Focault's Pendulum

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October 2020

1 Introduction

Foucault's pendulum is a simple device named after French physicist Léon Foucault and conceived as an experiment to demonstrate the Earth's rotation. In this report, sever analysis of this phenomena will be explained, developed, and proved.

In order to demonstrate the characteristics of this experiment, several models will be employed. A first approach will consider only small oscillations and no friction nor active forces, and therefore, the problem will be reduced to a simple pendulum described by the ODE version of the wave equation. A second approach will still consider relatively small oscillations, but it will introduce some important forces that appear due to Earth's rotational motion around its axes. The introduction of these forces and new terms will be crucial to the understanding of a more complex model that will appear afterwards. When considering this new model, it is very important to understand the apparent forces created due to relative rotation of Earth's reference frame with respect to its axis. No longer will the problem be reduce to a such simple case, but also will have to account for the equations of motion on a three-dimensional analysis and be aware of a new unknown variable (Tension). Consequently, it will be necessary to introduce a fourth equation relating the three space variables.

Before performing the experiment, several measurements and precautions have to be taken. To make a good approximation of the Earth's model, the length of the wire has to be much larger than the initial length of the state vector. Furthermore, air resistance damps the oscillation, so Foucault pendulum has to incorporate an electromagnetic or other drive to keep the bob swinging. Other imprecisions when constructing the pendulum can result in sudden directional changes and alter the model of rotation. Initial launch of the pendulum is also critical to avoid lateral swinging of the bob.

Moreover, the latitude point affects directly our planet's apparent period of oscillation. At the North Pole, the plane of oscillation of a pendulum remains pointing in the same direction while the Earth rotates underneath it, taking one sidereal day to complete a rotation. Same goes for the South Pole. On the other hand, the plane of oscillation of the Foucault pendulum is at all times co-rotating with the rotation of the Earth when considering the equator as the latitude point of reference. In this report, we will also consider what happens at Paris, midway in between the North Pole and the equator. We will find a combination of the both described effects.

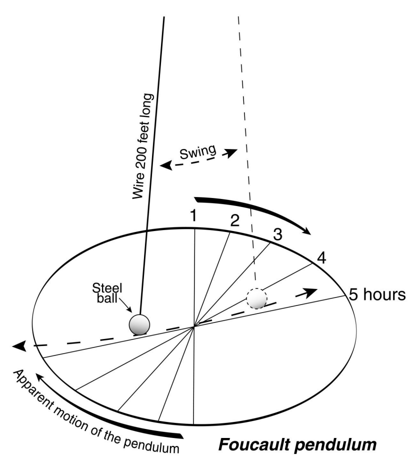

Figure 1: Focault's Pendulum

2 Methodology

2.1 Pendulum's Model

Initially, let's define an inertial reference frame $S0$:$\{0, B_0\}$, which has its origin at Earth's center point. In such frame, we define the right-handed basis $\{i_0, j_0, k_0\}$. The aim is to be able to define the position of the hanging point of the pendulum, A, the easiest way possible. To do so, a second non-inertial reference frame will be defined by taking a rotation of $\Omega \cdot t$ about the k_0 axis of reference frame $S0$. Then, this new reference frame $S1$:$\{P, B_1\}$, is translated to point A, such that $S1'$:$\{A, B_1\}$ with the right-handed basis $\{i_1, j_1, k_1\}$. Finally, by performing a rotation of $\phi = \frac{\pi}{2} - \lambda$ about the j_1 axis of $S1$, the new non-inertial reference frame $S2$:$\{A, B_2\}$ can be defined. Notice that by employing this new reference frame rather than the initial inertial one, simpler expressions can be obtained for the kinematics and dynamics of the bob. The relationship between all the before-mentioned basis, and its vector components are defined by the transformation:

$S2$:$\{A, B_2\}$

$$\{i_2, j_2, k_2\}$$

$S1'$:$\{A, B_1\}$

$$\vec{i_1} = sin\lambda\vec{i_2} + cos\lambda\vec{k_2}$$
$$\vec{j_1} = \vec{j_2}$$
$$\vec{k_1} = sin\lambda\vec{k_2} - cos\lambda\vec{i_2}$$

