

# Online frequency and amplitude tracking in structural vibrations under environment using APES spectrum postprocessing and Kalman filtering

Xuewen Yu<sup>a</sup>, Danhui Dan<sup>\*a,b</sup>

<sup>a</sup>*School of Civil Engineering, Tongji University, 1239 Siping Road, Shanghai, 200092, China*

<sup>b</sup>*Key Laboratory of Performance Evolution and Control for Engineering Structures of Ministry of  
Education, Tongji University, 1239 Siping Road, Shanghai, 200092, China*

---

## Abstract

Identifying time-varying frequency and amplitude online in real-life structural vibrations is an essential topic of data processing in structural health monitoring. This paper proposes a novel method for this task. We assume that structural vibration signals are stationary in a short time, thus a spectral analysis method called amplitude and phase estimation (APES) is conducted to obtain the amplitude spectrum at corresponding time window, and a postprocessing technique is proposed to extract the modal frequency and amplitude from the spectrum automatically. The extracted frequency and amplitude could be regarded as the average of the instantaneous frequency and instantaneous amplitude during the window. Due to the instability of measured structural vibrations and the uncertainty of spectral shapes under ambient excitation, Kalman filtering is introduced by taking the signal that reconstructed from the identified frequencies and amplitudes as the prediction to enhance the reliability and quality (signal-to-noise ratio) of the next measured signals. Numerical study is performed to inspect the performance of the proposed method. It is also employed to analyze the vibration signals of actual structures, i.e., a cable of a cable-stayed bridge, a hanger of

an arch bridge and the main girder of a suspension bridge. The results show its potential to track frequency and amplitude in structural vibrations under environmental measurements. The method is supposed to provide fundamental support for further information obtaining and high-level decision making for structural health monitoring systems.

*Keywords:* online frequency tracking, frequency-squeezing postprocessing, Kalman filtering, structural health monitoring

---

## 1. Introduction

In recent years, many large-scale civil structures such as bridges, buildings and dams have been equipped with structural health monitoring (SHM) systems, and massive monitoring data have been collected for condition evaluation and damage diagnosis [1, 2]. Frequency as a natural property of structures is one of the most fundamental and important indexes that should be concerned in processing monitoring data. As actual structures in service, especially for large-span bridges bearing varying live loads, exhibit nonstationary and non-linear dynamic characteristics, many researches have been focused on the identification of time-varying frequency.

The structural vibration response can be modeled as the superposition of multiple amplitude modulated (AM) and frequency modulated (FM) components:

$$x(t) = \sum_{k=1}^K x_k(t) = \sum_{k=1}^K a_k(t) \cos(\phi_k(t)), \quad (1)$$

where  $a_k(t)$  and  $\phi_k(t)$  represent instantaneous amplitude (IA) and instantaneous phase (IP), respectively. The phase can be written as  $\phi_k(t) = \int_0^t \omega_k(t)dt + \theta_k$ , and then the instantaneous frequency (IF) is generally defined as the first derivative of the phase function [3]:

$$\omega_k(t) = \frac{d}{dt} \phi_k(t). \quad (2)$$

As for structure response, in Eqs. (1) and (2), the amplitude is time-varying mainly due to the randomness of ambient excitation and the energy dissipation; and the frequency

is time-varying mainly due to the uncertainty (e.g., constraint uncertainty caused by temperature, mass uncertainty caused by passing vehicles) and the degeneration (e.g., stiffness decrease caused by section damage, material corrosion, etc.) of structure itself. Uncertainty results in the fluctuations of frequency around a referring value (i.e., natural frequency) while degeneration results in the permanent natural frequency change that cannot recovery automatically. These fluctuations and evolution laws of frequency and amplitude can be used for fatigue analysis, damage detection [4, 5], etc. Also, they can serve as the extracted underlying features for data-driven structural diagnosis techniques [6].

There are several commonly studied ways for IF identification. One class of method is based on mode decomposition. First, the multi-component signal is decomposed into mono-components called intrinsic mode functions (IMFs) in relative literature, which can be realized by empirical mode decomposition (EMD) [7], analytical mode decomposition (AMD) [8], empirical wavelet transform (EWT) [9], variational mode decomposition (VMD) [10], etc. For more discussions and comparisons on those or other mode decomposition methods, we recommend the review works [11, 12, 13, 14] to interested readers. Second, Hilbert transform (HT) is generally employed to obtain the analytical form  $z_k(t)$  of IMF  $x_k(t)$  [15]:

$$z_k(t) = x_k(t) + iv_k(t), \quad v_k(t) = \mathcal{H}(x_k(t)) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x_k(\tau)}{t - \tau} d\tau, \quad (3)$$

where  $\mathcal{H}$  denotes the HT. The IA is obtained as the envelop of the analytical signal,  $a_k(t) = \sqrt{x_k^2(t) + v_k^2(t)}$ , and the phase is calculated as  $\phi_k(t) = \arctan(v_k(t)/x_k(t)$ . According to Eq. (2), it's able to obtain the frequency as

$$\omega_k(t) = \frac{d}{dt} \left( \arctan \left( \frac{\tilde{u}_k(t)}{u_k(t)} \right) \right) = \frac{x_k(t)\dot{v}_k(t) - \dot{x}_k(t)v_k(t)}{x_k^2(t) + v_k^2(t)}. \quad (4)$$

Using Eq. (4) to calculate IF avoids phase unwrapping instead of using Eq. (2) directly after obtaining the phase. For this kind of technical route, A. Baccigalupi and A. Liccardo

[16], P. Ni et al. [17], Y. Xin et al. [18] respectively employed EMD, VMD and AMD to decompose the measured multi-component responses into limited number of IMFs, and the IF was identified by using HT according to Eqs.(2) and (4). Due to the end effect introduced by HT resulting Gibbs phenomenon in IF function, P. F. Pai and R. Zhong et al. proposed the sliding-window tracking method [19] and the conjugate-pair decomposition method [20] for IF identification after the acquisition of IMFs by EMD. In addition, A. Cicone et al. [21] developed a new definition for IF by mapping the IMF  $x_k(t)$  to  $\theta_k(t)$  in polar coordinate to alleviate the end effect and gained better ability to characterize the local property of a signal. Besides, if the IMF is to be a regular harmonic, the energy scheme, Teager-Kaiser algorithm (TKA) [22], is also usually used, which is applicable to online computing but sensitive to noise.

The second class of method for IF identification is based on the time-frequency analysis (TFA) and consists of two steps as well. First, by TFA approaches, the original 1D signal is transformed into a 2D time-frequency (TF) plane. Then the embedded IF trajectories  $\omega_k(t)$  (i.e., frequency value at each instant) in the TF plane need to be extracted for the purpose of further quantitative analysis such as natural frequency based cable force calculating [23].

For the first step, the signal processing community provides a wealth of TFA tools for disclosing IF trajectories in the TF plane. Widely used include the linear TFA methods such as short time Fourier transform (STFT) and wavelet transform (WT), and quadratic TFA methods such as Winger-Vile distribution (WVD) and its variants, Cohen class distributions [24, 25]. Due to the principle of Heisenberg uncertainty, the TF representation (TFR) or spectrogram is unable to achieve high resolution in both the time and frequency directions. Many efforts have been conducted to improve the readability of TFR to present more compact and clearer IF trajectories, and this is helpful to the frequency extraction in the next step. Thus, parametric TFA methods such as chirplet transform (CT) [26] and polynomial

CT (PCT) [27], which can be regarded as preprocessing techniques, were proposed by specifying the parametric form of TF atoms to match the IF's variation law of a nonstationary signal. Besides, several postprocessing techniques developed recently can greatly concentrate the energy dispersed around the IF traces in the TF plane. Reassignment method (RM) is one of the most adopted postprocessing schemes, which relocates the coefficients of STFT along both frequency- and time-axis by replacing them with centroids of the local energy distribution [28]. And one drawback of RM is that it doesn't support signal reconstruction. Synchrosqueezing transform (SST) proposed by Dubechies et al. [29] can be seen as a special case of RM by squeezing TF coefficients only along frequency-axis and it's allowable for signal reconstruction. Differ from RM and SST squeezing spectrogram, synchroextracting transform (SET) proposed by G. Yu [30] only extracts the coefficients of STFT most related to IF trajectories and removes the others, greatly enhancing the concentration of TFR. Let's denote the TFR results by those TFA methods as

$$\text{TFR}_{\text{method}}(t, \omega), \quad \text{method} \in \{\text{STFT}, \text{WT}, \text{SST}, \dots\}, \quad (5a)$$

$$\text{i.e.,} \quad \text{TFR}_{\text{STFT}}(t, \omega) = \int_{-\infty}^{\infty} x(\tau) g^*(\tau - t) e^{-j\omega\tau} d\tau. \quad (5b)$$

For the second step, the post-task of extracting IF from the TFR is generally resolved based on the fact that coefficients at the locations of IF trajectories are greater than that at other positions around [31, 32], i.e.,

$$\hat{\omega}_k(t) = \arg \max_{\omega} \text{TFR}_{\text{method}}(t, \omega), \quad (6)$$

so it's also called ridge detection [33, 34]. To keep the identified IF curves continuous and smooth, many ridge detection algorithms such as dynamic path optimization [35, 36], image processing technique [37], Markov chain Monte Carlo algorithm [33], maximum gradient method [38], ant colony optimization [39], etc., have been presented recently.

Of course, these two classes of methods mentioned above are not completely independent.

On the one hand, TFA methods that support signal reconstruction can also be used for mode decomposition. On the other hand, the decomposed IMFs by mode decomposition methods can also be transferred into TF plane for further IF extraction. These reviewed methods are summarized in Fig. 1.

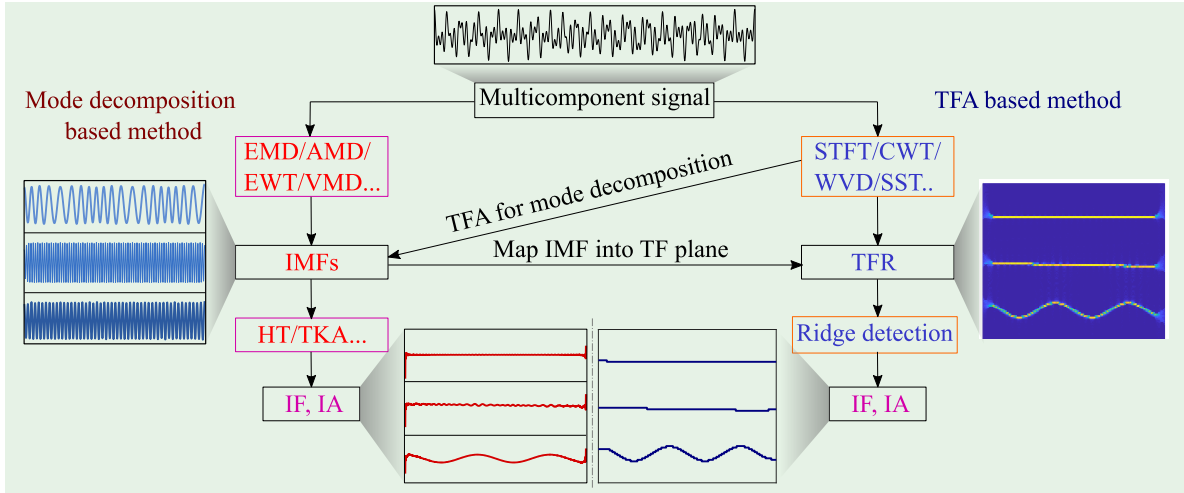


Figure 1: Review (incomplete) of existing methods of IF identification.

Existing studies provide powerful and plenty of ways for IF identification. However, nowadays, in the context of SHM, we are not only required to identify time-varying parameters for indicating structural conditions, but also realize this task in an online and real-time manner in order to support timelier anomaly detection and decision-making [40, 41]. While existing methods as shown in Fig. 1 are mostly realized offline with batch processing. To be adaptive to online data analysis, some modifications to them are necessary. Nevertheless, when handling with operational data, especially for the data generated by large-scale civil structures, there are some problems that need to be faced with for above methods, either realized offline or supposedly online. In an online monitoring environment, there exist very weak excitation periods which means a low signal-to-noise-ratio (SNR). As we can see, since the noise is dispersed among the decomposed components except the residual term, the identified IF according to Eqs. (2) and (4) will be extremely affected in low SNR situations. A low SNR also leads the IF trajectory in TF plane submerged in the jungle of noise, in-

creasing the difficulty of ridge extraction. And because of the randomness of excitation, the modal components participating in vibration and the number of components are different in different periods. This becomes a further obstacle for some mode decomposition methods and ridge detection algorithms that need to specify the number of signal components in advance. Additionally, existing methods are most verified on numerical and experimental signals, lacking validity proof directly applied to actual long-time continuous signals.

This paper presents a novel approach for online frequency and amplitude identification of time-varying structural vibrations. The approach consists of two aspects, one is the automatic natural frequency and amplitude extracting from the spectrum obtained by the method of amplitude and phase estimation (APES) [42] that possesses higher resolution than FT-based methods; another is embedding this extracting algorithm for a single analytical step into a Kalman filtering framework to obtain more robust and reliable continuous tracking results. Both numerical and real-world signals prove the effectiveness of this proposed approach.

The rest of this paper is organized as follows. Section 2 defines the problem of concern and Section 3 gives a full description of the proposed solving methodology. In Section 4, synthetic signals are used to illustrate the performance of the proposed method. In Section 5, three long vibration signals that each lasts for 24 hours belong to three actual bridge structures are analyzed and the results validate the method well. The last Section 6 is the conclusion of the whole work.

## 2. Problem definition

The signal recorded by the sensor of an SHM system is the discretization of realistic structural vibration, suppose the sampling frequency is  $F_s$ , then the signal value at instant  $t$  is denoted as  $x_t = x(t) := x(t \cdot 1/F_s)$ .

This paper focuses on the online identification of frequency and amplitude and the data processing scheme is shown in Fig. 2. Unlike offline data processing, first, only past data are available; Second, assuming the start time is  $t = 0$ , once a sufficiently long data series (denote as  $\mathbf{x}_0 := [x_0, x_1, \dots, x_N]^T$  in Fig. 2(a)), we called computing frame, has been generated, the analysis of this computing frame has to start immediately; Third, the computing frame in the previous step will be updated by the new samples, called updating block, forming a new computing frame and following the analysis of it. Here, the length of computing frame is fixed as  $N$ , and denote the length of updating block as  $S \in [1, N)$ . Thus, as illustrated in Fig. 2(a), the online data processing is in the manner of updating the  $N$  lengthed computing frame with  $S$  newest samples on and on. Each computing frame  $\mathbf{x}_k := [x_{kS}, x_{kS+1}, \dots, x_{kS+N-1}]^T$  is immediately analyzed to obtain the frequency and amplitude. In our paper, this is realized by using APES [42] to estimate the amplitude spectrum first and then using a postprocessing technique to extract the natural frequency and amplitude, which is introduced in Section 3.1.

