Passive-seismic Inversion of SH-Wave Input Motions in a Domain Truncated by WABC

Bruno P. Guidio¹, Boris Jeremic², Leandro P. Guidio³, and Chanseok Jeong^{4,*}

¹Postdoctoral Researcher, School of Engineering and Technology, Central Michigan University, Mount Pleasant, MI 48859, USA; ORCID: [0000-0003-1943-8806](https://orcid.org/0000-0003-1943-8806)

²Professor, Department of Civil and Environmental Engineering, University of California, Davis, CA 95616, USA

³Graduate Student, Institute of Mathematics, Statistics, and Computing, The University of Campinas, Campinas, SP 13083, Brazil; ORCID: [0000-0001-7554-6567](https://orcid.org/0000-0001-7554-6567)

⁴Assistant Professor, School of Engineering and Technology, Central Michigan University, Mount Pleasant, MI 48859, USA; ORCID: [0000-0002-0488-8559](https://orcid.org/0000-0002-0488-8559)

*Corresponding author: jeong1c@cmich.edu

Abstract

This paper introduces a new inversion method for the reconstruction of complex, incoherent SH incident wavefield in a domain that is truncated by a wave-absorbing boundary condition (WABC), using a partial differential equation (PDE)-constrained optimization method. In numerical examples, dynamic traction at the WABC mimics seismic incident wavefield. Estimated traction is discretized over space and time, and the discretized values are reconstructed by using seismic motion data that are sparsely made by sensors on the top surface of a domain and a vertical array. The discretize-then-optimize (DTO) approach is used in the mathematical modeling and numerical implementation, and the finite element method (FEM) is applied to solve state and adjoint problems. The numerical results show that incident, inclined plane waves, cannot be fully reconstructed if using only the top surface sensors. In order to improve the inversion performance, a vertical array of sensors on the side boundary of a domain should be included. Second, a sufficiently large number of sensors must be employed to improve the algorithm's inversion performance. Third, the minimizer suffers less from solution multiplicity when it identifies lower frequency traction (e.g., a realistic seismic signal). Fourth, the larger value of the inversion error in the reconstructed traction does not necessarily translate to an error of the same order of magnitude in the corresponding reconstructed wave responses in the computational domain. Fifth, our presented inversion algorithm's accuracy is not compromised by the material complexity of a background domain. Lastly, the error in the reconstructed traction and the error in the corresponding wave responses grow when the noise of a larger level is added to the measurement data, but not in the same proportion. By extending the presented method into realistic 3D settings, this algorithm can indicate where large amplitudes of stress waves (i.e., weak points) occur in built environments and soils in a domain of interest during seismic events.

1 Introduction

The ability to properly reconstruct response of infrastructure systems to actual earthquake loads, which are difficult to identify, would allow for improvement in the inspection of existing infrastructures that are hit by earthquakes as well as the design of new infrastructures. In order to properly reconstruct the response of infrastructure objects during earthquakes, the so-called earthquake soil-structure interaction (ESSI) systems, engineers need knowledge about earthquake wavefields that are exciting ESSI systems. In other words, engineers, analysts need to be able to identify incident seismic wavefield in a domain adjacent, within a few hundred meters, to the ESSI system. By using such reconstructed, realistic seismic wavefields, engineers can assess the impact of earthquakes on ESSI systems. It is important to note that the reconstructed incident seismic wavefield will contain a realistic combination of body and surface waves with broad frequencies and complex wave incoherence. To reconstruct seismic wavefield in a given ESSI domain, engineers could perform the following types of analysis.

The deconvolution algorithm has been used for the identification of an incoming seismic wave signal into a one-dimensional (1D) soil column by using the transfer function of a soil column and surficial measurement data (Ju, 2013; Mejia and Dawson, 2006; Poul and Zerva, 2018a,b). Namely, Mejia and Dawson (2006) showed the procedure to back-calculate an incident wave signal that induces seismic wave responses in a 1D soil column by using the deconvolution method and the SHAKE code (Schnabel, 1972) as a 1D seismic wave solver. We note that, although the deconvolution has been widely used in geotechnical earthquake engineering, it is effective on an individual soil column only when soil stratification is horizontally uniform, and when 1 component (1C) incoming seismic waves propagate vertically. However, it is well known that seismic waves typically consist of (a) 3 component (3C) body waves, which consist of primary, compressional, P waves, and secondary, shear, S waves, and (b) 3C surface waves, which consist of Rayleigh waves, Love waves, and other types of irregular surface waves. Because incoming seismic waves, consisting of body and surface waves, are in general, highly incoherent (not vertically propagating), the deconvolution cannot reconstruct these 3C incoming waves.

In addition, one could identify seismic-source profiles at hypocenter in a regional-scale domain. For instance, Akcelik et al. (2002) presented a method to invert for a simplified seismic source time signal in a large 3D domain that includes a source at a fault. This method requires forward and inverse wave simulations in a very large domain (e.g., an area of 100 km-by-100 km with a depth of 10 km). Upon characterizing the seismic source via inversion in a low-frequency scale (e.g., $f < 1$ Hz), seismologists could compute the low-frequency wave motions propagated from the source to a near-surface, where the real interest is. However, it is infeasible to accurately characterize the material properties of such a large domain. Thus, the reconstructed wave responses from this approach could be highly inaccurate. Moreover, the computational cost of such a large-scale source-inversion problem is typically too large to identify the seismic source profile with higher frequency components. In other words, the discretization spacing of finite element or finite difference model in such a regional-scale source-inversion simulation cannot be sufficiently small to make it efficient and accurate. Thus, in a practical sense, such methodology cannot reconstruct seismic wavefield with enough accuracy and with a frequency range of interest.

As an alternative to the aforementioned conventional methods (i.e., the deconvolution in a 1D soil column and the source-characterization method in a large-scale domain), we present a new numerical method to identify arbitrary, incoherent incoming seismic waves in a multi-dimensional truncated domain by using sparse seismic measurement. To fill this algorithmic gap, we are investigating a new numerical method, where we could discretize the spatial and temporal distribution of an estimated incident wave and calculate the discretized parameters of the estimated wave that can minimize the discrepancy between the measured data at sensor locations and their computed counterparts induced by the estimated wave. Since the number of discretized parameters is quite large (e.g., millions), the PDE-constrained optimization-based inversion framework could be favorable in this problem because it allows semi-analytical calculation of a gradient of an objective functional with respect to a large set of control parameters.

The PDE-constrained optimization framework has advanced the inverse problems associated with elastodynamic wave motions, and the associated applications span from material and scatterer characterization to dynamic-input identification. For examples, the geotechnical site characterization methodologies have been investigated in a soil domain that is truncated by Perfectly-Matched-Layers (PML) Fathi et al. (2015b); Kang and Kallivokas (2010b); Kucukcoban and Kallivokas (2011, 2013); Mashayekh et al. (2018); Poul and Zerva (2018c) by using state-adjoint-control equation approaches Fathi et al. (2015a, 2016); Kallivokas et al. (2013); Kang and Kallivokas (2010a, 2011); Kucukcoban et al. (2019). The Gauss-Newton-based full-waveform inversion method, which is based on a gradient vector and a Hessian matrix, had also been studied for characterizing the material profiles in a truncated two-dimensional solid domain Pakravan et al. (2016); Tran and McVay (2012) for the application of the site characterization. Askan et al. (2007, 2010) and Askan and Bielak (2008) described anelastic full-waveform seismic inversion algorithms for estimating seismic velocity and attenuation properties of large heterogeneous basins from surface observations of ground motions. Seidl et al. (2020, 2019) used the PDE-constrained optimization to estimate both shear modulus and traction boundary condition distributions in a joint-inversion problem for the biomechanical imaging (BMI) application. Hamdaoui et al. (2019) utilized an adjoint method to identify the shear modulus and the loss factor of a viscoelastic sandwich structure. Lamert et al. (2018) investigated a method that utilizes elastodynamic full-waveform inversion to localize disturbances (e.g., concrete remnants) ahead of a tunnel. Reuber et al. (2020) presented an adjoint formulation that uses principal stress directions as data in the misfit function to identify rheological parameters (e.g., density, viscosity) of lithosphere and mantle. Wang and Brigham (2017) presented an inverse material characterization algorithm that combines direct inversion and optimization-based inversion to estimate the spatial distribution of elastic modulus in

Figure 1: The problem configuration, showing an exemplary domain with a homogeneous material of its wave speed V_{s1} and inclusions with their wave speeds V_{s2} and V_{s3} .

plate structures from incomplete response field measurements. [Kraft et al. \(2016\)](#) studied an adjoint state approach for the identification of vehicle parameters (e.g., stiffness of the suspension) from on-track measurements for calibration of railway vehicle models. [Lloyd and Jeong \(2018\)](#) has shown the adjoint-control equation approach to identify the profiles of moving wave sources of an arbitrary number in a heterogeneous 1D solid. The discretize-then-optimize (DTO) approach—deriving the adjoint and control equations in discrete forms based on a discrete objective functional—has also advanced the field of the identification of dynamic wave sources in solids [Walsh et al. \(2013\)](#). An inverse-scattering problem—the field of detecting complex geometries of wave-scattering objects [Guzina et al. \(2003\)](#); [Jeong et al. \(2009\)](#); [Nguyen-Tuan et al. \(2019\)](#)—has also been studied by the PDE-constrained optimization framework.

To investigate the feasibility to identify arbitrary, incoherent incoming seismic waves in a truncated domain by using sparse seismic measurement by using the PDE-constrained optimization framework, [Jeong and Seylabi \(2018\)](#) studied a full-waveform source-inversion method to identify an incoming seismic wave in a 1D semi-infinite solid. [Guidio and Jeong \(2020\)](#) also presented the mathematical modeling and numerical implementation of a new inverse-source method to identify the spatial and temporal distributions of targeted traction on the boundary of a 2D heterogeneous, bounded, undamped solid by using the sparse measurement of anti-plane wave motions without any prior information about targeted traction. The work showed that the DTO-based inversion leads to the same performance as the optimize-then-discretize (OTD) counterpart, but the former is more straightforward in terms of the mathematical modeling than the latter. The work also shows that using the Tikhonov (TN) regularization does not improve the accuracy of the inversion.

Continuing the aforementioned papers [Guidio and Jeong \(2020\)](#); [Jeong and Seylabi \(2018\)](#), the presented paper discusses a numerical method that can fully reconstruct the comprehensive profiles (i.e., both spatial and temporal distributions) of complex, incoherent seismic incident wavefields propagating into a 2D truncated domain of anti-plane shear wave motions.

2 Problem Definition

This study explores a new full-waveform inversion method for reconstructing the spatial and temporal distributions of dynamic traction in a 2D reduced domain from data of wave responses measured at sparsely-located sensors. Here, the semi-extent of the domain is truncated by using a wave-absorbing boundary condition (WABC). Sensors are located horizontally on the top surface and vertically at the right-hand side of a domain. The problem configuration is shown in Fig. 1.