$S0$:$\{0, B_0\}$

$$\vec{i_0} = sin\lambda\vec{i_2} + cos\lambda\vec{k_2}$$
$$\vec{j_0} = \vec{j_2}$$
$$\vec{k_1} = sin\lambda\vec{k_2} - cos\lambda\vec{i_2}$$
$$\downarrow$$
$$\vec{i_0} = (sin\lambda \cos\Omega t)\vec{i_2} - sin\Omega t\vec{j_2} + (cos\lambda \cos\Omega t)\vec{k_2}$$
$$\vec{j_0} = (sin\lambda \sin\Omega t)\vec{i_2} + \cos\Omega t\vec{j_2} + (cos\lambda \cos\Omega t)\vec{k_2}$$
$$\vec{k_0} = -\cos\lambda\vec{i_2} + \sin\lambda\vec{k_2}$$

From this transformations, the following change of basis matrices could be defined: $[2R_1]$, from $S1$ to $S2$, and $[1R_0]$, from $S0$ to $S1$. Alternatively, the transformation of the vectors in basis $S0$ in to vectors in $S2$ can be written as $[2R_0] = [2R_1][1R_0]$

The position of the bob with respect to $S0$ is defined as:

$$r_0^A = R_\otimes \cos\lambda t \cos\Omega t\vec{i_0} + R_\otimes \cos\lambda t \sin\Omega t\vec{j_0} + R_\otimes \sin\lambda t\vec{k_0} = R_\otimes\vec{k_2}$$

The kinematics of an arbitrary point in space P with respect to reference frame $S2$ are written as:

$$r_2^P = x_2\vec{i_2} + y_2\vec{j_2} + z_2\vec{k_2}$$
$$v_2^P = \dot{x}_2\vec{i_2} + \dot{y}_2\vec{j_2} + \dot{z}_2\vec{k_2}$$
$$a_2^P = \ddot{x}_2\vec{i_2} + \ddot{y}_2\vec{j_2} + \ddot{z}_2\vec{k_2}$$

However, defining any other basis rather than cartesian coordinates$\{i_2, j_2, k_2\}$, will turn out different, as it would be necessary to account for the derivatives of the basis vectors. For instance, in polar coordinates $\{e_r, e_\phi, e_z\}$

$$\frac{de_r}{dt} = \dot{\phi}e_\phi$$

and

$$\frac{de_\phi}{dt} = -\dot{\phi}e_r$$

Dynamics of the pendulum:
Newton's equation for a mass m (the pendulum bob) written in the inertial reference frame $S0$ reads:

$$m\boldsymbol{a_2^P} = m\boldsymbol{g} - \boldsymbol{T}$$

where $T = -\frac{Tx_2}{l}\boldsymbol{i_2} - \frac{Ty_2}{l}\boldsymbol{j_2} - \frac{Tz_2}{l}\boldsymbol{k_2}$ is the tension force of the cable. However, $S2$ is non-inertial, so we need to define the apparent forces denoted by Newton's equation of the relative motion:

$$m\boldsymbol{a_2^P} = m\boldsymbol{a_0^P} - m\boldsymbol{a_A^0} - 2m\boldsymbol{\Omega_{20}} \times \boldsymbol{v_0^P} - m\boldsymbol{\Omega_{20}} \times (\boldsymbol{\Omega_{20}} \times \boldsymbol{v_0^P})$$

$$m\boldsymbol{a_2^P} = m\boldsymbol{g} + \boldsymbol{T} - m\boldsymbol{a_A^0} - 2m\boldsymbol{\Omega_{20}} \times \boldsymbol{v_0^P} - m\boldsymbol{\Omega_{20}} \times (\boldsymbol{\Omega_{20}} \times \boldsymbol{v_0^P})$$

So as we can see, $\vec{F} = m\boldsymbol{g} - \boldsymbol{T}$, and $F_{\vec{inertial}}$ accounts for the rest of the terms. Let's now define each of the terms composing the force due to the non-inertial reference frame:

