Visible Light Positioning with Multiple Transmitters and Multiple Receiving Sensors

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Abstract

Visible light communications (VLC) based positioning systems will form an important part of the future generation wireless communication systems because they offer higher accuracy for indoor positioning as compared to radio frequency based systems. In this paper, we have used non Bayesian statistical signal processing techniques for the hybrid time-of-arrival/received-signal-strength and hybrid time-difference-of-arrival/received-signal-strength based positioning. These hybrid measurements are combined with the following fusion algorithms: *weighted least squares* and the *best linear unbiased estimator*. These two fusion algorithms are compared in terms of the *mean Euclidean error* as a function of various parameters such as signal-to-noise ratio, transmitter arrangement and synchronization error. Even though the performance of the weighted least squares algorithm is better, the best linear unbiased estimator is still an attractive algorithm for systems that require a lower complexity.

Index Terms

Best linear unbiased estimator, mean Euclidean error, received-signal-strength, time-difference-of-arrival, time-of-arrival, visible light communications based positioning, weighted least squares

I. INTRODUCTION

Light-emitting-diode (LED) based visible light communications (VLC) has attracted a lot of attention in recent years [1]. This is primarily because such systems can provide illumination as well as high speed data rates to the users located indoors. Furthermore, such VLC systems can also help to position the receiver terminals indoors with sufficient accuracy as opposed to WiFi based

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localization that suffers from interference and fading due to multipath [2]. The accurate positioning performance provided by VLC systems is very important for future applications in sixth generation wireless systems such as robot navigation, as well as location based services and advertisements among many others [1].

A. Literature review

The focus of this paper is on utilizing statistical signal processing techniques in order to localize the receiver in an indoor visible light communications setting. Therefore, for the purpose of literature review, we have focused mainly on research that has approached visible light positioning from a signal processing perspective.

The authors in [2] have proposed Epsilon, which is a visible light positioning system based on the received signal strength measurements. Epsilon uses binary frequency shift keying signaling and channel hopping in order to avoid collisions from the uncoordinated LED sources. The positioning accuracy is reported to be 0.4 meters, 0.7 meters and 0.8 meters for different rooms/locations.

The authors in [3] derive the *Cramer-Rao Lower Bound* (CRLB) for the time-of-arrival (TOA) based transmitter/receiver distance estimation for windowed sinusoidal signals. The CRLB is derived under the assumption that the transmitter/receivers are synchronized in time. The authors in [4] improve upon this work by deriving the CRLB's for TOA, received-signal-strength (RSS), and the hybrid TOA-RSS systems, and they consider both the synchronous and asynchronous scenarios in their work. Moreover, the *maximum likelihood estimators* (MLE) are also proposed for the RSS and TOA based distance estimation. Furthermore, they also consider the effect of the sampling time on the CRLB. The same authors devised the positioning algorithm in [5] by extending the work in [4]. They consider a *direct positioning* algorithm in which the position is estimated by maximizing a likelihood function, and a *two-step positioning* algorithm in which the maximum likelihood estimates of time-delay and RSS are combined using a nonlinear least squares algorithm. Additionally, they consider the positioning for synchronous as well as asynchronous systems and derive the CRLB curves for different positioning scenarios. In their work, they conclude that the direct positioning yields more accurate results than the two-step system. However, the better accuracy of direct positioning comes with the extra cost of higher computational complexity.

The focus of [6] is to consider the effect of multipath on the TOA based positioning. They have modeled the delay profile as a Gamma function, and analyzed the effect of the shape parameter and

the scale parameter of the Gamma function on the *Generalized Cramer-Rao Lower Bound* (GCRLB). However, in most VLC scenarios, the non-line-of-sight (NLOS) component of the received signal is insignificant compared to the line-of-sight (LOS) signal [3], [4], and hence the application of [6] is limited to the situations where the NLOS portion of the signal is significantly larger.

The authors in [7] have presented an angel-of-arrival based positioning algorithm that employs least squares to achieve localization. They have also employed a Kalman filter assuming that the motion model in two dimensions is a Gaussian process. However, the average error of their localization algorithm is 0.2 meters on average, which may be considerable for applications that require more accuracy (for example robots trying to enter a narrow passageway etc.). In [8], the authors have proposed positioning based on one LED source and multiple receivers, in which the multiple receivers in a room cooperate during the process of localization. However, their algorithm is only feasible (achieves an accuracy of about 0.05 meters) when the receivers lie in certain circular patterns. Finally, in [9], the authors have proposed a machine learning based localization algorithm. In their paper, the first task is to carry out the RSS based fingerprinting followed by training of classifiers based on these fingerprinting results via machine learning techniques. The output of the classifiers are fused using robust least squares algorithms, and a *singular valued decomposition* (SVD) based method is proposed to deal with the numerical stability problem in case the prediction matrix is singular. However, fingerprinting and training of classifiers is a tedious and time consuming process, especially if the whole building is to be mapped for the purpose of fingerprinting. Moreover, the SVD based method also incurs significant computational complexity [9].

Regarding beam tracking in free-space optical communications, we refer the interested readers to [10]–[21]

B. Contributions of this paper

We primarily build on the work [5] in order to devise positioning algorithms based on visible light communication systems. The work presented in our paper is novel because we propose multiple optical sensors associated with a given receiver instead of positioning with just one sensor. In this sense, we are fusing more independent pieces of information (the observations for any given sensor are generated independently of the other sensors). This results in an increase in the *Fisher information* of the estimation problem, or, in other words, a reduction in the CRLB. This implies that the variance of the proposed estimators for a multiple sensor problem will be much smaller than a single sensor

case. Hence, the larger the number of the receiver sensors, the more accurately we can position the receiver.

Secondly, based on the received-signal-strength, we weight the measurements obtained for each transmitter. We used two types of weighting schemes in this regard: i) An RSS based and ii) a *binary* weighting scheme for the sources that have the largest signal strength at the receiver. Moreover, we consider three types of LED transmitter arrangements in a room, and compare their performance by simulations.

In order to “fuse” the measurements obtained for each transmitter/receiver pair, we use two algorithms. The measurements for the fusion algorithms are the hybrid maximum likelihood estimates of time-of-arrival (TOA), time-difference-of-arrival (TDOA) and received-signal-strength (RSS) corresponding to each sensor/transmitter pair. The first of these fusion algorithms is the *weighted nonlinear least squares* (WLS) that fuses the hybrid TOA-RSS and the hybrid TDOA-RSS measurements in order to localize the receiver.

