors and propagation length. What’s more, the input modes obtained from the QO launcher also have different incident angles. During the multi-mode design practice, the optimized tilting angle α_i and the offset amount z_i can only be obtained for specific problem and according to the following experiences,

- a) design a single-mode (e.g., TE_{22,6}) single-frequency mirror system;
- b) propagate different modes from the launcher using the design mirror system from a);
- c) determine the output desired FGB tilting angles and offset amounts for different modes by fitting the tilted- and offset-FGBs with the output beams from b);
- d) adjust the tilting angles and offset amounts during the following multi-mode multi-frequency design if necessary.

Typical values of the tilting and offset have been listed in Table 8.1, after [1].

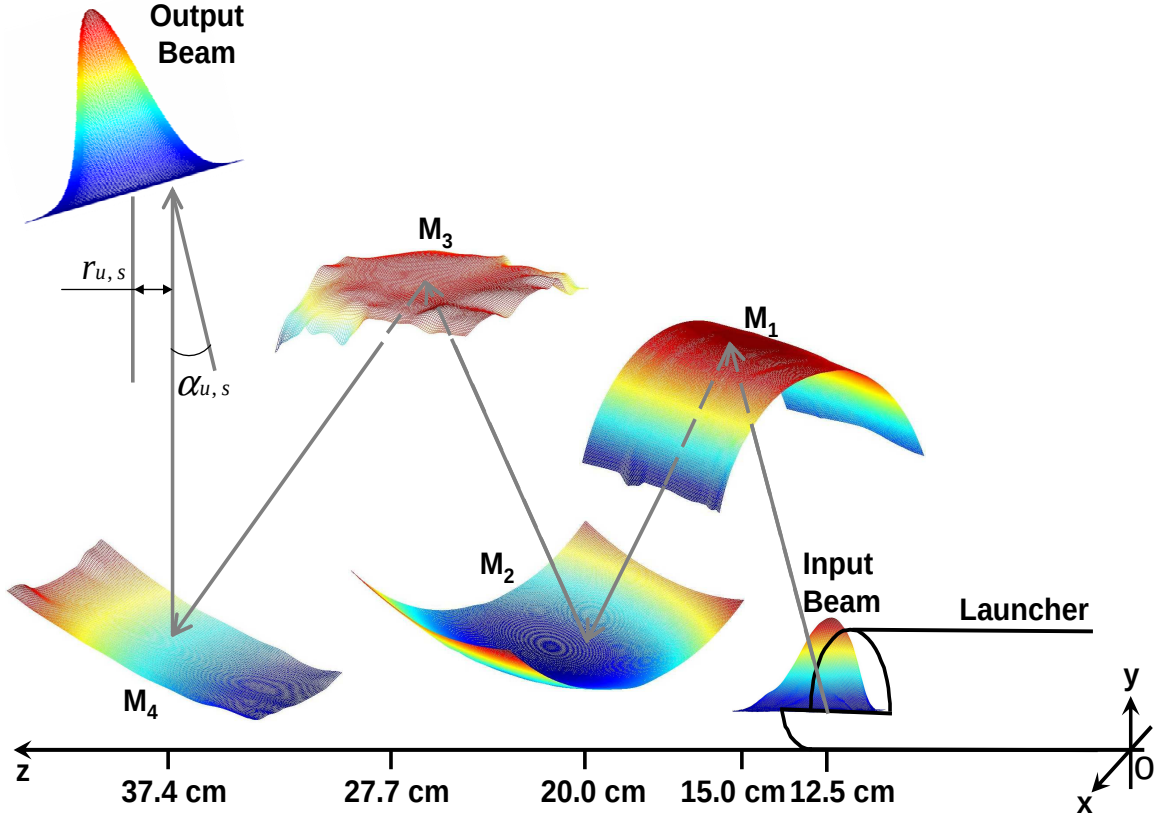


Figure 8.1: Illustration of tilting and offset of the multi-mode multi-frequency design. $\alpha_{u,s}$ and $r_{u,s}$ ($u = x, z$, i denotes different modes) are the tilting angle and the offset amount for multi-mode multi-frequency design. Note that both $\alpha_{u,i}$ and $r_{u,s}$ have been exaggerated, typically $|\alpha_{u,s}| < 1^\circ$ and $r_{u,s} < 1\text{cm}$. Usually, tiltings and offsets at both $\hat{x}$ and $\hat{z}$ directions are required.

Chapter 9

COMPUTER SIMULATION

To test the efficiency of the design procedure introduced in this article, a sub-THz beam-shaping 4-mirror system has been designed for a single-mode ($\text{TE}_{22,6}/110$ GHz), 1.25 MW CPI gyrotron application to shape the single-mode ($\text{TE}_{22,6}/110$ GHz) input beam into the target FGB. In this particular sub-THz beam-shaping mirror system design, the beam wavelength $\lambda \sim 2.7$ mm, which makes the iterative GOPC method feasible. The configuration of the sub-THz beam-shaping 4-mirror system has been shown in Figure 1.2. M_1 (in front of the QO launcher) and M_2 can be considered as “quasi-cylindrical” PEC mirror surfaces, with radii $\rho_1 = 6.5$ cm ($\sim 24\lambda$) and $\rho_2 = 7.4$ cm ($\sim 27\lambda$) respectively; while M_3 and M_4 (in front of the diamond window) can be considered as “quasi-planar” PEC mirror surfaces. So the TI-FFT algorithm can be used to speed up the sub-THz beam-shaping mirror system design.

9.1 Single-mode Single-frequency Design

First, the single-mode ($\text{TE}_{22,6}$) single-frequency (110GHz) beam-shaping mirror system was designed to test the efficiency of the described design procedure. The pseudocodes for the design procedure has been shown in Chapter 2: Section 2.2.

Table 9.1 shows $\chi^{(\ell)}$ for each iteration ℓ ($N_\ell = 16$). Note that the convergence for the first 4 iterations is fast ($\chi^{(4)} = 99.67\%$); while the convergence after the 4th iteration becomes relatively slow: the first maximum value of $\chi^{(\ell)}$ appears at the 14th iteration ($\chi^{(14)} = 99.85\%$).

The single-mode ($\text{TE}_{22,6}/110$ GHz) input beam (mainly E_ϕ) from the QO launcher is shown in Figure 9.1a) and the computer simulated output beam (mainly E_x , solid lines) on the diamond window for the last iteration ($\ell = N_\ell = 16$) is shown in Figure 9.1b), together with the target FGB as a comparison. Also,

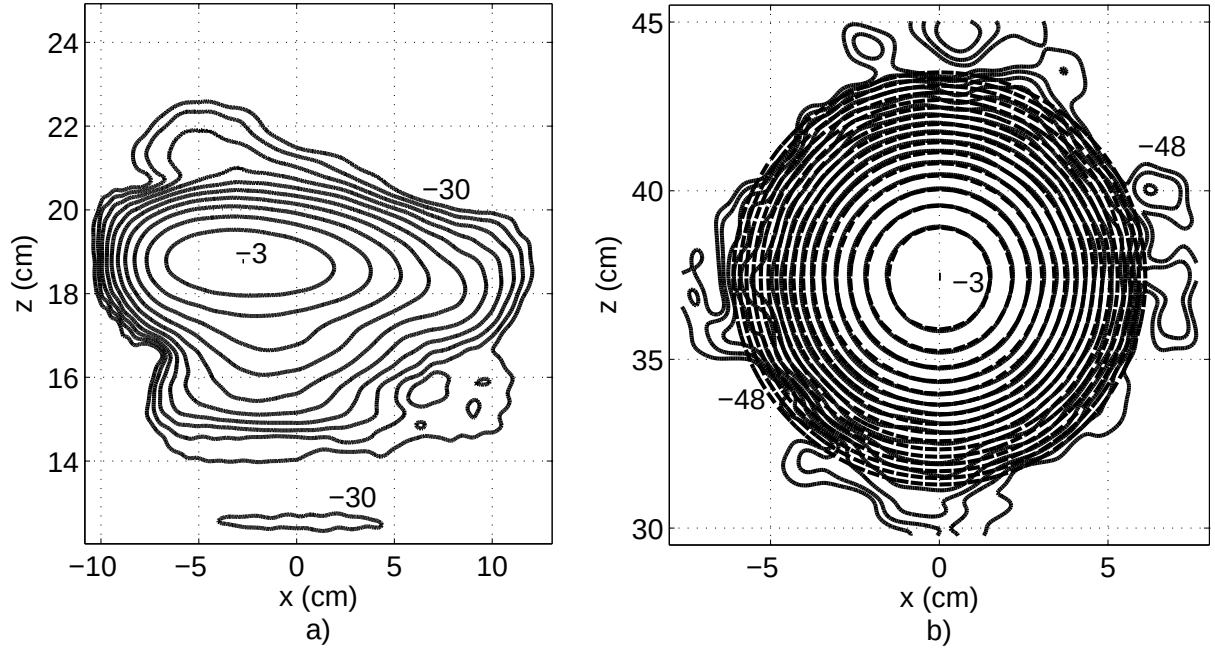


Figure 9.1: The input/output beams for a single-mode ($\text{TE}_{22,6}/110$ GHz), 1.25 MW CPI gyrotron application: a) the input beam magnitude pattern ($|E_\phi|$): dB contours are in 3 dB decrements, from -3 dB down to -30 dB. Values beyond -30 dB are close to the measurement noise floor and not shown; and b) the output beam magnitude pattern ($|E_x|$): the comparison between the computer-simulated output beam (solid lines) and the target FGB (dashed lines) was made, from -3 dB down to -48 dB, in 3 dB decrements.

Table 9.1: Single-mode Single-frequency Convergence for Iteration ℓ ($N_\ell = 16$)

ℓ	1	2	3	4	5	6	7	8
$\chi^{(\ell)} (\%)$	89.02	97.93	99.39	99.67	99.76	99.79	99.81	99.83
ℓ	9	10	11	12	13	14	15	16
$\chi^{(\ell)} (\%)$	99.83	99.83	99.84	99.84	99.84	99.85	99.85	99.85

Table 9.2: The CPU Time Comparison (16 Iterations)

$N \times N$	$t_{\text{TI}}(\text{sec.})$	$t_{\text{DI}}(\text{sec.})$	$t_{\text{DI}}/t_{\text{TI}}$
192×192	128.192	32001.024	43.685
256×256	1988.384	123363.008	62.042
320×320	2762.592	297961.984	107.856
512×512	7908.736	2066931.968	261.348

in Table 9.2, the comparison of the CPU time between the TI-FFT algorithm (t_{TI}) and the direct integration method (t_{DI}) [4],[6] has been shown for different computational grid ($N \times N$), from which the efficiency of the TI-FFT algorithm can be seen. What's more, the TI-FFT algorithm allows for a large sampling spacing limited only by the Nyquist sampling rate and there is no singularity problem because the TI-FFT algorithm works in the spectral domain. All work was done in Matlab 6.5 (release 13) on a Dell Pentium IV, 3.2 GHz PC with 2 GB RAM.

9.2 Multi-mode Multi-frequency Design

The design procedure for the multi-mode multi-frequency mirror system is similar to that of the single-mode single-frequency mirror, except the multi-mode optimization, tiltings and offsets of the output desired FGB (refer to Chapter 8), which is presented below,

(*MULTI-MODE MULTI-FREQUENCY PROGRAM BEGINS*)

Make initial guesses for all 4 mirror surfaces (M_1 , M_2 , M_3 and M_4) and choose the total iteration number N_ℓ (usually $N_\ell = 15 \sim 25$ is required);

Obtain the phase θ_s^i ($s = 1, 2, 3, \dots N$, for N modes) of the each input beam through the IPR technique and denote the input beam as $\mathbf{E}^i = \mathbf{E}^i e^{j\theta^i}$;

For $\ell = 1$ to N_ℓ (iteration loop begins)

Forward-propagate the all input beams $\mathbf{E}_s^i$ onto M_1 to obtain $(|\mathbf{E}_{1,s}^{f,(\ell)}|, \theta_1^{f,(\ell)})$;

Forward-propagate $\mathbf{E}_{1,s}^{f,(\ell)}$ onto M_2 to obtain $(|\mathbf{E}_{2,s}^{f,(\ell)}|, \theta_2^{f,(\ell)})$;

Forward-propagate $\mathbf{E}_{2,s}^{f,(\ell)}$ onto M_3 to obtain $(|\mathbf{E}_{3,s}^{f,(\ell)}|, \theta_3^{f,(\ell)})$;

Forward-propagate $\mathbf{E}_{3,s}^{f,(\ell)}$ onto M_4 to obtain $(|\mathbf{E}_{4,s}^{f,(\ell)}|, \theta_4^{f,(\ell)})$;

Back-propagate the target FGB with different titling and offset values (refer to Chapter 8) onto M_4 to obtain $(|\mathbf{E}_{4,s}^{b,(\ell)}|, \theta_4^{b,(\ell)})$;

Correct M_4 surface through the multi-mode optimization method (refer to Chapter 8), with the help of $(\delta\theta_{4,s} = \theta_{4,s}^{f,(\ell)} - \theta_{4,s}^{b,(\ell)})$;

Back-propagate the target FGB with different tilting and offset values onto M_3 and update $(|\mathbf{E}_{3,s}^{b,(\ell)}|, \theta_3^{b,(\ell)})$;

Correct M_3 surface using the the multi-mode optimization method (refer to Chapter 8), with the help of $(\delta\theta_{3,s} = \theta_{3,s}^{f,(\ell)} - \theta_{3,s}^{b,(\ell)})$;

Back-propagate the desired FGB with different tilting and offset values onto M_2 and update $(|\mathbf{E}_{2,s}^{b,(\ell)}|, \theta_2^{b,(\ell)})$;

Correct M_2 surface using the the multi-mode optimization method (refer to Chapter 8), with the help of $(\delta\theta_{2,s} = \theta_{2,s}^{f,(\ell)} - \theta_{2,s}^{b,(\ell)})$;

Back-propagate the target FGB onto M_1 and update $(|\mathbf{E}_{1,s}^{b,(\ell)}|, \theta_1^{b,(\ell)})$;

Correct M_1 surface using the the multi-mode optimization method (refer to Chapter 8), with the help of $(\delta\theta_{1,s} = \theta_{1,s}^{f,(\ell)} - \theta_{1,s}^{b,(\ell)})$;

Forward-propagate $\mathbf{E}_s^i$ (input beams) onto diamond window to obtain $\mathbf{E}_s^{o,(\ell)}$ (output beams);

Exit iteration loop when the criterion $\chi^{(\ell)}$ defined in (2.1) is met or $\ell = N_\ell$.

$\ell = \ell + 1$ (iteration loop continues)

(MULTI-MODE MULTI-FREQUENCY PROGRAM ENDS)

During the multi-mode multi-frequency design, different weighting scheme has been tried, 1) even weighting with $w_i = \frac{1}{2}$ for all modes; 2) amplitude weighting with $w_i = \frac{|\mathbf{E}_i|}{\sum_i |\mathbf{E}_i|}$; and 3) power weighting with $w_i = \frac{|\mathbf{E}_i|^2}{\sum_i |\mathbf{E}_i|^2}$. The convergency behavior of TE_{22,6} each weighting scheme has been shown in Table

Table 9.3: Multi-mode Multi-frequency Convergence of $\text{TE}_{22,6}$ for Iteration ℓ ($N_\ell = 32$): even weighting with $w_i = \frac{1}{2}$

ℓ	1	2	3	4	5	6	7	8
$\chi^{(\ell)} (\%)$	73.30	95.50	96.94	96.62	96.90	97.31	97.31	97.40
ℓ	9	10	11	12	13	14	15	16
$\chi^{(\ell)} (\%)$	97.57	97.57	97.61	97.72	97.72	97.75	97.86	97.86
ℓ	17	18	19	20	21	22	23	24
$\chi^{(\ell)} (\%)$	97.88	97.94	97.94	97.97	97.96	97.96	97.99	98.04
ℓ	25	26	27	28	29	30	31	32
$\chi^{(\ell)} (\%)$	98.04	98.06	98.08	98.08	98.10	98.11	98.12	98.12

9.3 - Table 9.5, from which we can see the importance of the weighting scheme: the power weighting is a little better than the amplitude weighting and they are both better than the even weighting scheme. So, in the rest of the simulation, we will only give results for the power weighting scheme.