-Force due to the acceleration of the origin: $-m\boldsymbol{a_A^0}$
-Euler's Force = 0 (We simplify the model so that the Earth rotation is considered constant).
-Centrifugal Force due to rotation of $S1 = -m\boldsymbol{\Omega_{20}} \times (\boldsymbol{\Omega_{20}} \times \boldsymbol{v_0^P})$
-Coriolis Force $= -2m\boldsymbol{\Omega_{20}} \times \boldsymbol{v_0^P}$

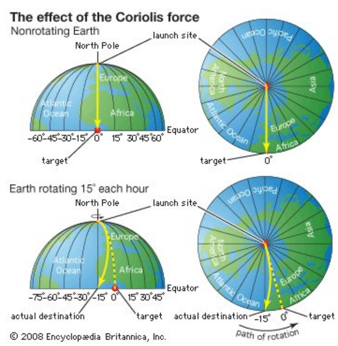

Figure 2: Effect of Coriolis Force on rotational reference frame

If we project Newton's equation along $\boldsymbol{i_2}, \boldsymbol{j_2}, \boldsymbol{k_2}$ we obtain the three scalar equations of motion of the bob in the relative frame:

$$\begin{cases} m\ddot{x_2} = -\frac{Tx_2}{l} + 2m\boldsymbol{\Omega_{20}} sin\lambda \dot{y_2} + \boldsymbol{W_{eff,x_2}} \\ m\ddot{y_2} = -\frac{Ty_2}{l} + 2m\boldsymbol{\Omega_{20}}(sin\lambda \dot{x_2} + \cos\lambda \dot{y_2}) + \boldsymbol{W_{eff,y_2}} \\ m\ddot{z_2} = -\frac{Tz_2}{l} + 2m\boldsymbol{\Omega_{20}} cos\lambda \dot{y_2} + \boldsymbol{W_{eff,z_2}} \end{cases}$$

2.2 Effective Weight definition

Notice that in this case, the centrifugal force is much less than the fictitious force due to the acceleration of point A (the origin of S2), which is much less than the weight. Therefore, $F_{centripetal} << F_{acc} << \boldsymbol{W}$. The projections of the effective way are defined below:

$$\boldsymbol{i_2} : \boldsymbol{W_{eff,x_2}} = m\Omega^2(x \sin^2\lambda + \sin\lambda\cos\lambda(R_\otimes + k))$$

$$\boldsymbol{j_2} : \boldsymbol{W_{eff,y_2}} = m\Omega^2 y$$

$$\boldsymbol{k_2} : \boldsymbol{W_{eff,z_2}} = m\Omega^2(\cos^2\lambda(R_\otimes + k) + x\sin\lambda\cos\lambda)$$

Note that absolute value is represented. An additional equation must be introduced since the tension force is unknown:

$$x_2^2 + y_2^2 + z_2^2 = l^2$$

2.3 Equilibrium positions

It is crucial to discuss the effect of each of the terms that compose the effective weight. The largest term in magnitude will determine the effect of $\boldsymbol{W_{eff}}$. The terms that have sines and cosines in it, are much smaller than those whose terms are multiplied by an entire number. For instance, a term multiplied by a trigonometric function will reduce its effect when not considering maximum or minimum values. In this case, the contribution of gravity force $\boldsymbol{g}$ will dominate over the rest in order to be able to neglect other forces such as Coriolis Force. This will be the reason why in the equilibrium position of the bob, we can neglect the effects of $\boldsymbol{W_{eff,x_2}}$ and $\boldsymbol{W_{eff,y_2}}$. Notice that the force exerted by gravity $(-mg\boldsymbol{k_2})$ acts on the z direction. Now we can define the equilibrium position of the bob:

$$\begin{cases} 0 = -\frac{Tx_{2,eq}}{l} \\ 0 = -\frac{Ty_{2,eq}}{l} \\ 0 = -\frac{Tz_{2,eq}}{l} + \boldsymbol{W_{eff,z_2}} \end{cases}$$