The second proposed algorithm “tracks” the position of the receiver in that it estimates the change in the position of the receiver. If the change in the position at the current instant of time is small from the position at the previous instant, the nonlinear equations in range and time can be linearized over that small change by the Taylor Series approximation. This results in a linear model for which we use a low complexity estimator, such as the *best linear unbiased estimator* (BLUE), which will fuse the maximum likelihood estimates of the time-delay at the receiver corresponding to the transmitters with the highest RSS. Therefore, the BLUE also fuses the hybrid TOA-RSS measurements in order to estimate the change in position.

The process of fusion of TOA /TDOA and RSS estimates using the two algorithms described above is a similar idea to the two-step positioning system described in [5]. However, the fusion algorithms devised in this paper are different than those discussed in [5], and yield a better mean Euclidean error performance as argued in the final part of the paper. Moreover, we left out the direct positioning algorithm in our paper because of the higher complexity associated with it [5].

Furthermore, we analyze the effect of the following factors on the positioning performance of both the proposed algorithms:

- 1) signal-to-noise ratio,
- 2) total number of sensors located on the receiver,

Fig. 1. An example of positioning via Triangulation. The transmitters are present at locations T_1 , T_2 and T_3 . The diameters of the circles represent the estimated time-delays corresponding to each transmitter. These estimated time-delays are based on the TOA or TDOA or RSS based measurements. For the ideal case when the time-delay estimates are exactly equal to the true value (shown by solid circles), the position where the three circles intersect (shown by black dot) represents the exact location of the receiver. However, because of noise, the estimates from transmitter T_2 are not accurate enough (represented by the dashed red circle), and the three circles do not intersect at one point causing an error in the position estimate (erroneous position is shown by the blue dot).

- 3) the total number of LED transmitters in a given location and their arrangement,
- 4) and the time synchronization error.

C. Model assumptions

For visible light based position estimation, we make the following assumptions for our model:

- 1) Each transmitter is assigned a unique frequency (or wavelength) band in which it can transmit. Similarly, the receiver can differentiate the signals from different transmitters based on their wavelengths, and filter those into separate channels for processing.
- 2) A pilot channel is dedicated solely for the purpose of positioning.
- 3) A rectangular pulse is used in order to position the terminal. However, the discussion is general, and holds for any pulse shape.
- 4) It is assumed that the power from the non-line-of-sight components of the received light is negligible for the purpose of positioning [3], [4]. Hence, only the line-of-sight (LOS) component is considered for our positioning system.

II. PROBLEM FORMULATION AND MEASUREMENT MODELS FOR VISIBLE LIGHT POSITIONING

Let us assume that there are N_T light-emitting-diode (LED) transmitters located at points $\mathbf{p}_t^{(i)}, i = 1, 2, \dots, N_T$, and each $\mathbf{p}_t^{(i)} \in \mathbb{R}^3$. Let us further assume that the receiver is made up of

M sensors, and those sensors are arranged symmetrically on the device (or a robot etc.) that needs to be tracked. If the location of sensor m is $\mathbf{p}_r^{(m)} \in \mathbb{R}^3$ for $m = 1, \dots, M$, then, roughly, the center of the device (the point that we want to track) is the sample average:

$$\mathbf{p}_r \triangleq \frac{1}{M} \sum_{m=1}^M \mathbf{p}_r^{(m)}, \quad (1)$$

where $\mathbf{p}_r$ is the location of the center of the receiver.

If we assume that the height z of the object from the ground is fixed, then we only want to track the x and y coordinates of its position on the floor. Then $\mathbf{p}_m$ —and therefore $\mathbf{p}_r$ —is just a function of x and y because z is a known fixed constant.

The received signal intensity (for the LOS scenario) at the m th sensor due to the i th transmitter is given by

$$\lambda_r^{(im)}(t) = \alpha_{im} \lambda_s^{(i)}(t - \tau_{im}) + \lambda_0^{(m)}, \quad (2)$$

where t inside some observation interval $[0, T]$, α_{im} is the attenuation factor of the optical channel, $\lambda_s^{(i)}(t)$ is the transmitted signal intensity of the i th transmitter, and $\lambda_r^{(im)}(t)$ is the received signal intensity at the m th receiver sensor due to the i th transmitter. The parameter $\lambda_0^{(m)}$ is a constant that represents the noise intensity at the m th receiver sensor due to thermal effect and background radiation. The variable τ_{im} represents the TOA parameter [4], and it represents the delay introduced by the i th transmitter m th sensor channel

$$\tau_{im} \triangleq T_0 + \frac{R_{im}}{c}, \quad (3)$$

for $i = 1, 2, \dots, N_T, m = 1, \dots, M$. In (3), R_{im} represents the distance between the i th transmitter and the m th receiver sensor, c represents the speed of light in air in meters/sec, and T_0 is the time at which all the transmitters begin the transmission of the pulse. If the receiver is synchronized with the transmitters, then T_0 will be considered a known quantity.

We now denote the channel attenuation factor and the time delay for the i th transmitter/ m th receiver pair by α_{im} and τ_{im} , respectively. The quantity α_{im} is modeled as [3]

$$\alpha_{im} = \frac{\ell + 1}{2\pi} \cos^\ell(\phi) \cos(\theta) \frac{A_r}{R_{im}^2}, \quad (4)$$

where ℓ is the Lambertian order, A_r is the area of the photo-detector at the receiver and ϕ and

Fig. 2. Geometry of the transmitter (LED) and receiver (optical sensor) in the visible light positioning problem.

θ represent the irradiation and incidence angles, respectively. However, when the LED transmitter is looking downwards, and the receiver is pointing upwards—which is usually the case as shown in Figure 2— then

$$\phi = \theta \implies \cos(\phi) = \cos(\theta) = h/R_{im},$$

where h represents the height of the LED transmitter with respect to the VLC receiver [4]. In that case [4],

$$\alpha_{im} = \frac{\ell + 1}{2\pi} h^{\ell+1} A_r R_{im}^{-\ell-3}. \quad (5)$$

A. Measurements for position estimation

In this section, we assume that the positioning system uses a pilot channel dedicated for the purpose of positioning. Hence, a positioning pulse of width T_p is transmitted every T seconds (or an integer multiple of T seconds) in order to estimate the position of the receiver.

Analyzing (2), we see that there are two unknowns in that equation: the delay τ_{im} , and the attenuation factor α_{im} . Both these factors will depend on the distance of a given sensor from a particular transmitter. Hence, we can use the measurements based on τ_{im} and α_{im} to our advantage as we estimate the position of a receiver.

In order to estimate the position of the receiver, we first form the parameter vector (with respect to the i th transmitter) as $\boldsymbol{\theta}_i \triangleq [\alpha_{i1} \ \alpha_{i2} \ \cdots \ \alpha_{iM} \ \tau_{i1} \ \tau_{i2} \ \cdots \ \tau_{iM}]^\top$. Furthermore, based on the estimate $\hat{\boldsymbol{\theta}}_i$ of the vector $\boldsymbol{\theta}_i$ for all i , we can fuse these estimates in order to estimate the coordinates of the receiver position in two dimensions.