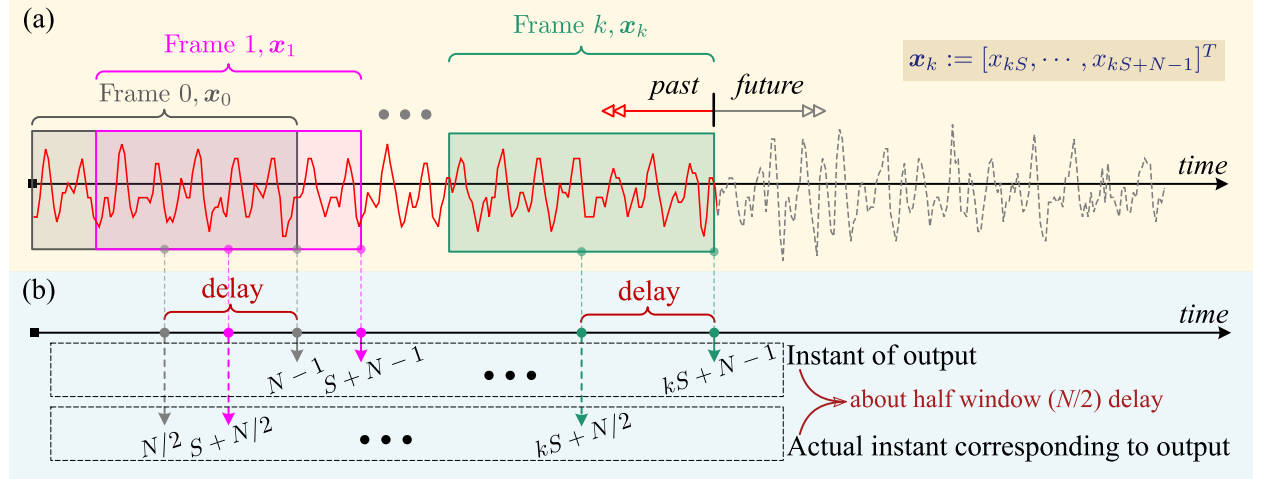


Figure 2: Online data processing scheme. (a) Fixed length computing frame and block updating model. (b) Illustration of analytical time-lag.

As shown in Fig. 2(b), the analytical result is outputted at the end of the computing frame, i.e., instants at  $[N-1, S+N-1, \dots, kS+N-1, \dots]$ . The outputting frequency

is  $F_s/S$ Hz. Since APES assumes the signal is stationary, the amplitude spectrum obtained by APES characterizes the average properties of the signal during the frame length, which means, the identified parameters at the end of the computing frame are actually around the center of it, i.e., instants at  $[N/2, S + N/2, \dots, kS + N/2, \dots]$ . Therefore, there exists an analysis delay about half the length of the computing frame. As for TFA methods which perform offline, the IF trajectory will also be delayed for half the window interval unless zeros are padded on both ends [43], so the analysis delay in online analysis can also be attributed to the cold start. More precisely, we define the analysis delay as the difference between the time of obtaining the parameter and the time that it actually locates, and take the time cost for data transmission and computation into accounts, then the total analysis delay  $\tau_{analysis}$  can be expressed as

$$\tau_{analysis} = \tau_{method} + \tau_{computation} + \tau_{transmission} \approx \tau_{method}, \quad (7)$$

where  $\tau_{method}$  represents the lag introduced by the method itself, e.g., in this paper,  $\tau_{method} \approx N/2$  as mentioned above,  $\tau_{transmission}$  and  $\tau_{computation}$  represent the lags caused by data transmission and computation. Among them,  $\tau_{method} \gg \tau_{computation} \gg \tau_{transmission}$ , the analysis delay  $\tau_{analysis}$  is dominated by  $\tau_{method}$ .

In order to enhance the real-time performance of identification, a method that a small computing frame length  $N$  is available should be adopted. This is also one of the points that we employ APES for spectra estimate in this paper. For vibration signals of civil structures, the analysis delay is several seconds of proposed method (longer for lower frequency structures). What's more, to ensure the continuity of data processing and avoid accumulation, it has to be satisfied that

$$\tau_{computation} + \tau_{transmission} < S/F_s. \quad (8)$$

So,  $S$  is a trade-off between the output frequency  $F_s/S$  and the threshold of computation

cost  $S/F_s$ .

### 3. Methodology

#### 3.1. Automatic extraction of frequency and amplitude within a single computing frame

For current computing frame, first of all, we transform it into frequency domain by APES method (Section 3.1.1), and then the frequency and amplitude are extracted from the frequency domain automatically by the proposed method in Section 3.1.2.

##### 3.1.1. Recap of APES method

APES was proposed by L. Li and P. Stoica in 1996 [42], which is a nonparametric adaptive filter-bank method with higher resolution than FT based methods [44]. Besides, it is able to adopt shorter computing frame for exact frequency and amplitude estimation [45]. Thus, it is employed for spectrum estimate in this paper and is briefly introduced as follows.

Consider a discrete one-dimensional (1D) time series  $\{x(n)\}_{n=0}^{N-1}$ , for a frequency  $\omega$  (in Hz) of interest, it can be decomposed into a sinusoidal term with amplitude  $\alpha(\omega)$  and a residual term  $\varepsilon_\omega(n)$  that contains all interference caused by sinusoidal components at other frequencies as well as the random noise:

$$x(n) = \alpha(\omega)e^{j2\pi\omega n} + \varepsilon_\omega(n), \quad n = 0, 1, \dots, N-1. \quad (9)$$

Arrange the  $N$  consecutive samples in an  $M \times L$  Hankel matrix:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \cdots & \mathbf{y}_{L-1} \end{bmatrix} = \begin{bmatrix} x(0) & x(1) & \cdots & x(L-1) \\ x(1) & x(2) & \cdots & x(L) \\ \vdots & \vdots & \ddots & \vdots \\ x(M-1) & x(M) & \cdots & x(N-1) \end{bmatrix}, \quad (10)$$

where  $M \leq N/2$  is a user-defined parameter, and  $L = N + 1 - M$ .

In the matched-filterbank interpretation of APES, the core is to design a data dependent finite-impulse-response (FIR) filter  $\mathbf{h}(\omega) \in \mathbb{C}^{M \times 1}$  that passes the sinusoid with frequency  $\omega$  undistorted and suppresses sinusoids with any other frequency as much as possible [46]:

$$\min_{\mathbf{h}(\omega), \alpha(\omega)} \sum_{l=0}^{L-1} |\mathbf{h}^H(\omega) \mathbf{y}_l - \alpha(\omega) e^{j2\pi\omega l}|^2 \quad s.t. \quad \mathbf{f}_M^H(\omega) \mathbf{h}(\omega) = 1, \quad (11)$$

where

$$\mathbf{f}_M(\omega) = \begin{bmatrix} 1 & e^{j2\pi\omega} & \dots & e^{j2\pi\omega(M-1)} \end{bmatrix}^T, \quad (12)$$

and  $(\cdot)^H$  and  $(\cdot)^T$  denote conjugate transpose and transpose of a vector or matrix, respectively.

The closed solution to Eq. (5) is

$$\mathbf{h}(\omega) = \frac{\mathbf{Q}(\omega)^{-1} \mathbf{f}_M(\omega)}{\mathbf{f}_M^H(\omega) \mathbf{Q}(\omega)^{-1} \mathbf{f}_M(\omega)}, \quad (13)$$

$$\alpha(\omega) = \mathbf{h}^H(\omega) \mathbf{g}(\omega) = \frac{\mathbf{f}_M^H(\omega) \mathbf{Q}(\omega)^{-1} \mathbf{g}(\omega)}{\mathbf{f}_M^H(\omega) \mathbf{Q}(\omega)^{-1} \mathbf{f}_M(\omega)}, \quad (14)$$

where

$$\mathbf{g}(\omega) = \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}_l e^{-j2\pi\omega l} = \frac{1}{L} \mathbf{Y} \mathbf{f}_L^*(\omega), \quad (15)$$

$$\mathbf{Q}(\omega) = \frac{1}{L} \mathbf{Y} \mathbf{Y}^H - \mathbf{g}(\omega) \mathbf{g}^H(\omega) = \mathbf{R} - \mathbf{g}(\omega) \mathbf{g}^H(\omega), \quad (16)$$

and  $(\cdot)^*$  denotes conjugate of a vector or matrix.

For more details about the derivation of APES, please refer to literature [44, 46]. To obtain the amplitude  $\alpha(\omega)$  over the whole frequency domain  $\boldsymbol{\omega} := [\omega_1, \omega_2, \dots, \omega_K]^T$ , i.e., the amplitude spectrum  $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_K]^T := [\alpha(\omega_1), \alpha(\omega_2), \dots, \alpha(\omega_K)]^T$ , repeatedly computing of Eq. (14) for every frequency point is needed and it's very time-consuming. We recommend the matrix implementation of it when actually coding, which is listed in Algorithm 1. And the appropriate parameters selection is summarized in Section 3.3.1.

---

**Algorithm 1** APES.

---

- 1: **Input:** Data series  $\{x(n)\}_{n=0}^{N-1}$ , calculating frequency  $\boldsymbol{\omega} = [\omega_1, \omega_2, \dots, \omega_K]^T$ , filter order  $M$  and sampling rate  $F_s$
  - 2: Calculate parameter  $L = N + 1 - M$ , matrices  $\mathbf{F}_M = [\mathbf{f}_M(\omega_1), \mathbf{f}_M(\omega_2), \dots, \mathbf{f}_M(\omega_K)]$  and  $\mathbf{F}_L = [\mathbf{f}_L(\omega_1), \mathbf{f}_L(\omega_2), \dots, \mathbf{f}_L(\omega_K)]$  by Eq. (12)
  - 3: Construct Hankel matrix  $\mathbf{Y}$  by Eq. (10) and calculate  $\mathbf{R} = L^{-1}(\mathbf{Y}\mathbf{Y}^H)$
  - 4: Calculate vectors:  $\mathbf{a} = \text{diag}(\mathbf{F}_M^H \mathbf{R}^{-1} \mathbf{F}_M)$ ,  $\mathbf{b} = L^{-1} \text{diag}(\mathbf{F}_M^H \mathbf{R}^{-1} \mathbf{Y} \mathbf{F}_L^*)$  and  $\mathbf{c} = L^{-2} \text{diag}(\mathbf{F}_L^T \mathbf{Y}^H \mathbf{R}^{-1} \mathbf{Y} \mathbf{F}_L^*)$
  - 5: Calculate amplitude  $\boldsymbol{\alpha} = \mathbf{b} \div (\mathbf{a} - \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{b}^*)$ , here ' $\div$ ' and ' $\times$ ' denote dot products (element-wise)
  - 6: **Output:** Frequency  $\boldsymbol{\omega}$  and amplitude  $\boldsymbol{\alpha}$
- 

### 3.1.2. Postprocessing for extracting frequency and amplitude

In our previous work [45], we realized a block-wise recursive APES (BRAPES) for time-varying signals analysis, which is analogous to STFT. Meanwhile, a frequency squeezing postprocessing (FSP) technique was proposed to concentrate the spectral lines distribution and obtain more compact TF representation. On the ground of FSP, here we are going to further propose a method of automatically extracting natural frequencies and amplitudes from the APES amplitude spectrum.

FSP consists of three steps: (1) amplitude pretreatment, (2) frequency squeezing and (3) amplitude restoration and zero setting.

First, pretreat the APES spectrum  $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_K]^T$  by normalization and exponentiation:

$$\bar{\alpha}_i = (\alpha_i / \max(\boldsymbol{\alpha}))^m, \quad i = 1, 2, \dots, K, \quad m > 1. \quad (17)$$

Second, squeeze uniformly distributed spectral lines by considering consecutive  $2P + 1$  spectral lines  $(\bar{\alpha}_{i-P}, \dots, \bar{\alpha}_i, \dots, \bar{\alpha}_{i+P})$  centered on frequency  $\omega_i$  and replacing the value of

$\omega_i$  with the centroid of the  $2P + 1$  spectral lines in frequency direction:

$$\omega_i^{n+1} = \frac{\sum_{k=\max(1,i-P)}^{\min(i+P,K)} \omega_k^n \bar{\alpha}_k}{\sum_{k=\max(1,i-P)}^{\min(i+P,K)} \bar{\alpha}_k}, \quad i = 1, 2, \dots, K, \quad P \geq 1. \quad (18)$$

Eq. (18) is going to move every spectral line at frequency  $\omega_i$  toward the local highest peak. With a proper choice of  $P$ , by iteratively conducting Eq. (18), the original spectral lines will be gradually concentrated around the positions of natural frequencies. Here, the convergence criteria is set as: the frequency difference between adjacent steps should be less than a threshold  $\delta_s$ , i.e.,  $\|\boldsymbol{\omega}^{n+1} - \boldsymbol{\omega}^n\|_2 \leq K\delta_s$ . A maximum step is also set to enforce the stop of iterations.

Third, restore the amplitude that was changed in step 1 by assigning the original amplitude vector  $\boldsymbol{\alpha}$  to the squeezed frequency vector  $\hat{\boldsymbol{\omega}}$  in subscript order. Meanwhile, set the amplitudes at both ends of the frequency vector  $\hat{\boldsymbol{\omega}}$  and the amplitudes corresponding to frequencies at cluster edges as zeros (see the enlargement (2) in Fig. 3(c)) to obtain more clear result presentation. That is,

$$\hat{\alpha}_i = \begin{cases} 0, & i \in \Omega = \{1, K\} \cup \{(j, j+1) \mid \hat{\omega}_{j+1} - \hat{\omega}_j > \delta_\omega\} \\ \alpha_i, & i \in \{1, 2, \dots, K\} \setminus \Omega \end{cases} \quad (19)$$

where  $\delta_\omega$  is a threshold used to judge the edges of frequency clusters.

The pseudo-code of FSP is listed in Algorithm 2 and the appropriate parameters selection is summarized in Section 3.3.1.

Figs. 3(a)~(c) shows the processing effect of FSP. As we can see, the original spectrum obtained by APES in which the spectral lines are uniformly spaced (Fig. 3(b)) are relocated after FSP in which the spectral lines are gathered around the natural frequencies (Fig. 3(c)). In addition, looking at the enlargement (1) in Fig. 3(c), slight frequency deviation is observed, which is introduced by frequency squeezing in Eq. (18).

Although FSP is available to make the spectrum appear ideal in terms of visual presen-

---

**Algorithm 2** FSP.

---

- 1: **Input:** Frequency  $\boldsymbol{\omega} = \boldsymbol{\omega}^1 = [\omega_1, \omega_2, \dots, \omega_K]^T$  and amplitude  $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_k]^T$  by Algorithm 1, order  $m$ , step size  $P$ , thresholds  $\delta_s$  and  $\delta_\omega$ , max iterations  $I$ ,  $n = 0$ ,  $\boldsymbol{\omega}^0 = \mathbf{0}$
  - 2: Amplitude pretreatment by Eq. (17)
  - 3: Frequency squeezing:
  - 4: **while**  $\|\boldsymbol{\omega}^{n+1} - \boldsymbol{\omega}^n\|_2 > K\delta_s$  **and**  $n < I$  **do**
  - 5:    $n = n + 1$
  - 6:   Moving frequency by Eq. (18)
  - 7: **end while**
  - 8: Amplitude restoration and zero setting by Eq. (19)
  - 9: **Output:** processed frequency  $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}^{n+1}$  and amplitude  $\hat{\boldsymbol{\alpha}}$
- 

tation, for the purpose of qualitative online analysis of the law of frequency variation, we have to extract the natural frequencies and amplitudes from estimated spectrum. This is usually realized by peak analysis with a proper selection of parameters such as minimum peak separation, minimum peak height, etc. However, when handling with realistic data online, those pre-fixed parameters are not suitable for all spectrum shapes, which means not every picked peak corresponds to a natural frequency all the time.