Therefore, the equilibrium position is located in the $\boldsymbol{-k_2}$ direction and is expressed as: $z_{2,eq} = \frac{mgl}{T}$. An additional equation must be introduced since the tension is still an unknown:

$$x_{2,eq}^2 + y_{2,eq}^2 + z_{2,eq}^2 = l^2$$

2.4 Model approximations

If we consider small oscillations around the local vertical direction, then $\theta << 1$, so that $z_2 \simeq -l$, and therefore $\ddot{z}_2 \simeq 0$. Moreover, considering $\boldsymbol{W_{eff}} \simeq mg$ as explained in the previous section, and taking a Coriolis Force much smaller than the effective weight, it can be stated that the equation for z_2:

$$m\ddot{z}_2 = -\frac{T(-l)}{l} - mg \simeq 0 \longrightarrow T \simeq mg$$

Now that the tension is not an unknown anymore, two specific types of problems can be encountered: One in which the apparent force are not considered, and the other in which Coriolis Force has to be included.

2.4.1 No apparent forces

The differential equations result in a simple harmonic oscillator

$$\ddot{x}_2 = -\frac{Tx_2}{l} = -\frac{g}{l}x_2$$

$$\ddot{y}_2 = -\frac{Ty_2}{l} = -\frac{g}{l}y_2$$

2.4.2 Inclusion of Coriolis Forces

An additional term due to the rotation and translation of the reference frame appears. This particular equation is the so-called Focault's Pendulum equation with Small Oscillations.

$$\ddot{x}_2 = -\frac{Tx_2}{l} = -\frac{g}{l}x_2 + 2\boldsymbol{\Omega_{20}}\dot{y}_2 sin\lambda$$

$$\ddot{y}_2 = -\frac{Ty_2}{l} = -\frac{g}{l}y_2 + 2\boldsymbol{\Omega_{20}}\dot{x}_2 sin\lambda$$

2.5 Solution Approaches

2.5.1 Analytical Solution

Manual computation of the solution for the simple pendulum using the general method of solving homogeneous second order ODEs with constant coefficients (g, l). The Homogeneous solution will be approximated by a polynomial of the form $x_h(t), y_h(t) \sim e^{kt}$. Applying this condition to our differential equation, the characteristic equation is obtained:

$$e^{kt}(k^2 + \frac{g}{l}) = 0 \longrightarrow k = \boldsymbol{i}\sqrt{\frac{g}{l}}$$

$$x_h(t), y_h(t) = Ce^{\boldsymbol{i}\sqrt{\frac{g}{l}}t} + De^{-\boldsymbol{i}\sqrt{\frac{g}{l}}t} \longrightarrow x_h(t), y_h(t) = 2Re(Ce^{\boldsymbol{i}\sqrt{\frac{g}{l}}t})$$

$$x_h(t), y_h(t) = A\cos\sqrt{\frac{g}{l}}t + B\sin\sqrt{\frac{g}{l}}t$$

Now, applying initial conditions of the pendulum: $x_{20} = 1; \dot{x_{20}} = y_{20} = \dot{y_{20}} = 0$

$$x_h(t) = \cos\sqrt{\frac{g}{l}}t \qquad \dot{x}_h(t) = -\sqrt{\frac{g}{l}}\sin\sqrt{\frac{g}{l}}t$$

$$y_h(t) = 0 \qquad \dot{y}_h(t) = 0$$

2.5.2 Numerical Solution

Solving the equation using MATLAB code *ode*45. To do so, we define the state vector $\boldsymbol{X}$ which includes the position and velocity for the planar case, in both simple and Coriolis cases. Then the derivatives vector is defined such that $\frac{d\boldsymbol{X}}{dt} = \boldsymbol{F(X)}$. Finally, it is necessary to provide the initial state vector, or in other words, the initial conditions, $\boldsymbol{X_0}$.

$$\boldsymbol{X} = \begin{bmatrix} x_2 \\ \dot{x}_2 \\ y_2 \\ \dot{y}_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{F(X)} = \begin{bmatrix} \dot{x_2} = X_2 \\ \ddot{x}_2 = f_x(\boldsymbol{X}) \\ \dot{y_2} = X_4 \\ \ddot{y}_2 = f_y(\boldsymbol{X}) \end{bmatrix} \quad \text{with} \quad \boldsymbol{X_0} = \begin{bmatrix} x_{20} \\ \dot{x}_{20} \\ y_{20} \\ \dot{x}_{20} \end{bmatrix}$$

Furthermore, it can be checked that the numerical solution provided by the computer will be optimal since the pendulum's length is much larger than x_{20} (good approximation).