Fig. 3. The transmitted pilot pulse from the i th transmitter reaches the m th sensor with a delay of τ_{im} seconds, and channel attenuation factor α_{im} . The time starts at time T_0 .

1) Measurement of time-delay with maximum likelihood estimation

In this section, we use the maximum likelihood principle in order to generate the measurements $\hat{\tau}_{im}$ for $i = 1, \dots, N_t$, $m = 1, \dots, M$, in order to estimate the position of the receiver. In this regard, we employ photomultiplier tube receivers, which are commonly used for the detection of signals in optical communications.

Furthermore, we can estimate τ_i (the average time it takes for the light signal from the i th transmitter to reach the center of the receiver) based on $\hat{\tau}_{im}$. Since, the maximum likelihood estimate commutes over a linear transformation, the maximum likelihood estimate $\hat{\tau}_i$ is simply $\hat{\tau}_i = \frac{1}{M} \sum_{m=1}^M \hat{\tau}_{im}$.

The values $\hat{\tau}_i$ will form the measurements/observations for the WLS or BLUE based fusion algorithm that we will discuss later in this paper.

Let us assume that the m th sensor of the receiver receives the following positioning pulse by the i th transmitter after the conversion of the optical signal into electric charge [4]:

$$x_{im}(t) = \alpha_{im} R_p r(t - \tau_{im}) + w_{im}(t), \quad (6)$$

where $r(t)$ is the transmitted optical signal and $w_{im}(t)$ is a zero mean *additive white Gaussian noise* (AWGN) process (with two-sided power spectral density $N_0/2$) that takes into account the effect of *shot noise*, the *thermal noise* and the *background radiation*. The factor R_p is the *responsivity* of the photo detector [4]. The signal $x_{im}(t)$ is then filtered by a low pass filter of frequency cut-off B Hz, where B is the effective bandwidth of the pulse $s(t)$. Let us denote $R_p r(t)$ by $s(t)$. The resulting filtered signal is then discretized in time to obtain

$$x_{im}[n] = \alpha_{im} s(n\Delta - \tau_{im}) + w_{im}[n]. \quad (7)$$

In (7) Δ is the sampling interval, and it is set equal to $1/(2B)$ seconds, the *Nyquist rate*. Moreover, $x_{im}[n] \triangleq x'_{im}(n\Delta)$ and $w_{im}[n] \triangleq w'_{im}(n\Delta)$ where $x'_{im}(t)$ is the filtered signal, and $w'_{im}(t)$ is the filtered noise bandlimited to B Hz. It can be shown that $w_{im}[n]$ is a zero mean white Gaussian noise process with variance $\sigma^2 = N_0B$ [?]. Equation (7) can be further simplified as

$$x_{im}[n] = \alpha_{im}s[n - D_{im}] + w_{im}[n], \quad n = 0, 1, \dots, N - 1, \quad (8)$$

where D_{im} is an integer, and represents the delay in discrete time, and $N \triangleq T/\Delta$ is the observation interval in discrete time. We assume that the sampling period Δ is small enough such that $\tau_{im} \approx D_{im}\Delta$.

The signal $x_{im}[n]$ is now passed through a *discrete-time matched filter* of impulse response $s^*[-n]$ to obtain

$$y_{im}[n] = x_{im}[n] * s^*[-n] = \alpha_{im}s[n - D_{im}] * s^*[-n] + w_{im}[n] * s^*[-n] \quad (9)$$

$$= \alpha_{im}r_{ss}[n - D_{im}] + r_{ws}[n], \quad n = 0, 1, \dots, N - 1, \quad (10)$$

where $r_{ss}[n]$ and $r_{ws}[n]$ are the autocorrelation and cross-correlation functions, respectively, and $*$ represents the discrete-time convolution operator. Furthermore, $r_{ws}[n]$ is a zero mean Gaussian process with variance $E(r_{ws}^2[n]) = \sigma^2 E_s$ where $E_s \triangleq \sum_{n=-\infty}^{\infty} s^2[n]$. The $N \times N$ covariance matrix of $r_{ws}[n]$ is

$$\mathbf{C} \triangleq \sigma^2 \begin{bmatrix} r_{ss}[0] & r_{ss}[1] & \cdots & r_{ss}[N-1] \\ r_{ss}[-1] & r_{ss}[0] & \cdots & r_{ss}[N-1] \\ \vdots & \vdots & \ddots & \vdots \\ r_{ss}[-(N-1)] & r_{ss}[-(N-2)] & \cdots & r_{ss}[0] \end{bmatrix}.$$

Our objective is to compute the maximum likelihood estimates $\hat{D}_{im}$ using (10), and based on these estimates, compute the maximum likelihood estimates $\hat{\tau}_{im}$ using $\hat{\tau}_{im} = \Delta \hat{D}_{im}$. Furthermore, $\hat{\tau}_i$ is just a sample average of $\hat{\tau}_{im}$'s.

Finally, the maximum likelihood estimate of the range R_{im} is, $\hat{R}_{im} = c\hat{\tau}_{im}$.

The maximum likelihood estimate of D_{im} based on the observations in (10) is defined as

$$\hat{D}_{im} = \arg \max_{D_{im}} p(\mathbf{y}_{im} | D_{im}) = \arg \max_{D_{im}} \frac{1}{\sqrt{(2\pi)^N |\mathbf{C}|}} \exp \left(-\frac{1}{2} (\mathbf{y}_{im} - \boldsymbol{\mu}_{im})^T \mathbf{C}^{-1} (\mathbf{y}_{im} - \boldsymbol{\mu}_{im}) \right), \quad (11)$$

where $\mathbf{y}_{im} \triangleq \begin{bmatrix} y_{im}[0] & y_{im}[1] & \cdots & y_{im}[N-1] \end{bmatrix}^\top$, and

$$\boldsymbol{\mu}_{im} \triangleq \begin{bmatrix} \alpha_{im} r_{ss}[0 - D_{im}] & \alpha_{im} r_{ss}[1 - D_{im}] & \cdots & \alpha_{im} r_{ss}[N-1 - D_{im}] \end{bmatrix}^\top.$$

Solving (11) further, the maximum likelihood estimate of D_{im} can be written in a compact form [?]:

$$\hat{D}_{im} = \arg \max_n (\alpha_{im} r_{ss}[n - D_{im}] + r_{ws}[n]), \quad (12)$$

and therefore, the estimate $\hat{D}_{im}$ is just the instant at which the output of the matched filter is maximized during the observation interval T .