Fig. 9.2 - Fig. 9.5 show the final designed mirror surfaces and Fig. 9.6 - Fig. 9.11 show the corresponding field patterns. We can see that mirror 1 is a perturbed cylinder (with radius about $r_1 = 5.6\text{cm}$); mirror 2 is a perturbed parabolical shape (with radius $r_{2,x} = 7.4\text{cm}$ in $\hat{\mathbf{x}}$ direction and radius $r_{2,z} = 40\text{cm}$ in $\hat{\mathbf{z}}$ direction); mirror 3 is a perturbed planar surface and mirror 4 is a perturbed planar surface with 24° tilting. Since the input field from the launcher mainly contains $\hat{\phi}$ -component, the major fields on mirror 1 and mirror 2 are thus $\hat{\mathbf{x}}$ -component and $\hat{\mathbf{z}}$ -component. The major field on mirror 3 and mirror 4 is mainly $\hat{\mathbf{x}}$ -component because the desired output field is $\hat{\mathbf{x}}$ -component (FGB) and mirror 1 and mirror 2 have almost shaped the input field $\hat{\phi}$ -component into $\hat{\mathbf{x}}$ -component.

9.3 Summary of Beam-shaping Mirror Design Computer Simulation

We have done both the single-mode single-frequency and multi-mode multi-frequency mirror design computer simulation based on the previous described

Table 9.4: Multi-mode Multi-frequency Convergence of TE_{22,6} for Iteration ℓ ($N_\ell = 30$): amplitude weighting $w_i = \frac{|\mathbf{E}_i|}{\sum_i |\mathbf{E}_i|}$

ℓ	1	2	3	4	5	6	7	8
$\chi^{(\ell)} (\%)$	73.30	95.51	97.09	95.14	95.39	96.36	96.36	96.52
ℓ	9	10	11	12	13	14	15	16
$\chi^{(\ell)} (\%)$	97.03	97.03	97.13	97.43	97.43	97.52	97.74	97.74
ℓ	17	18	19	20	21	22	23	24
$\chi^{(\ell)} (\%)$	97.82	97.98	97.98	98.01	98.13	98.13	98.14	98.22
ℓ	25	26	27	28	29	30	31	32
$\chi^{(\ell)} (\%)$	98.22	98.23	98.29	98.29	98.30	98.34	98.35	98.35

Table 9.5: Multi-mode Multi-frequency Convergence of TE_{22,6} for Iteration ℓ ($N_\ell = 30$): power weighting $w_i = \frac{|\mathbf{E}_i|^2}{\sum_i |\mathbf{E}_i|^2}$

ℓ	1	2	3	4	5	6	7	8
$\chi^{(\ell)} (\%)$	73.30	95.51	97.09	95.14	95.39	96.36	96.36	96.53
ℓ	9	10	11	12	13	14	15	16
$\chi^{(\ell)} (\%)$	97.03	97.03	97.13	97.43	97.43	97.51	97.74	97.74
ℓ	17	18	19	20	21	22	23	24
$\chi^{(\ell)} (\%)$	97.80	97.97	97.97	98.01	98.14	98.14	98.16	98.25
ℓ	25	26	27	28	29	30	31	32
$\chi^{(\ell)} (\%)$	98.25	98.26	98.33	98.33	98.34	98.38	98.38	98.39

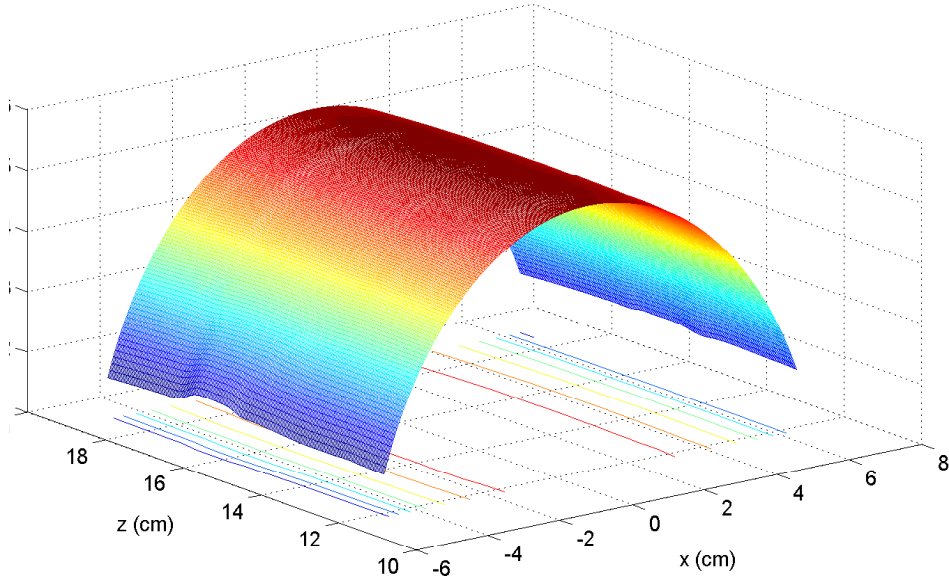


Figure 9.2: Surface plot of mirror 1.

procedure for QO gyrotron applications. The novel methods, algorithms and theories greatly facilitate the design procedure and make complicate sub-THz beam-shaping mirror system designs feasible. The design procedure is fast due to the efficient TI-FFT algorithm for beam propagations and scatterings and it is also accurate due to the use of the iterative GOPC method for PEC mirror surface corrections. A practical sub-THz beam-shaping 4-mirror system design for a single-mode ($\text{TE}_{22,6}/110$ GHz), 1.25 MW CPI gyrotron has been shown and the coupling between the designed output beam and the target FGB is up to 99.85%.

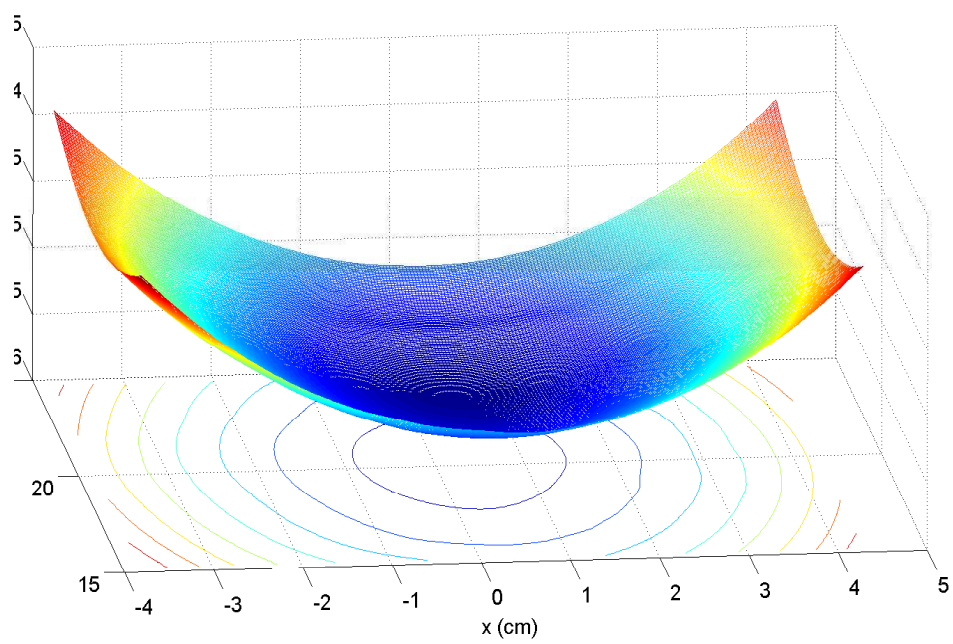


Figure 9.3: Surface plot of mirror 2.

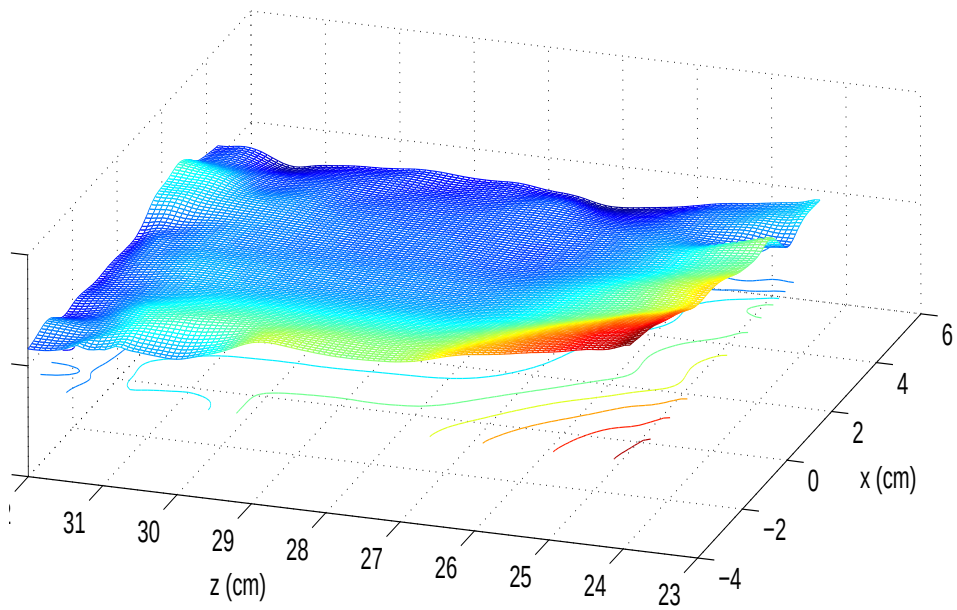


Figure 9.4: Surface plot of mirror 3.

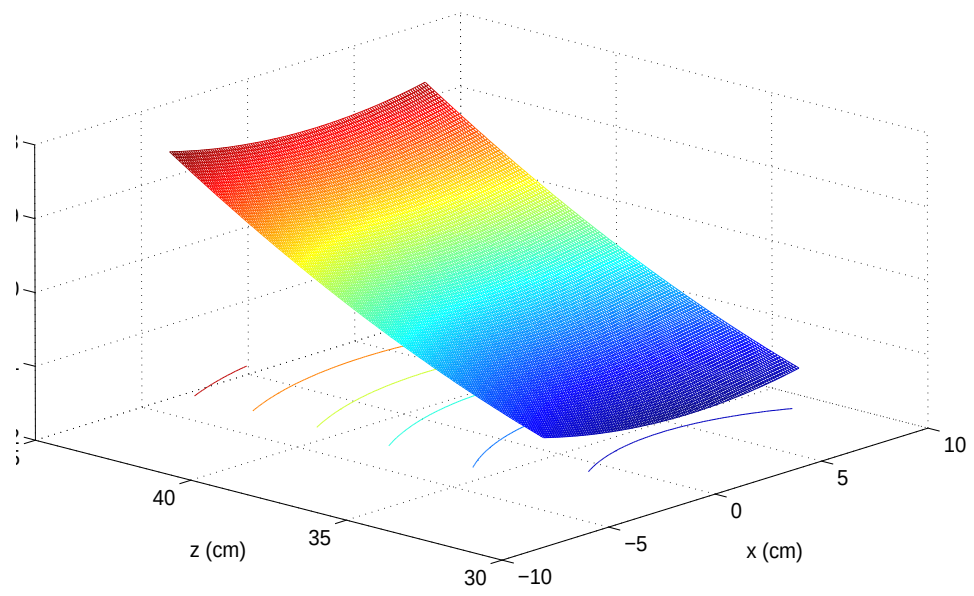


Figure 9.5: Surface plot of mirror 4.

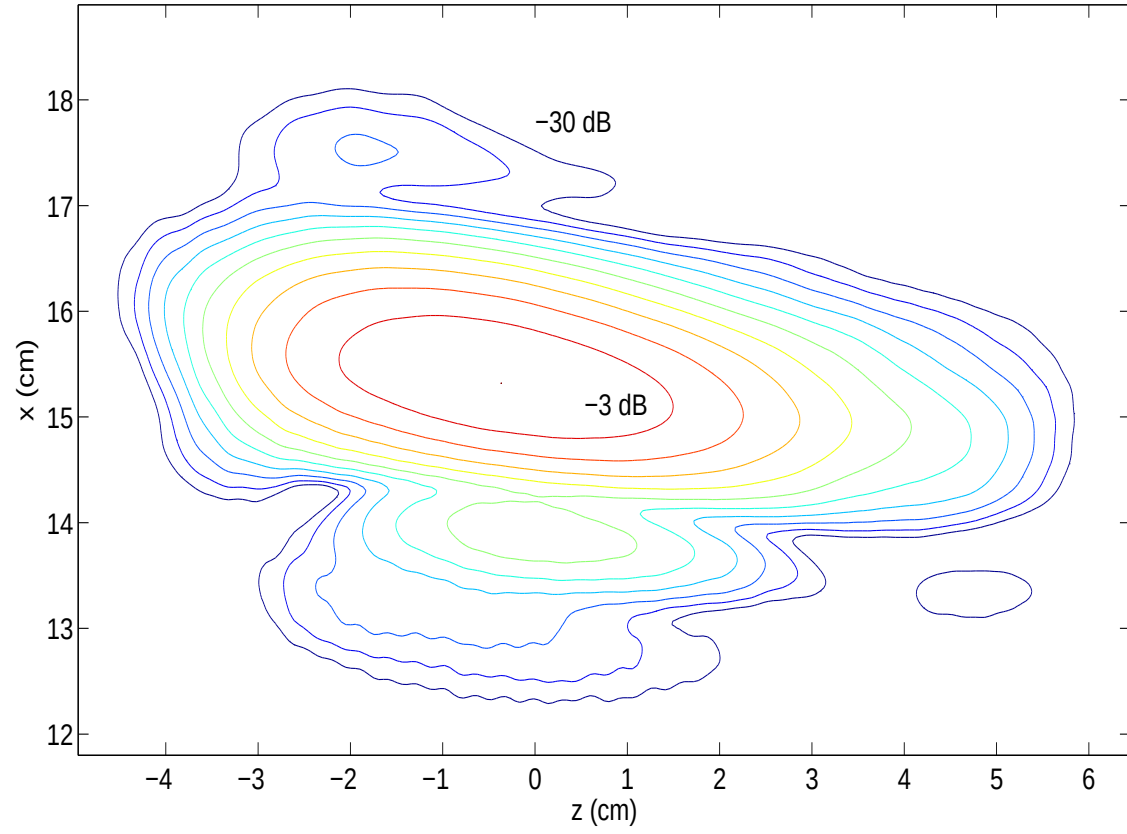


Figure 9.6: Contour magnitude plot of of $\hat{\mathbf{x}}$ -component on mirror 1 ($|\mathbf{E}_{1,x}|$), in -3 dB decrements.