3 Results and discussion

3.1 Comparison between analytical and numerical solutions

3.1.1 Analytical Solution. Position and velocity graphs

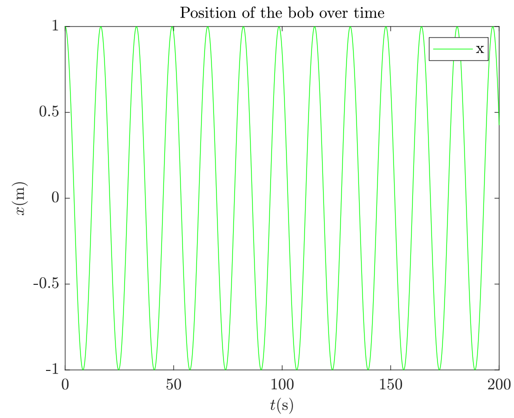

Figure 3: Position of the bob vs Time

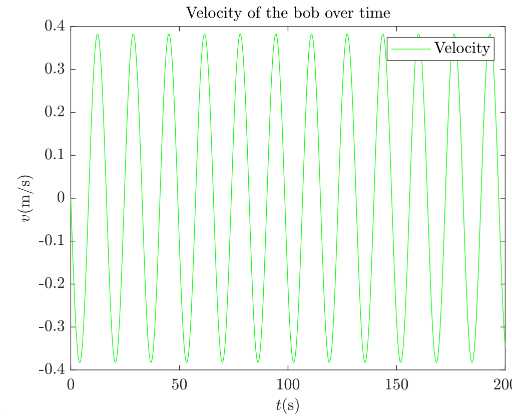

Figure 4: Velocity of the bob vs Time

3.1.2 Numerical Solution. Position and velocity graphs

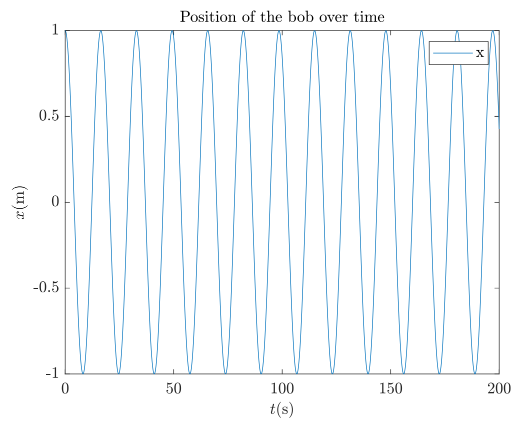

Figure 5: Position of the bob vs Time

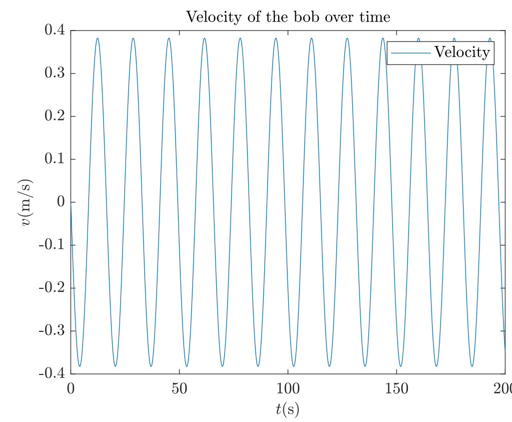

Figure 6: Velocity of the bob vs Time

3.1.3 Comparison Between the solutions

Since manual computations done in the analytical solution derive into approximations due to the method of resolution, there will be an error while approximating the values of the position and velocity that increases over time.