2) Estimation of signal strength with maximum likelihood estimation

For hybrid RSS-TOA based position estimation, we will need to estimate the received signal strength, or, the received signal attenuation factor α_{im} along with the time-delay estimates $\hat{D}_{im}$. Based on the estimated values of the signal strength, we will use the best η sources, where η is an integer and $\eta \geq 3$. Thus we use three or more sources that suffer the smallest attenuation in order to localize the receiver through *triangulation/multilateration*.

In order to estimate the attenuation factor α_{im} , one option is to use a sufficient statistic in order to estimate α_{im} that does not depend on the knowledge of D_{im} or τ_{im} . One way of eliminating the dependence on D_{im} is to average the observation y_{im} in (10). Thus, we will determine the maximum likelihood estimate of α_{im} based on a sample average of y_{im} during the observation interval $(-T, T)$:

$$\bar{y}_{im} \triangleq \frac{1}{2N-1} \sum_{n=-(N-1)}^{N-1} y_{im}[n] = \frac{1}{2N-1} \sum_{n=-(N-1)}^{N-1} \alpha_{im} r_{ss}[n - D_{im}] + \frac{1}{2N-1} \sum_{n=-(N-1)}^{N-1} r_{ws}[n] \quad (13)$$

$$= \frac{\alpha_{im}}{2N-1} \sum_{n=-(N-1)}^{N-1} r_{ss}[n] + \frac{1}{2N-1} \sum_{n=-(N-1)}^{N-1} r_{ws}[n] = \alpha_{im} \bar{r}_{ss} + \bar{r}_{ws} \quad (14)$$

where $\bar{r}_{ss}$ and $\bar{r}_{ws}$ represent the sample averages for $r_{ss}[n]$ and $r_{ws}[n]$, respectively. The first equality in (14) is true if N is large enough since the sum of $r_{ss}[n - D_{im}]$ is the same as the sum of $r_{ss}[n]$ over $(2N-1)$ terms when N is large. It should also be noted that $\bar{r}_{ws}$ is a zero mean Gaussian random variable. It can be shown easily that the variance of $\bar{r}_{ws}$ is

$$\sigma_{ws}^2 \triangleq E((\bar{r}_{ws})^2) = \frac{1}{(2N-1)^2} \sum_{i=1}^{(2N-1)} \sum_{j=1}^{(2N-1)} [\mathbf{C}_{ws}]_{ij} \quad (15)$$

where $[\mathbf{C}_{ws}]_{ij}$ is the (i, j) element of the $(2N-1) \times (2N-1)$ covariance matrix $\mathbf{C}_{ws}$ of the vector

$$\begin{bmatrix} r_{ws}[-(N-1)] & r_{ws}[-(N-2)] & \cdots & r_{ws}[0] & r_{ws}[1] & \cdots & r_{ws}[N-1] \end{bmatrix}^\top.$$

Using (14), the maximum likelihood estimate of α_{im} is

$$\hat{\alpha}_{im} = \arg \max_{\alpha_{im}} \frac{1}{\sqrt{2\pi\sigma_{ws}^2}} \exp \left(-\frac{1}{2\sigma_{ws}^2} (\bar{y}_{im} - \alpha_{im} \bar{r}_{ss})^2 \right). \quad (16)$$

Taking the derivative of the natural logarithm of the function in (16) with respect to α_{im} , and setting that equal to zero yields the maximum likelihood estimate: $\hat{\alpha}_{im} = \frac{\bar{y}_{im}}{\bar{r}_{ss}}$. If, however, $\hat{\alpha}_{im} < 0$, then we set $\hat{\alpha}_{im}$ equal to zero¹.

III. FUSION OF MEASUREMENTS

Now that we have generated the measurements $\hat{\alpha}_{im}$ and $\hat{\tau}_{im}$, we are ready to fuse these measurements using the two algorithms discussed below in order to position the receiver. In the following analysis, we assume that the true position of the receiver is (x_r, y_r) , and the estimate output by the fusion device is $(\hat{x}_r, \hat{y}_r)$.

A. Weighted least squares estimate of position

We use the *weighted least squares* (WLS) approach for the nonlinear position estimation problem. In this regard, we first define

$$\hat{\alpha}_i \triangleq \frac{1}{M} \sum_{m=1}^M \hat{\alpha}_{im}. \quad (17)$$

Furthermore, we define the set Q which contains η (η being a positive integer) largest $\hat{\alpha}_i$'s (corresponding to the η best sources). Thus, if $[\hat{\alpha}^{(1)} \ \hat{\alpha}^{(2)} \ \cdots \ \hat{\alpha}^{(N_T)}]^\top$ is the ordered array of attenuation factors $\hat{\alpha}_i$ such that $\hat{\alpha}^{(1)} > \hat{\alpha}^{(2)} > \cdots > \hat{\alpha}^{(N_T)}$, then $Q \triangleq \{\hat{\alpha}^{(1)}, \hat{\alpha}^{(2)}, \dots, \hat{\alpha}^{(\eta)}\}$ where $0 < \eta \leq N_T$, and $\eta \in \mathbb{Z}^+$.

In order to compute the WLS estimate, we minimize the function $J(\mathbf{l}_r)$ with respect to vector $\mathbf{l}_r$:

$$\hat{\mathbf{l}}_r = \arg \min_{\mathbf{l}_r} J(\mathbf{l}_r) = \arg \min_{\mathbf{l}_r} (\mathbf{r} - \mathbf{h}(\mathbf{l}_r))^\top \mathbf{W} (\mathbf{r} - \mathbf{h}(\mathbf{l}_r)), \quad (18)$$

where $\mathbf{l}(r)$ is the location vector of the receiver sensors defined by $\mathbf{l}_r \triangleq [\mathbf{p}_r^{(1)} \ \mathbf{p}_r^{(2)} \ \cdots \ \mathbf{p}_r^{(M)}]^\top$, and $\mathbf{p}_r^{(m)}$ is the location of the m th receiver sensor. The nonlinear function $\mathbf{h}(\mathbf{l}_r)$, which is an $(N_t \cdot M) \times 1$ vector, is defined as

¹The value of α_{im} equal to zero maximizes the likelihood function over the positive values of α_{im} (the $\alpha_{im} > 0$ is the admissible range) since the likelihood function is quadratic in α_{im} .

$$\mathbf{h}(\mathbf{l}_r) \triangleq \left[\|\mathbf{p}_t^{(1)} - \mathbf{p}_r^{(1)}\|_2 \quad \|\mathbf{p}_t^{(2)} - \mathbf{p}_r^{(1)}\|_2 \quad \dots \quad \|\mathbf{p}_t^{(N_T)} - \mathbf{p}_r^{(1)}\|_2 \quad \|\mathbf{p}_t^{(1)} - \mathbf{p}_r^{(2)}\|_2 \quad \dots \quad \|\mathbf{p}_t^{(N_T)} - \mathbf{p}_r^{(N_T)}\|_2 \right]^\top, \quad (19)$$

where $\mathbf{p}_t^{(i)}$ is the position of the i th transmitter, and $\|\mathbf{x}\|_2$ represents the ℓ_2 norm of a given vector $\mathbf{x}$.