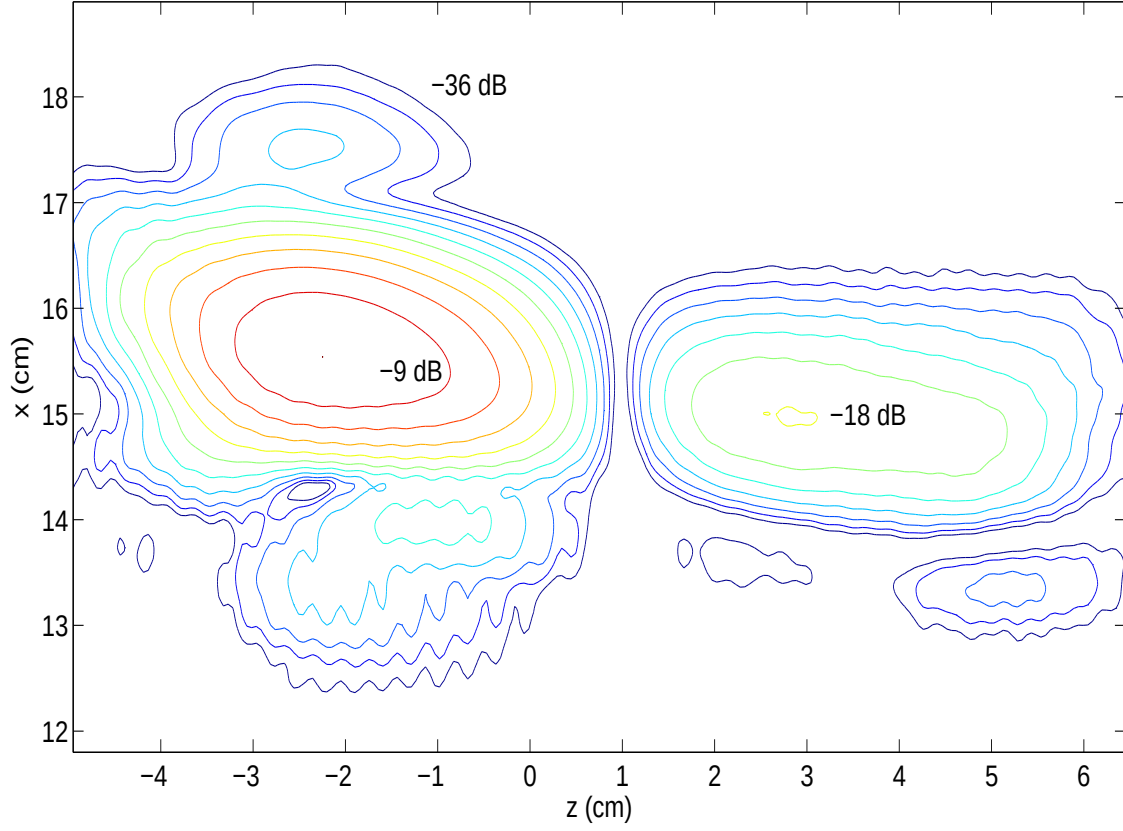


Figure 9.7: Contour magnitude plot of of $\hat{y}$ -component on mirror 1 ($|\mathbf{E}_{1,y}|$), in -3 dB decrements.

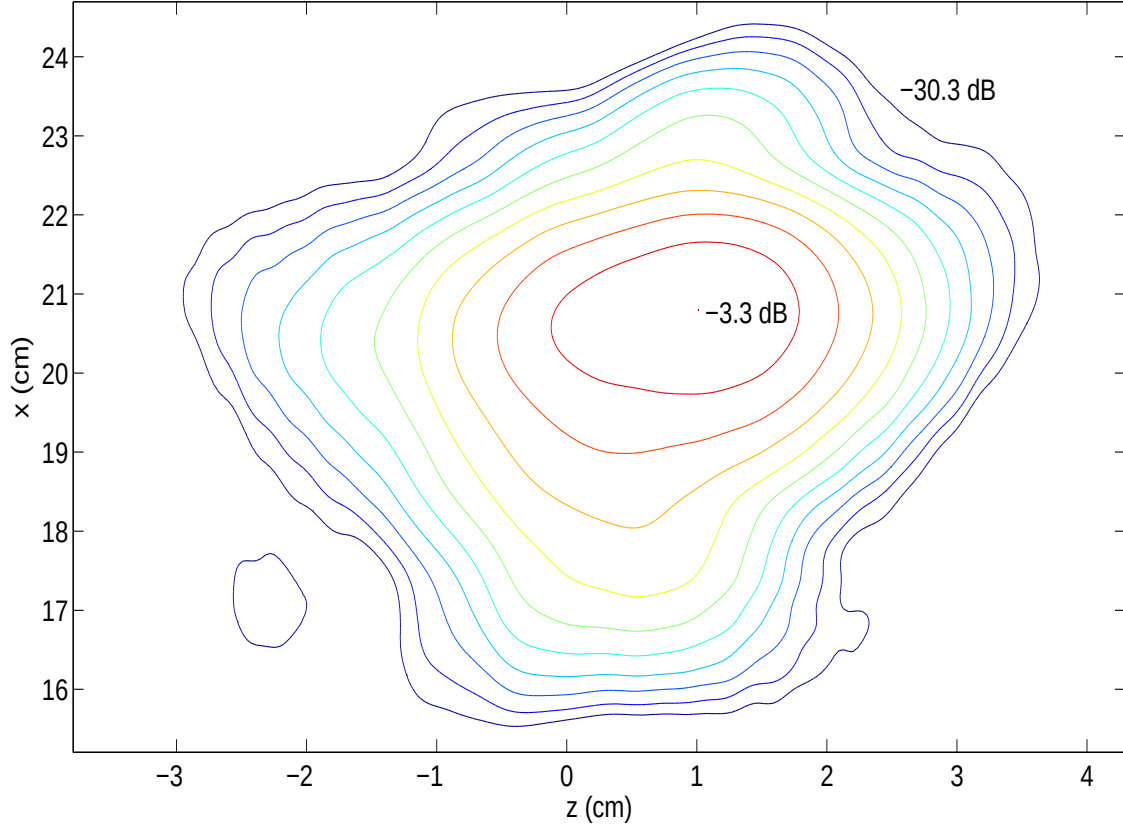


Figure 9.8: Contour magnitude plot of of $\hat{\mathbf{x}}$ -component on mirror 2 ($|\mathbf{E}_{2,x}|$), in -3 dB decrements.

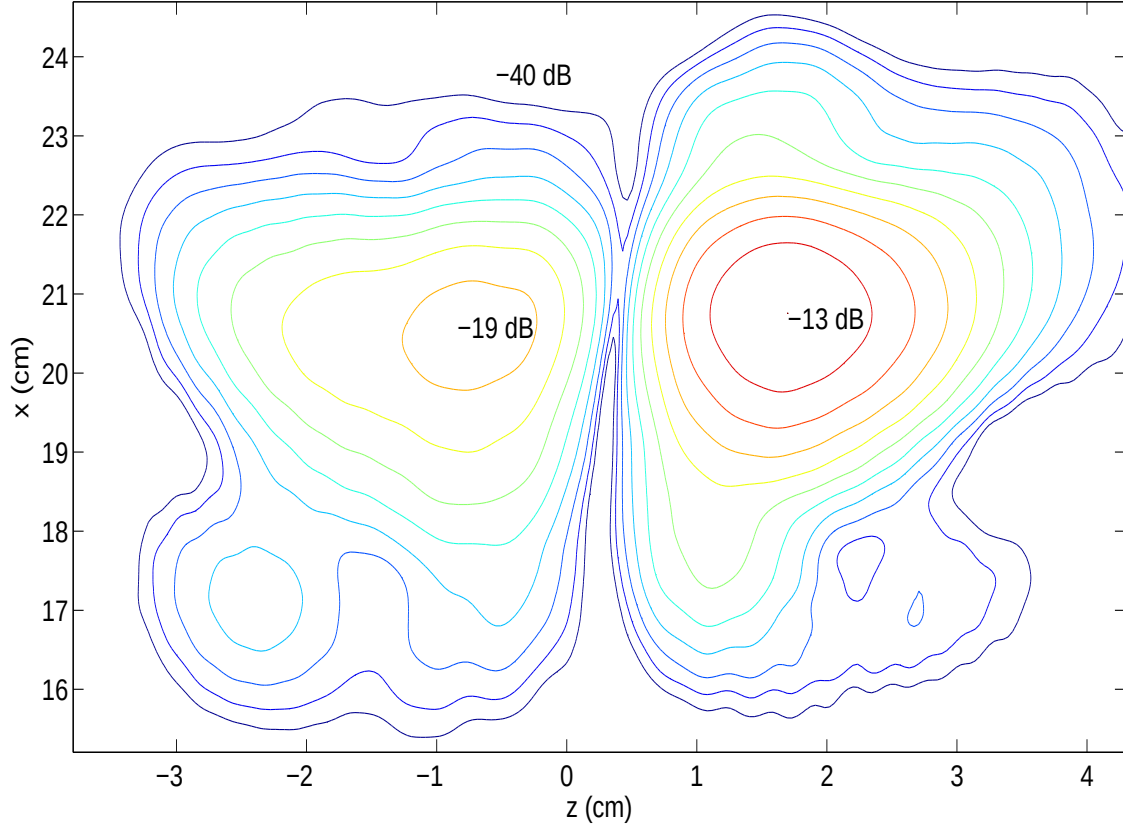


Figure 9.9: Contour magnitude plot of of $\hat{y}$ -component on mirror 2 ($|\mathbf{E}_{2,y}|$), in -3 dB decrements.

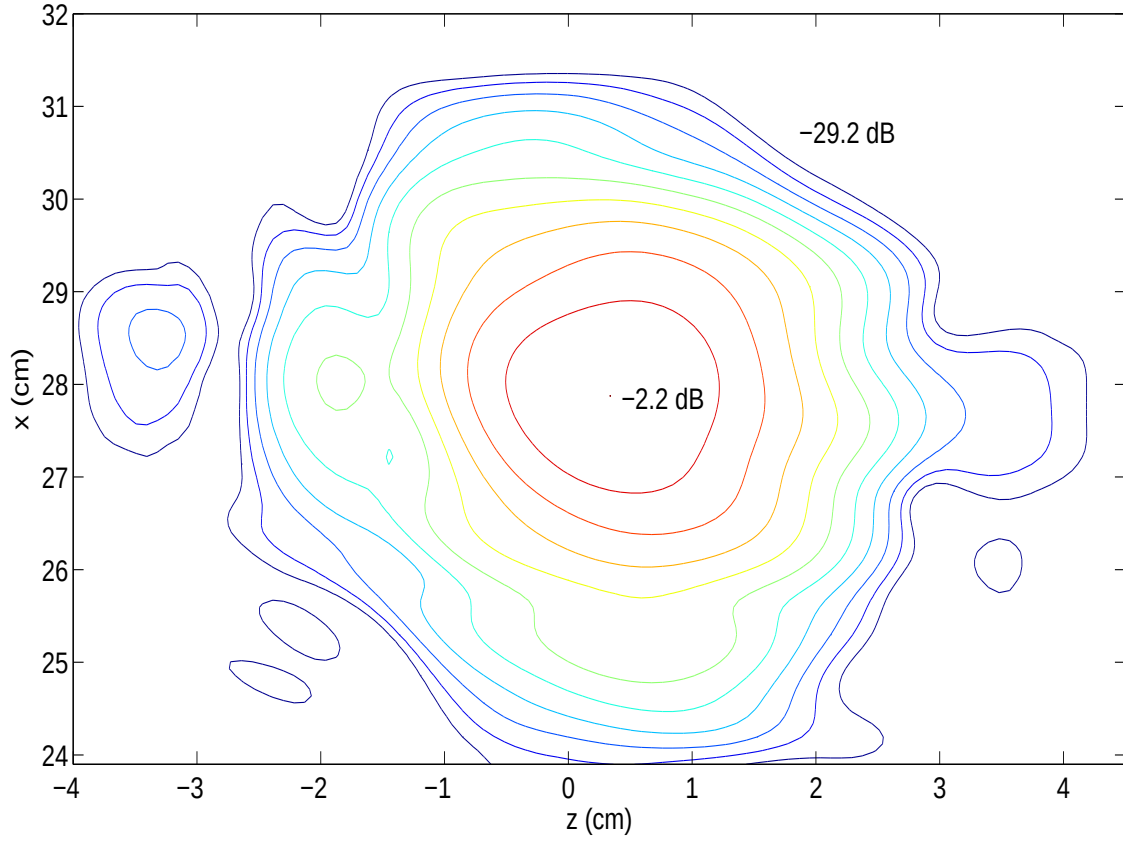


Figure 9.10: Contour magnitude plot of of $\hat{x}$ -component on mirror 3 ($|\mathbf{E}_{3,x}|$), in -3 dB decrements.

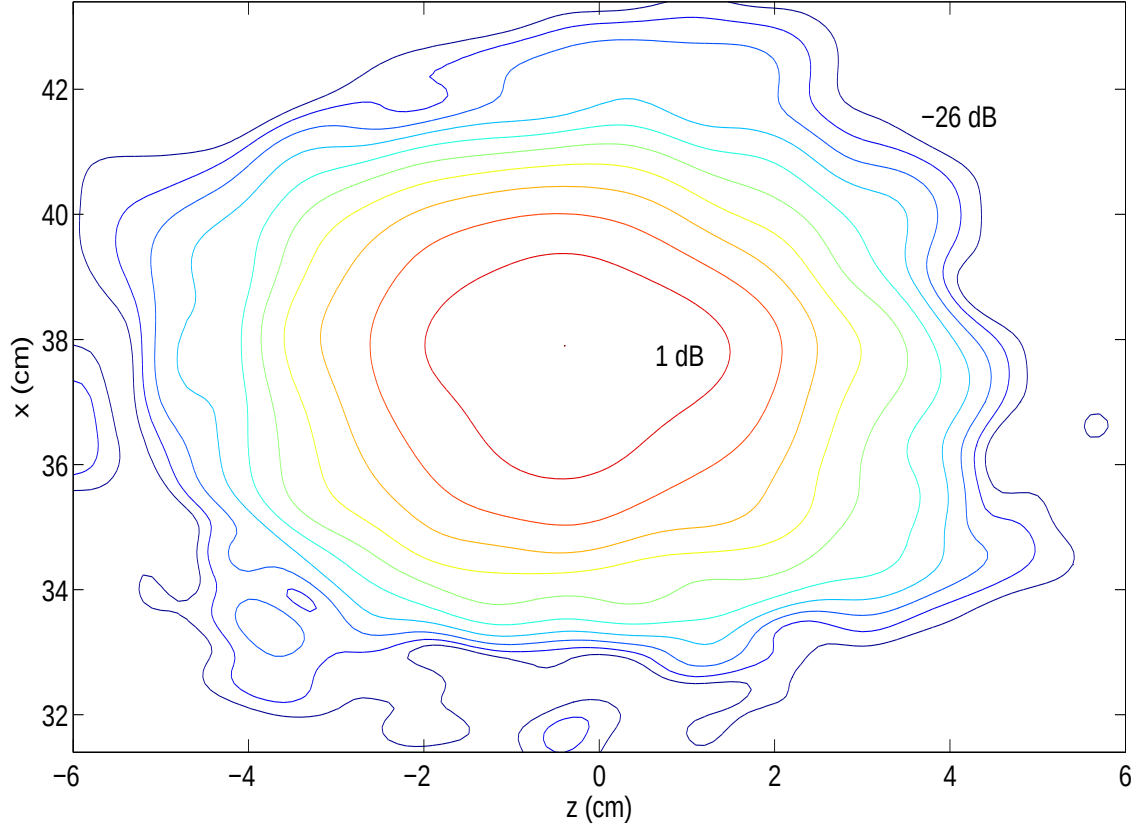


Figure 9.11: Contour magnitude plot of of $\hat{\mathbf{x}}$ -component on mirror 4 ($|\mathbf{E}_{4,x}|$), in -3 dB decrements.

Chapter 10

EXPERIMENT AND PROBE COMPENSATION

To test the efficiency of the beam-shaping PEC mirror system, the cold-test (milli-Watts radiation power) system has been built up at the Microwave Research Lab, in the Electrical and Computer Engineering, University of Wisconsin, Madison.