2.1 The governing equation

The governing equation for the anti-plane shear wave propagation in an undamped solid domain is (for brevity, the spatial and/or temporal dependencies of variables are omitted):

$$\nabla \cdot (G \nabla u) - \rho \frac{\partial^2 u}{\partial t^2} = 0, \quad \text{on } \Omega \times J, \quad (1)$$

where $u = u(x, y, t)$ denotes the displacement field in the anti-plane (z) direction of the wave motion of a solid particle (i.e., SH wave motion); x, y , and t denote horizontal and vertical coordinates and time; $G(x, y)$ and $\rho(x, y)$ denote the shear modulus and the mass density of the solid; Ω denotes the domain, and $J = [0, T]$ is the time interval of interest. The solid is subject to a traction-free condition on the top surface (Γ_t):

$$\frac{\partial u}{\partial y}(x, 0, t) = 0, \quad 0 \leq x \leq L. \quad (2)$$

The absorbing boundary conditions are presented on the bottom (Γ_b), left (Γ_l) and right (Γ_r) boundaries, respectively as:

$$\frac{\partial u}{\partial y}(x, D, t) = \frac{f(x, D, t)}{G} - \frac{1}{V_s} \frac{\partial u}{\partial t}, \quad 0 \leq x \leq L, \quad (3)$$

$$\frac{\partial u}{\partial x}(0, y, t) = \frac{f(0, y, t)}{G} - \frac{1}{V_s} \frac{\partial u}{\partial t}, \quad D \leq y \leq 0, \quad (4)$$

$$\frac{\partial u}{\partial x}(L, y, t) = -\frac{1}{V_s} \frac{\partial u}{\partial t}, \quad D \leq y \leq 0, \quad (5)$$

where D is the y -coordinate of Γ_b ; L is the x -coordinate of Γ_r ; and $V_s(x, y)$ is the shear wave speed of the medium. We note that $f(x, y, t)$ denotes the dynamic traction of shear stress applied in the z -direction on the boundaries, and $f(x, y, t)$ is used for modeling the incoming seismic wavefield arriving at the WABC boundary¹:

$$f(x, y, t) = \frac{2G}{V_s} \frac{\partial u^l}{\partial t}, \quad (6)$$

where u^l denotes the incident seismic wavefield. We consider that u^l impinges the domain at Γ_l and Γ_r with an inclined angle. The zero initial-value conditions are presented as the system is initially at rest:

$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = 0. \quad (7)$$

3 Forward Wave Modeling

This section presents the finite element modeling for obtaining the numerical solutions of the governing equation (1).

3.1 Finite element method to solve the state problem

From the governing equation (1), the weak form of the state problem is cast in order to find $u(t) \in U$ for all $v \in \mathcal{V}$:

$$\begin{aligned} \int_{\Omega} \nabla v \cdot (G \nabla u) \, d\Omega + \int_{\Gamma_r} v \frac{G}{V_s} \frac{\partial u}{\partial t} \, d\Gamma + \int_{\Gamma_l} v \frac{G}{V_s} \frac{\partial u}{\partial t} \, d\Gamma + \int_{\Gamma_b} v \frac{G}{V_s} \frac{\partial u}{\partial t} \, d\Gamma \\ + \int_{\Omega} v \rho \frac{\partial^2 u}{\partial t^2} \, d\Omega = \int_{\Gamma_b} v f(x, y, t) \, d\Gamma + \int_{\Gamma_l} v f(x, y, t) \, d\Gamma, \end{aligned} \quad (8)$$

where v denotes a test function. The function spaces, U and $\mathcal{V}$, are defined as:

$$U = \{u : u \in H^1\}, \quad \mathcal{V} = \{v : v \in H^1\} \quad (9)$$

¹We note that Bielak's Domain Reduction Method (DRM) [Bielak et al. \(2003\)](#) can be used for modeling the wave responses in a truncated domain induced by an incident wavefield, impinging a WABC of a domain. In this work, we are testing our inverse algorithm in a truncated domain by using the boundary conditions (3-6) as an alternative to the DRM.

Next, the test and trial function are approximated by the standard finite-element approximation as follows:

$$u(x, y, t) \simeq \psi(x, y)^T \mathbf{u}(t), \quad v(x, y) \simeq \mathbf{v}^T \psi(x, y), \quad (10)$$

where $\psi(x, y)$ denotes a vector of global basis functions that is made of shape functions in the local coordinate of each element in the domain; $\mathbf{v}$ denotes a vector of nodal solution of the test function; and $\mathbf{u}(t)$ denotes the nodal solution vector of the state problem at time t . Then, the weak form (8) reduces to:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t), \quad (11)$$

where $(\dot{\cdot})$ and $(\ddot{\cdot})$ denotes, respectively, the first and second-order derivative of its subtended variable with respect to t ; $\mathbf{M}$ denotes a global mass matrix; $\mathbf{C}$ denotes a global dumping matrix; $\mathbf{K}$ denotes a global stiffness matrix; and $\mathbf{F}$ denotes a global force vector. The vectors and matrices are defined as:

$$\begin{aligned} \mathbf{K} &= \int_{\Omega} G \left(\frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial y} \right) d\Omega, \\ \mathbf{C} &= \int_{\Gamma_r} \frac{G}{V_s} \psi \psi^T d\Gamma + \int_{\Gamma_l} \frac{G}{V_s} \psi \psi^T d\Gamma + \int_{\Gamma_b} \frac{G}{V_s} \psi \psi^T d\Gamma, \\ \mathbf{M} &= \int_{\Omega} \rho \psi \psi^T d\Omega, \\ \mathbf{F} &= \int_{\Gamma_b} \psi f d\Gamma + \int_{\Gamma_l} \psi f d\Gamma. \end{aligned} \quad (12)$$

3.2 Time integration of the state problem

We solve the time-dependent discrete form (11) in every discrete j -th time step as:

$$\mathbf{M}\ddot{\mathbf{u}}_j + \mathbf{C}\dot{\mathbf{u}}_j + \mathbf{K}\mathbf{u}_j = \mathbf{F}_j. \quad (13)$$

Applying the initial value-conditions (7) onto (13), the solution vector at the initial time step is obtained by solving the following:

$$\mathbf{M}\ddot{\mathbf{u}}_0 = \mathbf{F}_0, \quad (14)$$

By using the implicit Newmark time integration, this work solves the system of equations for each time step. The solution vector of a next time step is related to that of a previous time step as:

$$\dot{\mathbf{u}}_j = \frac{2}{\Delta t} (\mathbf{u}_j - \mathbf{u}_{j-1}) - \dot{\mathbf{u}}_{j-1}, \quad (15)$$

and

$$\ddot{\mathbf{u}}_j = \frac{4}{(\Delta t)^2} (\mathbf{u}_j - \mathbf{u}_{j-1}) - \frac{4}{\Delta t} \dot{\mathbf{u}}_{j-1} - \ddot{\mathbf{u}}_{j-1}, \quad (16)$$

where Δt denotes the size of a time step. By plugging (15) and (16) into (13), it turns into the following:

$$\left[\frac{4}{(\Delta t)^2} \mathbf{M} + \frac{2}{\Delta t} \mathbf{C} + \mathbf{K} \right] \mathbf{u}_j = \mathbf{F}_j + \mathbf{M} \left(\frac{4}{(\Delta t)^2} \mathbf{u}_{j-1} + \frac{4}{\Delta t} \dot{\mathbf{u}}_{j-1} + \ddot{\mathbf{u}}_{j-1} \right) + \mathbf{C} \left(\frac{2}{\Delta t} \mathbf{u}_{j-1} + \dot{\mathbf{u}}_{j-1} \right). \quad (17)$$

By using (17), this work solves for $\mathbf{u}_j$. Then, the values of $\dot{\mathbf{u}}_j$ and $\ddot{\mathbf{u}}_j$ can be updated by using $\mathbf{u}_j$ and, respectively, (15) and (16).

3.3 Compact form of the state problem

Equations (14) to (17) can be cast in the following compact form:

$$\mathbf{Q}\hat{\mathbf{u}} = \hat{\mathbf{F}}, \quad (18)$$

where the vector $\hat{\mathbf{u}}$ corresponds to the space-time discretization of $u(x, y, t)$ for $(x, y) \in \Omega$ and $t \in [0, T]$, induced by $f(x, y, t)$; and $\hat{\mathbf{F}}$ is a global force vector corresponding to all the time steps. These vectors are built as:

$$\hat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_0 \\ \dot{\mathbf{u}}_0 \\ \ddot{\mathbf{u}}_0 \\ \vdots \\ \mathbf{u}_N \\ \dot{\mathbf{u}}_N \\ \ddot{\mathbf{u}}_N \end{bmatrix}, \quad \hat{\mathbf{F}} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{F}_0 \\ \vdots \\ \mathbf{F}_N \\ 0 \\ 0 \end{bmatrix}, \quad (19)$$

where the total number of time steps is defined as N . Finally, the matrix $\mathbf{Q}$ is the discrete forward operator defined as:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{I} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{K} & \mathbf{C} & \mathbf{M} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 & \mathbf{Keff} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 \mathbf{I} & \mathbf{I} & 0 & -a_1 \mathbf{I} & \mathbf{I} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ a_0 \mathbf{I} & a_2 \mathbf{I} & \mathbf{I} & -a_0 \mathbf{I} & 0 & \mathbf{I} & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 & \mathbf{Keff} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_1 \mathbf{I} & \mathbf{I} & 0 & -a_1 \mathbf{I} & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_0 \mathbf{I} & a_2 \mathbf{I} & \mathbf{I} & -a_0 \mathbf{I} & 0 & \mathbf{I} \end{bmatrix}, \quad (20)$$

where:

$$a_0 = \frac{4}{(\Delta t)^2}, \quad a_1 = \frac{2}{\Delta t}, \quad a_2 = \frac{4}{\Delta t},$$

$$\mathbf{Keff} = a_0 \mathbf{M} + a_1 \mathbf{C} + \mathbf{K}, \quad \mathbf{L}_1 = -a_0 \mathbf{M} - a_1 \mathbf{C}, \quad \mathbf{L}_2 = -a_2 \mathbf{M} - \mathbf{C}, \quad \mathbf{L}_3 = -\mathbf{M}. \quad (21)$$

In (20), the 1st and 2nd rows correspond to the initial conditions $\mathbf{u}_0 = 0$ and $\dot{\mathbf{u}}_0 = 0$; the 3rd row is equivalent to (14); the 4, 7, 10, ...-th rows correspond to (17); the 5, 8, 11, ...-th rows are (15); and the 6, 9, 12, ...-th rows are (16). Instead of the time marching scheme, by solving the matrix system of (18) "all at once", we could obtain the forward wave solutions. However, to save the computational cost, we opt for using the time marching scheme for solving (18).

4 Inverse Modeling

This section presents mathematical modeling for identifying the temporal and spatial distributions of unknown dynamic traction $f(x, y, t)$ —mimicking unknown incoming seismic wavefield per (6)—based on measured wave responses while the properties of the system, such as the extents and the material properties of the solid, are assumed to be known. Specifically, the components of global force vectors corresponding to targeted traction will be identified.

To this end, the presented work studies the discretize-then-optimize (DTO) approach because, in contrast to the OTD counterpart, the DTO method involves quite a compact mathematical modeling procedure to side-impose the governing wave equation to an objective functional and to derive the derivatives of a Lagrangian functional with respect to adjoint and control variables. Namely, an objective functional is formulated in a discrete form, and, then, the discrete optimality conditions are derived and implemented.

4.1 Control parameters

Under this inversion method, we identify $P_{k,j}$ —a component of $\hat{\mathbf{F}}$ corresponding to γ_k and t_j —as control parameters, where γ_k is the k -th discrete node on the boundaries Γ_l and Γ_b , and t_j is the j -th time step. The discrete numbering of γ_k increases counter-clockwise from the top-left corner of the domain. Fig. 2 shows the discrete numbering k in $P_{k,j}$ with a reduced number of discrete nodes. We note that the components of $\hat{\mathbf{F}}$ that do not correspond to discrete nodes on Γ_l and Γ_b vanish, and they will not be inverted for.

Figure 2: Discrete numbering k in P_{kj} corresponding the discrete node γ_k on the boundaries Γ_l and Γ_b .