1) Synchronous case

For the synchronous case, the quantity $\mathbf{r}$ is defined to be the $(N_t \cdot M) \times 1$ vector $\begin{bmatrix} \hat{R}_{11} & \hat{R}_{21} & \dots & \hat{R}_{N_t 1} & \hat{R}_{12} & \hat{R}_{22} \end{bmatrix}$ where $\hat{R}_{im}$ is the estimate of the range between the i th transmitter and the m th sensor. The $(N_t \cdot M) \times (N_t \cdot M)$ diagonal matrix $\mathbf{W}$ (matrix of weights) is by definition

$$\mathbf{W} \triangleq \begin{bmatrix} \omega_{11} & 0 & \dots & 0 \\ 0 & \omega_{12} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_{N_t M} \end{bmatrix}, \quad (20)$$

where $\omega_{im} \triangleq \frac{\hat{\alpha}_{im}}{\sum_{i=1}^{\eta} \sum_{m=1}^M \hat{\alpha}_{im}}$ if $\hat{\alpha}_i \in Q$, and ω_{im} is zero otherwise. In other words, we use the η “best” LED sources which suffer the smallest attenuation at the receiver, and then weight them according to their received signal strength.

We can now rewrite (18) as

$$\hat{\mathbf{l}}_r = \arg \min_{\mathbf{l}_r} \sum_{i=1}^{N_T} \sum_{m=1}^M \left(\hat{R}_{im} - \|\mathbf{p}_t^{(i)} - \mathbf{p}_r^{(m)}\|_2 \right)^2 \omega_{im}. \quad (21)$$

2) Asynchronous case

For the asynchronous case, we define the *time-difference-of-arrival* (TDOA) estimates as

$$\hat{t}_{im} \triangleq \hat{\tau}_{im} - \hat{\tau}_{(i-1)m}, \quad i = 2, 3, \dots, N_T, \forall m. \quad (22)$$

The time delay estimate can be written as $\hat{\tau}_{im} = T_0 + \tau_{im} + \epsilon_{im}$, where T_0 is the time at which the transmitters transmit the pulse, and ϵ_{im} is the estimation error. Hence,

$$\hat{t}_{im} = T_0 + \tau_{im} + \epsilon_{im} - (T_0 + \tau_{(i-1)m} + \epsilon_{(i-1)m}) = \tau_{im} - \tau_{(i-1)m} + (\epsilon_{im} - \epsilon_{(i-1)m}), \quad (23)$$

Fig. 4. Geometry of the visible light positioning problem for the best linear unbiased estimator.

and the dependence on T_0 is eliminated. The weighted least squares estimate of the receiver position based on the TDOA is

$$\hat{\mathbf{l}}_r = \arg \min_{\mathbf{l}_r} \sum_{i=2}^{N_T} \sum_{m=1}^M \left([\hat{R}_{im} - \hat{R}_{(i-1)m}] - [\|\mathbf{p}_t^{(i)} - \mathbf{p}_r^{(m)}\|_2 - \|\mathbf{p}_t^{(i-1)} - \mathbf{p}_r^{(m)}\|_2] \right)^2 \omega_{im}, \quad (24)$$

where $\omega_{im} = 1$ if the corresponding $\hat{\alpha}_i$ is in Q , and zero otherwise. Hence, we are using a binary weighting scheme for the asynchronous case.

Finally, the estimate of the center of the receiver, $\hat{\mathbf{p}}_r$, for both the synchronous and asynchronous schemes, is given by

$$\hat{\mathbf{p}}_r \triangleq \frac{1}{M} \sum_{m=1}^M \hat{\mathbf{p}}_r^{(m)}.$$

B. Best linear unbiased estimate (BLUE) of position (tracking)

Using a *linear model* (linearity holds if the change in the position of the receiver is very small with respect to the sampling rate), we will obtain the position information as the *best linear unbiased estimate* (BLUE) of the true position based on the measurements $\hat{\tau}_i$.

The measurements $\hat{\tau}_i$ can be written as

$$\hat{\tau}_i = T_0 + R_i/c + \epsilon_i, \quad i = 1, 2, \dots, N_T, \quad (25)$$

where ϵ_i is the measurement noise associated with $\hat{\tau}_i$. It will be reasonable to assume that ϵ_i is uncorrelated with ϵ_j for $i \neq j$ and $i, j = 1, 2, \dots, N_T$. The variance of ϵ_i is denoted by σ_i^2 . The quantity R_i represents the actual distance of the i th transmitter to the center of the receiver at the actual location (x_{n+1}, y_{n+1}) at time $n + 1$. The geometry of this problem is depicted in Figure 4.

The nominal position $(\hat{x}_n, \hat{y}_n)$ is the estimate of the true receiver location (x_n, y_n) at the discrete-time n . At time instant $n + 1$, the distance R_i of the receiver from the i th source is by definition $R_i \triangleq \sqrt{(x_{n+1} - x_i)^2 + (y_{n+1} - y_i)^2}$. The Taylor expansion of R_i about the nominal distance R_{ni} (the distance at instant n) gives

$$R_i = R_{ni} + \frac{x_n - x_i}{R_{ni}} \delta x_{n+1} + \frac{y_n - y_i}{R_{ni}} \delta y_{n+1} + (\text{higher order terms}). \quad (26)$$

Let us assume that the nominal position estimate $(\hat{x}_n, \hat{y}_n)$ is accurate enough (close to (x_n, y_n)), and that the pulse transmission rate $1/T$ is large enough compared to the movements of the receiver. Furthermore, let us assume that $\hat{R}_{ni} \approx R_{ni}$, that is, we have at our disposal a good estimate of the range R_{ni} . Under this assumption, δx_{n+1} and δy_{n+1} will be small enough for linearization approximation to hold:

$$R_i \approx R_{ni} + \frac{x_n - x_i}{R_{ni}} \delta x_{n+1} + \frac{y_n - y_i}{R_{ni}} \delta y_{n+1}. \quad (27)$$