10.1 Experiment Setup

The cold-test experiment setup is shown in Figure 10.1, which can be explained as follows,

- 1) Signal generation: the 110 GHz carrier generated by the Gunn oscillator is modulated by the 24 KHz sine wave;
- 2) Signal analysis (monitor): the generated signal is passed through one port of the 3-port directional coupler, then demodulated by the 10th-order mixer, after which it is analyzed by the spectrum analyzer;
- 3) Mode generation: the TE_{22,6} mode (110 GHz) is excited at the input side of the launcher;
- 4) Launcher conversion: the TE_{22,6} mode is converted through the launcher (with cut and dimpled wall);
- 5) Beam-shaping mirror system: the output beam from the launcher is further shaped into the Gaussian-like beam;
- 6) Beam measurement: the Gaussian-like beam is measured through the tapered rectangular probe;

7) Date acquisition: the measured data is collected and plotted in the computer that is connected to the measurement probe.

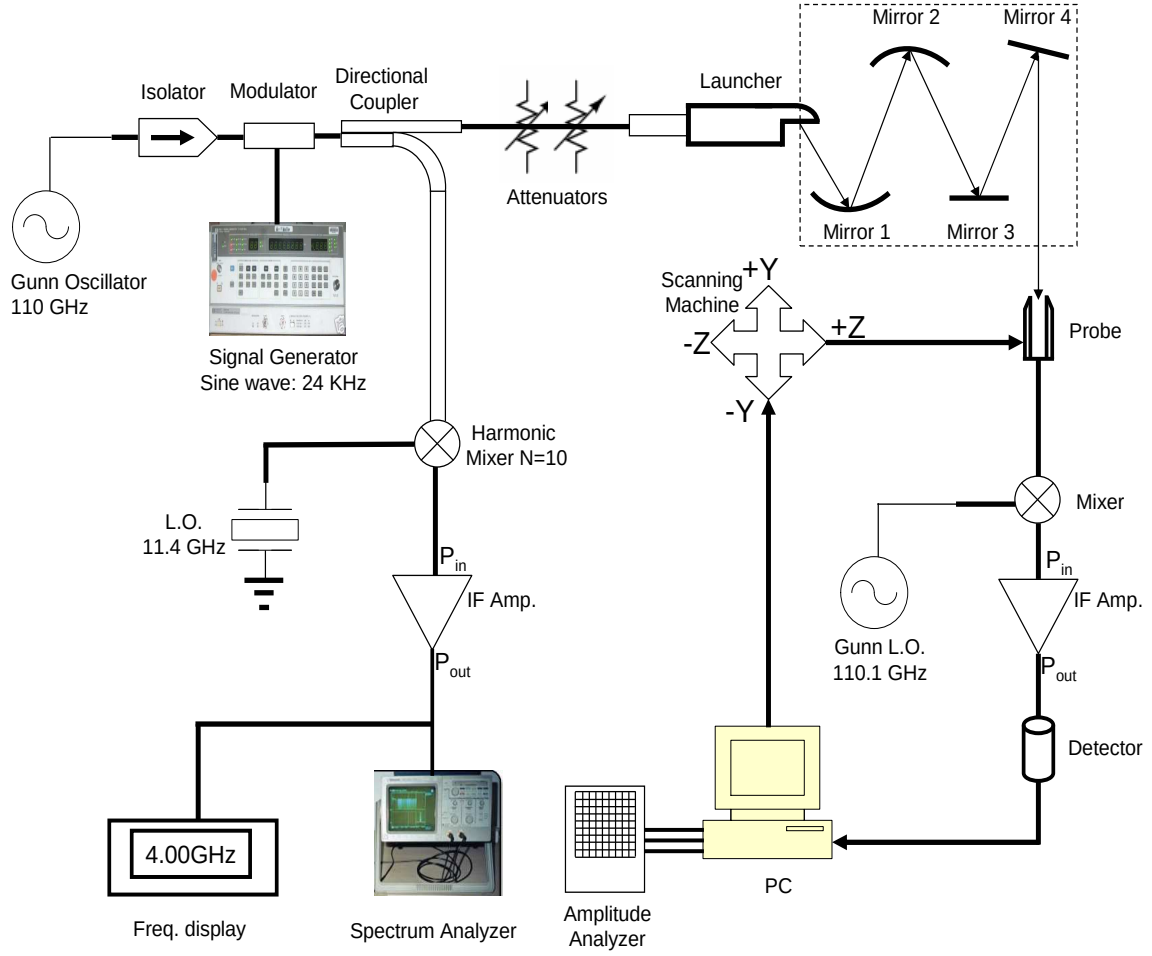


Figure 10.1: The experiment setup for cold-test of the beam-shaping PEC mirror system:

10.2 Probe Compensation

It is well-known that antenna near-field measurement are more or less affected by the presence of the measuring device [27], the probe in our case. Probe compensation is a must in principle for non-ideal probe (ideal probe include “elemental electric current probe”, “elemental magnetic current probe” [49]).

This is because the true far-field pattern or the Fourier spectrum of an antenna under test (AUT) is distorted by the interaction of the near-field phase fronts between the AUT and probe antennas in near-field measurements. There are mainly 3 types of probe compensation according to scanning surface : Planar, Cylindrical and Spherical probe compensations. Specific compensation theory (planar, cylindrical and spherical probe compensation) works well for specific configuration [27]. In each type of compensation, radiating fields are measured with probes of know far-field pattern. Usually, two sets of data are required in order to extract the two independent components of the AUT far field E_θ and E_ϕ from two independent equations. The two measurements could be done by two completely different probes or by the same probe with different rotation angles (usually 0° and 90°). For probe with rotation-free far-field pattern, it can be shown that the two equations are actually dependent on each other and only one measurement is required to do the compensation. Whats more, for high frequency up to 110 GHz, it is extremely difficult to measure the phase directly and the conventional IPR technique 3 can be applied on the measured near-field pattern here (note that rigorously the conventional IPR is not correct since the measured near-field magnitude is not exactly equal to the true near-field magnitude due to the presence of the probe).

Several things will be addressed in this section. First, we will present the connection between Friis formula and probe compensation; then we will give the general expressions for any type of combinations of AUT-tilting, AUT-rotation and probe-tilting, probe-rotation; what's more, we will show that when there is constraint on the AUT (Antenna-Under-Test) radiation components ($\hat{x}$ -/ $\hat{y}$ -component or θ -/ ϕ -component), one only need to make only one measurement in order to do the compensation; and finally, the theory will be applied to measured near-field data radiating from the QO launcher designed for the 110GHz TE_{22,6} mode gyrotron.

10.2.1 The Planar Scanning Scheme

Although cylindrical and spherical scanning are desirable for some geometry-complicate system, planar scanning is popular whenever it is possible due to its simplicity and not time-cost. So we only deal with the planar probe compensation.

In spherical coordinate, when the two measurements are done without probe rotation and with probe rotation of 90° , the well-known planar probe compensation theory gives [27],

$$E_{\phi}^{\text{AUT}}(k_x, k_y)E_{\phi}^p(k_x, -k_y) - E_{\theta}^{\text{AUT}}(k_x, k_y)E_{\theta}^p(k_x, -k_y) = \frac{C \cos \theta}{R^2}P(k_x, k_y) \exp(jk_z z) \quad (10.2.1.1)$$

$$E_{\phi}^{\text{AUT}}(k_x, k_y)E_{\phi}^p(-k_y, -k_x) - E_{\theta}^{\text{AUT}}(k_x, k_y)E_{\theta}^p(-k_y, -k_x) = \frac{C \cos \theta}{R^2}Q(k_x, k_y) \exp(jk_z z) \quad (10.2.1.2)$$

Where, E_{θ}^{AUT} and E_{ϕ}^{AUT} are θ and ϕ components of the radiating far field of the AUT; E_{θ}^p and E_{ϕ}^p are θ and ϕ components of the radiating far field of the probe; $P(k_x, k_y)$ and $Q(k_x, k_y)$ are the fourier transforms of the two measurements without probe rotation and with probe rotation of 90° , z is the distance between the AUT and the probe. From 10.2.1.1-10.2.1.2, the two unknown quantities E_{θ}^{AUT} and E_{ϕ}^{AUT} can be solved since there are also two independent equations.

In rectangular coordinate, 10.2.1.1-10.2.1.2 can be reformulated into

$$-E_x^{\text{AUT}}(k_x, k_y)E_x^p(-k_y, -k_x) + E_y^{\text{AUT}}(k_x, k_y)E_y^p(-k_y, -k_x) \quad (10.2.1.3)$$

$$-E_z^{\text{AUT}}(k_x, k_y)E_z^p(-k_y, -k_x) = \frac{C \cos \theta}{R^2}F(k_x, k_y) \exp(jk_z z)$$

$$-E_x^{\text{AUT}}(k_x, k_y)E_x^p(k_x, -k_y) + E_y^{\text{AUT}}(k_x, k_y)E_y^p(k_x, -k_y) \quad (10.2.1.4)$$

$$-E_z^{\text{AUT}}(k_x, k_y)E_z^p(k_x, -k_y) = \frac{C \cos \theta}{R^2}G(k_x, k_y) \exp(jk_z z)$$

10.2.2 The Friis Formula and the Probe Compensation

In the far-field regime, letting $z \rightarrow \infty$, (10.2.1.1) reduces to

$$V_{\text{meas.}}^{\text{AUT}}(R) \propto j \frac{kR^2}{C} \frac{\exp(-jkR)}{R} \left[E_{\phi}^{\text{AUT}}(k_x, k_y)E_{\phi}^p(k_x, -k_y) - E_{\theta}^{\text{AUT}}(k_x, k_y)E_{\theta}^p(k_x, -k_y) \right] \quad (10.2.2.5)$$

$$P_{\text{meas.}}^{\text{AUT}}(R) \propto \left| \frac{kR}{C} \right|^2 \left| E_{\phi}^{\text{AUT}}(k_x, k_y)E_{\phi}^p(k_x, -k_y) - E_{\theta}^{\text{AUT}}(k_x, k_y)E_{\theta}^p(k_x, -k_y) \right|^2 \quad (10.2.2.6)$$

where $V_{\text{meas.}}^{\text{AUT}}(R)$ and $P_{\text{meas.}}^{\text{AUT}}(R)$ denote the measured far-field voltage and power by the probe. Now, rewrite (10.2.2.6) as follows,

$$\begin{aligned}
P_{\text{meas.}}^{\text{AUT}}(R) \propto & \left| \frac{kR}{C} \right|^2 \left(\left| E_{\phi}^{\text{AUT}}(k_x, k_y) \right|^2 + \left| E_{\theta}^{\text{AUT}}(k_x, k_y) \right|^2 \right) \left(\left| E_{\phi}^p(k_x, -k_y) \right|^2 + \left| E_{\theta}^p(k_x, -k_y) \right|^2 \right) \\
& \times \left| \frac{E_{\phi}^{\text{AUT}}(k_x, k_y) E_{\phi}^p(k_x, -k_y)}{\sqrt{\left(\left| E_{\phi}^{\text{AUT}}(k_x, k_y) \right|^2 + \left| E_{\theta}^{\text{AUT}}(k_x, k_y) \right|^2 \right) \left(\left| E_{\phi}^p(k_x, -k_y) \right|^2 + \left| E_{\theta}^p(k_x, -k_y) \right|^2 \right)}} \right. \\
& \left. - \frac{E_{\theta}^{\text{AUT}}(k_x, k_y) E_{\theta}^p(k_x, -k_y)}{\sqrt{\left(\left| E_{\phi}^{\text{AUT}}(k_x, k_y) \right|^2 + \left| E_{\theta}^{\text{AUT}}(k_x, k_y) \right|^2 \right) \left(\left| E_{\phi}^p(k_x, -k_y) \right|^2 + \left| E_{\theta}^p(k_x, -k_y) \right|^2 \right)}} \right|^2
\end{aligned} \tag{10.2.2.7}$$

Now,

$$\begin{aligned}
\left| E_{\phi}^{\text{AUT}}(k_x, k_y) \right|^2 + \left| E_{\theta}^{\text{AUT}}(k_x, k_y) \right|^2 &= \frac{P^t}{4\pi R^2} D^{\text{AUT}} \\
\left| E_{\phi}^p(k_x, -k_y) \right|^2 + \left| E_{\theta}^p(k_x, -k_y) \right|^2 &= \frac{Q^p}{4\pi R^2} D^p \\
D^{\text{AUT}} &= \frac{4\pi R^2 \left(\left| E_{\phi}^{\text{AUT}}(k_x, k_y) \right|^2 + \left| E_{\theta}^{\text{AUT}}(k_x, k_y) \right|^2 \right)}{P^t} \\
D^p &= \frac{4\pi R^2 \left(\left| E_{\phi}^p(k_x, -k_y) \right|^2 + \left| E_{\theta}^p(k_x, -k_y) \right|^2 \right)}{Q^p} \\
Q^p &= R^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \left[\left| E_{\phi}^p(k_x, -k_y) \right|^2 + \left| E_{\theta}^p(k_x, -k_y) \right|^2 \right]
\end{aligned} \tag{10.2.2.8}$$

where D^{AUT} and D^p are the directivities of the AUT and the probe; P^t is the total transmitted power.

Also, define the Polarization Factor (PLF) as

$$\begin{aligned}
PLF &\equiv \hat{\rho}^{\text{AUT}}(k_x, k_y) \cdot \hat{\rho}^p(k_x, -k_y) \\
\hat{\rho}^{\text{AUT}}(k_x, k_y) &= \frac{\hat{\phi} E_{\phi}^{\text{AUT}}(k_x, k_y) + \hat{\theta} E_{\theta}^{\text{AUT}}(k_x, k_y)}{\sqrt{\left(\left| E_{\phi}^{\text{AUT}}(k_x, k_y) \right|^2 + \left| E_{\theta}^{\text{AUT}}(k_x, k_y) \right|^2 \right)}}
\end{aligned}$$

$$\hat{\rho}^p(k_x, -k_y) = \frac{\hat{\phi}E_\phi^p(k_x, -k_y) - \hat{\theta}E_\theta^p(k_x, -k_y)}{\sqrt{\left(|E_\phi^p(k_x, -k_y)|^2 + |E_\theta^p(k_x, -k_y)|^2\right)}} \quad (10.2.2.9)$$

So, we have,

$$PLF \equiv \hat{\rho}^{\text{AUT}}(k_x, k_y) \cdot \hat{\rho}^p(k_x, -k_y) \quad (10.2.2.10)$$

$$= \left| \frac{E_\phi^{\text{AUT}}(k_x, k_y)E_\phi^p(k_x, -k_y)}{\sqrt{\left(|E_\phi^{\text{AUT}}(k_x, k_y)|^2 + |E_\theta^{\text{AUT}}(k_x, k_y)|^2\right)} \left(|E_\phi^p(k_x, -k_y)|^2 + |E_\theta^p(k_x, -k_y)|^2\right)} - \frac{E_\theta^{\text{AUT}}(k_x, k_y)E_\theta^p(k_x, -k_y)}{\sqrt{\left(|E_\phi^{\text{AUT}}(k_x, k_y)|^2 + |E_\theta^{\text{AUT}}(k_x, k_y)|^2\right)} \left(|E_\phi^p(k_x, -k_y)|^2 + |E_\theta^p(k_x, -k_y)|^2\right)} \right|^2$$

From (10.2.2.8)-(10.2.2.10), the measured power $P_{\text{meas.}}^{\text{AUT}}(R)$ in (10.2.2.7) reduces to

$$P_{\text{meas.}}^{\text{AUT}}(R) \propto \left(\frac{k}{C}\right)^2 Q^p \frac{P^t}{(4\pi R)^2} D^{\text{AUT}} D^p PLF \quad (10.2.2.11)$$

It is clear that (10.2.2.11) is actually the Friis transmission formula except for the constant term $\left(\frac{k}{C}\right)^2 Q^p$.