4.2 Discrete Lagrangian functional

As a minimization problem, we seek the values of the control parameters (i.e., P_{kj} for all k and j) that correspond to a minimum of a discrete objective functional, which is defined as:

$$\hat{\mathcal{L}} = \frac{1}{2}(\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T \bar{\mathbf{B}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m), \quad (22)$$

where $\hat{\mathbf{u}}$ corresponds to the space-time discretization of a forward simulation variable induced by a set of estimated P_{kj} ; $\hat{\mathbf{u}}_m$ is the space-time discretization of $u_m(x, y, t)$ induced by a set of P_{kj} of to targeted force vector. In this computational study, we synthetically create the measured displacement field, $u_m(x, y, t)$, by using our FEM solver with a pseudo target force vector. In (22), $\bar{\mathbf{B}}$ is defined as $\Delta t \mathbf{B}$, where $\mathbf{B}$ is a square matrix, of which components are all 0 except for those of the diagonal, having values of all 1, if they correspond to the degrees of freedom of sensor locations. In this work, the measured data is obtained from sparsely-distributed sensors.

To tackle the minimization problem, the following Lagrangian functional $\hat{\mathcal{A}}$ is built by imposing the discrete form of the forward wave problem, using a Lagrange multiplier vector $\hat{\boldsymbol{\lambda}}$, onto $\hat{\mathcal{L}}$:

$$\hat{\mathcal{A}} = \frac{1}{2}(\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T \bar{\mathbf{B}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m) - \hat{\boldsymbol{\lambda}}^T (\mathbf{Q}\hat{\mathbf{u}} - \hat{\mathbf{F}}), \quad (23)$$

where $\hat{\boldsymbol{\lambda}}$ is the discrete (space-time) Lagrange multiplier:

$$\hat{\boldsymbol{\lambda}} = [\boldsymbol{\lambda}_0^T, \dot{\boldsymbol{\lambda}}_0^T, \ddot{\boldsymbol{\lambda}}_0^T, \dots, \boldsymbol{\lambda}_N^T, \dot{\boldsymbol{\lambda}}_N^T, \ddot{\boldsymbol{\lambda}}_N^T]^T. \quad (24)$$

We show that the objective functional (22) is quadratic and convex (please see the Appendix A) so that the first-order optimality conditions of the Lagrangian functional (23) could be sufficient for obtaining an inversion solution. Thus, we do not need to include any regularization in the presented work: our previous work Guidio and Jeong (2020), considering a 2D bounded domain, also showed that the Tikhonov (TN) regularization does not improve the performance of inverting dynamic traction on a boundary of 2D bounded solid domain. However, as shown in the previous work, because of the sparsity of the measurement data, the inversion solution may not be unique in the presented work unless a sufficiently large number of sensors measure wave motions: We discuss this aspect in Example 2 in the presented work.

4.3 The first-order optimality conditions

To identify unknown, target control parameters, the following optimality conditions should be satisfied:

$$\frac{\partial \hat{\mathcal{A}}}{\partial \hat{\boldsymbol{\lambda}}} = 0 \quad \text{The first condition (state problem),} \quad (25)$$

$$\frac{\partial \hat{\mathcal{A}}}{\partial \hat{\mathbf{u}}} = 0 \quad \text{The second condition (adjoint problem),} \quad (26)$$

$$\frac{\partial \hat{\mathcal{A}}}{\partial \hat{\mathbf{F}}} = 0 \quad \text{The third condition (control problem).} \quad (27)$$

The first condition, (25), the vanishing derivative of $\hat{\mathcal{A}}$ with respect to $\hat{\lambda}$, will be automatically satisfied when we solve the discrete forward problem (18), which is also referred to as the discrete state equation:

$$\frac{\partial \hat{\mathcal{A}}}{\partial \hat{\lambda}} = -\mathbf{Q}\hat{\mathbf{u}} + \hat{\mathbf{F}} = 0. \quad (28)$$

As the second condition, (26), the derivate of $\hat{\mathcal{A}}$ with respect to $\hat{\mathbf{u}}$ should vanish, and it will be automatically satisfied when we solve the following discrete adjoint problem:

$$\frac{\partial \hat{\mathcal{A}}}{\partial \hat{\mathbf{u}}} = \underbrace{\frac{\partial \hat{\mathcal{L}}}{\partial \hat{\mathbf{u}}}}_a - \underbrace{\frac{\partial (\hat{\lambda}^T \mathbf{Q}\hat{\mathbf{u}})}{\partial \hat{\mathbf{u}}}}_b = 0. \quad (29)$$

The part a in (29) can be written as (see the derivation in Appendix B):

$$a = \bar{\mathbf{B}}(\hat{\mathbf{u}} - \hat{\mathbf{u}}_m). \quad (30)$$

The part b in (29) can be written as:

$$\begin{aligned} b &= -\frac{\partial}{\partial \hat{\mathbf{u}}} \underbrace{(\hat{\lambda}^T \mathbf{Q}\hat{\mathbf{u}})}_{\text{scalar}} = -\frac{\partial}{\partial \hat{\mathbf{u}}} (\hat{\lambda}^T \mathbf{Q}\hat{\mathbf{u}})^T \\ &= -\frac{\partial}{\partial \hat{\mathbf{u}}} ((\mathbf{Q}\hat{\mathbf{u}})^T \hat{\lambda}) = -\frac{\partial}{\partial \hat{\mathbf{u}}} (\hat{\mathbf{u}}^T \mathbf{Q}^T \hat{\lambda}) \\ &= -\mathbf{Q}^T \hat{\lambda}. \end{aligned} \quad (31)$$

Due to (30) and (31), (29) can be written as:

$$\frac{\partial \hat{\mathcal{A}}}{\partial \hat{\mathbf{u}}} = -\underbrace{\mathbf{Q}^T \hat{\lambda} + \bar{\mathbf{B}}(\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)}_{\text{adjoint equation}} = 0. \quad (32)$$

Because (32) includes transpose of $\mathbf{Q}$, we solve it by marching backwards in time. Namely, from the last two rows of $\mathbf{Q}^T$, we obtain the final conditions:

$$\ddot{\lambda}_N = 0, \quad \dot{\lambda}_N = 0, \quad (33)$$

respectively; and the third row from the bottom of $\mathbf{Q}^T$ results in the following equation:

$$\mathbf{Keff}^T \lambda_N = \Delta t \mathbf{B}(\mathbf{u}_N - \mathbf{u}_{mN}) + a_1 \dot{\lambda}_N + a_0 \ddot{\lambda}_N, \quad (34)$$

which can be solved for λ_N . For time steps $j = N-1, N-2, \dots, 1$, we first update the followings:

$$\ddot{\lambda}_j = \mathbf{M}^T \lambda_{j+1} - \ddot{\lambda}_{j+1}, \quad (35)$$

$$\dot{\lambda}_j = (a_2 \mathbf{M}^T + \mathbf{C}^T) \lambda_{j+1} - \dot{\lambda}_{j+1} - a_2 \ddot{\lambda}_{j+1}. \quad (36)$$

Then, solve the following:

$$\mathbf{Keff}^T \lambda_j = \Delta t \mathbf{B}(\mathbf{u}_j - \mathbf{u}_{mj}) + a_1 \dot{\lambda}_j + a_0 \ddot{\lambda}_j + (a_0 \mathbf{M}^T + a_1 \mathbf{C}^T) \lambda_{j+1} - a_1 \dot{\lambda}_{j+1} - a_0 \ddot{\lambda}_{j+1}. \quad (37)$$

Eventually, the first three rows of $\mathbf{Q}^T$ lead to the followings:

$$\mathbf{M}^T \ddot{\lambda}_0 = \mathbf{M}^T \lambda_1 - \ddot{\lambda}_1, \quad (38)$$

$$\dot{\lambda}_0 = -\mathbf{C}^T \ddot{\lambda}_0 + (a_2 \mathbf{M}^T + \mathbf{C}^T) \lambda_1 - \dot{\lambda}_1 - a_2 \ddot{\lambda}_1, \quad (39)$$

$$\lambda_0 = -\mathbf{K}^T \ddot{\lambda}_0 + (a_0 \mathbf{M}^T + a_1 \mathbf{C}^T) \lambda_1 - a_1 \dot{\lambda}_1 - a_0 \ddot{\lambda}_1 + \Delta t \mathbf{B}(\mathbf{u}_0 - \mathbf{u}_{m0}). \quad (40)$$

As the third condition, (27), the vanishing derivative of $\hat{\mathcal{A}}$ with respect to $\hat{\mathbf{F}}$ is tantamount to the following control equation:

$$\frac{\partial \hat{\mathcal{A}}}{\partial \hat{\mathbf{F}}} = \hat{\lambda} = 0, \quad (41)$$

which implies that the inversion solution of $\hat{\mathbf{F}}$ should lead to the vanishing $\hat{\lambda}$. In other words, if the inversion solution of $\hat{\mathbf{F}}$ matches its target, $\hat{\mathbf{u}}$ will match $\hat{\mathbf{u}}_m$ at sensor locations. Thus, the driving force term, $\bar{\mathbf{B}}(\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)$ in (32) vanishes so that $\hat{\lambda}$ will vanish per (32). Eq. (41) also implies that $\frac{\partial \hat{\mathcal{A}}}{\partial(\xi=P_{kj})} = \frac{\partial \hat{\mathcal{L}}}{\partial(\xi=P_{kj})}$ is the component of the vector $\hat{\lambda}$ corresponding to the global node numbering and the time step of $\xi = P_{kj}$. In other words, (41) becomes:

$$\nabla_{(\xi=P_{kj})}\hat{\mathcal{A}} = \nabla_{(\xi=P_{kj})}\hat{\mathcal{L}} = \begin{cases} \ddot{\lambda}(\gamma_k, t_j) & \text{when } j = 1, \\ \lambda(\gamma_k, t_j) & \text{when } j \geq 2, \end{cases} \quad (42)$$

where $\ddot{\lambda}(\gamma_k, t_j)$ or $\lambda(\gamma_k, t_j)$ denotes the component of $\hat{\lambda}$ corresponding to the degree of freedom and the timestep of P_{kj} . Eq. (42) implies that this work uses the solution of the adjoint problem for evaluating the gradient of $\hat{\mathcal{L}}$ with respect to ξ in (42) in a semi-analytical manner. In addition, it also implies that searching a set of the control parameters, ξ , that leads to the vanishing $\hat{\lambda}$ is equivalent to searching ξ leading to the zero-valued $\nabla_{\xi}\hat{\mathcal{L}}$. This process is equivalent to the gradient-based minimization. The gradient-based minimization process, shown in the Algorithm 1.

5 Numerical Implementation of the Inversion Process

By utilizing the semi-analytically evaluated, reduced² gradient vector $\nabla_{\xi}\hat{\mathcal{L}}$, this work iteratively updates a set of estimated control parameters by using the gradient-based minimization scheme as follows:

- First, we use pseudo-traction $f(x, y, t)$, which corresponds to an inclined incident wave, to create a pseudo targeted force vector $\hat{\mathbf{F}}$ and generate synthetic measured data $\hat{\mathbf{u}}_m$ at sensors.
- Then, $\hat{\mathbf{u}}$ is obtained by using estimated $\hat{\mathbf{F}}$ that is constructed by estimated control parameters ξ (i.e., P_{kj}).
- Next, the adjoint problem is solved (i.e., computing $\hat{\lambda}$) by using $\hat{\mathbf{u}}$ that is calculated in the previous step.
- By using the adjoint solution, we compute the gradient of the objective functional, $\nabla_{\xi}\hat{\mathcal{L}}$.
- The conjugate-gradient method [Guidio and Jeong \(2020\)](#) determines the best search direction, and an optimal step length is calculated by Newton's method [Guidio and Jeong \(2020\)](#).
- Lastly, the gradient-based minimization scheme updates the approximate control parameters ξ using the search direction and optimal step length.

The numerical optimizer repeats the steps (b) to (f) above, and a set of these steps is counted as an inversion iteration. Throughout inversion iterations, our numerical optimizer iteratively calculate the values of estimated control parameters that lead to the vanishing gradient of the objective functional. The comprehensive method for the numerical optimizer is outlined in the Algorithm 1.