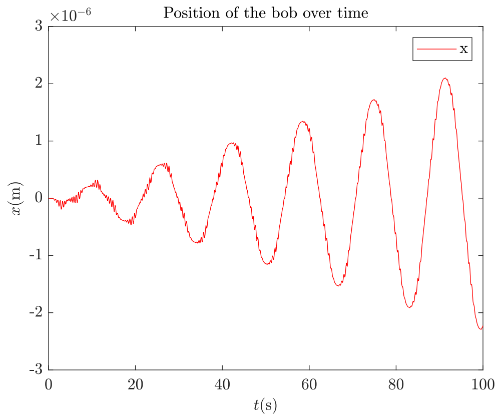

Figure 7: Error(Diference) between numerical and analytical solution

3.2 Numerical solution for the Focault's Pendulum in the North Pole

Due to the appearance of the Coriolis terms in both differential equations, these equations are not uncoupled anymore and cannot be solved in an analytical way. Therefore, Focault's Pendulum solutions will be described in terms of three different latitudes: Paris, the North Pole and the Equator.
In the case of the North Pole, the earth and the pendulum are not exerting much influence on each other due to the 90 degrees angle with respect to the equator. That is, moving in the same direction as the Earth axis, and thus, there is no relative motion between them.

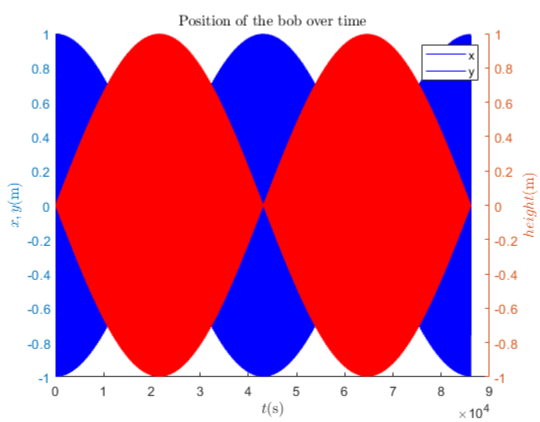

Figure 8: Focault's Pendulum position in the North Pole

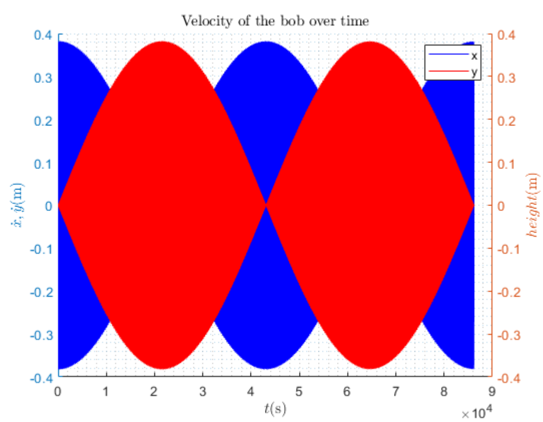

Figure 9: Focault's Pendulum velocity in the North Pole

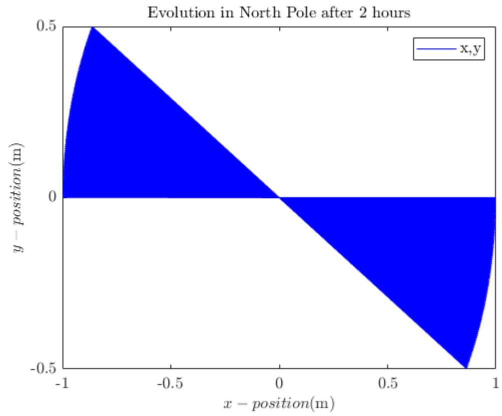

Figure 10: Focault's Pendulum evolution after 2 hours in North Pole

3.3 Numerical solution for the Focault's Pendulum in Paris

Paris makes and approximate angle of 48.51º with respect to Earth's latitude. Therefore, pendulum is not rotating in the same axis as the Earth, and there exists a relative motion between them. That causes a variation in the motion of the pendulum.