In this paper, we proposed an approach for automatically extracting frequency and amplitude based on the original and the processed spectra. First, it's found that after the processing of APES spectrum by FSP, the spectral lines are redistributed into several groups (Fig. 3(c)), each group corresponds to a potential location of natural frequency. So, we separate the frequency domain into  $P$  sub-intervals. In each sub-interval, the spectral lines are clustered within the maximum distance less than a threshold  $\delta_\Omega$ , as shown in the following equation:

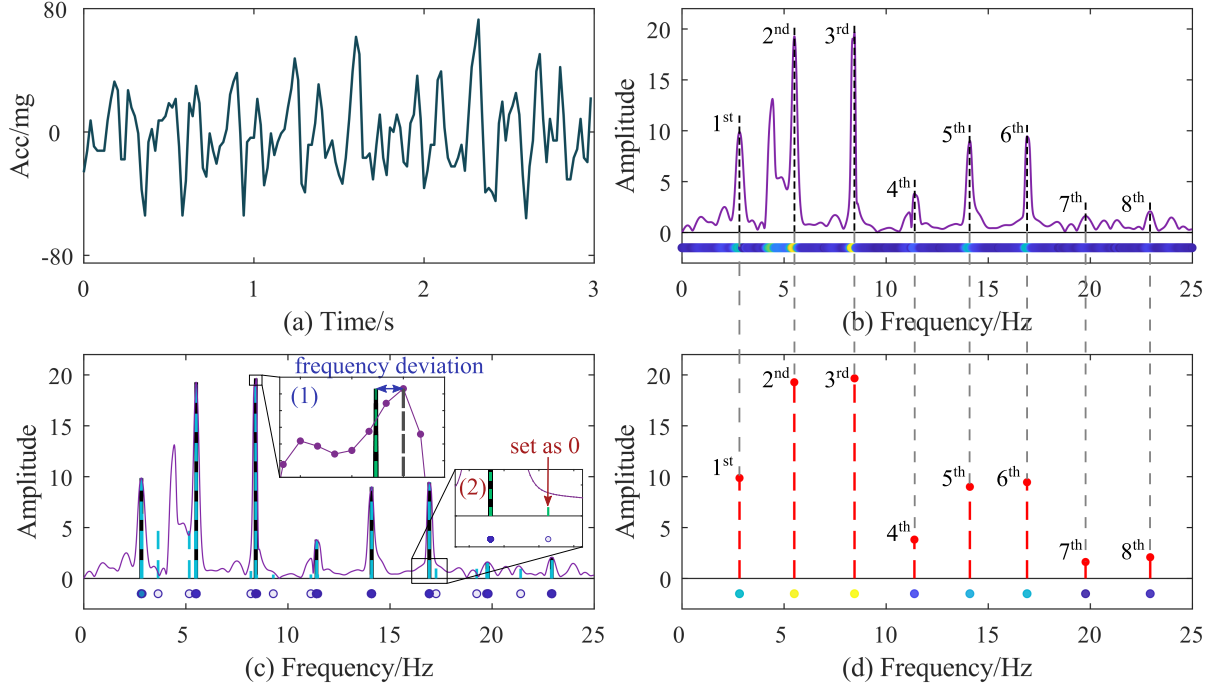


Figure 3: FSP and the extraction of frequency and amplitude. (a) Data series. (b) APES spectrum of (a). (c) FSP result of (b): original spectrum (purple line); spectral lines by FSP (black line); spectral lines by FSP without zero setting of amplitude (blue dashed line). (d) Extracted natural spectral lines by Algorithm 3. The points under the spectral lines in (b)~(d) show the distribution of frequencies.

$$\begin{cases} \Omega = \{1, 2, \dots, K\} = \Omega_1 \cup \dots \cup \Omega_i \cup \dots \cup \Omega_P, \\ \Omega_i = \{j \mid |\hat{\omega}_j - \hat{\omega}_{i_0}| \leq \delta_\Omega, j \in \Omega\}, \text{ where } i_0 = \begin{cases} 1, & i = 1, \\ \max(\Omega_{i-1}) + 1, & i \geq 2. \end{cases} \end{cases} \quad (20)$$

Note that the frequency  $\hat{\omega}$  still keeps monotonically increasing after FSP, i.e.,  $\hat{\omega}_1 \leq \hat{\omega}_2 \leq \dots \leq \hat{\omega}_K$ , and the number of sub-intervals  $P$  (i.e., the number of extracted frequency and amplitude) is automatically determined.

Second, we locate the maximum amplitude of each group  $\Omega_i$ :

$$\Omega_{max} = \left\{ m_i \mid m_i = \arg \max_m (\hat{\alpha}_m), m \in \Omega_i, i = 1, 2, \dots, P \right\}, \quad (21)$$

and index  $m_i$  of the maximum amplitude will be regarded as the position of natural frequency if it satisfies 1)  $\hat{\alpha}_{m_i} \neq 0$  and 2) the distance between original frequency and processed

frequency  $|\hat{\omega}_{m_i} - \omega_{m_i}|$  is less than a threshold  $\delta_d$ , i.e.,

$$\Omega_s = \{i \mid \hat{\alpha}_i \neq 0 \ \& \ |\hat{\omega}_i - \omega_i| < \delta_d, i \in \Omega_{max}\}. \quad (22)$$

The index set of  $\Omega_s$  in Eq.(22) is still just the potential locations where natural frequencies are, because there is no guarantee that the spectral lines will not cluster at the lobes caused by noise, especially in the case of low SNR.

Finally, extracting frequencies and amplitudes from the original frequency vector  $\boldsymbol{\omega}$  and amplitude vector  $\boldsymbol{\alpha}$  according to the indexes set  $\Omega_s$ :

$$\begin{cases} \boldsymbol{\omega}_s = \{\omega_i \mid i \in \Omega_s\}, \\ \boldsymbol{\alpha}_s = \{\alpha_i \mid i \in \Omega_s\}. \end{cases} \quad (23)$$

Here, the subscript ' $s$ ' in  $\Omega_s$ ,  $\boldsymbol{\omega}_s$  and  $\boldsymbol{\alpha}_s$  means 'sparse', because most non-natural frequencies and amplitudes have been removed. And we will denote the natural frequencies and amplitudes as  $\boldsymbol{\omega}_n$  and  $\boldsymbol{\alpha}_n$  in the following, obviously, ' $n$ ' means 'natural'. If the spectrum enjoys a well shape, each frequency extracted in Eq. (23) will correspond to a natural frequency, as shown in Fig. 3(d).

In addition, the frequency deviation in the processed spectrum  $(\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{\alpha}})$  by FSP as mentioned above is eliminated by Eq. (23), which relocates the biased peaks in processed spectrum to the true peaks in original APES spectrum  $(\boldsymbol{\omega}, \boldsymbol{\alpha})$ .

Actually, after the processing of FSP, the original spectral lines distributed with an equal spacing of  $\Delta\omega = \omega_{i+1} - \omega_i$  will be extremely compact (e.g., at the precision of 0.001Hz or smaller) at local peaks where probably the natural frequencies are. By using this feature, the following manner of cluster separation is suggested which is more efficient than Eq. (20):

$$\tilde{\omega}_i = \lfloor \hat{\omega}_i \times 10^n \rfloor / 10^n, \quad i = 1, 2, \dots, K, \quad (24a)$$

$$\hat{\omega} = \text{unique}(\tilde{\omega}) = \{\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_P\}, \quad P \ll K, \quad (24b)$$

$$\Omega_i = \{j \mid \tilde{\omega}_j = \hat{\omega}_i, j \in \Omega\}, \quad i = 1, 2, \dots, P. \quad (24c)$$

Eq. (24a) means the frequency  $\hat{\omega}_i$  is truncated to the  $n$ th decimal place, and  $\lfloor \cdot \rfloor$  represents round down. In Eq. (24b), operation  $\text{unique}(\cdot)$  means to delete the same elements in  $\tilde{\omega}$ , leaving only the  $P$  different elements. Eq. (24c) divides the original frequency interval into  $P$  sub-intervals.

The pseudo-code of frequency and amplitude extraction are given in Algorithm 3 and the appropriate parameters selection is summarized in Section 3.1.1.

---

**Algorithm 3** FSP based frequency and amplitude extraction.

---

- 1: **Input:** Original frequency  $\omega$  and amplitude  $\alpha$  by Algorithm 1, processed frequency  $\hat{\omega}$  amplitude  $\hat{\alpha}$  by Algorithm 2, threshold  $\delta_\Omega$  or order  $n$ , threshold  $\delta_d$
  - 2: Separate the frequency domain into  $P$  sub-intervals by Eq. (20) or Eq. (24)
  - 3: Find index of the maximum amplitude of each sub-intervals by Eq. (21) and determine if the index corresponds to a natural frequency by Eq. (22)
  - 4: Extract frequency and amplitude through the index by Eq. (23)
  - 5: **Output:** Extracted frequency  $\omega_s$  and amplitude  $\alpha_s$
- 

By now, using APES spectra estimate (Algorithm 1) and FSP based parameter extraction (Algorithms 2 and 3), it's able to automatically identify frequencies and amplitudes in a single step. However, when dealing with actual signals of structural vibrations for online continuous frequency and amplitude extracting, it's hard to guarantee that the structural vibrations in every computing step are well excited by the environment. So, there exist signal frames with very low SNR, resulting in poor natural frequency identification mixed

with non-natural frequency caused by noise. In order to improve the naive online frequency extracting results, i.e., each computing frame is processed separately, we apply Kalman filtering to the tracking framework, which is introduced in the following subsection.

### 3.2. Online frequency and amplitude tracking embedded with Kalman filtering

As shown in Fig. 2, denote the measured  $k$ th ( $k = 0, 1, \dots$ ) computing frame at time  $t_k := kS + N - 1$  (in unit  $1/F_s$  second,  $F_s$  is the sampling frequency in Hz) as  $\mathbf{x}_k := [x_{kS}, x_{kS+1}, \dots, x_{kS+N-1}]$ , and denote the procedure of frequency and amplitude extraction from  $\mathbf{x}_k$  using APES spectrum estimation and FSP postprocessing (Algorithms 1~3) as  $\text{FAE}(\cdot)$ :

$$(\boldsymbol{\omega}_{s,k}, \boldsymbol{\alpha}_{s,k}) = \text{FAE}(\mathbf{x}_k), \quad (25)$$

where,  $\boldsymbol{\omega}_{s,k}$  and  $\boldsymbol{\alpha}_{s,k}$  represent the estimated frequency and amplitude of the  $k$ th computing frame, respectively, and the name of operation 'FAE' means 'frequency and amplitude extraction'.

Assume that the extracted frequency and amplitude between adjacent computing frames ( $S$  samples spaced) do not change very much:

$$\boldsymbol{\omega}_{s,k+1} \cong \boldsymbol{\omega}_{s,k}, \quad \boldsymbol{\alpha}_{s,k+1} \cong \boldsymbol{\alpha}_{s,k}. \quad (26)$$

Therefore, according to Eq. (2), the vibration signal of  $(k+1)$ th computing frame can be written as

$$x_{k+1,n} = \text{Re} \left( \sum_{i=1}^{P_k+1} (\alpha_{s,k+1,i}) e^{j2\pi(\omega_{s,k+1,i})n} \right) \cong \text{Re} \left( \sum_{i=1}^{P_k} (\alpha_{s,k,i}) e^{j2\pi(\omega_{s,k,i})n} \right), \quad (27)$$

$$n = (k+1)S, (k+1)S+1, \dots, (k+1)S+N-1,$$

where  $\omega_{s,k,i}$  and  $\alpha_{s,k,i}$  represent the  $i$ th element of the estimated frequency vector  $\boldsymbol{\omega}_{s,k}$  and amplitude vector  $\boldsymbol{\alpha}_{s,k}$  at the  $k$ th computing frame, respectively;  $P_k \geq 1$  is the number of estimated parameters  $\boldsymbol{\omega}_{s,k}$  and  $\boldsymbol{\alpha}_{s,k}$ ;  $\text{Re}(\cdot)$  returns the real part of a complex number. The

constructed signal  $\mathbf{x}_{k+1}$  by Eq. (27) is a superposition of  $P_k$  harmonic components with constant oscillation frequency and amplitude. And we denote the construction operation expressed by Eq. (27) as

$$\mathbf{x}_{k+1} = \text{Construct}(\boldsymbol{\omega}_{s,k}, \boldsymbol{\alpha}_{s,k}). \quad (28)$$

Thus, taking the collected computing frame  $\mathbf{x}_k$  as state vector, the prediction and measurement equations can be written as

$$\begin{cases} \mathbf{x}_{k+1} &= \mathcal{F}(\mathbf{x}_k) + \mathbf{w}_k, \\ \mathbf{z}_k &= \mathbf{H}\mathbf{x}_k + \mathbf{v}_k, \end{cases} \quad (29)$$

where,  $\mathcal{F} : \mathbf{x}_k \rightarrow \mathbf{x}_{k+1}$  is realized by Eqs. (25) and (28), i.e.,  $\mathcal{F}(\mathbf{x}_k) = \text{Construct}(\text{FAE}(\mathbf{x}_k))$ ; measurement matrix  $\mathbf{H}$  is selected as unit matrix  $\mathbf{I}_{N \times N}$  to make a full observation;  $\mathbf{w}_k$  is the zero-mean process white noise with covariance matrix  $\text{E}[\mathbf{w}_k \mathbf{w}_k^T] = \mathbf{Q}_k$ ;  $\mathbf{v}_k$  is the zero-mean measurement white noise with covariance matrix  $\text{E}[\mathbf{v}_k \mathbf{v}_k^T] = \mathbf{R}_k$ ; and  $\text{E}[\cdot]$  stands for expectation.

As  $\mathcal{F}$  is a nonlinear function, generally, unscented kalman filter (UKF) [47] or extended kalman filter (EKF) [48] are resorted to for state estimation. For UKF, sigma points  $\boldsymbol{\chi}$  are generated first[49], and then perform unscented transform on these sigma points  $\boldsymbol{\chi}$  through the nonlinear function  $\mathcal{F}$  yielding a set of transformed points  $\boldsymbol{\mathcal{Y}} := \mathcal{F}(\boldsymbol{\chi})$  to compute the mean and covariance of the new estimate. In this procedure, function  $\mathcal{F}$  composed of the operations of  $\text{FAE}(\cdot)$  and  $\text{Construct}(\cdot)$  has to be conducted many times, which is extremely time-consuming and is not suitable for online estimate. EKF is more efficient compared to UKF, which handles nonlinearity by locally linearizing the system at the current estimate. To linearize  $\mathcal{F}(\mathbf{x}_k)$ , we have to calculate its Jacobian matrix at  $\mathbf{x}_k$ , i.e.,  $\frac{\partial \mathcal{F}(\mathbf{x})}{\partial \mathbf{x}}|_{\mathbf{x}_k}$ . However, this is unsolved observing the form of function  $\mathcal{F}$ . Considering that the state vector  $\mathbf{x}_k \in \mathbb{R}^{N \times 1}$  in Eq. (29) represents  $N$  consecutive observations of a vibration response, i.e., acceleration, at a measurement point of a structure, in this sense,  $\mathbf{x}_k$  itself is a series of

realizations of a single random state variable. It's reasonable to assume that the uncertainty  $P$  of state  $\mathbf{x}_k$  keeps the same after the mapping of  $\mathcal{F}$ , which means

$$\mathbf{F} := \frac{\partial \mathcal{F}(\mathbf{x})}{\partial \mathbf{x}} \big|_{\mathbf{x}_k} \cong \mathbf{I}_{N \times N}, \quad (30)$$

and the covariance matrices of process noise and measurement noise are reduced to a variance:  $\mathbf{Q}_k \rightarrow Q_k := \mathbb{E}[\mathbf{u}_k^T \mathbf{u}_k]$ ,  $\mathbf{R}_k \rightarrow R_k := \mathbb{E}[\mathbf{v}_k^T \mathbf{v}_k]$ .