5.1 Updating the estimated control parameters

Starting from the initially-guessed (zero-valued) control parameter vector ξ consisting of P_{kj} , the estimation of ξ can be modified iteratively as follows:

$$\xi_{(s+1)} = \xi_{(s)} + h_{(s)}\mathbf{d}_{(s)}, \quad (43)$$

where $h_{(s)}$ is a scalar-valued step size and $\mathbf{d}_{(s)}$ is a search-direction vector computed by the conjugate-gradient scheme [Guidio and Jeong \(2020\)](#), and s is the iteration index in the optimization process. For each iteration, s , the Newton's method [Guidio and Jeong \(2020\)](#); [Lloyd and Jeong \(2018\)](#) is used to determine the optimal step size, $h_{(s)}$.

6 Numerical Experiments

In this section, we investigate the performance of the presented inversion method in a domain truncated by using a WABC in consideration of various factors.

²The full gradient is $\left[\frac{\partial \hat{\mathcal{L}}}{\partial \lambda} (= 0); \frac{\partial \hat{\mathcal{L}}}{\partial \mathbf{u}} (= 0); \frac{\partial \hat{\mathcal{L}}}{\partial \xi} \right]$.

Algorithm 1 Inversion Algorithm

- 1: Set $TOL=10^{-7}$ and maximum iteration number = 1000.
  - 2: Build $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ matrices.
  - 3: Create synthetic measurement data $\hat{\mathbf{u}}_m$ from a forward wave solver using a targeted force vector $\hat{\mathbf{F}}$ corresponding to targeted traction $f(x, y, t)$.
  - 4: Set an iteration index $s = 1$ and initial control parameters ($\xi_{(s=1)} = 0$) and compute $\hat{\mathcal{L}}_{(s=1)}$.
  - 5: **while** ($\hat{\mathcal{L}}_{(s+1)} > TOL \times \hat{\mathcal{L}}_{(1)}$ and $s < (\text{maximum iteration number})$) **do**
  - 6:   Solve the discrete forward problem using $\xi_{(s)}$ and compute $\hat{\mathbf{u}}$.
  - 7:   Obtain $\hat{\lambda}$ by solving the discrete adjoint problem.
  - 8:   Compute the components of a gradient vector, $\mathbf{g}_{(s)} = \nabla_{\xi} \hat{\mathcal{L}}_{(s)}$.
  - 9:   Compute an optimal search direction $\mathbf{d}_{(s)}$ by using the conjugate-gradient scheme.
  - 10:   Compute an optimal step length $h_{(s)}$ by using Newton's method.
  - 11:   Update estimated control parameters as $\xi_{(s+1)} = \xi_{(s)} + \mathbf{d}_{(s)} h_{(s)}$ and compute $\hat{\mathcal{L}}_{(s+1)}$.
  - 12: $s \leftarrow s + 1$.
  - 13:   **if** ($s == (\text{maximum iteration number})$ and $\frac{\hat{\mathcal{L}}_{(s-10)} - \hat{\mathcal{L}}_{(s)}}{\hat{\mathcal{L}}_{(s-10)}} \times 100 \geq 1\%$) **then**
  - 14:   maximum iteration number = maximum iteration number + 1000.
  - 15:   **end if**.
  - 16: **end while**.
-

Figure 3: Material profile 2: a 2-layered solid with two inclusions.

The considered domain size is set to be $200 \text{ m} \times 60 \text{ m}$, and the mass density ρ is uniform as 3000 kg/m^3 in the domain. Two material profiles, which differ from each other in terms of wave speeds, are considered in the presented numerical experiments. The material profile 1 is of a homogeneous background solid domain with two inclusions as in Fig. 1 with shear wave speeds of $V_{s1} = 353.55 \text{ m/s}$, $V_{s2} = 800 \text{ m/s}$, and $V_{s3} = 1000 \text{ m/s}$. The material profile 2 is a 2-layered background solid with the same two inclusions, as shown in Fig. 3, with shear wave speeds of $V_{s1} = 353.55 \text{ m/s}$, $V_{s2} = 223.6 \text{ m/s}$, $V_{s3} = 800 \text{ m/s}$, and $V_{s4} = 1000 \text{ m/s}$.

The synthetic measured data $\hat{\mathbf{u}}_m$ is computed by using an element size of 0.5 m for a targeted force vector. To avoid an inverse crime³, we use an element size of 1 m for evaluating state solution $\hat{\mathbf{u}}$ and the adjoint solution $\hat{\lambda}$ for an estimated force vector. Δt of 0.001 s is used for computing all $\hat{\mathbf{u}}_m$, $\hat{\mathbf{u}}$, and $\hat{\lambda}$.

For a homogeneous background domain (i.e., the material profile 1), four targeted force vectors $\hat{\mathbf{F}}^{1,2,3,4}$ (corresponding to four different targeted dynamic tractions) are considered in the presented numerical experiments. The dynamic tractions are applied on the left and bottom boundaries of a homogeneous background domain, and they mimic incident plane wavefield propagating with an incident angle 45° . The time-dependent values of targeted $\hat{\mathbf{F}}^{1,2,3}$ corresponding to a node at a specific location $x = 100 \text{ m}$ and $y = -60 \text{ m}$ —Ricker wavelets with different frequency contents—are shown in Fig. 4(a-f). The signals corresponding to $\hat{\mathbf{F}}^{1,2,3}$ have the central frequency of 5, 10, 15 Hz, respectively. The total observation duration is 1.5 s in the inversion simulation of identifying targeted $\hat{\mathbf{F}}^{1,2,3}$. On the other hand, the time-dependent signal corresponding to $\hat{\mathbf{F}}^4$ at the same location and its frequency content are shown in Fig. 4(g,h). This signal is a portion of a registered ground motion signal during the 1994 Northridge earthquake from the Pacific Earthquake Engineering Research Center (PEER) ground motion database [PEER \(2000\)](#). The total observation duration is 6 s for the inversion simulation to identify the targeted $\hat{\mathbf{F}}^4$.

On the other hand, $\hat{\mathbf{F}}^5$ is a force vector due to dynamic traction on the left and bottom boundaries of a two-layered background domain (i.e., the material profile 2), and this signal mimics incident plane wavefield of an incident angle 45° in the bottom layer. The signal corresponding to $\hat{\mathbf{F}}^5$ at $x = 100 \text{ m}$ and $y = -60 \text{ m}$ is a Ricker wavelet with a central frequency of 5 Hz.

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³We should replicate a situation where the wave solver for computing $\hat{\mathbf{u}}$ is only approximating its true physics that governs $\hat{\mathbf{u}}_m$.

Figure 4: (a,c,e,g) The time signals of $\hat{\mathbf{F}}^{1,2,3,4}$ corresponding to a node at $x = 100$ m and $y = -60$ m, respectively; (b,d,f,h) the amplitude of Fourier Transform of each signal.

Table 1: Summary of all cases.

Case number	Material profile	Target force profile	Vertical array	# of sensors	Noise level	Terminal error $\mathcal{E}$	Error $\mathcal{E}^u$
1	1	$\hat{\mathbf{F}}^1$	N	50	0%	7.60%	0.89%
2	1	$\hat{\mathbf{F}}^1$	Y	65	0%	1.45%	0.16%
3	1	$\hat{\mathbf{F}}^1$	Y	32	0%	1.60%	0.18%
4	1	$\hat{\mathbf{F}}^1$	Y	16	0%	18.61%	8.84%
5	1	$\hat{\mathbf{F}}^2$	Y	65	0%	2.82%	0.62%
6	1	$\hat{\mathbf{F}}^3$	Y	65	0%	10.90%	3.45%
7	1	$\hat{\mathbf{F}}^4$	Y	65	0%	0.33%	0.00%
8	2	$\hat{\mathbf{F}}^5$	Y	65	0%	1.56%	0.16%
9	2	$\hat{\mathbf{F}}^5$	Y	65	1%	2.88%	0.18%
10	2	$\hat{\mathbf{F}}^5$	Y	65	2%	7.10%	0.20%
11	2	$\hat{\mathbf{F}}^5$	Y	65	5%	41.02%	0.40%
12	2	$\hat{\mathbf{F}}^5$	Y	65	10%	112.40%	0.86%

For the inversion process, the temporal and spatial intervals for the discretization are 0.001 s and 1 m, respectively, for computing wave responses induced by all the estimated $\hat{\mathbf{F}}$ in all the cases. Cases 1-6 and 8-11, which consider $\hat{\mathbf{F}}^{1,2,3,5}$, use 391,791 control parameters (i.e., 261 (over space) $\times$ 1501 (over time)), and Case 7, which consider $\hat{\mathbf{F}}^4$, uses 1,566,261 control parameters (i.e., 261 (over space) $\times$ 6001 (over time)). The control parameters are set as zero at the initial inversion iteration.

In what follows, six examples of numerical investigations are presented:

- The first example is aimed at comparing the performance of inverting P_{kj} , corresponding to $\hat{\mathbf{F}}^1$, between Case 1—utilizing only the sensors on the top surface (Γ_t) of the domain—versus Case 2—using a vertical array of sensors on the right boundary (Γ_r) with those on Γ_t .
- The second example tests the inversion performance of identifying $\hat{\mathbf{F}}^1$ with respect to the number of sensors on Γ_r and Γ_t in Cases 2 to 4.
- The third example is focused on analyzing the performance of inverse modeling to reconstruct targeted force vectors, of which time-dependent signals contain different central frequencies: Cases 2, 5, and 6 use $\hat{\mathbf{F}}^1$, $\hat{\mathbf{F}}^2$, and $\hat{\mathbf{F}}^3$, respectively, corresponding to tractions with central frequencies of 5, 10, and 20 Hz.
- The fourth example investigates the inversion performance of identifying $\hat{\mathbf{F}}^4$, which corresponds to a realistic seismic signal, in Case 7. Namely, the performance in Case 7 will be compared with that in Case 2.
- The fifth example examines the convergence of the estimated control parameters into targeted $\hat{\mathbf{F}}^5$ by using the material profile 2 (the 2-layered background solid with two inclusions) in Case 8.
- Lastly, the sixth example studies the effect of noise level in measurement data on the inversion performance of identifying $\hat{\mathbf{F}}^5$. In addition to Case 8, where a noise level of 0% is used, Case 9-12 are considered and correspond to noise levels of 1%, 2%, 5%, and 10%, respectively.

The summary of all the cases presented in this study is shown in Table 1.

For appraising the effectiveness to reconstruct $\hat{\mathbf{F}}$ in the numerical results, the following error norm between estimated force vector and its target is used:

$$\mathcal{E} = \frac{|\hat{\mathbf{F}}_{\text{target}} - \hat{\mathbf{F}}_{\text{estimate}}|^2}{|\hat{\mathbf{F}}_{\text{target}}|^2} \times 100[\%], \quad (44)$$

where we compare the components of $\hat{\mathbf{F}}_{\text{target}}$ and $\hat{\mathbf{F}}_{\text{estimate}}$ corresponding to the nodes on the boundaries where tractions are applied.⁴ For the sake of assessing the accuracy to reconstruct the wave responses

⁴More specifically, they are compared with each other at the nodes of the domain that is discretized for simulating the wave responses for $\hat{\mathbf{F}}_{\text{estimate}}$. Since a twice smaller element size is used for computing the wave responses for $\hat{\mathbf{F}}_{\text{target}}$, the components of $\hat{\mathbf{F}}_{\text{target}}$ at those nodes are multiplied twice when they are compared with $\hat{\mathbf{F}}_{\text{estimate}}$ in (44).

in the entire computational domain, the numerical results show the following error norm between the wave response solution vector at the j -th time step, $\mathbf{u}_j$, induced by final $\hat{\mathbf{F}}_{\text{estimate}}$ and its counterpart, $\mathbf{u}_{mj}$, induced by $\hat{\mathbf{F}}_{\text{target}}$ for all the time steps⁵:

$$\mathcal{E}^u = \sum_{j=1}^N \frac{|\mathbf{u}_{mj} - \mathbf{u}_j|^2}{|\mathbf{u}_{mj}|^2} \times 100[\%]. \quad (45)$$

6.1 The exemplary forward wave responses

Fig. 5(a-d) show the wave responses in the whole computational domain induced by targeted $\hat{\mathbf{F}}^1$ in Case 1 at $t = 0.3$ s, 0.4 s, 0.5 s, and 0.6 s, respectively. Fig. 6 presents the wave responses caused by targeted $\hat{\mathbf{F}}^5$ in Case 8, in which material profile 2 is employed, at the same time steps.