10.2.3 Tilting and rotation

In our case, both AUT and probe can be treated as aperture antenna. Whats more, they can be tilted (or rotated) with respect to the scanning plane (refer to Fig. 10.2 for description). The tilting and rotation of the AUT and probe can be related to the original coordinate (x_0, y_0, z_0) in the matrix form following 3 consecutive steps (refer to 10.3), i.e., 1) rotation around the original z_0 with angle α ; 2) rotation around the new coordinate y_1 after 1) with angle β ; and 3) rotation around the new coordinate z_2 after 2) with angle γ . The 3 transfer matrix $T_1^{3 \times 3}$, $T_2^{3 \times 3}$ and $T_3^{3 \times 3}$ are given as

$$T_1^{3 \times 3}(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (10.2.3.12)$$

$$T_2^{3 \times 3}(\beta) = \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix} \quad (10.2.3.13)$$

$$T_3^{3 \times 3}(\gamma) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (10.2.3.14)$$

and finally, the total transfer matrix from (x_0, y_0, z_0) to (x, y, z) is given as the product of $T_1^{3 \times 3}(\alpha)T_2^{3 \times 3}(\beta)T_3^{3 \times 3}(\gamma)$,

$$\begin{aligned} T^{3 \times 3}(\alpha, \beta, \gamma) &= T_1^{3 \times 3}(\alpha)T_2^{3 \times 3}(\beta)T_3^{3 \times 3}(\gamma) \\ &= \begin{pmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \beta \cos \gamma \\ -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \beta \sin \gamma \\ \cos \alpha \sin \beta & \sin \alpha \sin \beta & \cos \beta \end{pmatrix} \end{aligned} \quad (10.2.3.15)$$

From the Near-Field Far-Field (NF-FF) transform, it is known that the far-field is related to the Fourier transform of the near-field, in the original coordinate, we have,

$$\begin{pmatrix} E_{x0}^{\text{AUT}}(k_{x0}, k_{y0}) \\ E_{y0}^{\text{AUT}}(k_{x0}, k_{y0}) \\ E_{z0}^{\text{AUT}}(k_{x0}, k_{y0}) \end{pmatrix} = \frac{jk \exp(-jkR)}{R} S^{3 \times 2}(\theta_0, \phi_0) \begin{pmatrix} F_{x0}^{\text{AUT}}(k_{x0}, k_{y0}) \\ F_{y0}^{\text{AUT}}(k_{x0}, k_{y0}) \end{pmatrix} \quad (10.2.3.16)$$

$$S^{3 \times 2}(\theta_0, \phi_0) = \begin{pmatrix} \cos \theta & 0 \\ 0 & \cos \theta \\ \sin \theta \cos \phi & \sin \theta \sin \phi \end{pmatrix}$$

Similarly, for probe, we have

$$\begin{pmatrix} E_{x0}^p(k_{x0}, -k_{y0}) \\ E_{y0}^p(k_{x0}, -k_{y0}) \\ E_{z0}^p(k_{x0}, -k_{y0}) \end{pmatrix} = \frac{jk \exp(-jkR)}{R} S^{3 \times 2}(\theta_0, -\phi_0) \begin{pmatrix} F_{x0}^p(k_{x0}, -k_{y0}) \\ F_{y0}^p(k_{x0}, -k_{y0}) \end{pmatrix} \quad (10.2.3.17)$$

From (10.2.3.15), in the new coordinate (x, y, z) , we have,

$$\begin{aligned}
\begin{pmatrix} E_x^{\text{AUT}}(k_x, k_y) \\ E_y^{\text{AUT}}(k_x, k_y) \\ E_z^{\text{AUT}}(k_x, k_y) \end{pmatrix} &= T^{3 \times 3}(\alpha^{\text{AUT}}, \beta^{\text{AUT}}, \gamma^{\text{AUT}}) \begin{pmatrix} E_{x0}^{\text{AUT}}(k_{x0}, k_{y0}) \\ E_{y0}^{\text{AUT}}(k_{x0}, k_{y0}) \\ E_{z0}^{\text{AUT}}(k_{x0}, k_{y0}) \end{pmatrix} \quad (10.2.3.18) \\
&= \frac{jk \exp(-jkR)}{R} T^{3 \times 3}(\alpha^{\text{AUT}}, \beta^{\text{AUT}}, \gamma^{\text{AUT}}) S^{3 \times 2}(\theta_0, \phi_0) \begin{pmatrix} F_{x0}^{\text{AUT}}(k_{x0}, k_{y0}) \\ F_{y0}^{\text{AUT}}(k_{x0}, k_{y0}) \end{pmatrix} \\
&\quad \begin{pmatrix} E_x^p(k_x, -k_y) \\ E_y^p(k_x, -k_y) \\ E_z^p(k_x, -k_y) \end{pmatrix} = T^{3 \times 3}(\alpha, \beta, \gamma) \begin{pmatrix} E_{x0}^p(k_{x0}, -k_{y0}) \\ E_{y0}^p(k_{x0}, -k_{y0}) \\ E_{z0}^p(k_{x0}, -k_{y0}) \end{pmatrix} \quad (10.2.3.19) \\
&= \frac{jk \exp(-jkR)}{R} T^{3 \times 3}(\alpha, \beta, \gamma) S^{3 \times 2}(\theta_0, -\phi_0) \begin{pmatrix} F_{x0}^p(k_{x0}, -k_{y0}) \\ F_{y0}^p(k_{x0}, -k_{y0}) \end{pmatrix}
\end{aligned}$$

For the scanning plane defined on the original coordinate $x_0 - y_0$ plane and the AUT and probe tilted and rotated off the $x_0 - y_0$ scanning plane, substitute (10.2.3.18) and (10.2.3.19) into (10.2.1.3),

$$\begin{aligned}
&\begin{pmatrix} F_{x0}^{\text{AUT}}(k_{x0}^{\text{AUT}}, k_{y0}^{\text{AUT}}) \\ F_{y0}^{\text{AUT}}(k_{x0}^{\text{AUT}}, k_{y0}^{\text{AUT}}) \end{pmatrix}^T [S^{3 \times 2}(\theta^{\text{AUT}}, \phi^{\text{AUT}})]^T [T^{3 \times 3}(\alpha^{\text{AUT}}, \beta^{\text{AUT}}, \gamma^{\text{AUT}})]^T D \\
&\times T^{3 \times 3}(\alpha, \beta, \gamma) S^{3 \times 2}(\theta^p, \phi^p) \begin{pmatrix} F_{x0}^p(k_{x0}^p, -k_{y0}^p) \\ F_{y0}^p(k_{x0}^p, -k_{y0}^p) \end{pmatrix} = \tilde{C} \cos \theta F(k_{x0}, k_{y0}) \exp(jk_{z0}z) \quad (10.2.3.20)
\end{aligned}$$

where $\tilde{C} = -\frac{C \exp(j2kR)}{k^2}$ and the relation between the wave vectors is

$$\begin{pmatrix} k_x^{\text{AUT}} \\ k_y^{\text{AUT}} \\ k_z^{\text{AUT}} \end{pmatrix} = T^{3 \times 3}(\alpha^{\text{AUT}}, \beta^{\text{AUT}}, \gamma^{\text{AUT}}) \begin{pmatrix} k_{x0} \\ k_{y0} \\ k_{z0} \end{pmatrix} \quad (10.2.3.21)$$

$$\begin{pmatrix} k_x^p \\ k_y^p \\ k_z^p \end{pmatrix} = T^{3 \times 3}(\alpha^p, \beta^p, \gamma^p) \begin{pmatrix} -k_{x0} \\ k_{y0} \\ k_{z0} \end{pmatrix} \quad (10.2.3.22)$$

What's more, $(\theta^{\text{AUT}}, \phi^{\text{AUT}})$ and (θ^p, ϕ^p) are defined as

$$\theta^{\text{AUT}} = \arccos \left[\frac{k_z^{\text{AUT}}}{k} \right], \quad \phi^{\text{AUT}} = \arctan \left[\frac{k_y^{\text{AUT}}}{k_x^{\text{AUT}}} \right] \quad (10.2.3.23)$$

$$\theta^p = \arccos \left[\frac{k_z^p}{k} \right], \quad \phi^p = \arctan \left[\frac{k_y^p}{k_x^p} \right] \quad (10.2.3.24)$$

and D is give as

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (10.2.3.25)$$

10.2.4 Special cases

Now we will give formulas for some special cases, i.e., 1) without any rotation; 2) rotation around x_0 -axis; 3) rotation around y_0 axis; and 4) rotation around z_0 axis.

1) without any rotation

$$\begin{aligned} P(k_{x0}, k_{y0}) = & F_{x0}^{\text{AUT}}(k_{x0}, k_{y0}) F_{x0}^p(k_{x0}, -k_{y0}) \frac{k_{z0}^2 + k_{x0}^2}{k_{z0}} + F_{y0}^{\text{AUT}}(k_{x0}, k_{y0}) F_{y0}^p(k_{x0}, -k_{y0}) \frac{-k_{z0}^2 - k_{y0}^2}{k_{z0}} \\ & - F_{x0}^{\text{AUT}}(k_{x0}, k_{y0}) F_{y0}^p(k_{x0}, -k_{y0}) \frac{k_{x0} k_{y0}}{k_{z0}} + F_{y0}^{\text{AUT}}(k_{x0}, k_{y0}) F_{x0}^p(k_{x0}, -k_{y0}) \frac{k_{x0} k_{y0}}{k_{z0}} \end{aligned} \quad (10.2.4.26)$$

2) rotation around x_0 -axis

$$\begin{aligned} P(k_{x0}, k_{y0}) = & F_{x0}^{\text{AUT}}(k_{x0}, k_y^{\text{AUT}}) F_{x0}^p(k_{x0}, k_y^p) \frac{k_z^{\text{AUT}} k_z^p + k_{x0}^2 \cos(\alpha^{\text{AUT}} + \alpha^p)}{k_{z0}} \\ & + F_{y0}^{\text{AUT}}(k_{x0}, k_y^{\text{AUT}}) F_{y0}^p(k_{x0}, k_y^p) \frac{-k_{z0}^2 - k_{y0}^2}{k_{z0}} - F_{y0}^{\text{AUT}}(k_{x0}, k_y^{\text{AUT}}) F_{x0}^p(k_{x0}, k_y^p) \frac{k_y^p k_{x0}}{k_{z0}} \end{aligned}$$

$$-F_{x0}^{\text{AUT}}(k_{x0}, k_y^{\text{AUT}})F_{y0}^p(k_{x0}, k_y^p)\frac{k_{x0}k_y^{\text{AUT}}}{k_{z0}} \quad (10.2.4.27)$$

3) rotation around y_0 -axis

$$\begin{aligned} P(k_{x0}, k_{y0}) = & F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_{y0})F_{y0}^p(k_x^p, -k_{y0})\frac{-k_z^{\text{AUT}}k_z^p - k_{y0}^2 \cos(\alpha^{\text{AUT}} - \alpha^p)}{k_{z0}} \\ & + F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_{y0})F_{x0}^p(k_x^p, -k_{y0})\frac{k_{z0}^2 + k_{x0}^2}{k_{z0}} + F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_{y0})F_{x0}^p(k_x^p, -k_{y0})\frac{k_{y0}k_x^{\text{AUT}}}{k_{z0}} \\ & - F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_{y0})F_{y0}^p(k_x^p, -k_{y0})\frac{k_x^p k_{y0}}{k_{z0}} \end{aligned} \quad (10.2.4.28)$$

4) rotation around z_0 -axis

a) when both the AUT and the probe rotate,

$$\begin{aligned} P(k_{x0}, k_{y0}) = & F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}})F_{x0}^p(k_x^p, k_y^p)\frac{k_z^{\text{AUT}}k_z^p \cos(\alpha^{\text{AUT}} + \alpha^p) + k_x^{\text{AUT}}k_x^p}{k_{z0}} \\ & + F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}})F_{y0}^p(k_x^p, k_y^p)\frac{k_y^{\text{AUT}}k_y^p - k_z^{\text{AUT}}k_z^p \cos(\alpha^{\text{AUT}} + \alpha^p)}{k_{z0}} \\ & + F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}})F_{x0}^p(k_x^p, k_y^p)\frac{k_y^{\text{AUT}}k_x^{\text{AUT}} - k_z^{\text{AUT}}k_z^p \sin(\alpha^{\text{AUT}} + \alpha^p)}{k_{z0}} \\ & + F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}})F_{y0}^p(k_x^p, k_y^p)\frac{k_x^p k_y^p - k_z^{\text{AUT}}k_z^p \sin(\alpha^{\text{AUT}} + \alpha^p)}{k_{z0}} \end{aligned} \quad (10.2.4.29)$$

b) when only the probe rotates,

$$P(k_{x0}, k_{y0}) = P_{\text{Co-Polarization}} \cos \alpha^p + P_{\text{Cross-Polarization}} \sin \alpha^p \quad (10.2.4.30)$$

$$\begin{aligned}
P_{\text{Co-Polarization}} &= F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{x0}^p(k_x^p, k_y^p) \frac{k_{z0}^2 + k_{x0}^2}{k_{z0}} \\
&\quad + F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{y0}^p(k_x^p, k_y^p) \frac{-k_{z0}^2 - k_{y0}^2}{k_{z0}} \\
&\quad - F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{y0}^p(k_x^p, k_y^p) \frac{k_{x0}k_{y0}}{k_{z0}} + F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{x0}^p(k_x^p, k_y^p) \frac{k_{x0}k_{y0}}{k_{z0}}
\end{aligned} \tag{10.2.4.31}$$

$$\begin{aligned}
P_{\text{Cross-Polarization}} &= -F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{x0}^p(k_x^p, k_y^p) \frac{k_{x0}k_{y0}}{k_{z0}} \\
&\quad - F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{y0}^p(k_x^p, k_y^p) \frac{k_{x0}k_{y0}}{k_{z0}} \\
&\quad - F_{x0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{y0}^p(k_x^p, k_y^p) \frac{k_{z0}^2 + k_{x0}^2}{k_{z0}} + F_{y0}^{\text{AUT}}(k_x^{\text{AUT}}, k_y^{\text{AUT}}) F_{x0}^p(k_x^p, k_y^p) \frac{-k_{z0}^2 - k_{y0}^2}{k_{z0}}
\end{aligned} \tag{10.2.4.32}$$

In particular, when the far-field of the probe is circularly symmetrical, i.e., $F_{x0}^p(k_x^p, k_y^p) = F_{x0}^p(k_{x0}, k_{y0})$ and $F_{y0}^p(k_x^p, k_y^p) = F_{y0}^p(k_{x0}, k_{y0})$, we have,

$$P(k_{x0}, k_{y0}) = P(k_{x0}, k_{y0})|_{\alpha^p=0^\circ} \cos \alpha^p + P(k_{x0}, k_{y0})|_{\alpha^p=90^\circ} \sin \alpha^p \tag{10.2.4.33}$$

which is expected due to the polarization effect of the AUT and the probe system.

10.2.5 Simulation Results

First, the probe compensation theory presented previously is applied to the FBG (waist $w = 20\lambda$) without tilting and a rectangular probe ($a = b = 2mm$) with $0^\circ/25^\circ$; Second, we consider the case where the FBG (waist $w = 20\lambda$) is titled 25° and a rectangular probe ($a = b = 2mm$) with $0^\circ/25^\circ$; and finally, the probe compensation is applied to the practical measured field from launcher.