Therefore, a standard liner Kalman filter is formalized as follows:

$$(\boldsymbol{\omega}_{s,k}, \boldsymbol{\alpha}_{s,k}) = \text{FAE}(\hat{\mathbf{x}}_{k|k}), \quad (31a)$$

$$\hat{\mathbf{x}}_{k+1|k} = \mathcal{F}(\hat{\mathbf{x}}_{k|k}) = \text{Construct}(\boldsymbol{\omega}_{s,k}, \boldsymbol{\alpha}_{s,k}), \quad (31b)$$

$$\tilde{\mathbf{y}}_{k+1} = \mathbf{z}_{k+1} - \hat{\mathbf{x}}_{k+1|k}, \quad (31c)$$

$$e_{k+1} = \|\hat{\mathbf{x}}_{k+1|k}\|_2^2 / (\|\hat{\mathbf{x}}_{k+1|k}\|_2^2 + \|\mathbf{z}_{k+1}\|_2^2), \quad (31d)$$

$$d_{k+1} = 1 - \hat{\mathbf{x}}_{k+1|k}^T \mathbf{z}_{k+1} / \sqrt{(\hat{\mathbf{x}}_{k+1|k}^T \hat{\mathbf{x}}_{k+1|k})(\mathbf{z}_{k+1}^T \mathbf{z}_{k+1})}, \quad (31e)$$

$$Q_{k+1} = (\rho(1 - e_{k+1}) + (1 - \rho)d_{k+1})\sigma_{\tilde{\mathbf{y}}_{k+1}}, \quad (31f)$$

$$R_{k+1} = (\rho e_{k+1} + (1 - \rho)(1 - d_{k+1}))\sigma_{\tilde{\mathbf{y}}_{k+1}}, \quad (31g)$$

$$P_{k+1|k} = P_{k|k} + Q_{k+1}, \quad (31h)$$

$$K_{k+1} = P_{k+1|k} / (P_{k+1|k} + R_{k+1}), \quad (31i)$$

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + K_{k+1} \tilde{\mathbf{y}}_{k+1}, \quad (31j)$$

$$P_{k+1|k+1} = (1 - K_{k+1})P_{k+1|k}. \quad (31k)$$

The process function  $\mathcal{F}$  is separated into two parts of Eqs. (31a) and (31b) to explicitly express the extraction of natural frequencies and amplitudes. Eq. (31c) computes the residual between prediction  $\hat{\mathbf{x}}_{k+1|k}$  and measurement  $\mathbf{z}_{k+1}$ . Eqs. (31d)~(31g) aim to computing the covariance  $Q_{k+1}$  of process noise and the covariance  $R_{k+1}$  of measurement noise adaptively based on the following considerations:

- (1) If the measurement  $\mathbf{z}_{k+1}$  enjoys a high energy (defined as  $\|\mathbf{z}_{k+1}\|_2^2$ ), which means the

structure is well excited and the measured signal has a high SNR, thus we should be inclined to trust the measurement to capture changes in the signal's spectral structure. Otherwise, we should be inclined to the prediction  $\hat{\mathbf{x}}_{k+1|k}$  to keep the continuity of identification results. This defines the energy metric  $e$  as Eq. (31d).

- (2) If the prediction  $\hat{\mathbf{x}}_{k+1|k}$  is similar to the measurement  $\mathbf{z}_{k+1}$ , we are inclined to the prediction, because it is reconstructed by extracted frequencies and amplitudes, meaning the noise term has been filtered. Otherwise, we should be inclined to the measurement  $\mathbf{z}_{k+1}$  to ensure the sensitivity to change perception. This defines the distance metric  $d$  as Eq. (31e).

Therefore,  $Q_{k+1}$  (process noise's variance) and  $R_{k+1}$  (measurement noise's variance) are designed relating to above energy metric  $e$  and distance metric  $d$  as shown in Eq. (31f) and Eq. (31g), where  $0 \leq \rho \leq 1$  is a weight between the energy metric and distance metric and  $\sigma_{\hat{\mathbf{y}}}$  is the variance of residual term  $\hat{\mathbf{y}}$ . Eqs. (31h) and (31i) are to obtain the Kalman gain according to the process uncertainty and the measurement uncertainty. Eq. (31j) gives the state estimate and Eq. (31k) gives the estimate of process uncertainty according to Riccati equation [50].

As shown in Eq. (31), we take the structural vibration response at a certain measured point in time-domain as the state variable, and the actually concerned natural frequency and amplitude in frequency-domain are exported from the state estimate. The frequency  $\omega_{s,k}$  and corresponding amplitude  $\alpha_{s,k}$  identified by Eq. (31a) or Eq. (25) are arranged in order of frequency from the smallest to the largest, i.e.,  $\omega_{s,k,1} < \omega_{s,k,2} < \dots < \omega_{s,k,P_k}$ . Due to the randomness of environment, there is no guarantee that all modes are excited efficiently always, and we couldn't make sure that the identified frequency  $\omega_{s,k}$  and amplitude  $\alpha_{s,k}$  from state estimate  $\hat{\mathbf{x}}_k$  at different steps are in alignment although with the embedding of Kalman filtering. Alignment here has two meanings: (1) the number  $P_k$  of identified

frequency  $\omega_{s,k}$  in each step keeps the same:  $P_k = P_{k+1}$ ; (2) at the same indexed position  $i$ , the element  $\omega_{s,k,i}$  of identified frequency vector  $\omega_{s,k}$  in each step is of the same order. If frequencies in different steps are not aligned all the time, it is hard to conduct further quantitative analysis but for result visualization.

Fig. 4 demonstrates the frequency alignment schema designed in this paper. First, the reference frequency  $\omega_r$  and the allowable variation range  $\delta_a$  have to be appointed in advance. For each reference position  $i$ , select the candidates from extracted frequency  $\omega_{s,k+1}$ , which are in the supposed interval  $(\omega_{r,i} - \delta_{a,i}, \omega_{r,i} + \delta_{a,i})$ :

$$\Omega_i = \{j \mid |\omega_{s,k+1,j} - \omega_{r,i}| < \delta_{a,i}, j \in \{1, 2, \dots, P_{k+1}\}\}, \quad i = 1, 2, \dots, P_r. \quad (32)$$

In the above equation,  $P_{k+1}$  means the number of extracted frequency  $\omega_{s,k+1}$  at the  $k+1$ th computing frame,  $P_r$  means the number of reference frequency  $\omega_r$ ,  $\Omega_i$  is the index set of candidates correspond to  $\omega_{r,i}$ .

Second, determine true natural frequencies and amplitudes from the candidates: if there is no candidate for  $\omega_{r,i}$ , i.e.,  $\Omega_i = \emptyset$ , the frequency will be copied from the last step, and the amplitude will be assigned as half of that in the last step; If the candidate is unique for  $\omega_{r,i}$ , the extracted frequency and amplitude will be directly set as the natural frequency and amplitude; If there are multiple candidates for  $\omega_{r,i}$ , first select the candidate with maximum amplitude,  $\Omega_{i,max} = \arg \max_j \{\alpha_{s,k+1,j}, j \in \Omega_i\}$ , and the candidate that is nearest to the frequency  $\omega_{n,k,i}$  in the last step,  $\Omega_{i,nearest} = \arg \min_j \{|\omega_{s,k+1,j} - \omega_{n,k,i}|\}, j \in \Omega_i\}$ , then if the maximum amplitude  $\alpha_{s,k+1,\Omega_{i,max}}$  is greater than twice the amplitude  $\alpha_{n,k,i}$  in the last step, the frequency and amplitude will be set according to the maximum candidate  $\Omega_{i,max}$ , otherwise, they are set according to the nearest candidate  $\Omega_{i,nearest}$ . Those are expressed in the following equation:

$$\omega_{n,k+1,i} = \begin{cases} \omega_{n,k,i} & \Omega_i = \emptyset, \\ \omega_{s,k+1,\Omega_i} & \text{card}(\Omega_i) = 1, \\ \begin{cases} \omega_{s,k+1,\Omega_{i,max}} & \alpha_{s,k+1,\Omega_{i,max}} \geq 2\alpha_{n,k,i} \\ \omega_{s,k+1,\Omega_{i,nearest}} & \text{otherwise} \end{cases} & \text{card}(\Omega_i) \geq 2, \end{cases} \quad i = 1, 2, \dots, P_r, \quad (33a)$$

$$\alpha_{n,k+1,i} = \begin{cases} 0.5\alpha_{n,k,i} & \Omega_i = \emptyset, \\ \alpha_{s,k+1,\Omega_i} & \text{card}(\Omega_i) = 1, \\ \begin{cases} \alpha_{s,k+1,\Omega_{i,max}} & \alpha_{s,k+1,\Omega_{i,max}} \geq 2\alpha_{n,k,i} \\ \alpha_{s,k+1,\Omega_{i,nearest}} & \text{otherwise} \end{cases} & \text{card}(\Omega_i) \geq 2, \end{cases} \quad i = 1, 2, \dots, P_r, \quad (33b)$$

where,  $\text{card}(\cdot)$  returns the number of elements in the set.

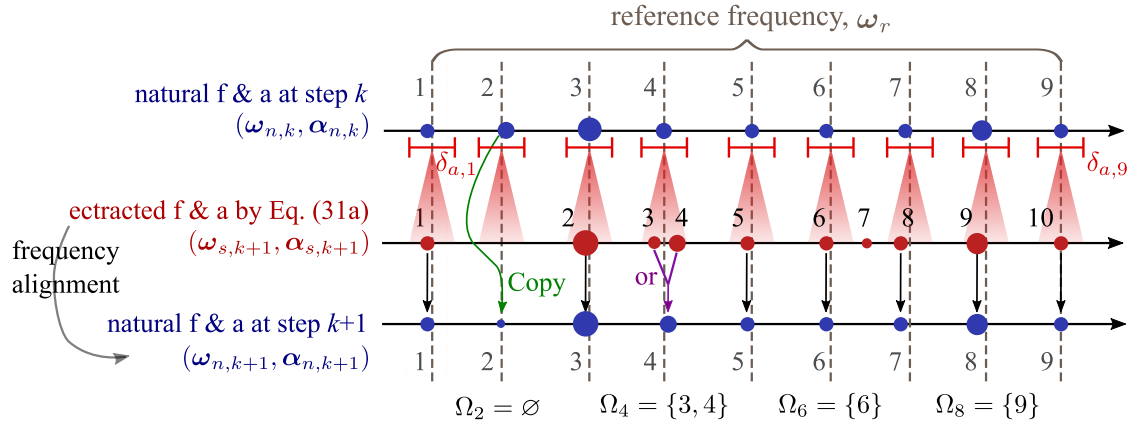


Figure 4: Illustration of frequency alignment in Eqs. 32 and 33. 'f & a' in the figure means 'frequency and amplitude'.

For the situation where the time-varying frequency fluctuates at a certain level, the reference value of frequency can be determined by the statistical average based on the analysis results of historical data, and it can be updated according to the latest analysis

results to capture the slow change. For the situation where the frequency shows obvious trend variation, we can set the extracted results in the last step as the reference value of frequency, i.e.,  $\omega_r \leftarrow \omega_{n,k}$ .

Algorithm 4 gives the pseudo-code of frequency and amplitude tracking embedded with Kalman filtering.

---

**Algorithm 4** Frequency and amplitude tracking using APES postprocessing and Kalman filtering.

---

- 1: **Input:** Initial frequency  $\omega_{n,0}$  and amplitude  $\alpha_{n,0}$ , updating block length  $S$ , weight  $\rho$ , reference frequency  $\omega_r$  and allowable frequency variation range  $\delta_a$
  - 2: **for**  $k = 0, 1, 2, \dots$  **do**
  - 3:   Predict the state  $\hat{\mathbf{x}}_{k+1|k}$  by Eq. (31b)
  - 4:   Obtain the measurement  $\mathbf{z}_{k+1} = [x_{kS}, x_{kS+1}, \dots, x_{N+kS-1}]$
  - 5:   Calculate the residual signal  $\tilde{\mathbf{y}}_{k+1}$  by Eq. (31c)
  - 6:   Calculate the variance of process noise  $Q_{k+1}$  and the variance of measurement noise  $R_{k+1}$  by Eqs. (31d)~(31g)
  - 7:   Calculate the gain  $K_{k+1}$  by Eqs. (31h)~(31i)
  - 8:   Update the state  $\hat{\mathbf{x}}_{k+1|k+1}$  by Eq. (31j) and the process uncertainty  $P_{k+1|k+1}$  by Eq. (31k)
  - 9:   Extracted the frequency  $\omega_{s,k+1}$  and amplitude  $\alpha_{s,k+1}$  by Eq. (31a), i.e., Algorithms 1~3
  - 10:   Frequency alignment by Eqs. (32) and (33) to obtain the natural frequency  $\omega_{n,k+1}$  and amplitude  $\alpha_{n,k+1}$ .
  - 11: **end for**
  - 12: **Output:** Time-varying natural frequency  $[\omega_{n,0}, \omega_{n,1}, \dots]$  and amplitude  $[\alpha_{n,0}, \alpha_{n,1}, \dots]$
-

### 3.3. Section summary

#### 3.3.1. Parameter settings in the algorithms

The proposed approach involves Algorithms 1~4. The automatic frequency and amplitude extraction in a single computing frame realized by Algorithms 1~3 is embedded with Kalman filtering for continuous computing frame analysis, and the whole process is written in Algorithm 4. What followed provides a specific description of the parameter settings in those Algorithms.

- Parameters in Algorithm 1: APES

- a. Computing frame length  $N$  should be selected according to the longest period of signal components, and  $N$  is recommended to be 5 to 10 times the longest period.
- b. Calculating frequency  $\boldsymbol{\omega} = [\omega_1, \omega_2, \dots, \omega_K]^T$  is assigned according to frequency interval of interest, and  $0 \leq \omega_i \leq F_s/2$ . Generally, the elements of frequency vector  $\boldsymbol{\omega}$  are uniformly distributed, and the spacing  $\Delta\omega := \omega_{k+1} - \omega_k$  is set according the requirement of precision, such as 0.01Hz, 0.001Hz.
- c. Filter order  $M$  should be selected as a trade-off between the resolution and the statistical stability, and it is suggested  $N/4 \leq M \leq N/2$  [44, 51].

- Parameters in Algorithm 2: FSP

- a. Order  $m$  (Eq. (17)) is used to take the  $m$ th power of the normalized amplitude, which is beneficial to promote convergence and effectiveness of frequency squeezing, and is suggested as  $30 \sim 60$ .
- b. Step size  $P$  (Eq. (18)) is a key parameter for frequency squeezing. A large value of  $P$  is helpful for spectrum concentration. However, a too large  $P$  will lead to the missing of natural frequency with relatively small amplitude. Generally, the frequency span composed of the  $2P + 1$  spectral lines shouldn't include more than 2 natural frequencies.

- c. Threshold  $\delta_\omega$  (Eq. (19)) is used as a criterion to judge the edges of frequency clusters.

Based on the observation that frequency points will be extremely gathered at the natural frequencies after several steps of frequency squeezing (Eq. 18), a small  $\delta_\omega$  should be selected and we recommend it as  $\delta_\omega = 0.01\Delta\omega$ , where  $\Delta\omega := \omega_{i+1} - \omega_i$ .

- d. Threshold  $\delta_s$  is used to stop iteration of frequency squeezing, a small value such as  $1e-4 \sim 1e-8$  could be set.
- e. Maximum iterations  $I$  is for compulsively stopping the iteration in case the frequency squeezing fails to converge in finite steps. In this paper, it is set as  $I = 50$ .

- Parameters in Algorithm 3: FSP based frequency and amplitude extraction

- a. Threshold  $\delta_\Omega$  (Eq. (20)) is used to separate the processed frequency  $\hat{\omega}$  into  $P$  sub-intervals in which the maximum distance between individuals is less than  $\delta_\Omega$ . And we recommend it as  $F_s/N$ , which is the precision of fast Fourier Transform (FFT).
- b. Order  $n$  (Eq. (24a)) is used to truncate decimal places, and it's suggested as  $n = \max(1, \lfloor \log_{10}(1/\Delta\omega) \rfloor)$ .
- c.  $\delta_d$  (Eq. (22)) determines a threshold to judge if the frequency deviation caused by FSP is acceptable, and it's also suggested to be  $F_s/N$ .