The plots at $t = 0.3$ s in Fig. 5(a) and Fig. 6(a) show that the dynamic input traction at the WABC mimics seismic incident, inclined plane waves. Fig. 6(a,b) shows the change in the incident angle of the wave caused by the difference of the shear wave speeds in the two-layered medium. This behavior complies with Snell's Law.

Fig. 5(b) and Fig. 6(b) display the waves passing through the first inclusion, which is located at 60 m from the left boundary. Fig. 5(c) and Fig. 6(c) show the wave motions passing through the second inclusion, which is 120 m from the left boundary. Finally, Fig. 5(d) and Fig. 6(d) exhibit the waves arriving at the right boundary.

Fig. 7 shows the comparison between the components of targeted $\hat{\mathbf{F}}^1$ and those of a vector $\frac{2G}{V} \frac{\partial \mathbf{u}}{\partial t}$ —per (6)—on Γ_l , Γ_b , and Γ_r in a homogeneous domain induced by the same $\hat{\mathbf{F}}^1$. As shown in Fig. 7(b), they are in excellent agreement with each other except for those that are attributed to the waves reflected from Γ_t , in turn, arriving at Γ_l and Γ_b . Therefore, our underlying rationale of (6) is justified.

6.2 The verification of the inverse modeling

The theoretical derivation of the adjoint and control equations and the numerical implementation are verified prior to the parametric studies. The gradient obtained by our semi-analytical DTO approach is compared with that from the finite difference (FD) procedure, which uses the following equation:

$$\frac{\partial \hat{\mathcal{L}}}{\partial \xi_{\text{FD}}} = \frac{\hat{\mathcal{L}}(\xi + h) - \hat{\mathcal{L}}(\xi)}{h}, \quad (46)$$

where ξ indicates a control parameter. In this verification, we used the targeted $\hat{\mathbf{F}}^1$, material profile 1, and only one sensor in the center of Γ_t . Fig. 8 reveals an outstanding agreement between the normalized gradient (i.e., $\nabla_{\xi} \hat{\mathcal{L}} / |\nabla_{\xi} \hat{\mathcal{L}}|$) that is calculated by using the presented DTO approach and that by the FD approach. Both of them are computed at the first inversion iteration for control parameters corresponding to a node at $x = 100$ m and $y = -60$ m. Therefore, the theoretical derivation and mathematical implementation are trustworthy.

6.3 Example 1: Investigating the inversion performance with or without a vertical array of sensors

In this example, we compare the performance of the presented inverse modeling of identifying a targeted set of P_{kj} of $\hat{\mathbf{F}}^1$ in Case 1, where we use the sensors only on the top surface Γ_t , with that in Case 2, where we add a vertical (e.g., at borehole) array of sensors on the boundary on the right side Γ_r . We used the material profile 1 and 50 sensors on Γ_t in both Cases. In Case 2, a vertical array of 15 sensors are added on Γ_r .

Fig. 9 shows the targeted set of P_{kj} and its reconstructed counterpart obtained by using sensors on only Γ_t versus both Γ_t and Γ_r , in Case 1 and 2, respectively. When Case 1 is tested, as shown in Fig. 9(b), the targeted control parameters on the bottom boundary Γ_b (especially, those at the right-bottom portion of Γ_b) cannot be fully recovered. However, by adding the vertical array of sensors on Γ_r , they are fully reconstructed (Fig. 9(c)).

⁵Again, the wave responses $\mathbf{u}_j$ and $\mathbf{u}_{mj}$ are compared with each other at the nodes of the domain that is discretized for simulating the wave responses for $\hat{\mathbf{F}}_{\text{estimate}}$.

Figure 5: (a-d) The wave motions of $u(x, y, t)$ [m], at $t = 0.3$ s, 0.4 s, 0.5 s, and 0.6 s, respectively, in Case 1.

Figure 6: (a-d) The wave motions of $u(x, y, t)$ [m], at $t = 0.3$ s, 0.4 s, 0.5 s, and 0.6 s, respectively, in Case 8.

Figure 7: (a) A set of P_{kj} corresponding to targeted traction and (b) $\frac{2G}{V} \frac{\partial u}{\partial t}$ of the wave responses induced by the targeted traction in N/m^2 in Case 1.

Figure 8: Comparison between the gradients generated by the presented inverse modeling and the FD approximation.

Figure 9: Example 1: (a) A set of P_{kj} corresponding to targeted traction and (b,c) The sets of reconstructed P_{kj} for Case 1 and 2 at the 1000th iteration. The dominant frequency of 5 Hz is used. The horizontal axis in the contour plot represents the nodes on the absorbing boundaries of the domain that is discretized for simulating the wave responses for $\hat{\mathbf{F}}_{\text{estimate}}$. Moreover, since the element size used to calculate the wave responses for $\hat{\mathbf{F}}_{\text{target}}$ is twice smaller, the targeted force components are twice multiplied at those nodes. We repeat this in the other contour plots where the targeted set of P_{kj} and its reconstructed counterpart are compared.

Figure 10: Example 1 - After 1000 iterations, the final value of $\mathcal{E}$ is 7.60% in Case 1, and that in Case 2 is 1.45%, which shows a reduction of 6.15% in $\mathcal{E}$.

Fig. 10 shows that, by using the vertical array of sensors in Case 2, the terminal value of $\mathcal{E}$ of the inversion becomes much smaller than that in Case 1. After 1000 iterations, the final value of $\mathcal{E}$ is 7.60% in Case 1, but that in Case 2 is 1.45%, which shows a reduction of 6.15% in $\mathcal{E}$. Thus, we suggest that using not only the sensors on the top surface but also a vertical array of sensors improves the inversion performance in the presented problem setting compared to the case using only the sensors on the top surface.

As shown in Table 1, even though the terminal error $\mathcal{E}$ is 7.60% in Case 1, the error $\mathcal{E}^u$ between the wave responses induced by targeted and final estimated force vector is only 0.89% in Case 1. It is because the wave induced by a particular set of P_{kj} at the right bottom corner passes Γ_r immediately after its onset.

6.4 Example 2: Investigating the inversion performance with respect to the number of sensors

In this example, we test the performance of identifying a targeted set of P_{kj} of $\hat{\mathbf{F}}^1$ with respect to the number of sensors on Γ_t and Γ_r . We considered Cases 2, 3, and 4, which use 65 sensors (50 on Γ_t and 15 on Γ_r), 32 sensors (25 on Γ_t and 7 on Γ_r), and 16 sensors (12 on Γ_t and 4 on Γ_r), respectively, and material profile 1.

Fig. 11 shows the targeted set of P_{kj} corresponding to $\hat{\mathbf{F}}^1$ and its counterparts obtained in Cases 2, 3, and 4. Fig. 11 shows that when Cases 2 and 3 are tested (64 and 32 sensors are used), the inversion algorithm can accurately reconstruct the targeted control parameters. However, in Case 4, where 16 sensors are used, the control parameters cannot be accurately recovered.

As shown in Fig. 12, while the terminal values of the errors $\mathcal{E}$ in Case 2 and 3, where 65 and 32 are used, are 1.45% and 1.6%, the error $\mathcal{E}$ is increased to 18.61% in Case 4, where 16 sensors are utilized.

Table 1 shows that the error $\mathcal{E}^u$ between the wave responses induced by the target and its final-estimated counterpart also grows as the number of sensors is decreased. For instance, the error $\mathcal{E}^u$ grows from 0.16% in Case 2 to 8.84% in Case 4. Fig. 13 shows the wave responses' snapshots in the entire computational domain obtained in Case 4 due to targeted and reconstructed control parameters. Although the two wave responses look overall alike in Fig. 13, the value of $\mathcal{E}^u$ in Case 4 is the highest among all the cases 1 to 12.

Thus, we suggest that the desirable ratio of the size of measurement data to the number of control parameters can be as small as 1:8—e.g., (32 sensors $\times$ 1500 timesteps):(261 nodes $\times$ 1500 timesteps)—in the presented example.

Figure 11: Example 2: (a) The set of P_{kj} corresponding to targeted traction and (b,c,d) the sets of reconstructed P_{kj} for Case 2, 3, and 4, respectively, at the 1000th iteration.

Figure 12: Example 2 - After 1000 iterations, the final value of $\mathcal{E}$ is 1.45% in Case 2, 1.60% in Case 3, and 18.61% in Case 4.

Figure 13: Example 2 - Case 4: Wave responses, $u(x, y, t)[\text{m}]$, in the domain induced by (left) the targeted set of P_{kj} and (right) its reconstructed counterpart at (a-b) 0.3 s, (c-d) 0.4 s, (e-f) 0.5 s, and (g-h) 0.6 s.

Figure 14: Example 3 - The terminal value of $\mathcal{E}$, 1.45%, for reconstructing P_{kj} in Case 2 is smaller than its counterparts, 2.82% in Case 5 and 10.90% in Case 6.

6.5 Example 3: Investigating the inversion performance with respect to the dominant frequency of the time signal of targeted traction

This example studies the accuracy of the inversion with respect to the dominant frequency of the time signal of targeted traction. We used 50 sensors on the top surface Γ_t , 15 sensors on the vertical array along Γ_r , and material profile 1. In addition to the Case 2 in Example 1, where the central frequency of targeted traction is 5 Hz, we examined Case 5 and 6, which consider targeted traction of central frequencies of 10 and 20 Hz, respectively.

Fig. 14 shows that, the higher frequency content the targeted traction contains, the higher terminal value of $\mathcal{E}$ is obtained. Namely, the terminal value of $\mathcal{E}$, 1.45%, for reconstructing $\hat{\mathbf{F}}^1$ in Case 2 is smaller than its counterparts 2.82% for identifying $\hat{\mathbf{F}}^2$ in Case 5 and 10.90% for reconstructing $\hat{\mathbf{F}}^3$ in Case 6. Thus, we suggest that the inversion modeling presented in this study suffers from solution multiplicity more when it identifies traction of a higher dominant frequency.

Due to the traction time signals' higher frequencies in Case 5 and 6, more inversion iterations are used for reaching convergence of inversion solution in these cases than Case 1 (i.e., 1000 iterations in Case 1 vs 2000 iterations in Cases 5 and 6).

Fig. 15(a,c,e) show sets of P_{kj} corresponding to targeted traction for Cases 2, 5, and 6, respectively, while Fig. 15(b,d,f) show their reconstructed counterparts. Figs. 16 to 18 also show the snapshots of the wave responses in the entire computational domain induced by the targeted $\hat{\mathbf{F}}^{1,2,3}$ and their reconstructed counterparts, respectively. All the figures present agreements between wave responses induced by the targeted and reconstructed force vectors. As shown in Table 1, $\mathcal{E}^u$, the error between wave responses induced by $\hat{\mathbf{F}}_{\text{target}}$ and its final-estimated counterpart, is smaller than the terminal value of error $\mathcal{E}$ for all Cases. For instance, in Case 6, the terminal value of $\mathcal{E}$ is 10.90%, but $\mathcal{E}^u$ is 0.33%. Thus, we suggest that the larger terminal value of $\mathcal{E}$ in the reconstructed force vector of a higher central frequency does not necessarily lead to $\mathcal{E}^u$ of the same order of magnitude due to inherent low-pass filtering of a typical FEM wave solver.