1) FGB case: $\alpha^{\text{AUT}} = 0^\circ$ and $\alpha^p = 0^\circ/25^\circ$

Fig. 10.4 - Fig. 10.6 show the probe compensation (values are shown as the probed field pattern minus the true field pattern $|E_{\text{probed}}| - |E_{\text{true}}|$) for 0° Gaussian beam (AUT with only $\hat{x}$ -component) tilting and 0° probe tilting, on plane $0.1L$, $0.5L$ and L , with $L = 2\frac{(2w)^2}{\lambda}$;

Fig. 10.7 - Fig. 10.9 show the probe compensation for 0° Gaussian beam (AUT) tilting and 25° probe tilting, on plane $0.1L$, $0.5L$ and L , with $L = 2\frac{(2w)^2}{\lambda}$. It is clear that the probe compensation is small for 0° tilted probe and is significant for 25° tilted probe.

2) FGB case: $\alpha^{\text{AUT}} = 25^\circ$ and $\alpha^p = 0^\circ/25^\circ$

Fig. 10.10 - Fig. 10.12 show the probe compensation for 25° Gaussian beam (AUT) tilting and 0° probe tilting, on plane $0.1L$, $0.5L$ and L , with $L = 2\frac{(2w)^2}{\lambda}$;

Fig. 10.13 - Fig. 10.15 show the probe compensation for 25° Gaussian beam (AUT) tilting and 25° probe tilting, on plane $0.1L$, $0.5L$ and L , with $L = 2\frac{(2w)^2}{\lambda}$. One can see that the probe compensation is larger (compared that of 25° tilted probe) for 0° tilted probe and is smaller for 25° tilted probe.

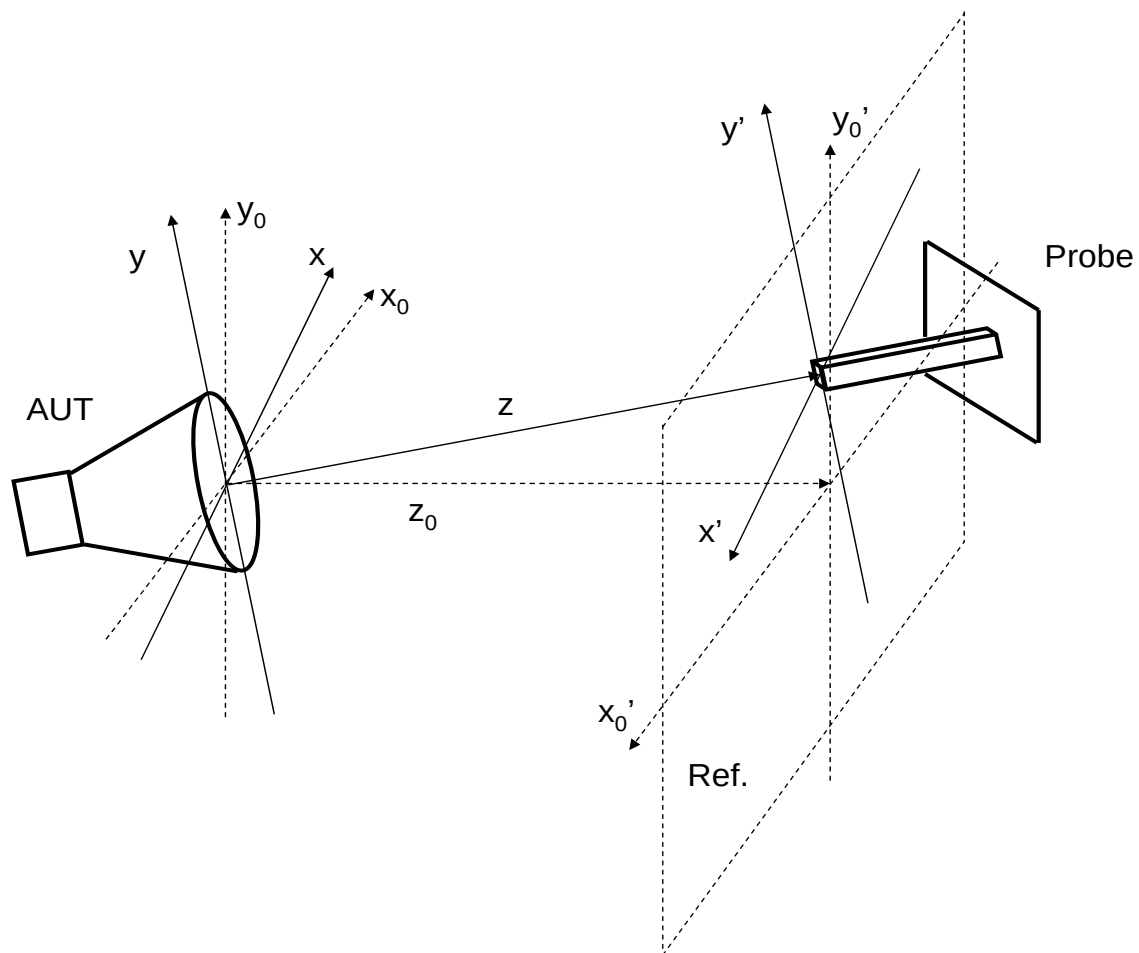


Figure 10.2: The illustration of the planar scanning measurement scheme with tilting and rotation of the AUT and probe.

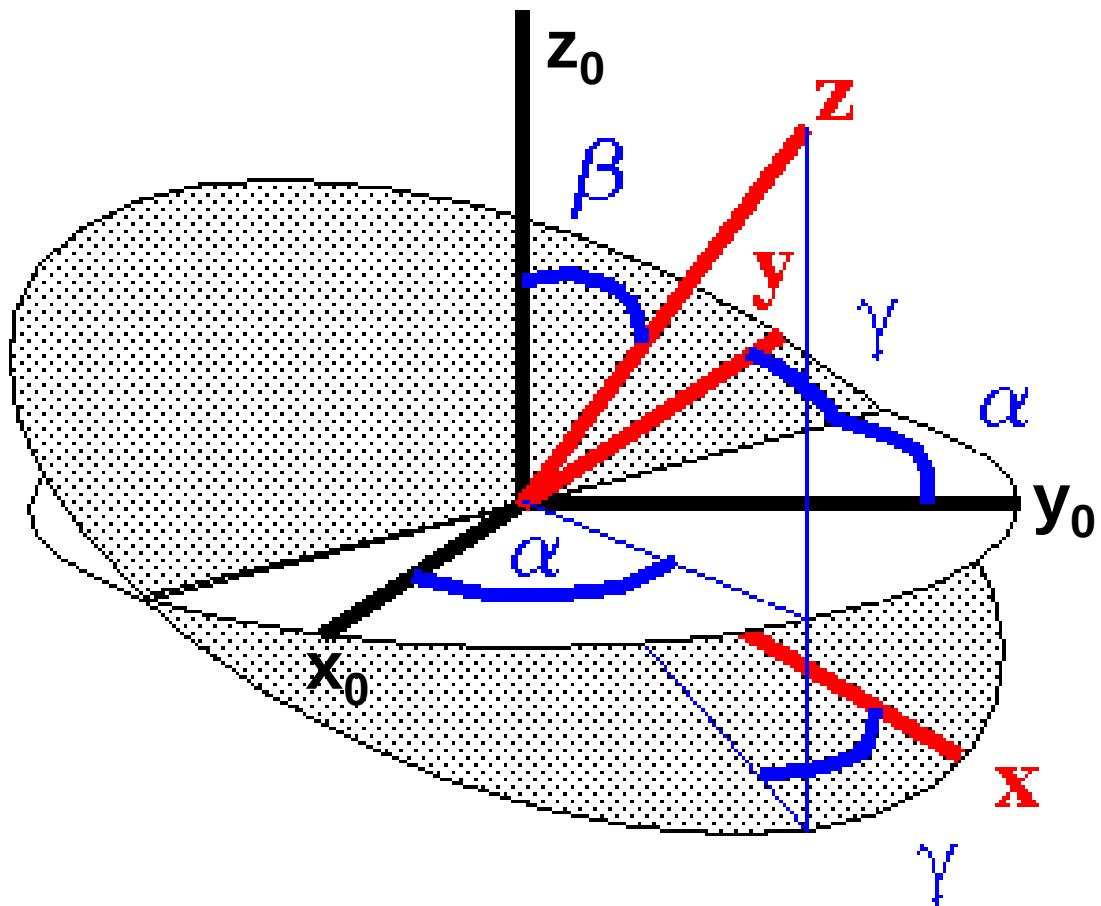


Figure 10.3: Euler angle for the tilting and rotation.

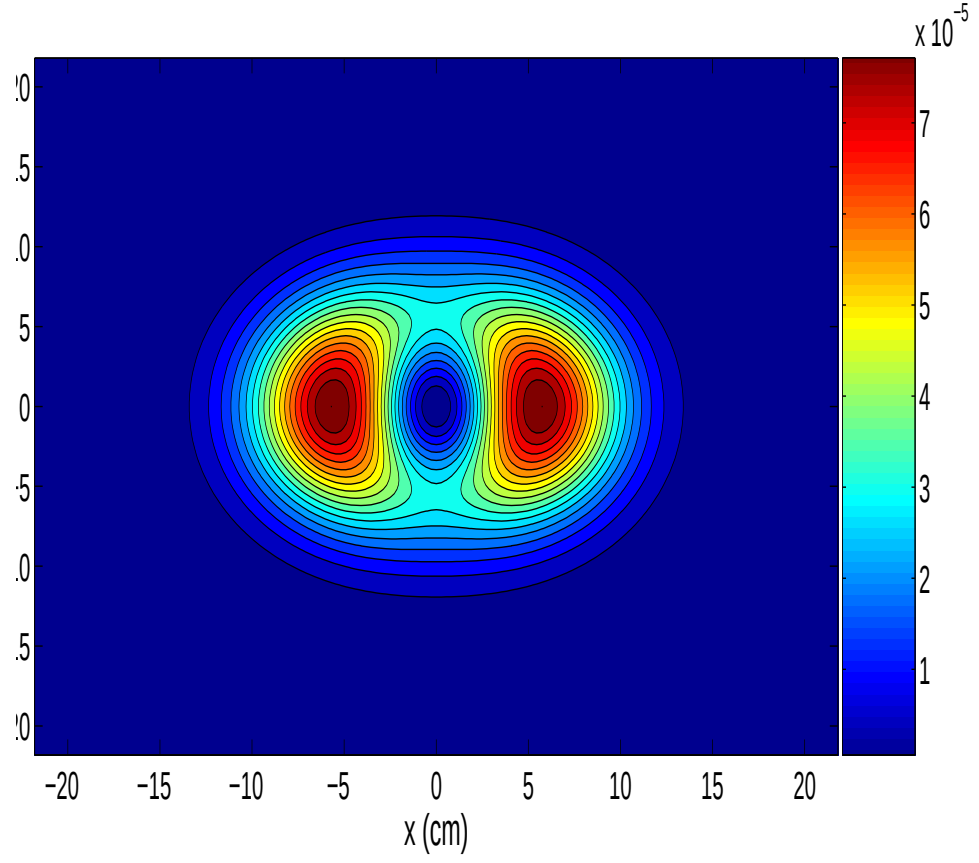


Figure 10.4: Probe compensation of $\alpha^{\text{AUT}} = 0^\circ$ and $\alpha^p = 0^\circ$, on plane $z = 0.1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

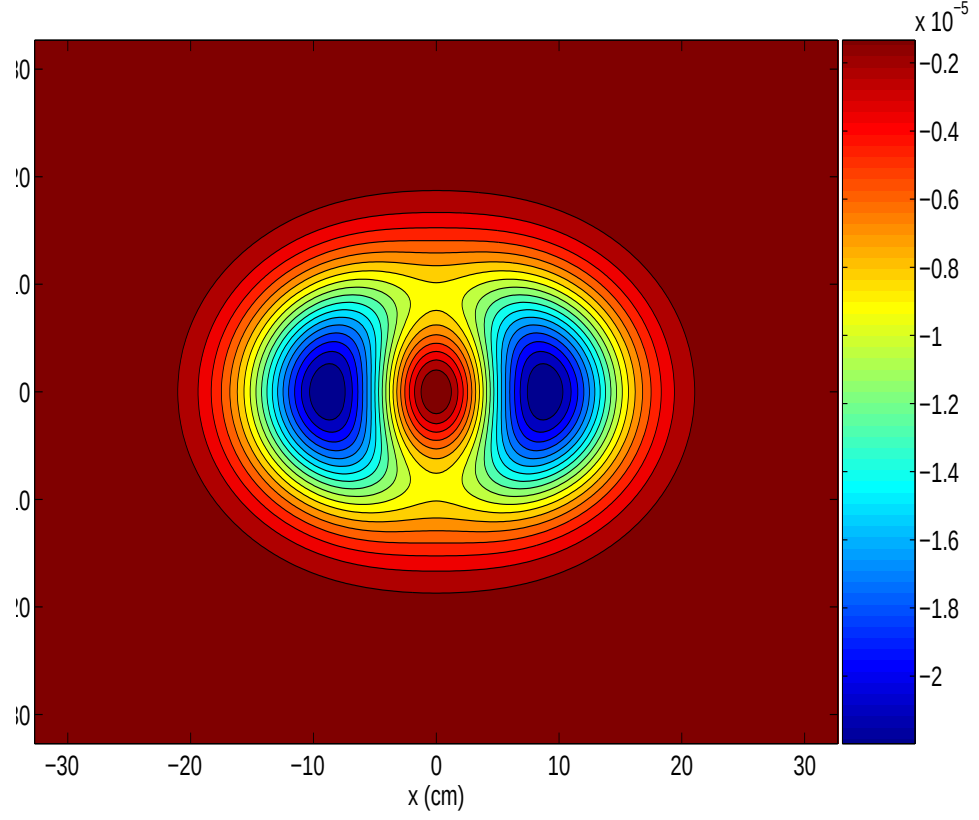


Figure 10.5: Probe compensation of $\alpha^{\text{AUT}} = 0^\circ$ and $\alpha^p = 0^\circ$, on plane $z = 0.5L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

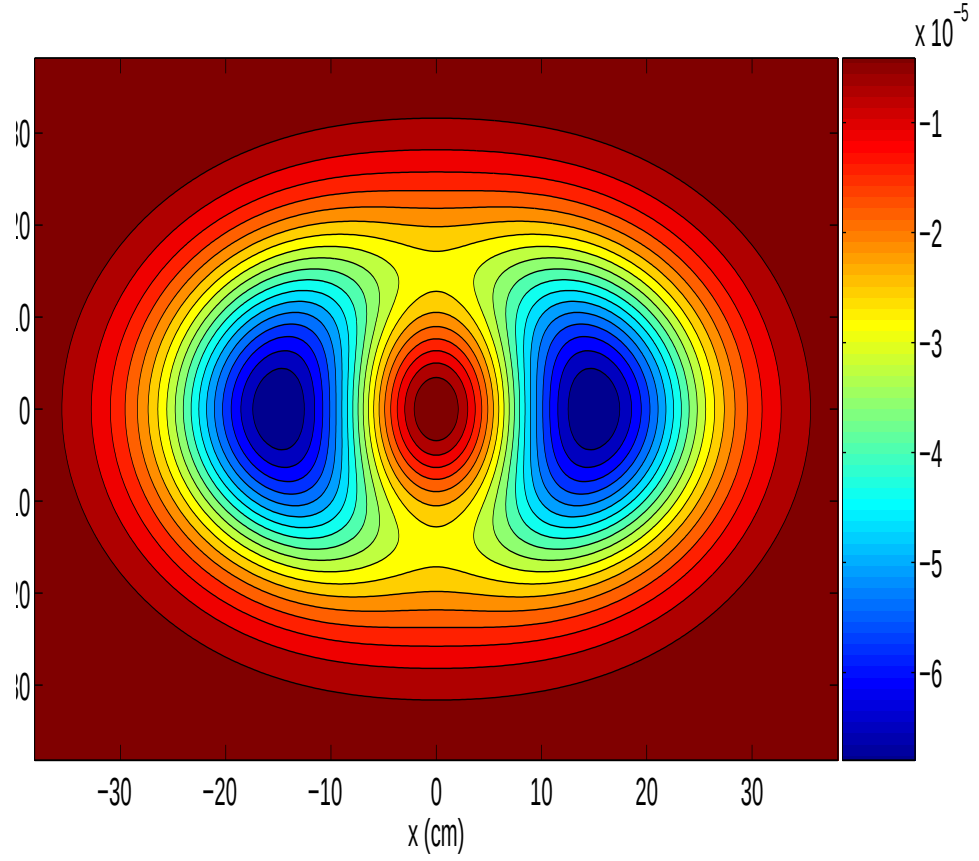


Figure 10.6: Probe compensation of $\alpha^{\text{AUT}} = 0^\circ$ and $\alpha^p = 0^\circ$, on plane $z = 1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