- Parameters in Algorithm 4: Frequency and amplitude tracking using APES postprocessing and Kalman filtering

- a. Initial frequency  $\omega_{n,0}$  and amplitude  $\alpha_{n,0}$  can be obtained from historical data, and  $\omega_{n,0}$  can also be assigned as the reference frequency  $\omega_r$ .
- b. Updating block length  $S$  controls the output frequency of results, i.e., a smaller  $S$  means a higher time resolution. Meanwhile, the computations in a single step should be completed within  $S$  samples, so  $S$  can't be too small (see in Eq.8). And we recommended it as  $1/10$  to  $1/5$  times the computing frame length  $N$ .

- c. Weight  $\rho$  (Eqs. (31f),(31g) balances the energy metric and distance metric for updating covariances of process noise and measurement noise, and it's 0.5 used in this paper.
- d.  $\delta_a$  is the allowable range of frequency variation with respect to reference value  $\omega_r$ . It could be a vector with the same length as the reference frequency vector  $\omega_r$ , that is, each order of reference frequency  $\omega_{r,i}$  has its own allowable variation range; or it could be a scalar, that is, all order reference frequencies have the same allowable variation range.

### 3.3.2. Usages

The proposed method consists of Algorithms 1~4 could be employed for different purposes in different ways, in addition to the online frequency and amplitude tracking, which is the most focus of this paper, as illustrated in Fig. 5.

1. Using Algorithms 1 and 2 (APES+FSP) for TFR obtaining. This combination is equivalent to SST in terms of result exhibition, as shown in Fig. 5(a). Because the spectral lines of a TFR slice are concentrated at natural frequencies by frequency squeezing, so we called the result squeezed TFR.
2. Using Algorithms 1~3 (defined as FAE in Eq. (25)) for sparse TFR obtaining. In this combination, the potential spectral lines corresponding to natural frequencies are further extracted on the base of squeezed TFR. That is, the number of spectral lines of each slice is reduced from  $K$  to  $P_k$ , and  $P_k \ll K$ , so we called the result here sparse TFR, as shown in Fig. 5(b).
3. Using Algorithms 1~4 (FAE+Kalman filtering(KL)) for online frequency and amplitude tracking. In which, the Kalman filtering enhances the reliability of sparse TFR and the adopted frequency alignment removes all non-natural spectral lines, as shown in Fig. 5(c).

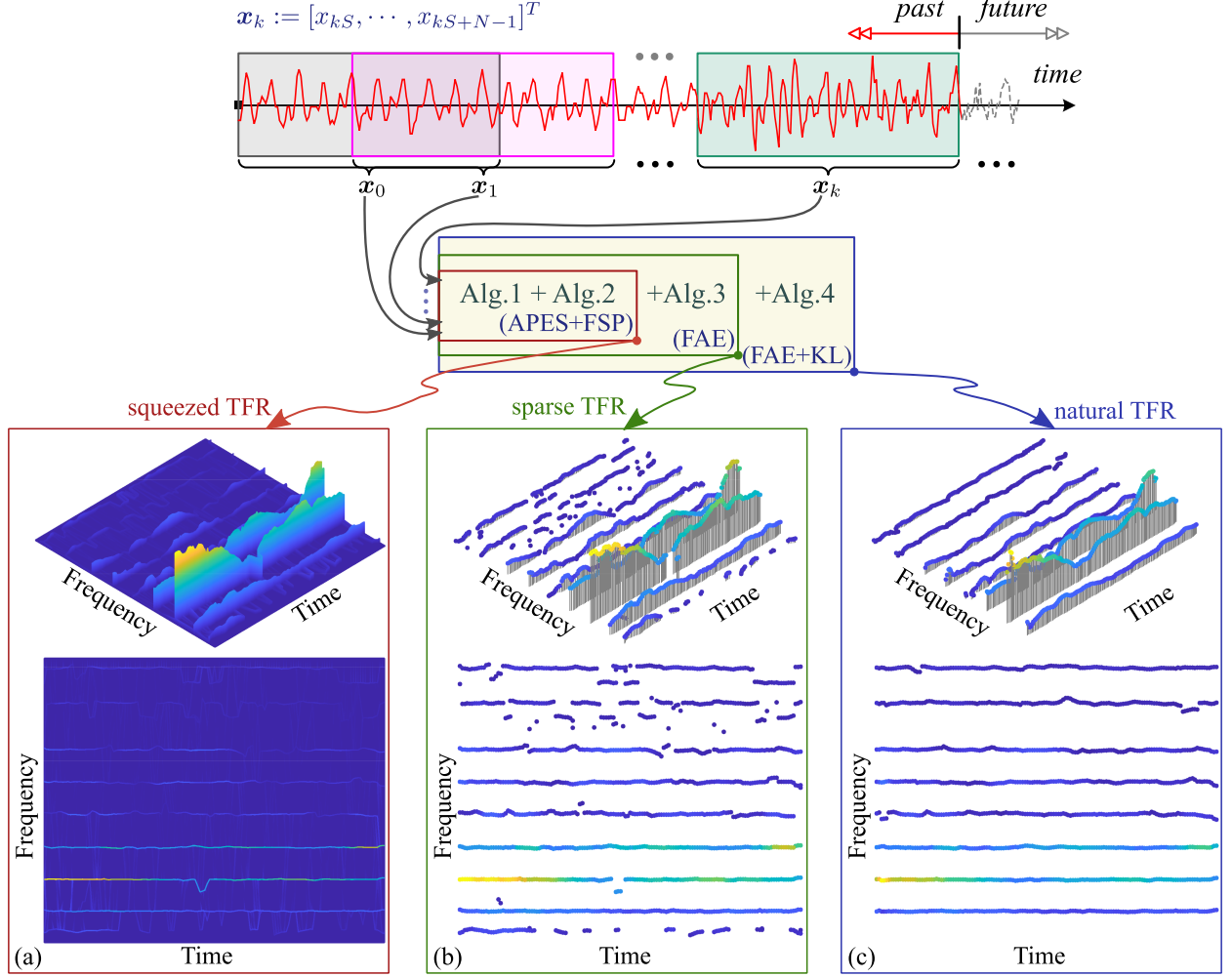


Figure 5: Usages of the proposed Algorithms. (a) Algorithms 1 and 2 for squeezed TFR. (b) Algorithms 1~3 for sparse TFR. (c) Algorithms 1~4 for natural frequency and amplitude tracking.

We illustrate the usages of 1 and 2 here is that sometimes the whole TFR plotting gives us more comprehensive and convincing results than the only surviving frequency trajectories. However, the latter supports further quantitative analysis.

### 3.3.3. Characteristics

The main characteristics of the proposed method are described below.

1. The method is applicable to online frequency and amplitude tracking with a small analysis lag and high resolution due to the superiority of APES. And it's data-dependent in two aspects: one is the data-dependent 'basis' of APES spectra analysis and the other is the adaptive noise variance setting in Kalman filtering.

2. The framework is easily altered and derives other specific approaches, such as replacing APES with other spectral analysis methods, adopting other frequency alignment strategies, establishing the state equation in Kalman filtering with other forms.
3. The signal reconstruction using the identified frequency and amplitude can be regarded as the filtering of estimated signal.
4. The frequency alignment requests that the different modes are well separated, this is true for most civil structures.

#### 4. Numerical verification

Because the true values of frequency and amplitude of realistic signals are unknown, before diving to them, we investigate a numerical example to explicitly illustrate the performance of proposed method on accuracy and noise robustness.

The considered synthetic signal consists of three components and an additive Gaussian white noise term:

$$\begin{aligned}
 x(t) &= x_1(t) + x_2(t) + x_3(t) + n(t), \quad 0 \leq t \leq 40, \\
 x_1(t) &= 3 \cos(2\pi t + 2 \sin(0.25\pi t)), \quad x_2(t) = 4e^{-0.05t} \cos(5\pi t - 0.005\pi t^2), \\
 x_3(t) &= \begin{cases} 3.0 \cos(8\pi t), & 0 \leq t < 10, 20 \leq t < 30, \\ 1.5 \cos(8\pi t), & 10 \leq t < 20, 30 \leq t < 40, \end{cases} \quad n(t) \sim \mathcal{N}(0, \sigma_n^2),
 \end{aligned} \tag{34}$$

where  $\sigma_n$  denotes the standard deviation of the white noise. In this section, a low noise level ( $\sigma_n = 0.5$ ) case in which SNR=21.52dB and a high noise level ( $\sigma_n = 2.0$ ) case in which SNR=-2.57dB are analyzed. And the SNR is calculated as

$$\text{SNR} = 10 \log_{10} \frac{\|signal\|_2^2}{\|noise\|_2^2}. \tag{35}$$

Fig. 6 plots the components, noise term, synthetic signal and IF laws defined in Eq. (34).

The IA functions can be seen from the envelopes of components in Figs. 6(a)~(c).

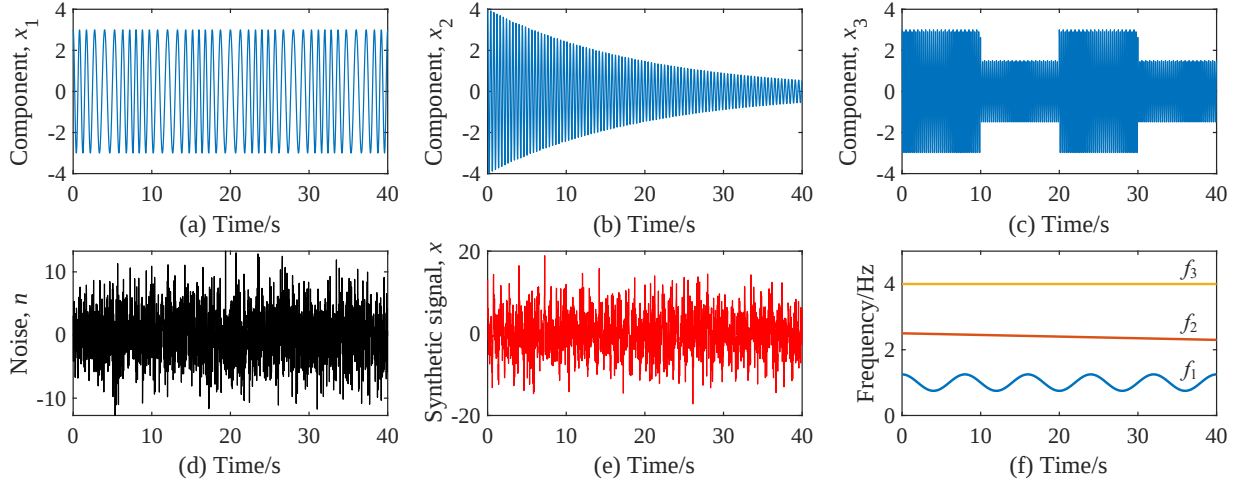


Figure 6: Numerical signal in Eq. (34). (a)~(c) Components 1~3. (d) Noise term (showed  $\sigma_n = 2.0$ ). (e) Synthetic signal. (f) IF law.

To validate the effectiveness of the method proposed, we compare it with two other advanced offline methods. One method is based on VMD, in which the multi-component signal is first decomposed with VMD; Then instead of using Eqs. (2) or (4) to calculate the IF, which performance will be greatly reduced with the emergence of noise, we conduct SST on the decomposed mono-components and ridges are extracted from the TFR as the estimate of IF functions. This technical route has also been reported in [52]. Another method is based on SST to get the TFR of the multi-component signal directly followed with the detection of ridges, and these ridges are projected as IA and IF functions.

Fig. 7 shows the analysis results corresponding to the case of low Gaussian white noise. In this situation, VMD enjoys remarkable signal decomposition (Fig. 7(a)), and the IF trajectories obtained by SST (Fig. 7(b)) and our method (Fig. 7(c)) are also very clear. Thus, the identified amplitudes and frequencies (Fig. 7(d)~(f)) are very close to the true values except for the amplitudes obtained by SST, which is distorted during the postprocessing. Of course, this can be avoided by locating the original amplitudes to the detected ridges.

Accordingly, Fig. 8 shows the analysis results corresponding to the case of high Gaussian white noise. As we can see, the noise extremely destroys the waveform of the decomposed

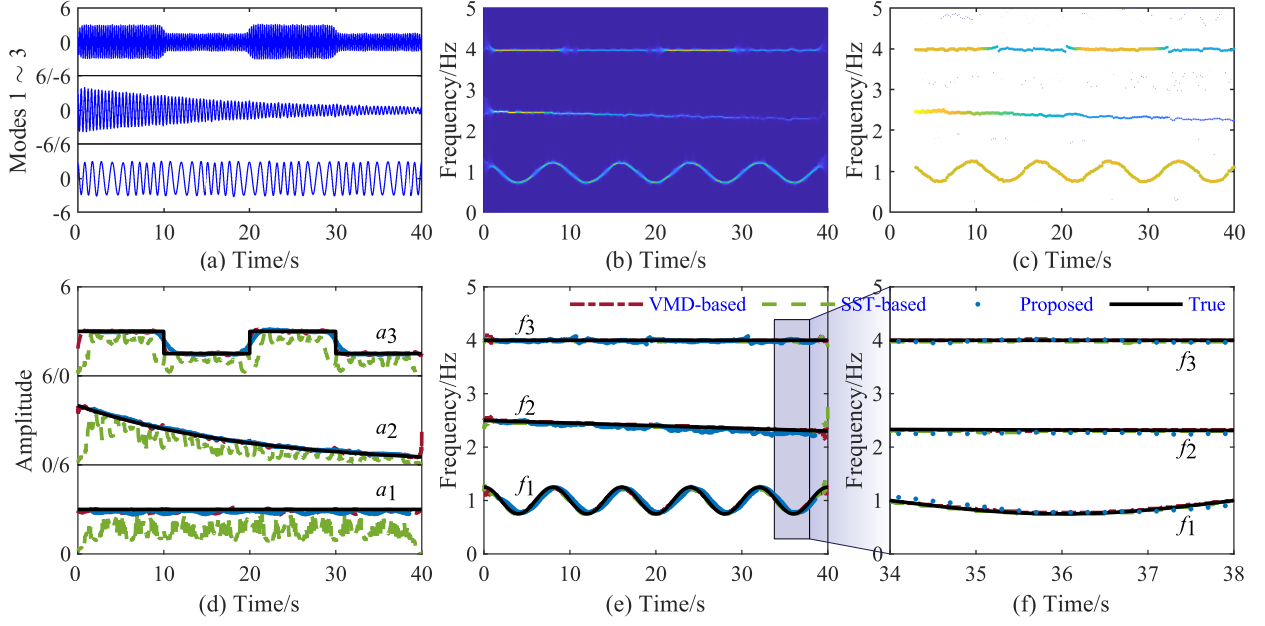


Figure 7: Analysis results of signal in Eq. (34) with a low noise level ( $\sigma_n=0.5$ ,  $\text{SNR}=21.52\text{dB}$ ). (a) Decomposed modes by VMD. (b) TFR by SST. (c) Sparse TFR by FAE, see in Fig. 5(b). (d) Amplitude obtained by VMD-, SST-based and proposed method. (e) Frequency obtained by VMD-, SST-based and proposed method. (f) Partial enlargement of (e).

mode by VMD (compare Fig. 8(a) with Fig. 6(a)~(c)) and contaminates the TFR obtained by SST (Fig. 8(b)) and our method (Fig. 8(c)) as well. The extraction of amplitude and frequency is affected accordingly (Figs. 8(d)~(f)) as it can be regarded as a further postprocessing of mode decomposition and TFR. To be specific, in Fig. 8(d), the amplitude directly obtained by ridge detection makes no-meaningful because it has been distorted by SST. Due to the noise dispersion among decomposed modes, in VMD-based method, the noise is obviously also introduced in the amplitude that is estimated by calculating the envelopes of these modes. But for the method proposed in this paper, it gains the closest amplitude estimation to true values. And in Figs. 8(e) and (f), the frequency in VMD-based approach (note that is obtained by conducting SST-based method on each decomposed IMF) performs better than that in SST-based method, i.e., it's found that decompose the multi-component signal into IMFs and then transform them into TF plane with TFA tools for

further ridges detection behaves better IF estimations than directly conduct TFA on the multi-component signal followed by ridges detection. However, the 2nd ( $f_2$ ) and the 3rd ( $f_3$ ) frequencies extracted by VMD- and SST-based methods at the end of the time window (Fig. 8(f)) are more deviate from the true positions, which is caused by the amplitude decrease of components  $x_2$  and  $x_3$  resulting in a lower SNR. Relatively, the estimates by the proposed method in this paper are closer to true values. Those advantages of higher accuracy and noise-robustness in frequency as well as amplitude extracting owe a good deal to, first, for method based on spectra analysis, the noise energy is widely dispersed throughout the whole frequency space, so the amplitude extracted from the spectrum is less affected by noise than the calculation in time domain; Second, the spectrum analyzed by a length of data series endows the extracted parameters more property of average value and better noise resistance; Third, as mentioned above, APES spectrum performs better than traditional FT with high-resolution and low side-lobes; Finally, the Kalman filtering embedded in our method enhances the ability in parameters tracking under the case of low signal energy, i.e., weak excitation in ambient.