6.6 Example 4: Examining the feasibility of the presented inverse modeling to reconstruct targeted traction of a realistic seismic signal

In this example, we study the performance of identifying a targeted set of P_{kj} of $\hat{\mathbf{F}}^4$, which is created by a realistic seismic signal (Case 7). In Case 7, we use the material profile 1 with 50 sensors on Γ_t and 15 sensors on Γ_r .

Fig. 19 shows the excellent agreement between the targeted and reconstructed P_{kj} in Case 7. The inversion performance in Case 7 is also compared with that in Case 2. Cases 2 and 7 use the same material and sensor configurations except different force vectors $\hat{\mathbf{F}}^1$ and $\hat{\mathbf{F}}^4$, respectively. Fig. 20 shows that the terminal value of $\mathcal{E}$, 0.33%, in Case 7 is smaller than that in Case 2 (1.45%). We note that, even though the signal to generate $\hat{\mathbf{F}}^4$ is more complex than the other corresponding to $\hat{\mathbf{F}}^1$, our inversion solver identifies $\hat{\mathbf{F}}^4$ more

Figure 15: Example 3: (a) Target and (b) Reconstructed P_{kj} in Case 2; (c) Target and (d) Reconstructed P_{kj} in Case 5; and (e) Target and (f) Reconstructed P_{kj} in Case 6, for Case 2 at the 1000th iteration, and Case 5-6 at the 2000th iteration.

Figure 16: Example 3 - Case 2: Wave responses, $u(x, y, t)[\text{m}]$, in the domain induced by (left) the targeted set of P_{kj} and (right) its reconstructed counterpart at (a-b) 0.3 s, (c-d) 0.4 s, (e-f) 0.5 s, and (g-h) 0.6 s.

Figure 17: Example 3 - Case 5: Wave responses, $u(x, y, t)[\text{m}]$, in the domain induced by (left) the targeted set of $P_{k,j}$ and (right) its reconstructed counterpart at (a-b) 0.3 s, (c-d) 0.4 s, (e-f) 0.5 s, and (g-h) 0.6 s.

Figure 18: Example 3 - Case 6: Wave responses, $u(x, y, t)[\text{m}]$, in the domain induced by (left) the targeted set of $P_{k,j}$ and (right) its reconstructed counterpart at (a-b) 0.3 s, (c-d) 0.4 s, (e-f) 0.5 s, and (g-h) 0.6 s.

Figure 19: Example 4: (a) A target set of P_{kj} and (b) its reconstructed counterpart corresponding to $\hat{\mathbf{F}}^4$ for Case 7 at the 1000th iteration.

Figure 20: Example 4 - $\mathcal{E}$ over iterations in Case 2 and Case 7.

accurately than $\hat{\mathbf{F}}^1$ because the signal in $\hat{\mathbf{F}}^4$ has a lower dominant frequency than $\hat{\mathbf{F}}^1$. This behavior is consistent with our finding from Example 3.

Table 1 shows that $\mathcal{E}^u$ between the wave responses induced by $\hat{\mathbf{F}}^4$ and its final-estimated counterpart is zero. In addition, Figs. 21 and 22 show the agreement between the wave responses induced by the target and those induced by its reconstructed counterpart in Case 7.

6.7 Example 5: Investigating the inversion performance in a 2-layered background domain

This example focuses on examining the performance of inverting a targeted set of P_{kj} of $\hat{\mathbf{F}}^5$ by considering a two-layered background domain with inclusions. The performance is compared with that using the single-layered, homogeneous background domain counterpart.

To this end, we used the material profile 2, a vertical array of 15 sensors on Γ_r , and 50 sensors on Γ_t in Case 8. It is compared with Case 2, in which the material profile 1 is used with $\hat{\mathbf{F}}^1$. The traction signals of both $\hat{\mathbf{F}}^5$ and $\hat{\mathbf{F}}^1$ have the same central frequency of 5 Hz. The sensor configuration in Case 8 is same as Case 2.

Fig. 23 shows the excellent agreement between the targeted P_{kj} and its reconstructed counterpart. Fig. 24 shows the value of $\mathcal{E}$ over iterations in Case 2 and 8 for material profiles 1 and 2, respectively. Despite using a more complex background domain in Case 8, the difference between the terminal values of $\mathcal{E}$ in Cases 2

Figure 21: Example 4 - Case 7: Wave responses, $u(x, y, t)$ [m], in the domain induced by (left) the targeted set of P_{kj} and (right) its reconstructed counterpart at (a-b) 1.0 s, (c-d) 2.5 s, (e-f) 3.0 s, and (g-h) 3.5 s.

Figure 22: Example 4 - Case 7: Wave responses, $u(x, y, t)$ [m], in the domain induced by (left) the targeted set of P_{kj} and (right) its reconstructed counterpart at (a-b) 3.75 s, (c-d) 4.5 s, (e-f) 5.25 s, and (g-h) 6.0 s.

Figure 23: Example 5: (a) A target set of P_{kj} and (b) its reconstructed counterpart corresponding to $\hat{\mathbf{F}}^5$ for Case 8 at the 1000th iteration.

Figure 24: Example 5 - The final value of $\mathcal{E}$, in Case 8, is 1.56%, and that in Case 2 is 1.45%.

and 8 is only 0.11%.

Table 1 shows that $\mathcal{E}^u$ is only 0.16% in Case 8, and it is as low as Case 2. Fig. 25 presents the wave responses in the domain induced by the targeted $\hat{\mathbf{F}}^5$ and its final-reconstructed counterpart in Case 8 at $t = 0.3$ s, 0.4 s, 0.5 s, and 0.6 s. In general, the responses are in excellent agreement with each other. Thus, we suggest that the accuracy of our presented inversion algorithm is not compromised by the material complexity of a background domain.

6.8 Example 6: Examining the effect of noisy data on the inversion performance

In this example, we compare the accuracy of the inversion with respect to the noise level of random noise that is added to measured signals $u_m(x, y, t)$ at sensor locations prior to the inversion. We considered the material profile 2 with 50 sensors on Γ_t and 15 sensors on Γ_r , and examined the inversion performance of identifying a targeted set of P_{kj} of $\hat{\mathbf{F}}^5$ in Cases 8-12, which consider the noise level of 0%, 1%, 2%, 5%, and 10%, respectively.

Fig. 26 shows the targeted P_{kj} and its reconstructed counterparts obtained in Cases 8-12. Fig. 26 shows a more significant difference between the target and estimated P_{kj} for a larger noise level. Fig. 27 reveals that the terminal value of $\mathcal{E}$ increases as the noise level increases. For instance, while the terminal value of $\mathcal{E}$ in Case 8, of a noise level 0%, is 1.56%, the final value of $\mathcal{E}$ in Case 12, where 10% of noise level is studied, is 112.40%.

Although the terminal value of $\mathcal{E}$ exponentially grows as the noise level grows, Table 1 shows that $\mathcal{E}^u$ increases in a much smaller ratio: $\mathcal{E}^u$ is less than 1% for all the Cases in this example. For instance, while the terminal value of $\mathcal{E}$ in Case 12 is 112.40%, $\mathcal{E}^u$ is only 0.86%. The snapshots of the wave responses in the entire computational domain induced by the targeted and estimated force vector in Case 12 are shown in Fig. 28, and they are in good agreement with each other. It is because the noise-like error in the recon-

Figure 25: Example 5 - Case 8: Wave responses, $u(x, y, t)$ [m], in the domain induced by (left) the targeted set of P_{kj} and (right) its reconstructed counterpart at (a-b) 0.3 s, (c-d) 0.4 s, (e-f) 0.5 s, and (g-h) 0.6 s.

Figure 26: Example 6: (a) A set of P_{kj} corresponding to targeted traction and (b-f) reconstructed sets of P_{kj} in Cases 8-12, corresponding to noise level of 0%, 1%, 2%, 5%, and 10%, respectively, after 1000 iterations.

structed P_{kj} does not affect the wave responses due to the low-pass filtering of the FEM solver.

Thus, we suggest that the terminal values of $\mathcal{E}$ and $\mathcal{E}^u$ increase as the noise level increases, but not in a similar ratio to each other. Furthermore, as discussed previously in Example 3, a large value of $\mathcal{E}$ does not necessarily mean a value of $\mathcal{E}^u$ of the same order of magnitude.

7 Conclusion

We present a robust mathematical and computational modeling that identifies complex, incoherent seismic incident wavefield in a 2D solid, truncated by a WABC. The PDE-constrained optimization method is utilized to tackle the inverse problem. The DTO approach is employed to solve the adjoint problem and to semi-analytically compute the gradient of the discrete objective functional in the inverse modeling procedure.

Numerical results depict the performance of this inversion approach in a domain enclosed by WABC. First, targeted traction—mimicking incident, inclined plane waves—can not be fully reconstructed if we use only the sensors on the top surface. It is because wave motions induced by a part of dynamic traction at the bottom boundary do not reach the top surface of the considered truncated domain. Thus, to improve the inversion performance, a vertical array of sensors on the side boundary of a domain should be adopted. Second, to optimize the algorithm's inversion efficiency, a sufficiently large number of sensors should be used. In the presented example, a desirable ratio of measurement data size to the number of the control parameters can be as small as 1:8. Third, due to the intrinsic low-pass filtering of the FEM wave solver, a

Figure 27: Example 6 - The terminal value of $\mathcal{E}$ comparing Cases 8-12 with respect to the noise level.

Figure 28: Example 6 - Case 12: Wave responses, $u(x, y, t)$ [m], in the domain induced by (left) the targeted set of P_{kj} and (right) its reconstructed counterpart at (a-b) 0.3 s, (c-d) 0.4 s, (e-f) 0.5 s, and (g-h) 0.6 s.

large value of the inversion error in the reconstructed traction does not necessarily lead to an error of the same order of magnitude in the corresponding reconstructed wave responses in the entire computational domain. Fourth, the minimizer suffers less from solution multiplicity when it identifies lower dominant frequency traction (e.g., a real earthquake signal). Fifth, the accuracy of our presented inversion algorithm is not compromised by the material complexity of a background domain. Lastly, the terminal values of $\mathcal{E}$ and error $\mathcal{E}^u$ grow as the noise level that is added to u_m is increased, but not in a similar ratio to each other.

The wave responses in the entire computational domain, caused by the targeted traction and the restored one, are in excellent agreement. Therefore, by extending the presented approach to practical 3D elastodynamic settings, this algorithm can determine where significant amplitudes of stress waves (or structural failures) occur in built-in environments of interest during seismic events.

The presented proposed method could also be extended in the 3D setting such that it could flexibly accommodate measurement types (e.g., full translational 3C motions, 6C (translational 3C with rotational 3C motions), vertical-only 1C motions, etc.). Thus, our method can fully take advantage of big data of seismic measurement data from modern infrastructures and built environments equipped with ubiquitous sensors.

Data Availability

Some or all data, models, or code generated or used during the study are available from the corresponding author by request.

- MATLAB code (.m format) of the presented inverse modeling that contains the optimization solver and the forward and adjoint wave solvers.
- MATLAB datasets (.mat format) of the presented numerical results.