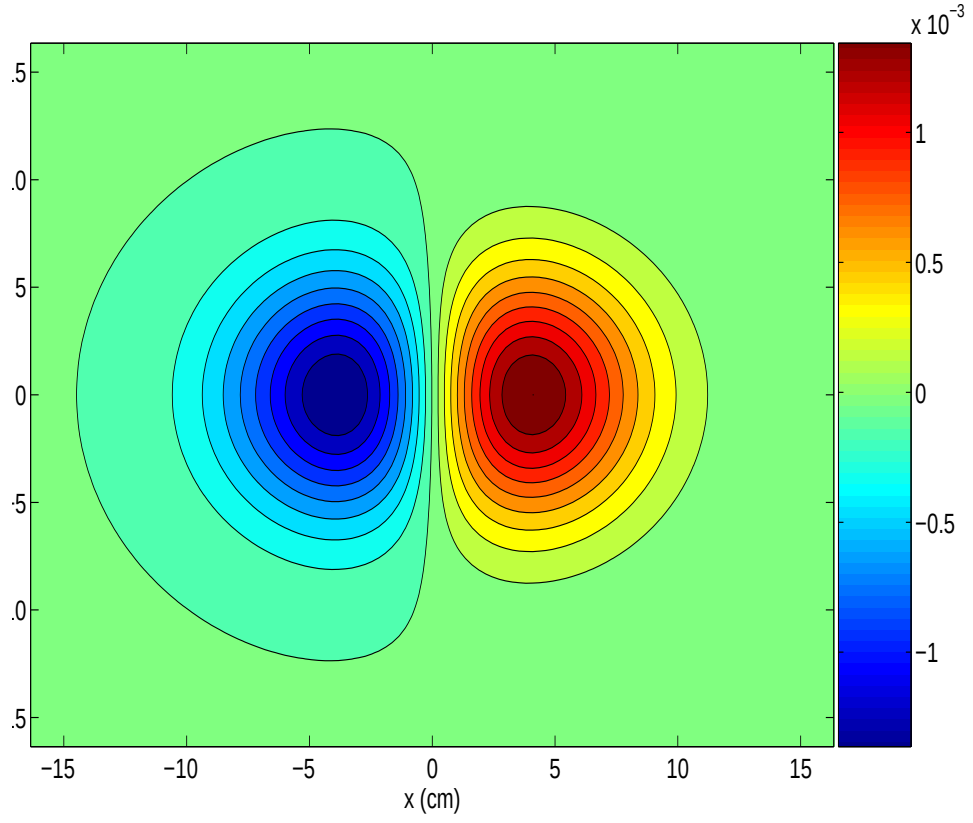


Figure 10.7: Probe compensation of $\alpha^{\text{AUT}} = 0^\circ$ and $\alpha^p = 25^\circ$, on plane $z = 0.1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

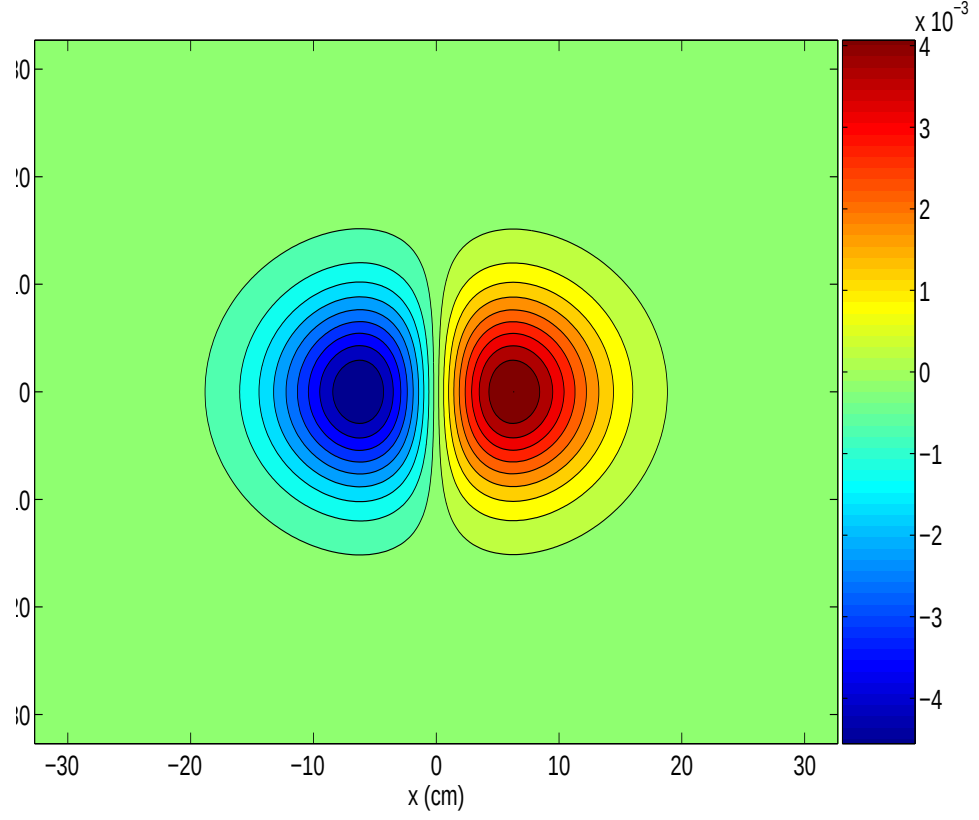


Figure 10.8: Probe compensation of $\alpha^{\text{AUT}} = 0^\circ$ and $\alpha^p = 25^\circ$, on plane $z = 0.5L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

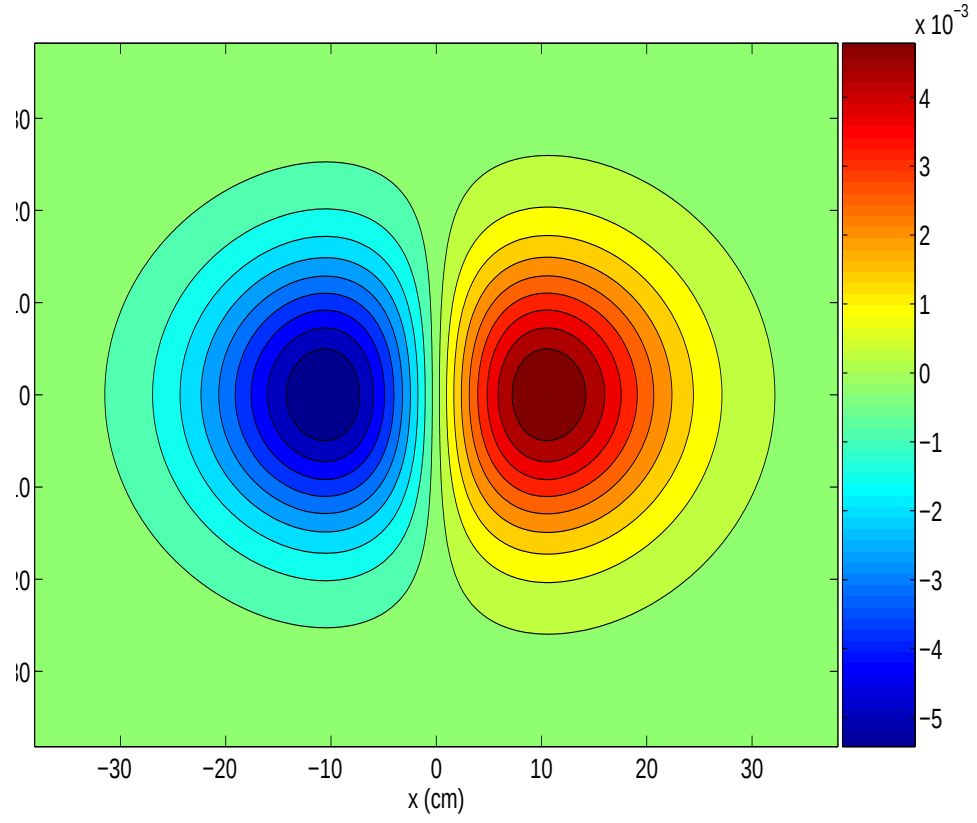


Figure 10.9: Probe compensation of $\alpha^{\text{AUT}} = 0^\circ$ and $\alpha^p = 25^\circ$, on plane $z = 1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

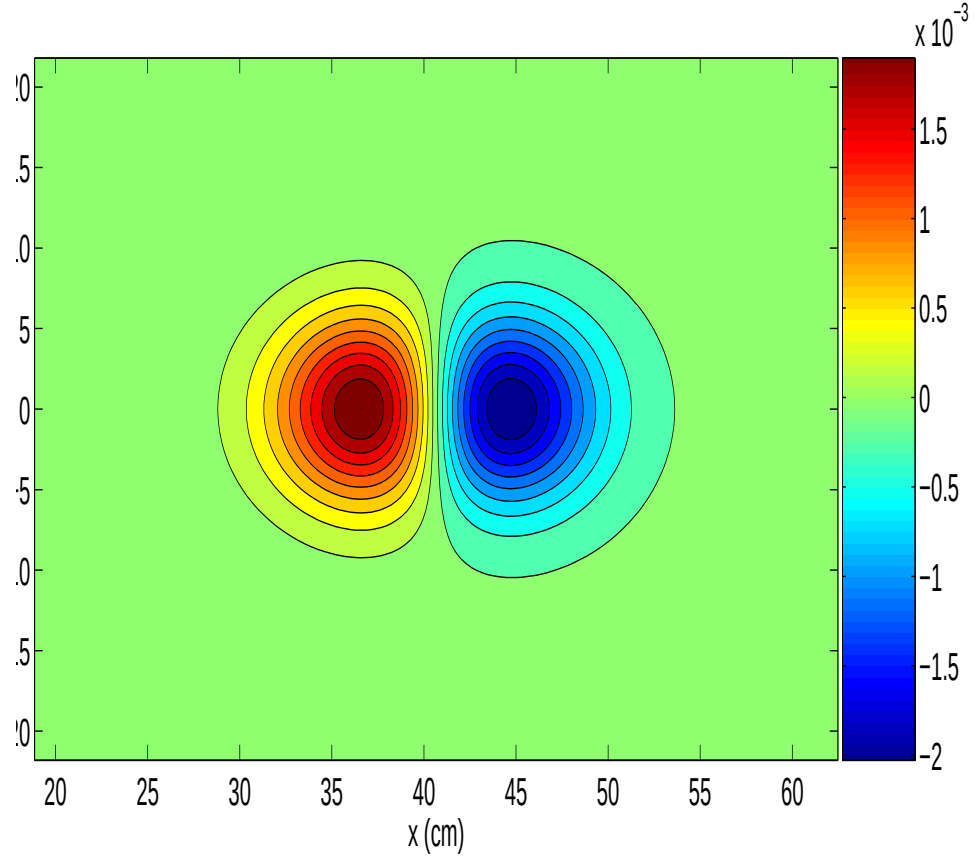


Figure 10.10: Probe compensation of $\alpha^{\text{AUT}} = 25^\circ$ and $\alpha^p = 0^\circ$, on plane $z = 0.1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

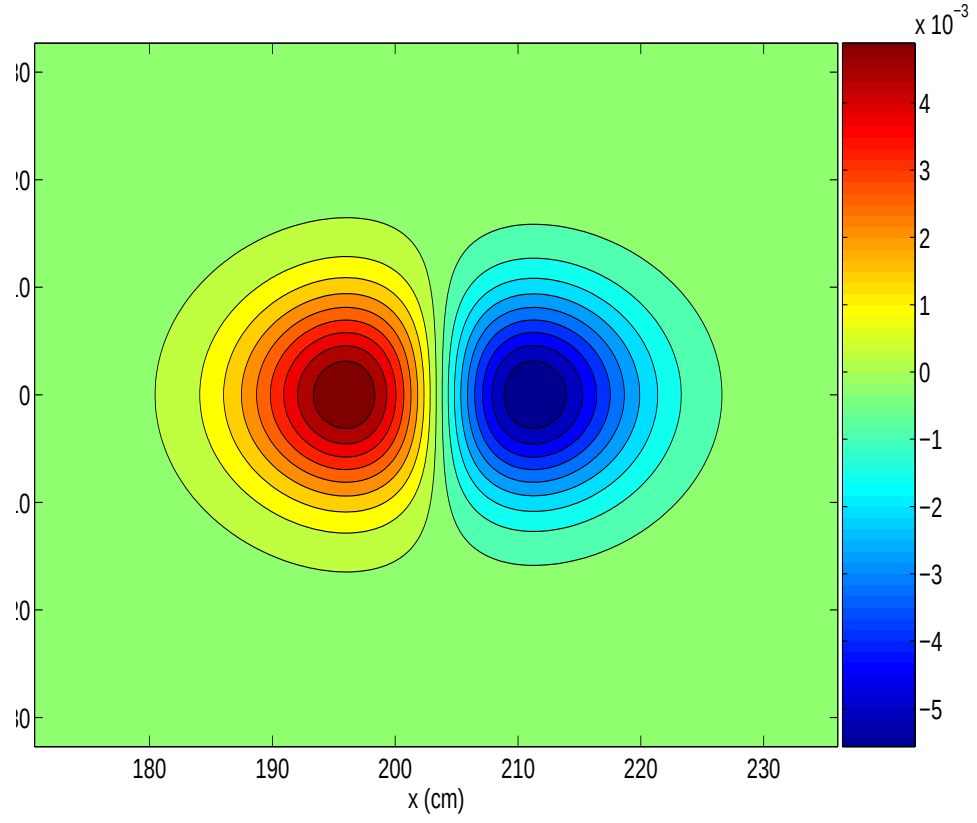


Figure 10.11: Probe compensation of $\alpha^{\text{AUT}} = 25^\circ$ and $\alpha^p = 0^\circ$, on plane $z = 0.5L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