Note that although the proposed method and the compared methods are both available to reveal the variations of parameters over time, the former is already applicable to online frequency and amplitude tracking as demonstrated in the scheme in Fig. 2 while the latter is still realized in batch processing.

The parameter settings of the algorithms in the above numerical cases are summarized in Table 1.

## 5. Realistic application

In this section, we will apply the proposed method to track the natural frequency and amplitude from the vibration signals of three actual structures recorded by their SHM sys-

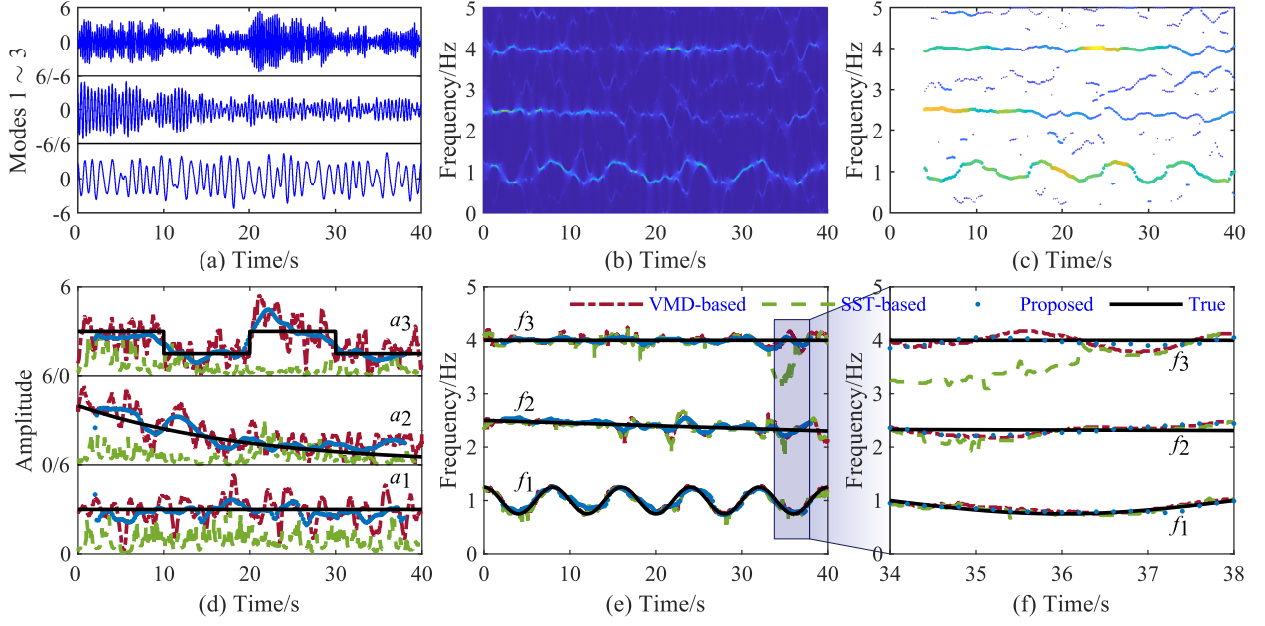


Figure 8: Analysis results of signal in Eq. (34) with a high noise level ( $\sigma_n=2.0$ ,  $\text{SNR}=-2.57\text{dB}$ ). (a) Decomposed modes by VMD. (b) TFR by SST. (c) Sparse TFR by FAE, see in Fig. 5(b). (d) Amplitude obtained by VMD-, SST-based and proposed method. (e) Frequency obtained by VMD-, SST-based and proposed method. (f) Partial enlargement of (e).

tems. They are the ambient vibrations of a cable of a cable-stayed bridge (Section 5.1), a hanger of a half-through arch bridge (Section 5.2) and the girder of a suspension bridge (Section 5.3), as shown in Fig. 9. In each case, the analyzed signal lasts for 24 hours. The parameter settings in the algorithms are also summarized in Table 1 and will not be described at the time of discussing these cases in detail.

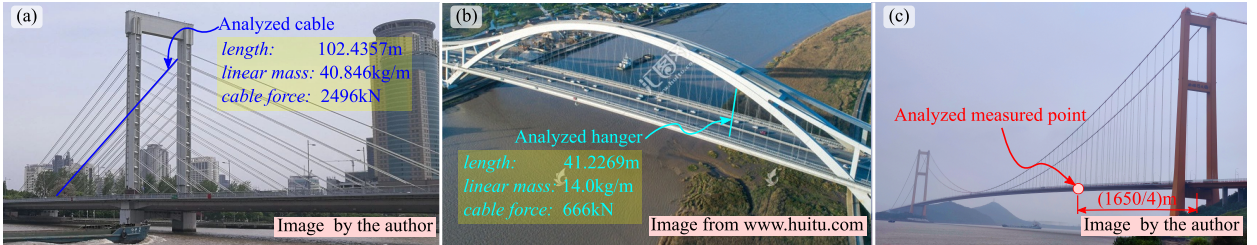


Figure 9: Actual structures that generate the vibrations. (a) A cable of a cable-stayed bridge, span layout: (105+97)m. (b) A hanger of a half-through arch bridge, span layout: (100+450+100)m. (c) The girder of a suspension bridge, span layout: (578+1650)m.

Table 1: Parameter setting of the proposed method in analyzing numerical and realistic cases.

| Alg. | Para.           | Sec. 4                | Sec. 5.1              | Sec. 5.2               | Sec. 5.3                                      |
|------|-----------------|-----------------------|-----------------------|------------------------|-----------------------------------------------|
| 1    | $N$             | 150                   | 150                   | 100                    | 4000                                          |
|      | $\omega$        | $[0, 0.01, \dots, 5]$ | $[0, 0.01, \dots, 8]$ | $[0, 0.01, \dots, 12]$ | $[0, 0.001, \dots, 0.5]$                      |
|      | $M$             | 37                    | 37                    | 37                     | 666                                           |
|      | $F_s$           | 50                    | 50                    | 50                     | 50                                            |
| 2    | $m$             | 50                    | 50                    | 50                     | 50                                            |
|      | $P$             | 25                    | 60                    | 100                    | 20                                            |
|      | $\delta_\omega$ | 1e-4                  | 1e-4                  | 1e-4                   | 1e-5                                          |
|      | $\delta_s$      | 1e-8                  | 1e-6                  | 1e-6                   | 1e-6                                          |
|      | $I$             | 50                    | 50                    | 50                     | 50                                            |
| 3    | $n$             | 2                     | 2                     | 2                      | 3                                             |
|      | $\delta_d$      | 0.3333                | 0.3333                | 0.5                    | 0.0125                                        |
| 4    | $S$             | 10                    | 15                    | 10                     | 100                                           |
|      | $\omega_r$      | $\omega_{n,k}$        | 1.25[1,2,3,4,5,6]     | 2.50[1,2,3,4]          | [.095,.133,.183,.230,<br>.275,.324,.435,.492] |
|      | $\omega_{n,0}$  | [1.1,2.49,4.0]        | 1.25[1,2,3,4,5,6]     | 2.50[1,2,3,4]          | [.095,.133,.183,.230,<br>.275,.324,.435,.492] |
|      | $\alpha_{n,0}$  | [3.0,3.71,3.0]        | [1,1,1,1,1,1]         | [1,1,1,1]              | [1,1,1,1,1,1,1,1]                             |
|      | $\rho$          | 0.5                   | 0.5                   | 0.5                    | 0.5                                           |
|      | $\delta_a$      | 0.5                   | 0.3125                | 0.625                  | 0.0238                                        |

### 5.1. Cable vibrations of a cable-stayed bridge

The vibration signals of a cable of a cable-stayed bridge (Fig. 9(a)) that last from 0:00 at midnight to the next day (Fig. 10(a)) is analyzed. The cable is 102.4357m long with a

linear mass of 40.846kg/m, and the cable force measured once is 2496kN. The first 6 orders of frequencies and amplitudes of interests are tracked by the proposed method.

Fig. 10(b) plots the time-varying frequencies, and the results during 2.5~2.52h and 8.5~8.52h are zoomed in Figs. 10(c) and (d). The signals corresponding to these two intervals represent the situations of relatively small and high vibrations, respectively. For the time between 2.5~2.52h at night, there is supposed to be less traffic on the bridge deck than the time between 8.5~8.52h at day, so the signal during the former interval has lower vibration energy. As we can see from Figs. 10(c) and (d), the proposed method is able to track the frequency continuously no matter under the low vibration energy or high. Actually, there still exist instants corresponding to some computing frames, where the shape of amplitude spectrum obtained by APES is not regular enough to exactly extract the natural frequencies and amplitudes of interest. Here a regular spectral shape means that the lobes at all the natural frequencies are higher than the lobes caused by noise. And the non-regular spectral shape is not only going to appear in the case of weak environmental excitation, but also in the case of high environmental excitation because not all modes are excited well. The non-regular spectral shape results in the raw extracted parameters not-aligned and then the frequency-alignment in Eq. (33) is activated to make the identified frequencies aligned and continuous. In the operation of frequency-alignment, if there are no candidates within the allowable variation range of reference value, the frequency in current step is inherited from the last step while the amplitude is taken as half of value in the last step. Thus, in Figs. 10(c) and (d), some straight segments of zero amplitude are observable.

Furthermore, Figs. 11 and 12 redraw the identified results in Figs. 10(c) and 10(d) in different panels separately according to the order, which provide a clearer view revealing the frequency fluctuations with respect to magnitudes of the identified amplitude. It's easy to find that, if the amplitude enjoys a continuous and moderately large magnitude

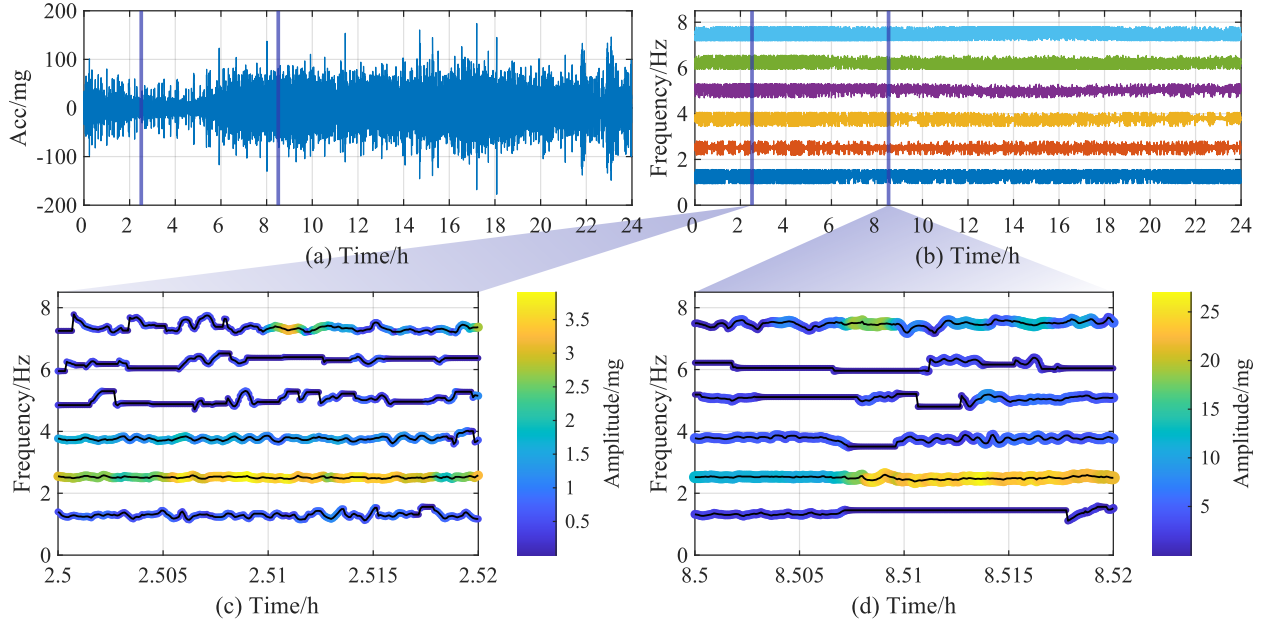


Figure 10: (a) Acceleration (Acc.) time history of a cable of a cable-stayed bridge in 24 hours. (b) Tracked first six natural frequencies. (c) Enlargement of (b) during 2.5~2.52h. (d) Enlargement of (b) during 8.5~8.52h.

(Figs. 11(b) and 12(b)), the corresponding frequency enjoys a smaller variance. Otherwise, if the amplitude is at a low level and changes quickly (Figs. 11(a), (f) and 12(f)), the corresponding frequency is more discrete. This is reasonable since our method assumes that the signal during the analytical time window of  $N$  length is stationary. So, the relatively large variation of frequency may be more likely to be caused by the fast or abruptly changing amplitude, not the variation of structural characteristic itself. This also suggests that the frequency results with large amplitudes are more reliable.

And the observations above inspired us to further plot the tracking results of frequency and amplitude in the manner as shown in Fig. 13. In which the amplitudes of each order are divided into ten intervals, and those time-frequency-amplitude (F-T-A) points belonging to the same certain amplitude interval will be assigned to a group, which is expressed by giving them the same color in the figure. From Fig. 13, it's found that the identified frequency-amplitude points with a relative low amplitude have a wider distribution in the

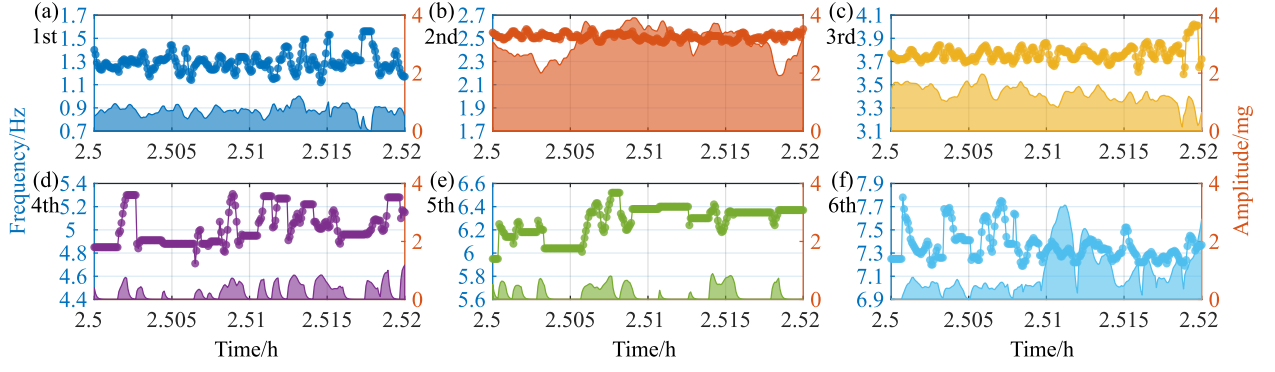


Figure 11: Frequency and amplitude of each order corresponding to the result in Fig. 10(c). (a)~(f) Results from 1st to 6th order.