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8 Nomenclature

Symbol	Comment
x, y, z	Horizontal, vertical, and anti-plane directions, respectively
t	Time
T	Final time of the total observation duration
$u(x, y, t)$	Displacement field in the anti-plane (z) direction of the wave motion
$G(x, y)$	Shear modulus
$V_s(x, y)$	Shear wave speed
$\rho(x, y)$	Mass density
Ω	Domain
J	Total observation duration
$\Gamma_t, \Gamma_b, \Gamma_l, \Gamma_r$	Top, bottom, left, and right boundaries of the domain
D, L	y -coordinate of Γ_b and x -coordinate of Γ_r
$f(x, y, t)$	Dynamic traction
u^l	Incident seismic wavefield
v	Test function
$U, \mathcal{V}$	Function spaces
H^1	Hilbert space
$\psi(x, y)$	Vector of global basis function
$\mathbf{v}$	Vector of nodal solution of test function
$\mathbf{u}(t)$	Solution vector of the state problem at t
$\dot{\mathbf{u}}(t), \ddot{\mathbf{u}}(t)$	First- and second-order derivatives of solution vector with respect to t
$\mathbf{M}, \mathbf{K}, \mathbf{C}$	Global mass, stiffness, and damping matrices
$\mathbf{F}$	Global force vector
j	The j -th time step; the j -th component in P_{kj}
Δt	Size of time step
$\mathbf{Q}$	Discrete forward operator
$\hat{\mathbf{u}}$	Space-time discretization of $u(x, y, t)$ for all the time steps
$\hat{\mathbf{F}}$	Global force vector corresponding to all the time steps
N	Total number of time steps
γ_k	The k -th discrete node on the absorbing boundary
k	Numbering of the discrete node γ_k ; the numbering of the k -th component in P_{kj}
P_{kj}	Component of $\hat{\mathbf{F}}$ corresponding to γ_k and t_j
$\hat{\mathcal{L}}$	Discrete objective functional
$u_m(x, y, t)$	Wave signal measured by a sensor at a location of x and y induced by a targeted traction
$\hat{\mathbf{u}}_m$	Space-time discretization of $u_m(x, y, t)$ for all the time steps
$\mathbf{B}$	Square matrix that is zero everywhere except on the diagonals that correspond to a degree of freedom
$\bar{\mathbf{B}}$	Block diagonal matrix
$\hat{\mathcal{A}}$	Discrete Lagrangian functional
$\hat{\boldsymbol{\lambda}}$	Lagrange multiplier vector for all the time steps
$\boldsymbol{\xi}$	A set of control parameters, $\boldsymbol{\xi} = P_{kj}$
$\mathbf{g}$	Gradient vector
$\mathbf{d}$	Search direction vector
h	Scalar-value step size
s	The s -th inversion iteration
$\mathcal{E}$	Error norm between estimated traction and its target
$\mathcal{E}^u$	Error norm between wave responses induced by targeted traction and its counterpart

where:

$$a_0 = \frac{4}{(\Delta t)^2}, a_1 = \frac{2}{\Delta t}, a_2 = \frac{4}{\Delta t},$$

$$\mathbf{K}_{\text{eff}} = a_0 \mathbf{M} + a_1 \mathbf{C} + \mathbf{K}, \quad \mathbf{L}_1 = -a_0 \mathbf{M} - a_1 \mathbf{C}, \quad \mathbf{L}_2 = -a_2 \mathbf{M} - \mathbf{C}, \quad \mathbf{L}_3 = -\mathbf{M}. \quad (50)$$

The vector $\hat{\mathbf{F}}$ is built as:

$$\hat{\mathbf{F}} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{F}_0 \\ \mathbf{F}_1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{F}_0 = [0 \ P_{4,0} \ 0 \ 0 \ P_{2,0} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad \mathbf{F}_1 = [0 \ P_{4,1} \ 0 \ 0 \ P_{2,1} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T, \quad (51)$$

where $P_{2,0}$ and $P_{2,1}$ are the components of a force vector at the node of $\gamma_k = 2$ and at the timesteps of $t = 0$ s and $t = 0.1$ s, respectively. Similarly, $P_{4,0}$ and $P_{4,1}$ are the components of a force vector at the node of $\gamma_k = 4$ and at the timesteps of $t = 0$ s and $t = 0.1$ s, respectively. In addition, the block diagonal matrix $\bar{\mathbf{B}}$ is defined as:

$$\bar{\mathbf{B}} = \Delta t \begin{bmatrix} \mathbf{B}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{B}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (52)$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are a 12x12 square matrix that is zero everywhere except for the 10-th diagonal component with its value of 1 that corresponds to the degree of freedom where the sensor is placed.

Based on (49) to (52), the part d in (48) can be written as:

$$\begin{aligned} d = & 1.47 \times 10^{-21} P_{2,0}^2 + 1.31 \times 10^{-21} P_{2,0} P_{4,0} + 2.94 \times 10^{-21} P_{2,0} P_{2,1} + 1.31 \times 10^{-21} P_{2,0} P_{4,1} \\ & + 2.92 \times 10^{-22} P_{4,0}^2 + 1.31 \times 10^{-21} P_{4,0} P_{2,1} + 5.84 \times 10^{-22} P_{4,0} P_{4,1} \\ & + 1.47 \times 10^{-21} P_{2,1}^2 + 1.31 \times 10^{-21} P_{2,1} P_{4,1} \\ & + 2.92 \times 10^{-22} P_{4,1}^2. \end{aligned} \quad (53)$$

As shown in (53), the part d in (48) is a quadratic function in terms of $P_{2,0}$, $P_{4,0}$, $P_{2,1}$, and $P_{4,1}$. We note that the part d in (48) is convex if the eigenvalues of its Hessian matrix are greater than or equal to 0. Here, the Hessian matrix of the part d in (48) is:

$$\begin{aligned} \mathbf{H} = & \begin{bmatrix} \frac{\partial^2 d}{\partial P_{2,0}^2} & \frac{\partial^2 d}{\partial P_{2,0} \partial P_{4,0}} & \frac{\partial^2 d}{\partial P_{2,0} \partial P_{2,1}} & \frac{\partial^2 d}{\partial P_{2,0} \partial P_{4,1}} \\ \frac{\partial^2 d}{\partial P_{4,0} \partial P_{2,0}} & \frac{\partial^2 d}{\partial P_{4,0}^2} & \frac{\partial^2 d}{\partial P_{4,0} \partial P_{2,1}} & \frac{\partial^2 d}{\partial P_{4,0} \partial P_{4,1}} \\ \frac{\partial^2 d}{\partial P_{2,1} \partial P_{2,0}} & \frac{\partial^2 d}{\partial P_{2,1} \partial P_{4,0}} & \frac{\partial^2 d}{\partial P_{2,1}^2} & \frac{\partial^2 d}{\partial P_{2,1} \partial P_{4,1}} \\ \frac{\partial^2 d}{\partial P_{4,1} \partial P_{2,0}} & \frac{\partial^2 d}{\partial P_{4,1} \partial P_{4,0}} & \frac{\partial^2 d}{\partial P_{4,1} \partial P_{2,1}} & \frac{\partial^2 d}{\partial P_{4,1}^2} \end{bmatrix} \\ = & \begin{bmatrix} 2.94 \times 10^{-21} & 1.31 \times 10^{-21} & 2.94 \times 10^{-21} & 1.31 \times 10^{-21} \\ 1.31 \times 10^{-21} & 5.84 \times 10^{-22} & 1.31 \times 10^{-21} & 5.84 \times 10^{-22} \\ 2.94 \times 10^{-21} & 1.31 \times 10^{-21} & 2.94 \times 10^{-21} & 1.31 \times 10^{-21} \\ 1.31 \times 10^{-21} & 5.84 \times 10^{-22} & 1.31 \times 10^{-21} & 5.84 \times 10^{-22} \end{bmatrix}, \end{aligned} \quad (54)$$

of which eigenvalues are obtained as:

$$|\mathbf{I}^* \lambda - \mathbf{H}| = \begin{vmatrix} \lambda - 2.94 \times 10^{-21} & 1.31 \times 10^{-21} & 2.94 \times 10^{-21} & 1.31 \times 10^{-21} \\ 1.31 \times 10^{-21} & \lambda - 5.84 \times 10^{-22} & 1.31 \times 10^{-21} & 5.84 \times 10^{-22} \\ 2.94 \times 10^{-21} & 1.31 \times 10^{-21} & \lambda - 2.94 \times 10^{-21} & 1.31 \times 10^{-21} \\ 1.31 \times 10^{-21} & 5.84 \times 10^{-22} & 1.31 \times 10^{-21} & \lambda - 5.84 \times 10^{-22} \end{vmatrix} = 0. \quad (55)$$

Namely, the eigenvalues are:

$$\lambda_{1,2} = 0, \quad \lambda_3 = 7.05 \times 10^{-21}, \quad \lambda_4 = 4.88 \times 10^{-25}. \quad (56)$$

The eigenvalues are non-negative so that the part d in (48) is convex, and, therefore, the objective functional $\hat{\mathcal{L}}$ is convex.

B Derivation of the first term of the adjoint problem

In this section, we show the derivation of the part a in (32).

$$a = \frac{\partial \hat{\mathcal{L}}}{\partial \hat{\mathbf{u}}} = \frac{\partial \frac{1}{2}(\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T \bar{\mathbf{B}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)}{\partial \hat{\mathbf{u}}} \quad (57)$$

To this end, we compute the derivative using the product rule:

$$a = \frac{1}{2} \frac{\partial (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T}{\partial \hat{\mathbf{u}}} \bar{\mathbf{B}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m) + \frac{1}{2} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T \bar{\mathbf{B}} \frac{\partial (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)}{\partial \hat{\mathbf{u}}}. \quad (58)$$

Next, we use the following rule for vector differentiation:

$$\frac{\partial (\mathbf{r}^T \mathbf{w})}{\partial \mathbf{r}} = \frac{\partial (\mathbf{w}^T \mathbf{r})}{\partial \mathbf{r}} = \mathbf{w}, \quad (59)$$

where $\mathbf{w}$ is treated as constant in the derivative. Therefore, by denoting $\mathbf{w}_1 = \bar{\mathbf{B}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)$ and $(\mathbf{w}_2)^T = (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T \bar{\mathbf{B}}$, and introducing them in (58), part a can be written as:

$$\begin{aligned} a &= \frac{1}{2} \frac{\partial (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T}{\partial \hat{\mathbf{u}}} \mathbf{w}_1 + \frac{1}{2} \mathbf{w}_2^T \frac{\partial (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)}{\partial \hat{\mathbf{u}}} \\ &= \frac{1}{2} \frac{\partial \{(\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)^T \mathbf{w}_1\}}{\partial \hat{\mathbf{u}}} + \frac{1}{2} \frac{\partial \{\mathbf{w}_2^T (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m)\}}{\partial \hat{\mathbf{u}}} \\ &= \frac{1}{2} \mathbf{w}_1 + \frac{1}{2} \mathbf{w}_2 \\ &= \frac{1}{2} \bar{\mathbf{B}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m) + \frac{1}{2} \bar{\mathbf{B}}^T (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m) \\ &= \frac{1}{2} (\bar{\mathbf{B}} + \bar{\mathbf{B}}^T) (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m). \end{aligned} \quad (60)$$

The block diagonal matrix $\bar{\mathbf{B}}$ is symmetric, then:

$$a = \bar{\mathbf{B}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}_m) \quad (61)$$