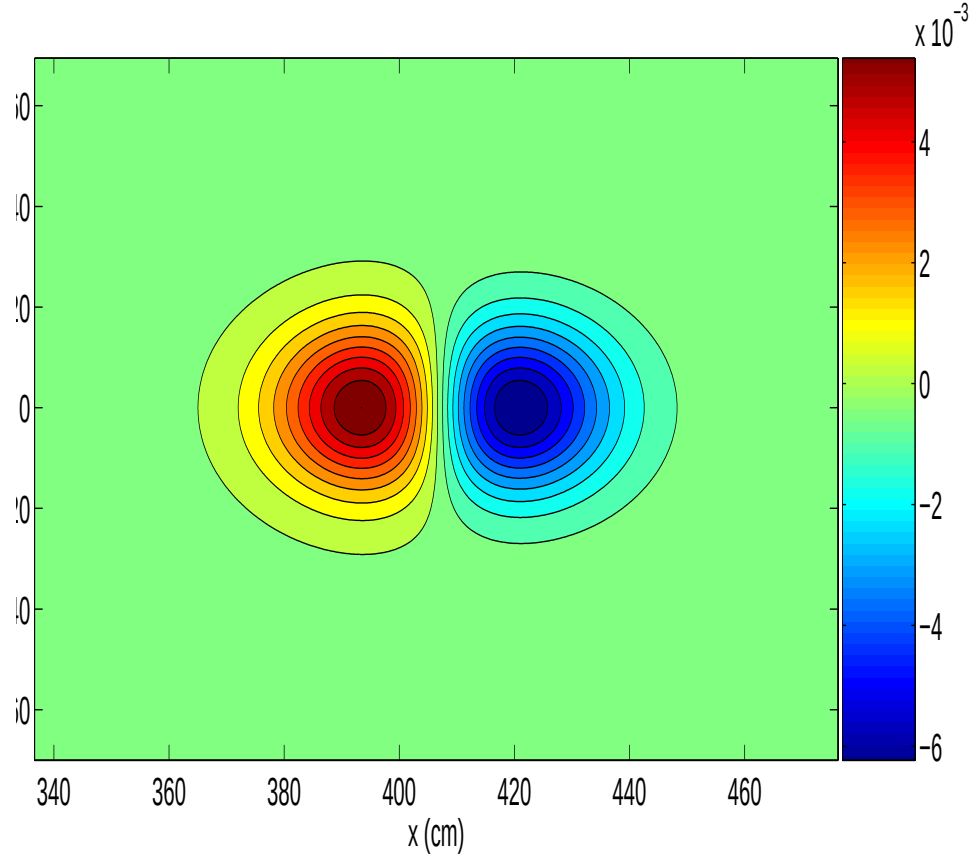


Figure 10.12: Probe compensation of $\alpha^{\text{AUT}} = 25^\circ$ and $\alpha^p = 0^\circ$, on plane $z = 1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

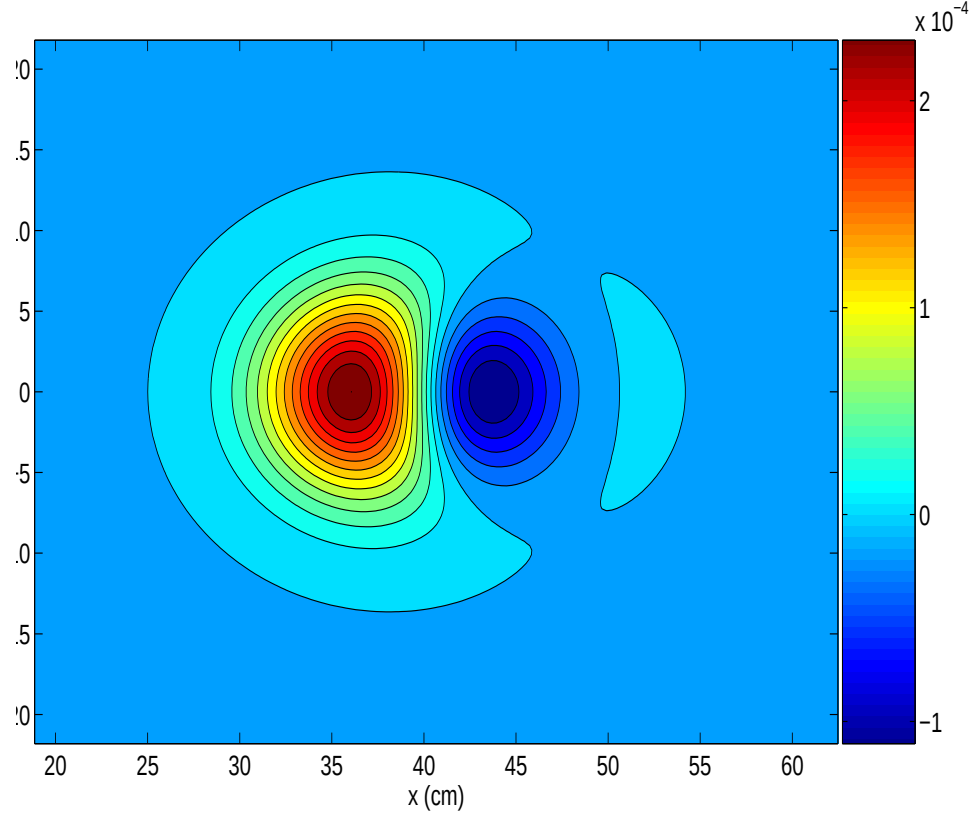


Figure 10.13: Probe compensation of $\alpha^{\text{AUT}} = 25^\circ$ and $\alpha^p = 25^\circ$, on plane $z = 0.1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

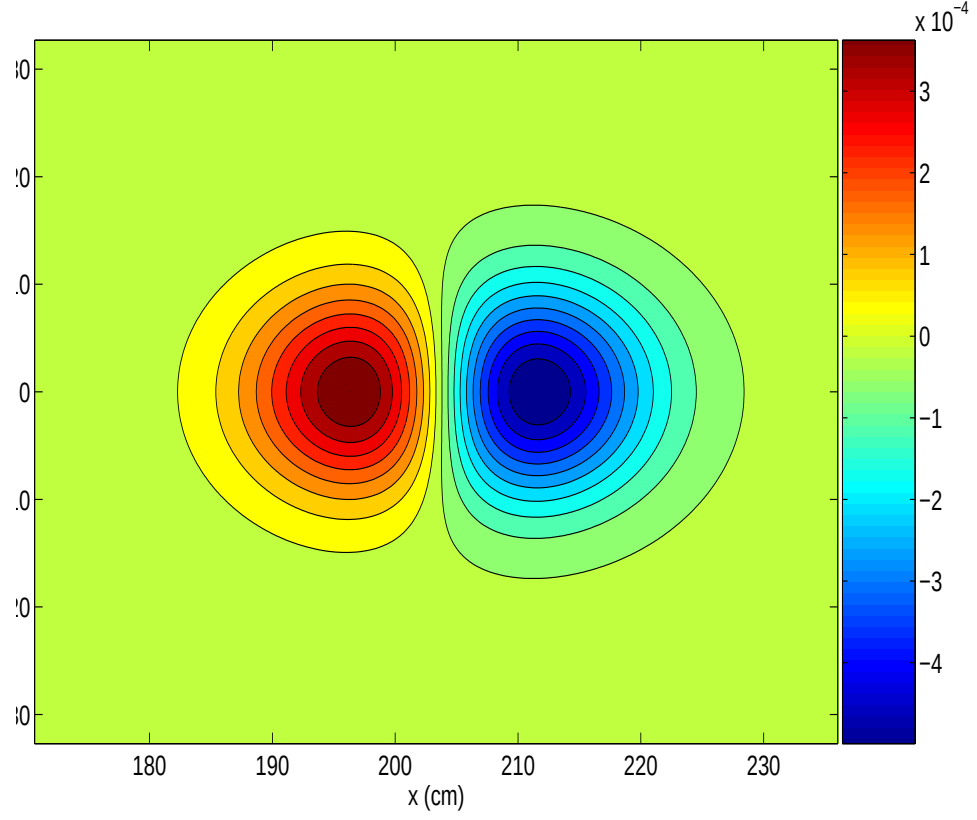


Figure 10.14: Probe compensation of $\alpha^{\text{AUT}} = 25^\circ$ and $\alpha^p = 25^\circ$, on plane $z = 0.5L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

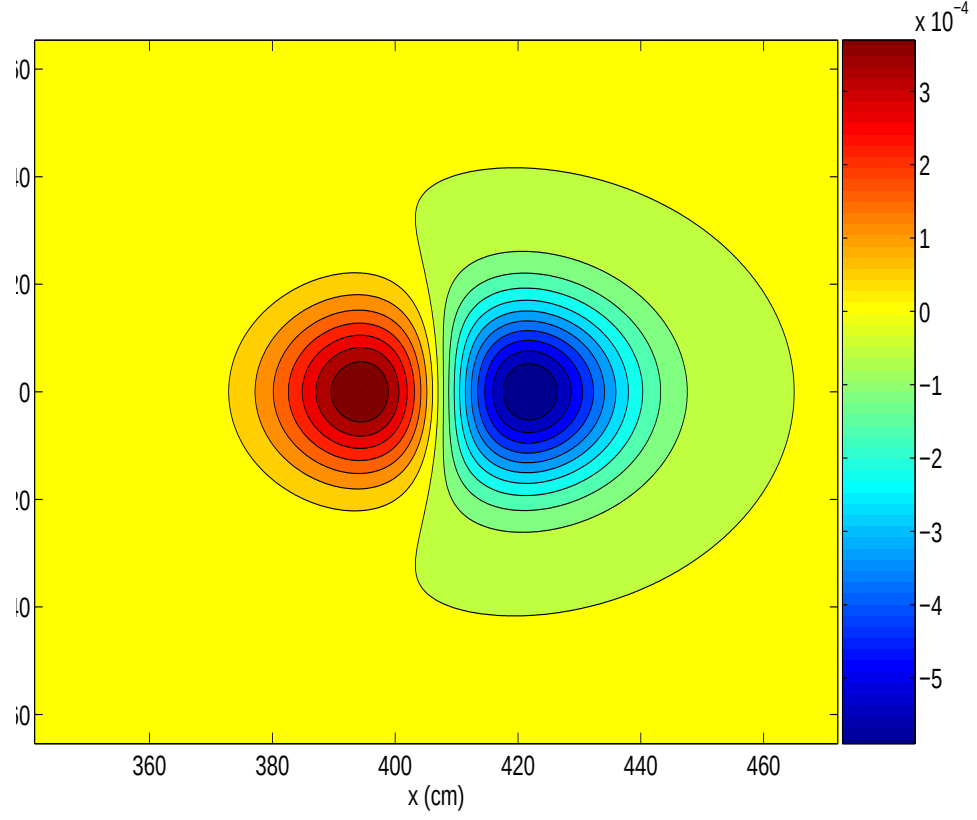


Figure 10.15: Probe compensation of $\alpha^{\text{AUT}} = 25^\circ$ and $\alpha^p = 25^\circ$, on plane $z = 1L$, where $L = 2\frac{(2w)^2}{\lambda}$ is the far-field distance.

Chapter 11

FUTURE PLAN

Based on the theory, algorithms and methods that have been developed in the previous Chapters, we will design and cold-test the multi-mode beam-shaping PEC mirror system. The modes we will design are $TE_{22,6}$ (110 GHz). 4 pieces of PEC mirrors will be used. The first two input mirrors M_1 and M_2 (“quasi-cylindrical”) are parabolic-like and the output beam from the QO dimpled-wall launcher is obtained from the Surf3d software. The two output mirrors M_3 and M_4 (“quasi-planar”) are designed using the measured input beam (output beam from M_2). After the beam-shaping PEC mirror system design, it is tested in the cold-test setup with 3 milli-Watts power.

Appendix A

Derivation of (5.1.3.25) in the Cylindrical Coordinate

$$\begin{aligned} \text{FT}_{2\text{D}} \left[G(\mathbf{R}) \right] &= \frac{1}{2\pi} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{e^{-jk|\mathbf{R}|}}{4\pi|\mathbf{R}|} e^{jk_x x} e^{jk_y y} dx dy \\ &= \frac{1}{2\pi} e^{jk_x x'} e^{jk_y y'} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{e^{-jk|\mathbf{R}|}}{4\pi|\mathbf{R}|} e^{jk_x(x-x')} e^{jk_y(y-y')} dx dy \end{aligned} \quad (\text{A.1})$$

Changing variables $x'' = x - x'$, $y'' = y - y'$ and $z'' = z - z'$, and replacing the dummy primed variables (x'', y'', z'') with the unprimed variables (x, y, z) , (A.1) becomes,

$$\text{FT}_{2\text{D}} \left[G(\mathbf{R}) \right] = \frac{1}{2\pi} e^{jk_x x'} e^{jk_y y'} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{e^{-jk|\mathbf{R}|}}{4\pi|\mathbf{R}|} e^{jk_x x} e^{jk_y y} dx dy \quad (\text{A.2})$$

In the cylindrical coordinate,

$$|\mathbf{R}| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \sqrt{(r_{\perp})^2 + z^2} \quad (\text{A.3})$$

where $r_{\perp} = x^2 + y^2$ and the following relation can be obtained from (A.3),

$$dr_{\perp} = \frac{|\mathbf{R}|}{r_{\perp}} d|\mathbf{R}| \quad (\text{A.4})$$

Now, express (A.2) in the cylindrical coordinate with the help of (A.4),

$$\begin{aligned}
\text{FT}_{2\text{D}} \left[G(\mathbf{R}) \right] &= \frac{1}{4\pi} e^{jk_x x'} e^{jk_y y'} \int_{|\mathbf{R}|=|z|}^{\infty} d|\mathbf{R}| e^{-jk|\mathbf{R}|} \left[\frac{1}{2\pi} \int_{\phi=0}^{2\pi} d\phi e^{-jk_{\perp} r_{\perp} \cos(\psi-\phi)} \right] \\
&= \frac{1}{4\pi} e^{jk_x x'} e^{jk_y y'} \int_{|\mathbf{R}|=|z|}^{\infty} d|\mathbf{R}| e^{-jk|\mathbf{R}|} J_0 \left(k_{\perp} \sqrt{|\mathbf{R}|^2 - z^2} \right) \\
&= \frac{-j}{4\pi k_z} e^{jk_x x'} e^{jk_y y'} e^{-jk_z |z-z'|} \tag{A.5}
\end{aligned}$$

where J_0 is the Bessel function of the first kind of order 0. Note that $\psi = \arctan \left[\frac{k_y}{k_x} \right]$ and $\phi = \arctan \left[\frac{y}{x} \right]$ have been used to derive (A.5).

Appendix B

2D Fourier Spectra for Stratton-Chu and Franz Formulas

The 2D Fourier spectrum of the Stratton-Chu formula is,

$$\begin{aligned} \text{FT}_{2\text{D}} \left[\mathbf{E}(\mathbf{r})|_{\text{Stratton-Chu}} \right] = \mathcal{G}^>(k_x, k_y, 0) \left\{ \frac{\omega\mu}{j} \mathbf{L} \left[\mathbf{J}_s(\mathbf{r}') \right] \right. \\ + \frac{j\mathbf{k}}{\epsilon} \mathbf{L} \left[\rho_s(\mathbf{r}') \right] + \hat{\mathbf{x}}j \left[k_y \mathbf{L} \left[n_y \mathbf{J}_{ms,z} \right] - k_z \mathbf{L} \left[n_z \mathbf{J}_{ms,y} \right] \right] \\ + \hat{\mathbf{y}}j \left[k_z \mathbf{L} \left[n_z \mathbf{J}_{ms,x} \right] - k_x \mathbf{L} \left[n_x \mathbf{J}_{ms,z} \right] \right] \\ \left. + \hat{\mathbf{z}}j \left[k_x \mathbf{L} \left[n_x \mathbf{J}_{ms,y} \right] - k_y \mathbf{L} \left[n_y \mathbf{J}_{ms,x} \right] \right] \right\} \end{aligned} \quad (\text{B.1})$$

The 2D Fourier spectrum of the Franz formula is,

$$\begin{aligned} \text{FT}_{2\text{D}} \left[\mathbf{E}(\mathbf{r})|_{\text{Franz}} \right] = \mathcal{G}^>(k_x, k_y, 0) \left\{ \frac{-j}{\omega\epsilon} \left[k^2 \mathbf{L} \left[\mathbf{J}_s(\mathbf{r}') \right] \right. \right. \\ \left. \left. - \mathbf{k} \sum_{v=x,y,z} \left(k_v \mathbf{L} \left[\mathbf{J}_{s,v}(\mathbf{r}') \right] \right) \right] + j\mathbf{k} \times \mathbf{L} \left[\mathbf{J}_{ms}(\mathbf{r}') \right] \right\} \end{aligned} \quad (\text{B.2})$$

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