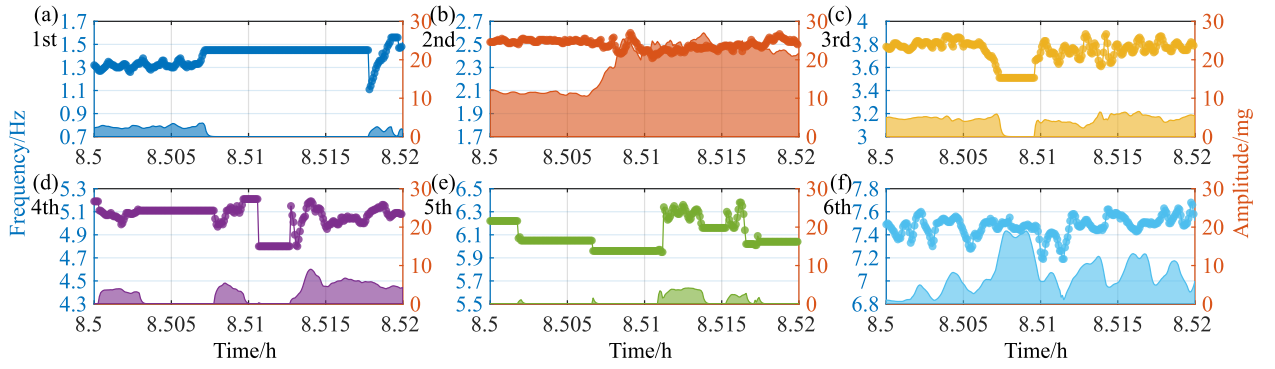


Figure 12: Frequency and amplitude of each order corresponding to the result in Fig. 10(d). (a)~(f) Results from 1st to 6th order.

frequency direction. And the variance of low-order frequency is greater than that of high-order frequency. Besides, the frequency of high order with relative high amplitude exhibit a trend (the 3rd, 4th, 6th order results in Fig. 13). That may be caused by the temperature variation, which is beyond the scope of this paper.

## 5.2. Hanger vibrations of a half-through arch bridge

In the last subsection, we have discussed the results of a cable vibration in detail. Here we give the analysis results of the 24-hour vibrations of a hanger of a half-through arch bridge (9(b)) in brief since the major conclusions are similar between them. The hanger is 41.2269m long with a linear mass of 14.0kg/m, and the cable force measured once is 666kN.

Fig. 14(a) is the time history of the hanger vibrations and Fig. 14(b) shows the obtained

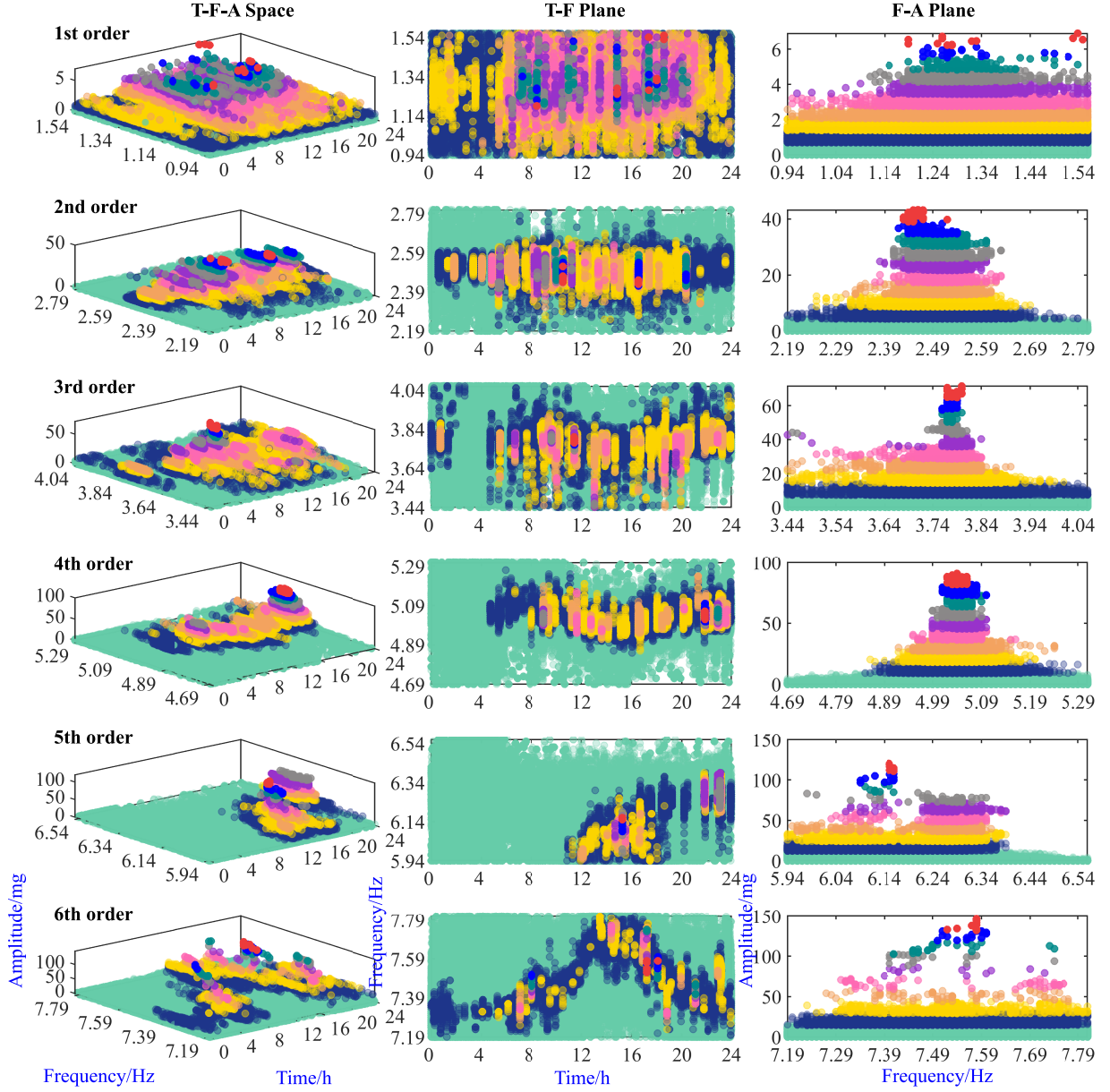


Figure 13: The results stratified by amplitude magnitude. The 1st column: T-F-A 3D view; the 2nd column: T-F plane view; the 3rd column: F-A Plane view. The 1st ~ 6th row: the results of the order from 1st to 6th.

frequencies of the first 4 modes. For a clearer view of the frequency changes, an enlargement of Fig. 14(b) in the time interval between 2.5h to 2.52h is given in Fig. 14(c). What's more, as what Figs. 11 and 12 do, Fig. 14(d) plots the frequency as well as the amplitude of each order separately, note that the limitations of the frequency and amplitude axes of each panel have been set to be consistent for comparison purpose.

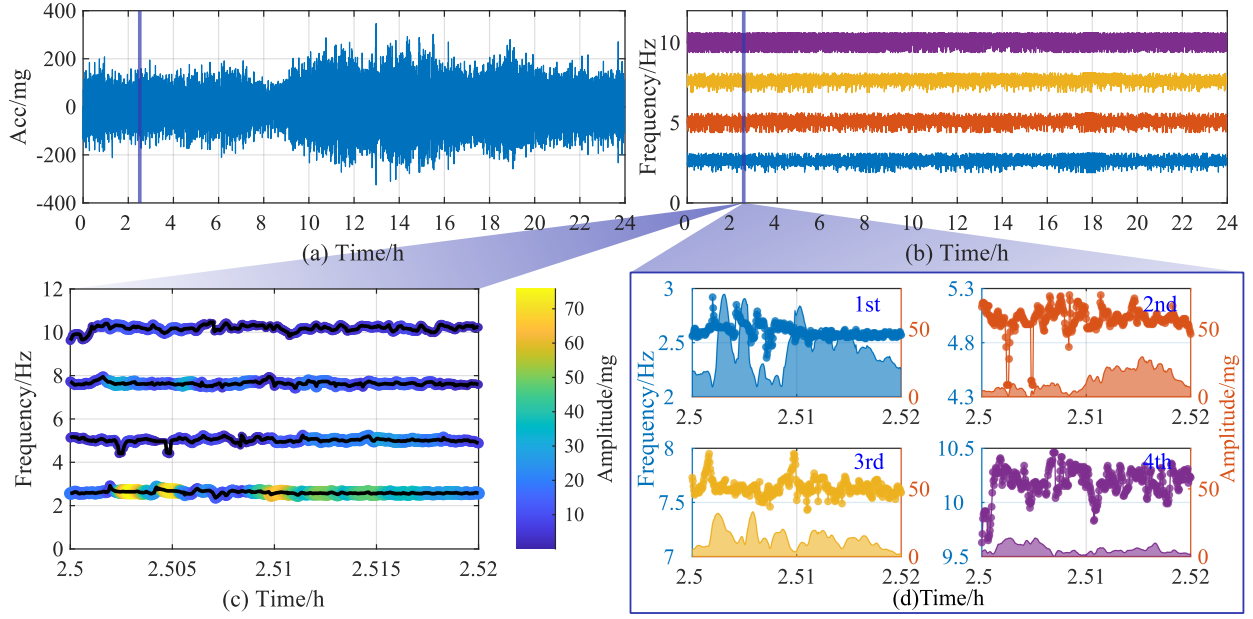


Figure 14: (a) Acceleration time history of a hanger of a half-through arch bridge in 24 hours. (b) Tracked first four natural frequencies. (c) Enlargement of (b) during 2.5~2.52h. (d) Frequency and amplitude of each order corresponding to the result in (c).

As we can see, for this about 41m hanger, which is more well excited than the cable in the last subsection (compare Fig. 14(a) with 10(a)), showing more convincing frequency identification results (compare Fig. 14(c) with 10(c)). And Fig. 14(d) provides more proof that the frequency variation may also be caused by the rapidly varying amplitude not only by the real variation of itself. And the former is attributed to the assumption that the amplitude is slowly varying in an analytical window used in spectral analysis. And this assumption is common in many existing methods no matter whether to mode decomposition or TFA.

### 5.3. Girder vibrations of a suspension bridge

Lastly, we analyze the vibration signals generated by the girder of a suspension bridge, and the signals are recorded by an accelerometer installed at 1/4 of the main span, as shown in 9(c). Compared with the frequency and amplitude tracking of the cable or hanger, whose vibrational modes are well separated by almost equal distances, the same objective for the long-span suspension bridge becomes more challenging because its dominated modes are

concentrated in the low frequency region and it's possible to happen mode crossover.

Similarly, the acceleration time history of the main girder is depicted in Fig. 15(a). And Fig. 15(b) is a scatter graph of the tracked results, showing the frequencies at different instants and the brighter color means larger amplitude corresponded. As we can see, the suspension bridge exhibits 8 close vertical modes under 0.5Hz, and the frequency trajectories fluctuate around the early measured natural frequencies, which can be obtained from the Table 1 of the paper by H. Li et, al. [53]. Also, we list the 8 measured natural frequencies here. They are 0.0953, 0.1328, 0.1825, 0.2301, 0.2757, 0.3237, 0.4351 and 0.4918Hz. Observe the interval marked in Fig. 15, it is found that the bridge experienced vortex-induced vibrations (VIVs). The occurrence of VIV can be observed from both the acceleration time history in Fig. 15(a) and the extracted frequencies and amplitudes in Fig. 15(b). And the latter gives us more information about the energy distribution among different modes and which mode is participating in the VIV.

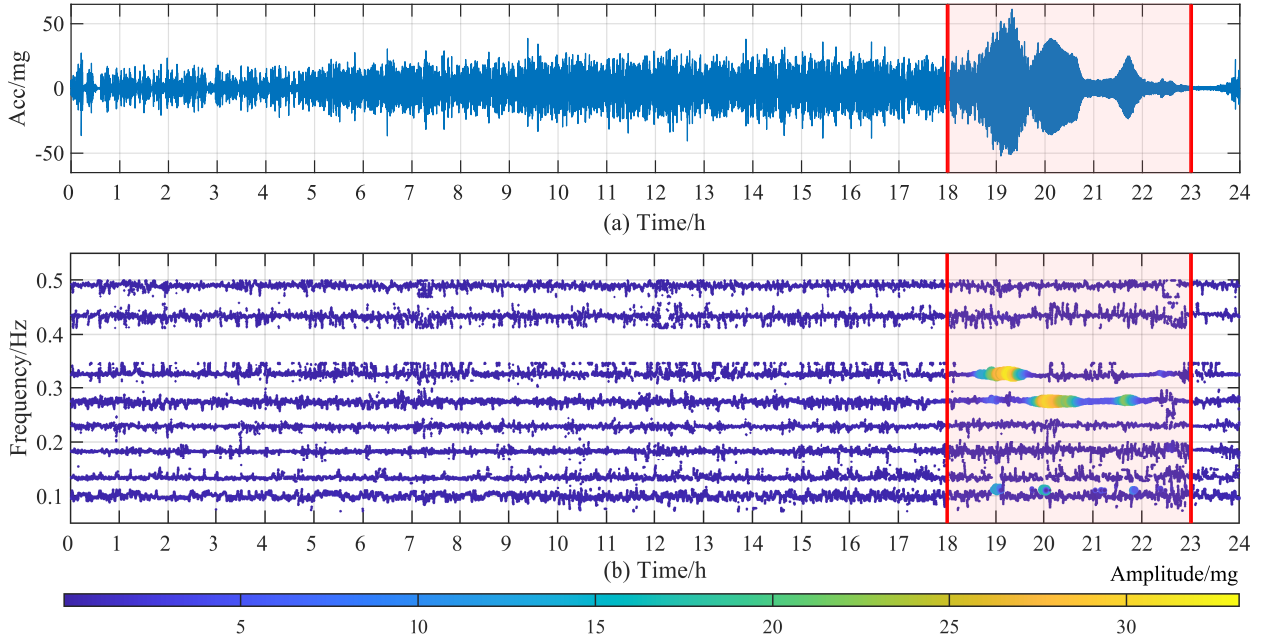


Figure 15: (a) Acceleration time history of the girder of a suspension bridge in 24 hours. (b) Tracked results in the low frequency zone.

We enlarge the results during the VIV as shown in Fig. 16(a) and plot the time-varying

amplitude in Fig. 16(b). It's interesting to find that there exists an interchange for the modes participating in the VIV. At the beginning, the vibration energy is concentrated on the 6th mode, then the 6th mode's energy is progressively reduced while the 5th mode's energy is steadily raised until the 5th mode becomes the dominant one. Similarly, at the end of the disappearance of VIV on the 5th mode, the vibration energy is concentrated on the 6th mode again for a short while. This should be the product of the interactions between structure and wind. Additionally, from Fig. 16, we can determine when the VIV happens, which mode is most exciting and how long the VIV lasts. These discoveries are going to offer a new fundamental understanding of VIV of large-scale suspension bridge for the monitoring, perception and vibration control. Of course, this is beyond the scope of this paper and will be one of the focuses of our future work.