If we assume that $\hat{R}_{ni} \approx R_{ni}$ and $(\hat{x}_n, \hat{y}_n) \approx (x_n, y_n)$, we have that

$$R_i \approx \hat{R}_{ni} + \frac{\hat{x}_n - x_i}{\hat{R}_{ni}} \delta x_{n+1} + \frac{\hat{y}_n - y_i}{\hat{R}_{ni}} \delta y_{n+1}, \quad (28)$$

$$(29)$$

and it follows that

$$\hat{R}_i = \hat{R}_{ni} + \frac{\hat{x}_n - x_i}{\hat{R}_{ni}} \delta x_{n+1} + \frac{\hat{y}_n - y_i}{\hat{R}_{ni}} \delta y_{n+1} + \epsilon_i. \quad (30)$$

$$(31)$$

Therefore, in terms of time-delay

$$\hat{\tau}_i \approx T_0 + \frac{\hat{R}_{ni}}{c} + \frac{\hat{x}_n - x_i}{\hat{R}_{ni}c} \delta x_{n+1} + \frac{\hat{y}_n - y_i}{\hat{R}_{ni}c} \delta y_{n+1} + \epsilon'_i \quad \forall i, \quad (32)$$

where $\epsilon'_i \triangleq \epsilon_i/c$. For a synchronous system, the quantity T_0 is known. For this case, let us define $\xi_i \triangleq \hat{\tau}_i - T_0 - \hat{R}_{ni}/c$, and the linear model is

$$\xi_i = \frac{\hat{x}_n - x_i}{\hat{R}_{ni}c} \delta x_{n+1} + \frac{\hat{y}_n - y_i}{\hat{R}_{ni}c} \delta y_{n+1} + \epsilon'_i, \quad \forall i. \quad (33)$$

In the matrix form, (33) can be rewritten as

$$\boldsymbol{\xi} = \mathbf{H}\boldsymbol{\theta} + \boldsymbol{\epsilon}, \quad (34)$$

where $\boldsymbol{\xi} \triangleq [\xi_1 \ \xi_2 \ \dots \ \xi_{N_T}]^\top$, $\boldsymbol{\theta} \triangleq [\delta x_{n+1} \ \delta y_{n+1}]^\top$, $\boldsymbol{\epsilon} \triangleq [\epsilon'_1 \ \epsilon'_2 \ \dots \ \epsilon'_{N_T}]^\top$ and

$$\mathbf{H} \triangleq \frac{1}{c} \begin{bmatrix} \frac{\hat{x}_n - x_1}{\hat{R}_{n1}} & \frac{\hat{y}_n - y_1}{\hat{R}_{n1}} \\ \frac{\hat{x}_n - x_2}{\hat{R}_{n2}} & \frac{\hat{y}_n - y_2}{\hat{R}_{n2}} \\ \vdots & \vdots \\ \frac{\hat{x}_n - x_{N_T}}{\hat{R}_{nN_T}} & \frac{\hat{y}_n - y_{N_T}}{\hat{R}_{nN_T}} \end{bmatrix}.$$

The best linear unbiased estimator of $\boldsymbol{\theta}$ is given by [?]

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^\top \mathbf{H})^{-1} \mathbf{H}^\top \boldsymbol{\xi}, \quad (35)$$

with the covariance matrix $\mathbf{C}_{\hat{\boldsymbol{\theta}}} = \sigma_i^2 \mathbf{I}(\mathbf{H}^\top \mathbf{H})^{-1}$, where $\mathbf{I}_2$ is the identity matrix, and $\hat{\boldsymbol{\theta}}$ is by definition $\begin{bmatrix} \hat{\delta x}_{n+1} & \hat{\delta y}_{n+1} \end{bmatrix}$

Based on the received signal strength estimates, the BLUE may be redefined (improved) as

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^\top \mathbf{W} \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{W} \boldsymbol{\xi}, \quad (36)$$

with $\mathbf{W}$ a diagonal matrix

$$\mathbf{W} \triangleq \begin{bmatrix} \omega_1 & 0 & \dots & 0 \\ 0 & \omega_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_{N_T} \end{bmatrix} \quad (37)$$

where $\omega_i \triangleq \frac{\hat{\alpha}_i}{\sum_{i=1}^{\eta} \hat{\alpha}_i}$ if $\hat{\alpha}_i \in Q$ and ω_i is zero otherwise.

The current estimate of the receiver position (at time instant $n+1$) is

$$\hat{x}_{n+1} = \hat{x}_n + \hat{\delta x}_{n+1}, \quad (38)$$

$$\hat{y}_{n+1} = \hat{y}_n + \hat{\delta y}_{n+1}. \quad (39)$$

IV. SIMULATIONS

A. Simulation Preliminaries

For simulations, we choose the following parameter values [4]: Responsivity $R_p = 0.4$ mA/mW, Lambertian order $\ell = 1$, height of the transmitter $h = 2.5$ meters, area of the photo detector $A_r = 1$ cm² and the spectral density of Gaussian noise is $N_0/2 = 1.336 \times 10^{-22}$ W/Hz [4]. If the bandwidth of the light pulse B is chosen to be 100 MHz (same as the modulation bandwidth of the LED), then this translates to a noise power of $\sigma^2 = \frac{N_0}{2} \times 2B = 2.672 \times 10^{-14}$ W. Such a modulation bandwidth also translates approximately to a minimum pulse duration of 1×10^{-8} seconds (if we ignore the sidelobes). Hence, $T_p \geq 10^{-8}$ seconds. The peak power (intensity) of the transmitted signal $r(t)$ in (6) ranges from 1 through 10 W/m² [3].

We use a rectangular pulse for the purpose of positioning: $r(t) = A \cdot \mathbb{1}_{[0, T_p]}(t)$, where $\mathbb{1}_E(\cdot)$ is the indicator function over a measurable set E .

The sampling interval Δ is set equal to $1/(2B)$ (inverse of the Nyquist rate) seconds.

For the rectangular pulse, the sampling frequency F_s is $2B$ where B is the effective bandwidth of the pulse. The bandwidth B can be set equal large enough to cover the first two or three sidelobes of the *sinc* function (which is the Fourier Transform of the rectangular pulse). Hence, for example, if B is to include the first three sidelobes, then $B = \frac{4}{T_p}$ which implies $F_s = 2B = \frac{8}{T_p}$. This leads to $\Delta = \frac{T_p}{8}$.

The performance criterion for positioning accuracy is the mean Euclidean error between the true position and the estimated position of the receiver. This is defined as the mean Euclidean distance (or the ℓ_2 norm) between the actual coordinates (x, y) of the position and the estimated coordinates $(\hat{x}, \hat{y})$. The Euclidean error is denoted by $\|\mathbf{e}\|_2$ where $\mathbf{e} \triangleq \begin{bmatrix} (x - \hat{x}) & (y - \hat{y}) \end{bmatrix}^\top$. The mean of the Euclidean error is computed in a Monte Carlo fashion by simulating the experiment many number of times. For the weighted least squares algorithm, we have set the number of simulations equal to 5000 for the purpose of computing the means. However, for the case of BLUE based positioning, the step size of the movement of the user is small, and at the same time, we want the user to traverse the entire room/premises in order to estimate the “average” positioning performance. Hence, the number of simulations for BLUE algorithm is set much higher at 50000.