References

- Akcelik, V., Biros, G., and Ghattas, O. (2002). Parallel multiscale Gauss-Newton-Krylov methods for inverse wave propagation. In *Supercomputing, ACM/IEEE 2002 Conference*, pages 41–41. IEEE.
- Askan, A., Akcelik, V., Bielak, J., and Ghattas, O. (2007). Full Waveform Inversion for Seismic Velocity and Anelastic Losses in Heterogeneous Structures. *Bulletin of the Seismological Society of America*, 97(6):1990–2008, ISSN: 0037–1106, DOI: [10.1785/0120070079](https://doi.org/10.1785/0120070079), <https://doi.org/10.1785/0120070079>.
- Askan, A., Akcelik, V., Bielak, J., and Ghattas, O. (2010). Parameter sensitivity analysis of a nonlinear least-squares optimization-based anelastic full waveform inversion method. *Comptes Rendus Mécanique*, 338(7):364–376, ISSN: 1631–0721, DOI: <https://doi.org/10.1016/j.crme.2010.07.002>, <http://www.sciencedirect.com/science/article/pii/S1631072110000963>. Inverse problems.
- Askan, A. and Bielak, J. (2008). Full Anelastic Waveform Tomography Including Model Uncertainty. *Bulletin of the Seismological Society of America*, 98(6):2975–2989, ISSN: 0037–1106, DOI: [10.1785/0120080138](https://doi.org/10.1785/0120080138), <https://doi.org/10.1785/0120080138>.
- Bielak, J., Loukakis, K., Hisada, Y., and Yoshimura, C. (2003). Domain reduction method for three-dimensional earthquake modeling in localized regions, Part I: Theory. *Bulletin of the Seismological Society of America*, 93(2):817–824.
- Fathi, A., Kallivokas, L. F., and Poursartip, B. (2015a). Full-waveform inversion in three-dimensional PML-truncated elastic media. *Computer Methods in Applied Mechanics and Engineering*, 296:39–72.
- Fathi, A., Poursartip, B., and Kallivokas, L. F. (2015b). Time-domain hybrid formulations for wave simulations in three-dimensional PML-truncated heterogeneous media. *International Journal for Numerical Methods in Engineering*, 101(3):165–198.
- Fathi, A., Poursartip, B., Stokoe II, K. H., and Kallivokas, L. F. (2016). Three-dimensional P- and S-wave velocity profiling of geotechnical sites using full-waveform inversion driven by field data. *Soil Dynamics and Earthquake Engineering*, 87:63–81.
- Guidio, B. and Jeong, C. (2020). Full-waveform inversion of incoherent dynamic traction in a bounded 2d domain of scalar wave motions. DOI: [10.31224/osf.io/2qpxv](https://doi.org/10.31224/osf.io/2qpxv), engrxiv.org/2qpxv.
- Guzina, B. B., Fata, S. N., and Bonnet, M. (2003). On the stress-wave imaging of cavities in a semi-infinite solid. *International Journal of Solids and Structures*, 40(6):1505–1523.
- Hamdaoui, M., Ledi, K., Robin, G., and Daya, E. (2019). Identification of frequency-dependent viscoelastic damped structures using an adjoint method. *Journal of Sound and Vibration*, 453:237–252, ISSN: 0022–460X, DOI: <https://doi.org/10.1016/j.jsv.2019.04.022>, <http://www.sciencedirect.com/science/article/pii/S0022460X19302330>.
- Jeong, C., Na, S.-W., and Kallivokas, L. F. (2009). Near-surface localization and shape identification of a scatterer embedded in a halfplane using scalar waves. *Journal of Computational Acoustics*, 17(03):277–308.
- Jeong, C. and Seylabi, E. E. (2018). Seismic input motion identification in a heterogeneous halfspace. *Journal of Engineering Mechanics*, 144(8):04018070, DOI: [10.1061/\(ASCE\)EM.1943-7889.0001495](https://doi.org/10.1061/(ASCE)EM.1943-7889.0001495).
- Ju, S. H. (2013). A Deconvolution scheme for determination of seismic loads in finite element analyses. *Bulletin of the Seismological Society of America*, 103(1):258–267, ISSN: 0037–1106, 1943–3573, DOI: [10.1785/0120120034](https://doi.org/10.1785/0120120034), <http://www.bssaonline.org/content/103/1/258>.
- Kallivokas, L. F., Fathi, A., Kucukcoban, S., Stokoe, K. H., Bielak, J., and Ghattas, O. (2013). Site characterization using full waveform inversion. *Soil Dynamics and Earthquake Engineering*, 47:62–82.
- Kang, J. W. and Kallivokas, L. F. (2010a). The inverse medium problem in 1D pml-truncated heterogeneous semi-infinite domains. *Inverse Problems in Science and Engineering*, 18(6):759–786, DOI: [10.1080/17415977.2010.492510](https://doi.org/10.1080/17415977.2010.492510).
- Kang, J. W. and Kallivokas, L. F. (2010b). Mixed unsplit-field perfectly matched layers for transient simulations of scalar waves in heterogeneous domains. *Computational Geosciences*, 14(4):623–648.

- Kang, J. W. and Kallivokas, L. F. (2011). The inverse medium problem in heterogeneous PML-truncated domains using scalar probing waves. *Computer Methods in Applied Mechanics and Engineering*, 200(1):265 – 283, ISSN: 0045-7825, DOI: <https://doi.org/10.1016/j.cma.2010.08.010>, <http://www.sciencedirect.com/science/article/pii/S0045782510002434>.
- Kraft, S., Puel, G., Aubry, D., and Funfschilling, C. (2016). Parameter identification of multi-body railway vehicle models — application of the adjoint state approach. *Mechanical Systems and Signal Processing*, 80:517 – 532, ISSN: 0888-3270, DOI: <https://doi.org/10.1016/j.ymssp.2016.04.037>, <http://www.sciencedirect.com/science/article/pii/S0888327016300723>.
- Kucukcoban, S., Goh, H., and Kallivokas, L. F. (2019). On the full-waveform inversion of lame parameters in semi-infinite solids in plane strain. *International Journal of Solids and Structures*, 164:104 – 119, ISSN: 0020-7683, DOI: <https://doi.org/10.1016/j.ijsolstr.2019.01.019>.
- Kucukcoban, S. and Kallivokas, L. F. (2011). Mixed perfectly-matched-layers for direct transient analysis in 2D elastic heterogeneous media. *Computer Methods in Applied Mechanics and Engineering*, 200:57–76.
- Kucukcoban, S. and Kallivokas, L. F. (2013). A symmetric hybrid formulation for transient wave simulations in PML-truncated heterogeneous media. *Wave Motion*, 50:57 – 79.
- Lamert, A., Nguyen, L. T., Friederich, W., and Nestorovia, T. (2018). Imaging disturbance zones ahead of a tunnel by elastic full-waveform inversion: Adjoint gradient based inversion vs. parameter space reduction using a level-set method. *Underground Space*, 3(1):21 – 33, ISSN: 2467-9674, DOI: <https://doi.org/10.1016/j.undsp.2018.01.006>, <http://www.sciencedirect.com/science/article/pii/S2467967417301083>. Computational Methods in Mechanized Tunneling.
- Lloyd, F. and Jeong, C. (2018). Adjoint equation-based inverse-source modeling to reconstruct moving acoustic sources in a 1D heterogeneous solid. *Journal of Engineering Mechanics*, 144(9):04018089, DOI: [10.1061/\(ASCE\)EM.1943-7889.0001508](https://doi.org/10.1061/(ASCE)EM.1943-7889.0001508).
- Mashayekh, H., Kallivokas, L. F., and Tassoulas, J. L. (2018). Parameter estimation in layered media using dispersion-constrained inversion. *Journal of Engineering Mechanics*, 144(11):04018099, DOI: [10.1061/\(ASCE\)EM.1943-7889.0001527](https://doi.org/10.1061/(ASCE)EM.1943-7889.0001527).
- Mejia, L. and Dawson, E. (2006). Earthquake deconvolution for FLAC. *FLAC and Numerical*.
- Nguyen-Tuan, L., Nanthakumar, S. S., and Lahmer, T. (2019). Identification of multiple flaws in dams using inverse analysis based on hydro-mechanical XFEM and level sets. *Computers and Geotechnics*, 110:211 – 221, ISSN: 0266-352X, DOI: <https://doi.org/10.1016/j.compgeo.2019.02.006>, <http://www.sciencedirect.com/science/article/pii/S0266352X19300400>.
- Pakravan, A., Kang, J. W., and Newton, C. M. (2016). A Gauss-Newton full-waveform inversion for material profile reconstruction in viscoelastic semi-infinite solid media. *Inverse Problems in Science and Engineering*, 24(3):393–421, ISSN: 1741-5977, DOI: [10.1080/17415977.2015.1046861](https://doi.org/10.1080/17415977.2015.1046861), <http://dx.doi.org/10.1080/17415977.2015.1046861>.
- PEER (2000). Pacific Earthquake Engineering Research Center (PEER) ground motion database. <http://ngawest2.berkeley.edu/site>.
- Poul, M. K. and Zerva, A. (2018a). Efficient time-domain deconvolution of seismic ground motions using the equivalent-linear method for soil-structure interaction analyses. *Soil Dynamics and Earthquake Engineering*, 112:138 – 151, ISSN: 0267-7261, DOI: <https://doi.org/10.1016/j.soildyn.2018.04.032>, <http://www.sciencedirect.com/science/article/pii/S0267726117305742>.
- Poul, M. K. and Zerva, A. (2018b). Nonlinear dynamic response of concrete gravity dams considering the deconvolution process. *Soil Dynamics and Earthquake Engineering*, 109:324 – 338, ISSN: 0267-7261, DOI: <https://doi.org/10.1016/j.soildyn.2018.03.025>, <http://www.sciencedirect.com/science/article/pii/S0267726117308825>.
- Poul, M. K. and Zerva, A. (2018c). Time-domain PML formulation for modeling viscoelastic waves with Rayleigh-type damping in an unbounded domain: Theory and application in ABAQUS. *Finite Elements in Analysis and Design*, 152:1 – 16, ISSN: 0168-874X, DOI: <https://doi.org/10.1016/j.finel.2018.08.004>, <http://www.sciencedirect.com/science/article/pii/S0168874X18300313>.

- Reuber, G. S., Holbach, L., Popov, A. A., Hanke, M., and Kaus, B. J. P. (2020). Inferring rheology and geometry of subsurface structures by adjoint-based inversion of principal stress directions. *Geophysical Journal International*, 223(2):851–861, ISSN: 0956–540X, DOI: [10.1093/gji/ggaa344](https://doi.org/10.1093/gji/ggaa344).
- Schnabel, P. B. (1972). SHAKE a computer program for earthquake response analysis of horizontally layered sites. *EERC report*, pages 72–12.
- Seidl, D. T., Oberai, A. A., and Barbone, P. E. (2019). The coupled adjoint-state equation in forward and inverse linear elasticity: Incompressible plane stress. *Computer Methods in Applied Mechanics and Engineering*, 357:112588, ISSN: 0045–7825, DOI: <https://doi.org/10.1016/j.cma.2019.112588>, <http://www.sciencedirect.com/science/article/pii/S0045782519304591>.
- Seidl, D. T., van Bloemen Waanders, B. G., and Wildey, T. M. (2020). Simultaneous inversion of shear modulus and traction boundary conditions in biomechanical imaging. *Inverse Problems in Science and Engineering*, 28(2):256–276, DOI: [10.1080/17415977.2019.1603222](https://doi.org/10.1080/17415977.2019.1603222), <https://doi.org/10.1080/17415977.2019.1603222>.
- Tran, K. T. and McVay, M. (2012). Site characterization using Gauss-Newton inversion of 2-D full seismic waveform in the time domain. *Soil Dynamics and Earthquake Engineering*, 43:16–24, ISSN: 02677261, DOI: [10.1016/j.soildyn.2012.07.004](https://doi.org/10.1016/j.soildyn.2012.07.004), <https://linkinghub.elsevier.com/retrieve/pii/S0267726112001613>.
- Walsh, T., Aquino, W., and Ross, M. (2013). Source Identification in Acoustics and Structural Mechanics using SIERRA/SD. Technical report, Sandia National Laboratories, <https://prod-ng.sandia.gov/techlib-noauth/access-control.cgi/2013/132689.pdf>.
- Wang, M. and Brigham, J. C. (2017). A generalized computationally efficient inverse characterization approach combining direct inversion solution initialization with gradient-based optimization. *Computational Mechanics*, 59(3):507–521, ISSN: 1432–0924, DOI: [10.1007/s00466-016-1362-3](https://doi.org/10.1007/s00466-016-1362-3).