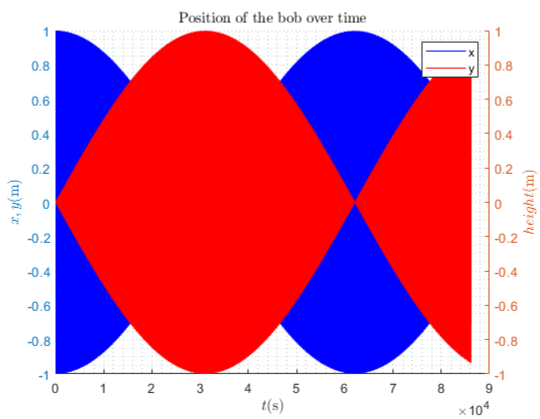

Figure 11: Focault's Pendulum position Paris

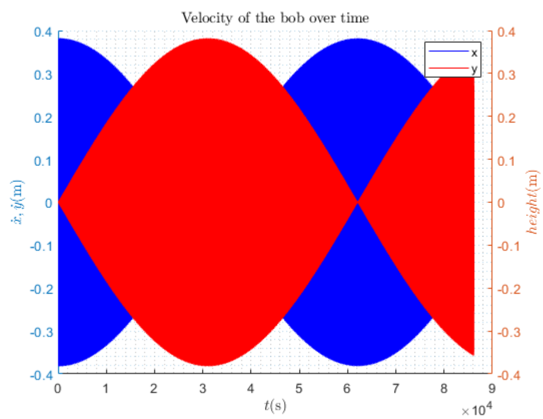

Figure 12: Focault's Pendulum velocity in Paris

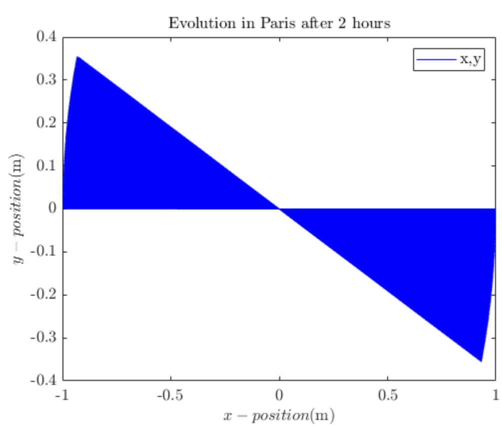

Figure 13: Focault's Pendulum evolution after 2 hours in Paris

3.4 Numerical solution for the Focault's Pendulum in the equator

The equator lies in the 0º latitude line, and therefore presents quite a particular motion for this pendulum case. The plane of rotation is coaxial with the Earth, and therefore is co-rotating with the planet's rotation.