For the weighted least squares estimates, we use a real number genetic algorithm in order to find the minimum of (21). The population size for the algorithm is set at 200, and number of generations

at 400. The crossover and mutation rates are set at 0.75 and 0.50, respectively. The reader may refer to [22] for more details about the genetic algorithms.

In order to run a given simulation for weighted least squares based positioning, we choose the true coordinate (x, y) from within the region of the room/location randomly (uniformly). Then, we produce estimates $(\hat{x}, \hat{y})$ based on the observations. For BLUE, we start at a random location in the room, and the quantities δx and δy are sampled independently of each other from a uniform distribution: $\mathcal{U}[-\nu, \nu]$. We refer to the quantity ν as the “step size.” The value of ν is set at 0.2 meters for most simulations on BLUE positioning.

We conduct simulations for different kinds of LED arrangements in a given location, and for different numbers of receiver sensors which are arranged in a symmetric pattern about the center of the receiver. The total number of sensors M is set equal to 2, 4, 6 and 8 for the purpose of performance comparison.

B. Simulation results and performance evaluation (weighted least squares)

1) Weighted least squares

In Figure 5 and Figure 6, we represent the normalized histograms of the error (difference between the x coordinate of the true position and its estimate using the weighted least squares approach) for different LED configurations and numbers of receiver sensors for the WLS estimate.

These histograms correspond to the following parameter values: $r(t) = 6$ W for Figure 5 and Figure 6(a) and $r(t) = 3$ W for Figure 6(b), T_p is 10^{-8} seconds, T is 4×10^{-8} seconds, R_p is 0.4, l is 1, h is 2.5 m, A_r is 0.01^2m^2 , N_0 is $2 \times 1.336 \times 10^{-22}$ W/Hz, B is $100/T_p$ Hz, σ_0 is $\sqrt{N_0 B}$ W, F_s is $2B$ Hz, Δ is $1/F_s$ seconds and c is 3×10^8 m/s. The center of the square shaped receiver is sampled from a two dimensional uniform distribution in the square region $\mathbb{1}_{[-3,3]}(x) \cdot \mathbb{1}_{[-3,3]}(y)$. The four sensors are present at each of the four corners, and the distance of each sensor from the center is 0.0707 meters. We used 5000 measurements for the creation of these histograms.

We now consider Figure 7 which shows the LED arrangement of the first kind, and its corresponding mean Euclidean error performance. The LED's are represented by blue dots, and the receiver sensors by red dots. We use the same values of the parameters that we have laid out in the second paragraph of Section IV-B, except that, now, we evaluate the positioning performance for a range of transmitted power levels. Moreover, we also analyze the effect of total number of receiver sensors on the performance. We varied the transmitted power from 2 W/m^2 through 10 W/m^2 in steps of

Fig. 5. Histogram of the measurement error along x -axis for the LED configuration shown in Figure 7(a) for the synchronous system. The total number of receiver sensors M is 4 and the peak signal power is 6 Watts per LED.

Fig. 6. Histogram of errors along x -axis. Figure (a) corresponds to LED configuration shown in Figure 8(a) and $M = 6$ and peak power of 6 Watts per LED. Figure (b) corresponds to LED configuration shown in Figure 9(a) with $M = 2$ and peak power of 3 Watts per LED. Both plots refer to the synchronous systems.

2 W/m². The area of the square shaped room is 36 m². The value of η was set equal to 3 (we used the three best sources out of 4). The mean Euclidean error performance is evaluated for both the synchronous and asynchronous systems.

As seen in Figure 7, the mean Euclidean error decreases as the LED power levels are increased. Moreover, as expected, the error performance improves as we increase the number of sensors on the receiver. We were able to achieve the mean Euclidean error of approximately 0.02 meters with 8 sensors when the peak transmitted power for each of the four LED's was 10 Watts.

Figure 8 shows another layout of the LED's for the room of similar dimensions and area as discussed previously, and the mean Euclidean error performance for this arrangement. The value of η was set

Fig. 7. LED arrangement of the first type and its mean Euclidean error performance.

Fig. 8. LED arrangement of the second type and its mean Euclidean error performance.

equal to 3. These performance curves are expected to be better than the system in Figure 7 since we now have an extra LED source in the middle. The improvement in performance is more pronounced when both the SNR and the number of sensors is small. We were able to localize the receiver to within a mean Euclidean error of 0.01 meters for $M = 8$ and peak power of 10 Watts.

Figure 9 represents the LED arrangement of the third type in which we have a total of 8 LED's arranged symmetrically in the room. The value of η was set equal to 4. We compare the mean Euclidean error of this system with the one shown in Figure 7 (containing 4 LED's), and find out that this system outperforms the system with 4 LED's even though the average power of the two

Fig. 9. LED arrangement of the third type and its mean Euclidean error performance.

systems is the same. With 8 sensors and a peak power of 5 Watts per LED, we were able to achieve the mean Euclidean error of around 0.007 meters.

Since the time synchronization error affects the time-of-arrival based positioning, we consider the effect of synchronization error in Figure 10. For a pulse of width T_p of 10 nanoseconds, we plotted the error curves as a function of the synchronization error² standard deviation that ranges from 0.2 to 1.0 nanoseconds. The synchronization error is assumed to be a zero-mean Gaussian random variable. Surprisingly, the error performance of the system with larger number of transmitters suffers more as the synchronization error becomes large. Moreover, as expected, the asynchronous systems are immune to synchronization errors.

C. Simulation results and performance evaluation (BLUE)

Figure 11 shows the mean Euclidean error performance for the two LED configurations for the step size ν of 0.2 meters. The other parameters concerning the histograms are the same as discussed in the second paragraph of Section IV-B. Figure 12 represents the mean Euclidean error of the two systems with different LED arrangements as a function of signal power. We note that the system with larger number of LED's outperforms the other system even though both systems have the same average power. This conclusion is similar to the the corresponding results obtained in the case of weighted least squares based positioning.

²This error corresponds to the error in the transmitter and receiver clock time references.

(a) Performance of the LED arrangement of the first type (4 LED's)

(b) Performance of the LED arrangement of the third type (8 LED's)

Fig. 10. Effect of synchronization error on the mean Euclidean error performance. The average power of both systems is the same (24 watts).

(a) Histogram of error for the LED arrangement of the first type (4 LED's) and $M = 4$.

(b) Histogram of error for the LED arrangement of the third type (8 LED's) and $M = 4$.

Fig. 11. Histograms of error in the x -dimension of BLUE systems for different LED configurations. Both systems have the same average power (24 Watts).