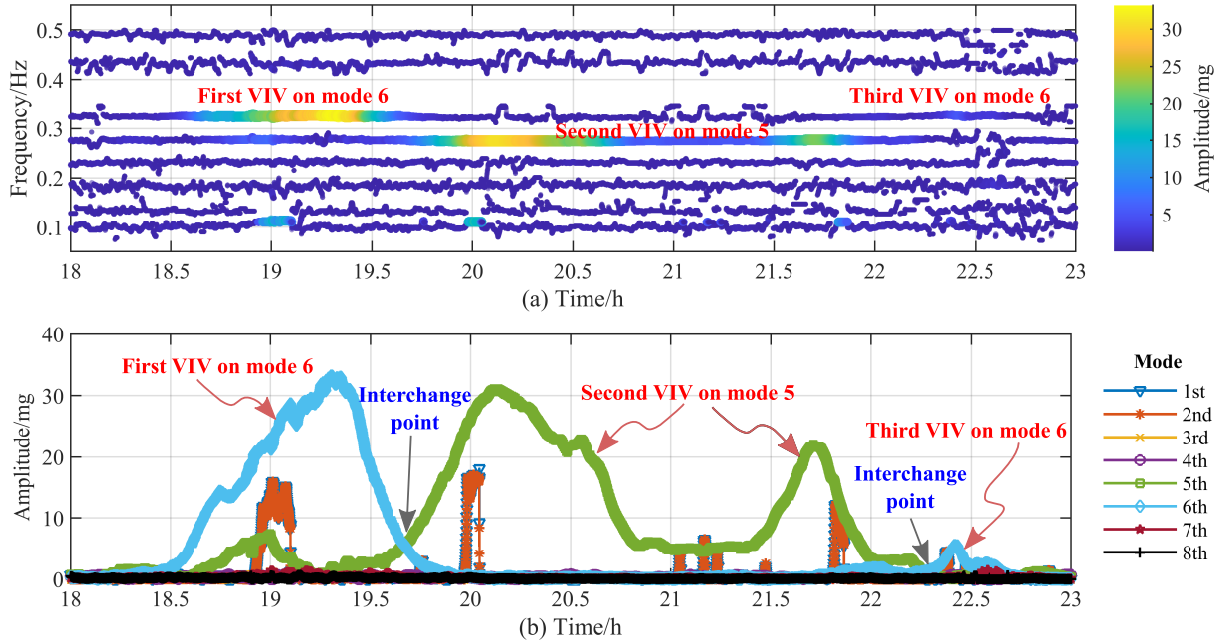


Figure 16: (a) Enlargement of the results in Fig. 16(b) during 18~23h where VIVs occurred. (b) The tracked amplitudes corresponding to the first 8 mode during 18~23h.

Furthermore, we pull our eyes back to the tracking results of the time-varying frequency (Figs. 15(b) and 15(a)). The proposed method is able to obtain the time series of frequency

and amplitude of each mode. However, for this main girder with close modes in the low frequency zone, under environmental excitation, the mode mixing or coinciding are inevitable in the identified frequencies. Thus, the frequency trajectories in Figs. 15(b) and 15(a) at some intervals appear turbulent. On the other hand, to what extent this 'turbulence' is a true reflection of structural characteristics is worth further investigation. This phenomenon is slighter in the cases of cable and hanger discussed above because their modes are more separated and their vibrations are more easily excited.

## 6. Conclusion

For the purpose of tracking natural frequency and amplitude of civil structures from their monitoring data, this paper develops a method based on APES spectrum postprocessing and Kalman filtering. Its effectiveness is validated through the analysis of numerical examples and the comparison with VMD- and SST-based methods, exhibiting more accurate and robust results in the case of low SNR. What's more, we employ the proposed method to the analyze the actual vibration signals of a cable of a cable-stayed bridge, a hanger of an arch bridge and the girder of a suspension bridge. While proving the ability of the proposed method to extract frequency and amplitude online in environmental vibrations, some main conclusions are reached as follows:

1. The randomness of ambient excitation will increase the variance of identified frequencies. This is because the severe amplitude modulation is going to submerge the information of structural oscillations and bring errors in the identified frequency positions. On the other hand, those frequencies with relatively large and slow-changing amplitudes are more reliable. In other words, the identified amplitude can be used as a measure of the reliability of the identified frequency.
2. For multi-component structural vibrations, it is not the total signal energy but the

energy distribution over all components that should be paid more attention. And this indicates another significance of time-varying amplitude tracking.

3. The proposed online data processing method shows the application potential for damage diagnosis and abnormal event detection suitable for SHM system, e.g., the VIV monitoring of suspension bridge discussed in the Section 5.3.

## Acknowledgments

This work is supported by the Opening Foundation of National Key Laboratory of Bridge Structural Health and Safety; the National Nature Science Foundation of China (Grant No. 5187849); the National Key R&D Program of China (2017YFF0205605).

## References

- [1] Y. Bao, Z. Chen, S. Wei, Y. Xu, Z. Tang, H. Li, The state of the art of data science and engineering in structural health monitoring, *Engineering* 5 (2019) 234–242.
- [2] S. Sony, S. Laventure, A. Sadhu, A literature review of next-generation smart sensing technology in structural health monitoring, *Structural Control and Health Monitoring* 26 (2019) e2321.
- [3] B. Boashash, Estimating and interpreting the instantaneous frequency of a signal. i. fundamentals, *Proceedings of the IEEE* 80 (1992) 520–538.
- [4] O. Salawu, Detection of structural damage through changes in frequency: a review, *Engineering structures* 19 (1997) 718–723.
- [5] M. Mousavi, D. Holloway, J. Olivier, A. H. Gandomi, Beam damage detection using synchronisation of peaks in instantaneous frequency and amplitude of vibration data, *Measurement* 168 (2021) 108297.

- [6] H. Pan, M. Azimi, F. Yan, Z. Lin, Time-frequency-based data-driven structural diagnosis and damage detection for cable-stayed bridges, *Journal of Bridge Engineering* 23 (2018) 04018033.
- [7] N. E. Huang, Z. Shen, S. R. Long, M. C. Wu, H. H. Shih, Q. Zheng, N.-C. Yen, C. C. Tung, H. H. Liu, The empirical mode decomposition and the hilbert spectrum for nonlinear and non-stationary time series analysis, *Proceedings of the Royal Society of London. Series A: mathematical, physical and engineering sciences* 454 (1998) 903–995.
- [8] G. Chen, Z. Wang, A signal decomposition theorem with hilbert transform and its application to narrowband time series with closely spaced frequency components, *Mechanical systems and signal processing* 28 (2012) 258–279.
- [9] J. Gilles, Empirical wavelet transform, *IEEE transactions on signal processing* 61 (2013) 3999–4010.
- [10] K. Dragomiretskiy, D. Zosso, Variational mode decomposition, *IEEE transactions on signal processing* 62 (2013) 531–544.
- [11] Z. Feng, D. Zhang, M. J. Zuo, Adaptive mode decomposition methods and their applications in signal analysis for machinery fault diagnosis: a review with examples, *IEEE access* 5 (2017) 24301–24331.
- [12] T. Liu, Z. Luo, J. Huang, S. Yan, A comparative study of four kinds of adaptive decomposition algorithms and their applications, *Sensors* 18 (2018) 2120.
- [13] M. Barbosh, P. Singh, A. Sadhu, Empirical mode decomposition and its variants: a review with applications in structural health monitoring, *Smart Materials and Structures* 29 (2020) 093001.

- [14] M. Civera, C. Surace, A comparative analysis of signal decomposition techniques for structural health monitoring on an experimental benchmark, *Sensors* 21 (2021) 1825.
- [15] M. Feldman, Hilbert transform in vibration analysis, *Mechanical systems and signal processing* 25 (2011) 735–802.
- [16] A. Baccigalupi, A. Liccardo, The huang hilbert transform for evaluating the instantaneous frequency evolution of transient signals in non-linear systems, *Measurement* 86 (2016) 1–13.
- [17] P. Ni, J. Li, H. Hao, Y. Xia, X. Wang, J.-M. Lee, K.-H. Jung, Time-varying system identification using variational mode decomposition, *Structural Control and Health Monitoring* 25 (2018) e2175.
- [18] Y. Xin, J. Li, H. Hao, Enhanced vibration decomposition method based on multisynchrosqueezing transform and analytical mode decomposition, *Structural Control and Health Monitoring* 28 (2021) e2730.
- [19] P. F. Pai, Online tracking of instantaneous frequency and amplitude of dynamical system response, *Mechanical Systems and Signal Processing* 24 (2010) 1007–1024.
- [20] R. Zhong, P. F. Pai, An instantaneous frequency analysis method of stay cables, *Journal of Low Frequency Noise, Vibration and Active Control* 40 (2021) 263–277.
- [21] A. Cicone, J. Liu, H. Zhou, Adaptive local iterative filtering for signal decomposition and instantaneous frequency analysis, *Applied and Computational Harmonic Analysis* 41 (2016) 384–411.
- [22] D. Vakman, On the analytic signal, the teager-kaiser energy algorithm, and other methods for defining amplitude and frequency, *IEEE Transactions on signal processing* 44 (1996) 791–797.

- [23] Y. Bao, Z. Shi, J. L. Beck, H. Li, T. Y. Hou, Identification of time-varying cable tension forces based on adaptive sparse time-frequency analysis of cable vibrations, *Structural Control and Health Monitoring* 24 (2017) e1889.
- [24] F. Hlawatsch, F. Auger, *Time-frequency analysis*, John Wiley & Sons, 2013.
- [25] L. Stanković, I. Djurović, S. Stanković, M. Simeunović, S. Djukanović, M. Daković, Instantaneous frequency in time–frequency analysis: Enhanced concepts and performance of estimation algorithms, *Digital Signal Processing* 35 (2014) 1–13.
- [26] S. Mann, S. Haykin, The chirplet transform: Physical considerations, *IEEE Transactions on Signal Processing* 43 (1995) 2745–2761.
- [27] Z. Peng, G. Meng, F. Chu, Z. Lang, W. Zhang, Y. Yang, Polynomial chirplet transform with application to instantaneous frequency estimation, *IEEE Transactions on Instrumentation and Measurement* 60 (2011) 3222–3229.
- [28] F. Auger, P. Flandrin, Improving the readability of time-frequency and time-scale representations by the reassignment method, *IEEE Transactions on signal processing* 43 (1995) 1068–1089.
- [29] I. Daubechies, J. Lu, H.-T. Wu, Synchrosqueezed wavelet transforms: An empirical mode decomposition-like tool, *Applied and computational harmonic analysis* 30 (2011) 243–261.
- [30] G. Yu, M. Yu, C. Xu, Synchroextracting transform, *IEEE Transactions on Industrial Electronics* 64 (2017) 8042–8054.
- [31] B. Boashash, Estimating and interpreting the instantaneous frequency of a signal. ii. algorithms and applications, *Proceedings of the IEEE* 80 (1992) 540–568.

- [32] V. Popović-Bugarin, S. Djukanović, Efficient instantaneous frequency estimation in high noise based on the wigner distribution, *Signal Processing* 157 (2019) 25–29.
- [33] R. A. Carmona, W. L. Hwang, B. Torr  sani, Multiridge detection and time-frequency reconstruction, *IEEE transactions on signal processing* 47 (1999) 480–492.
- [34] S. Meignen, D.-H. Pham, S. McLaughlin, On demodulation, ridge detection, and synchrosqueezing for multicomponent signals, *IEEE Transactions on Signal Processing* 65 (2017) 2093–2103.
- [35] C. Wang, W.-X. Ren, Z.-C. Wang, H.-P. Zhu, Instantaneous frequency identification of time-varying structures by continuous wavelet transform, *Engineering Structures* 52 (2013) 17–25.
- [36] D. Iatsenko, P. V. McClintock, A. Stefanovska, Extraction of instantaneous frequencies from ridges in time–frequency representations of signals, *Signal Processing* 125 (2016) 290–303.
- [37] L. Rankine, M. Mesbah, B. Boashash, If estimation for multicomponent signals using image processing techniques in the time–frequency domain, *Signal Processing* 87 (2007) 1234–1250.
- [38] J. Liu, X. Wei, R.-H. Qiu, J.-Y. Zheng, Y.-J. Zhu, I. Laory, Instantaneous frequency extraction in time-varying structures using a maximum gradient method, *Smart Struct. Syst* 22 (2018) 359–368.
- [39] M. Brajović, V. Popović-Bugarin, I. Djurović, S. Djukanović, Post-processing of time-frequency representations in instantaneous frequency estimation based on ant colony optimization, *Signal Processing* 138 (2017) 195–210.

- [40] Y. Kaya, E. Safak, Real-time analysis and interpretation of continuous data from structural health monitoring (shm) systems, *Bulletin of Earthquake Engineering* 13 (2015) 917–934.
- [41] S. Quqa, L. Landi, P. P. Diotallevi, Instantaneous modal identification under varying structural characteristics: A decentralized algorithm, *Mechanical Systems and Signal Processing* 142 (2020) 106750.
- [42] J. Li, P. Stoica, An adaptive filtering approach to spectral estimation and sar imaging, *IEEE Transactions on Signal Processing* 44 (1996) 1469–1484.
- [43] L. Stanković, M. Daković, T. Thayaparan, A real-time time-frequency based instantaneous frequency estimator, *Signal processing* 93 (2013) 1392–1397.
- [44] E. G. Larsson, J. Li, P. Stoica, High-resolution nonparametric spectral analysis: Theory and applications, in: *High-resolution and robust signal processing*, volume 4, Marcel Dekker, 2003, pp. 153–253.
- [45] X. Yu, D. Dan, Block-wise recursive apes aided with frequency-squeezing postprocessing and the application in online analysis of vibration monitoring signals, *Mechanical Systems and Signal Processing* 162 (2022) 108063. doi:<https://doi.org/10.1016/j.ymssp.2021.108063>.
- [46] P. Stoica, R. L. Moses, et al., *Spectral analysis of signals*, Pearson Prentice Hall Upper Saddle River, NJ, 2005.
- [47] S. J. Julier, J. K. Uhlmann, Unscented filtering and nonlinear estimation, *Proceedings of the IEEE* 92 (2004) 401–422.
- [48] S. J. Julier, J. K. Uhlmann, New extension of the kalman filter to nonlinear systems, in:

Signal processing, sensor fusion, and target recognition VI, volume 3068, International Society for Optics and Photonics, 1997, pp. 182–193.

- [49] R. Van Der Merwe, et al., Sigma-point Kalman filters for probabilistic inference in dynamic state-space models, Ph.D. thesis, OGI School of Science & Engineering at OHSU, 2004.
- [50] V. Kučera, The discrete riccati equation of optimal control, *Kybernetika* 8 (1972) 430–447.
- [51] P. Stoica, A. Jakobsson, J. Li, Matched-filter bank interpretation of some spectral estimators, *Signal Processing* 66 (1998) 45–59.
- [52] C. Wang, J. Zhang, H. P. Zhu, A combined method for time-varying parameter identification based on variational mode decomposition and generalized morse wavelet, *International Journal of Structural Stability and Dynamics* 20 (2020) 2050077.
- [53] H. Li, S. Laima, J. Ou, X. Zhao, W. Zhou, Y. Yu, N. Li, Z. Liu, Investigation of vortex-induced vibration of a suspension bridge with two separated steel box girders based on field measurements, *Engineering Structures* 33 (2011) 1894–1907.