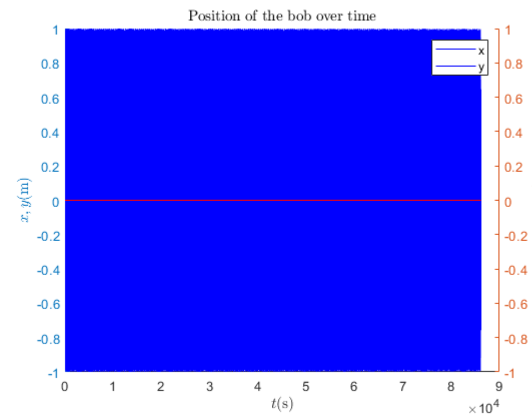

Figure 14: Focault's Pendulum position in the equator

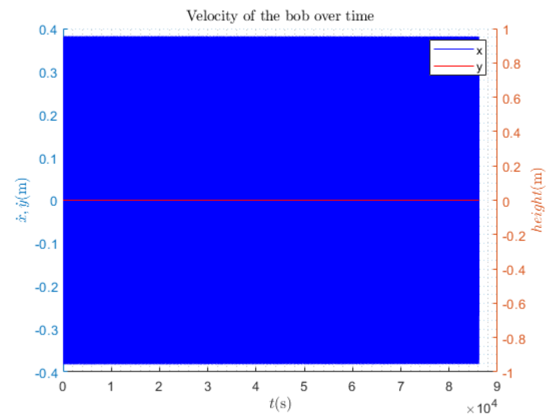

Figure 15: Focault's Pendulum velocity in the Equator

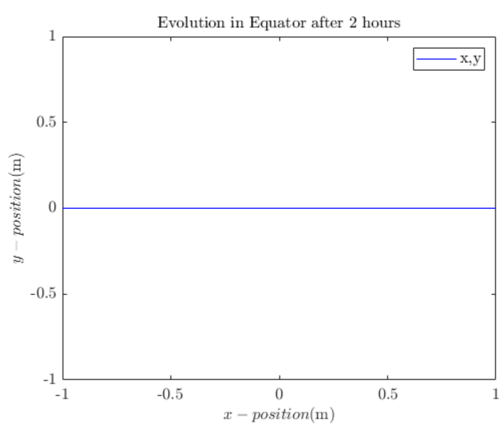

Figure 16: Focault's Pendulum evolution after 2 hours in the equator

4 Conclusion

To begin with, let's comment briefly about the different methods of solving the simple pendulum case. As explained before, simple pendulum neglects any kind of Coriolis force acting on the pendulum, and approaches the problem using the simple Harmonic Oscillator 2nd order ODE. The error between the analytical and numerical solution, as commented in Section 3.1.3, is due to the imprecision of performing manual computations, since the solution is an approximation. However, the numerical error might have some error for very large values of time, since the model is just proven to work considering Siderial rotation of the Earth. Now, the more developed case will be commented.

The main conclusion drawn for the inclusion of Coriolis force in the Focault's Model, is that both in the North Pole and in Paris, it can be found that the effect of this new force alters the oscillations of the pendulum. This force becomes larger, as latitude angle λ increases, and one can see that the plane of the pendulum's oscillation rotates over time(Figure 17). Therefore, effects in the North Pole, more oscillations are observed, as shown in Figure 8. The most relevant characteristic that the equator presented in section 3.4, was that the plane of oscillation was coaxial with the Earth. Consequently, this neglected the Coriolis force at this latitude, and it is the reason why no alteration can be seen when comparing with the solution of the simple pendulum. A closer approach on how this plane of rotation varies with time can be observed in Figure 10, Figure 13, and Figure 15; where it can be seen that the North Pole plane of oscillation changes grater than in Paris, and that is null in the equator. Moreover, the value of the latitude angle λ is responsible for the changer in rotation in the poles. as it can be seen in Figure 10, the northern hemisphere (positive angles) presents a counter-clockwise rotation. Had we taken the South Pole instead and a southern hemisphere city rather than Paris, we would have observed a clockwise rotation, as observed in Figure 18. In summary:

*If $\lambda = 0^{\text{o}}$, which is the case of the Equator, the function the pendulum does is the same as the simple pendulum case. This happens because the Coriolis force in this case is null. Therefore, the Y position remains constant and the same occurs for the X position.

* If $\lambda \neq 0^{\text{o}}$, which is the case of Paris and the North Pole. This time, the Y position is not null, and the pendulum oscillates along the XY-plane.

Although Focault's Pendulum presents a very well-posed problem on how Earth rotates, it presents some limitations. Realising the bob from a different position, or not taking care of avoiding lateral motion of the pendulum, can cause the model to fail and revert the study of the motion. Furthermore, closer electric and magentic sources should be removed to avoid adding new forces to the equations that alter the pendulum's motion, and do not resemble Focault's approach. This can be fixed by employing a magnetic sucker to absorb any king of electromagnetic interaction of the metallic bob with the sorroundings.

Finally, it is worth mentioning the real approach of the pendulum model. In Focault's representation, air resistance is neglected. However, this is not true for most of the cases. Air presents a friction force that must be analyzed in how it affects the motion of the pendulum. This air flow might be linear, turbulent, or mixed (analyzed with Reynold's Number), and has some drag. This would introduce two new forces to the equation.

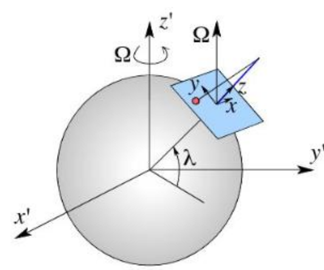

Figure 17: Oscillation Plane at an arbitrary point in northern hemisphere. Source: Sciencebits

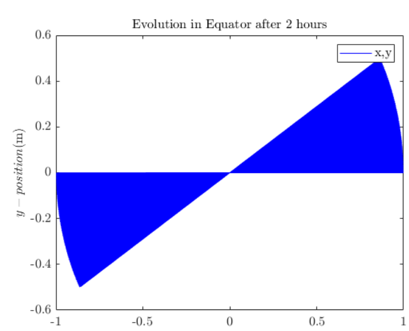

Figure 18: Focault's Pendulum evolution after 2 hours in the South Pole