The performance of the BLUE based positioning will also depend on the step size of the movement of the user (or how frequently we choose to update the position estimate). If the step size ν is not small enough, the Taylor Series approximation breaks down, and the performance will suffer. In order to analyze the effect of ν on the positioning performance, we measured the mean Euclidean error for different values of ν . The results are plotted in Figure 13.

Figure 14 depicts the effect of the synchronization error on the mean Euclidean error performance for the two LED configurations.

(a) Performance of the LED arrangement of the first type (4 LED's)

(b) Performance of the LED arrangement of the third type (8 LED's)

Fig. 12. Performance of BLUE systems for different LED configurations. Both systems have the same average power.

(a) Performance of the LED arrangement of the first type (4 LED's)

(b) Performance of the LED arrangement of the third type (8 LED's)

Fig. 13. Effect of step size on the mean Euclidean error. Both systems have the same average power.

D. Comparison of performance with the state-of-the-art algorithm

In this section, we compare the performance of the proposed weighted least squares (WLS) algorithm with the two-step positioning algorithm discussed in [5]. We consider the synchronous versions of the two algorithms. For the purpose of performance comparison, we use the same waveform as discussed in [5], which is given by

$$s(t) = A(1 - \cos(2\pi t/T_p))(1 + \cos(2\pi f_c t)) \cdot \mathbb{1}(t)_{[0, T_p]}. \quad (40)$$

(a) Performance of the LED arrangement of the first type (4 LED's)

(b) Performance of the LED arrangement of the third type (8 LED's)

Fig. 14. Effect of synchronization error on the mean Euclidean error performance. Both systems have the same average power.

The nonlinear least squares algorithm used by the authors in [5] for the two step positioning is

$$\hat{\mathbf{l}}_r^{\text{TS,syn}} = \arg \max_{\mathbf{l}_r} \sum_{i=1}^{N_T} \left(E_1^i \hat{\alpha}_i^2 (\hat{\tau}_i - \tau_i) + E_2^i (\hat{\alpha}_i - \alpha_i)^2 \right), \quad (41)$$

where $E_1^i \triangleq \int_0^{T_{p,i}} (s'_i(t))^2 dt$, and $E_2^i \triangleq \int_0^{T_{p,i}} (s_i(t))^2 dt$. In other words, E_2^i and E_1^i represent the energy in the i th transmitted signal and its derivative, respectively.

In our simulations, we used $T_p = 10^{-8}$ seconds and $f_c = 10^{10}$ Hz. The bandwidth of this signal B is $(10^{10} + 10^8)$ Hz, and $F_s = 2B$ Hz. We considered the first configuration (4 LED's) and third configuration (8 LED's) of the LED arrangement. All other parameters are the same as discussed in Section IV-A.

The mean Euclidean error for the two algorithms was computed by sampling the position uniformly in the room, and then computing the mean of the Euclidean errors. The number of simulations was set at 5000. We used $\eta = 3$ for the 4-LED configuration, $\eta = 4$ for the 5-LED arrangement, and $\eta = 6$ for the 8-LED case.

As shown in Figure 15 and Figure 16, the weighted least squares algorithm provides a better mean Euclidean error performance than the nonlinear least squares for all LED arrangements, and the difference in performance becomes more significant at higher signal-to-noise ratio.

(a) LED arrangement of the first type (4 LED's)

(b) LED arrangement of the third type (8 LED's)

Fig. 15. Comparison of the weighted least squares and the nonlinear least squares algorithms for the two types of LED configurations.

Fig. 16. Comparison of the weighted least squares and the nonlinear least squares algorithm for the LED configuration of the second type (5 LED's).

V. A COMPARATIVE ANALYSIS OF THE TWO FUSION ALGORITHMS

As mentioned previously, the weighted least squares and the best linear unbiased estimator fuse the maximum likelihood estimates of time-delay and received-signal-strength in order to produce position estimates. The major difference in the complexity of these algorithms lies in the difference of computation of (21) (WLS) and the evaluation of (35) (BLUE). As can be seen, the determination of the WLS solution is a nonlinear optimization problem, whereas the evaluation of BLUE is a linear one.

As can be verified by the simulation results, the mean Euclidean error performance of the WLS

estimator is better than BLUE's for every scenario discussed. Furthermore, the improvement in performance becomes more pronounced as the number of LED's in a location, or the number of sensors, become smaller.

The complexity of WLS solution largely depends on the dimensionality of the objective function which is to be minimized. The dimensionality depends on the number of sensors M located on the receiver. Hence, using more sensors incurs more computational resources if WLS used for positioning.

The complexity of BLUE largely depends on the step size ν . As discussed earlier, a smaller value of ν , which translates to a more frequent update of the estimate, is important in order to acquire a certain minimum accuracy required for positioning. However, a smaller ν also results in a more computationally expensive system. Hence, a trade-off between computational resource consumption and accuracy has to be reached in this regard.

Even though the performance of WLS is better than BLUE's, the former is a more computationally expensive algorithm. Hence, the better performance comes with the price of more computations each time the position is estimated. The complexity of a general real number genetic algorithm—required for determining the WLS estimates—has been discussed in detail in [10]. Comparing the genetic algorithm's complexity with that of evaluating a few matrix multiplications and inverses clearly lays bare the large difference in the computational overheads of the two algorithms.

Finally, the one dimensional errors in position for all the algorithms are distributed as (approximately) Gaussian—a fact that can be verified from the normalized histograms of error. This fact may be useful if we plan to use a linear Bayesian filtering (Kalman filter or Gaussian sum filter for example) approach to visible light positioning if either the WLS's or BLUE's positioning estimates are used as measurements for the filter.

VI. CONCLUSION

In this paper, we have looked at non Bayesian Statistical Signal Processing algorithms in the context of visible light positioning. We have discussed the implementation of the weighted least squares (WLS) and the best linear unbiased estimator (BLUE) which have been used to fuse the maximum likelihood estimates of time-delay and received-signal-strength. Moreover, we have compared their performance in terms of the mean Euclidean error, and have evaluated their pros and cons in terms of error performance and computational complexities. We conclude that the WLS provides a better

mean Euclidean error performance than BLUE for all scenarios. However, BLUE incurs a lower computational complexity as compared to WLS.

In light of the arguments of Section V, we propose a hybrid positioning system based on WLS and BLUE. The system will switch back and forth between the two algorithms depending on the signal-to-noise ratio, the number of transmitters/sensors employed and the synchronization errors. For instance, if the signal-to-noise ratio is high, or the number of transmitters/receiver sensors used is large, we may prefer BLUE over WLS in order to save on computational resources. However, for more adverse conditions, we may switch to WLS for more accuracy.

In future, we plan on proposing a Bayesian formulation of the visible light positioning problem, for which the output of one of the proposed algorithms in this paper may be used as measurements for the Bayesian tracking filter.

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