THIN ELASTIC PLATES EXPERIENCING LARGE DEFLECTIONS

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Thin elastic plates (homogeneous or composite) experiencing large deflections are considered. The deflections are much larger than the plate thickness. The geometrically nonlinear elasticity theory and the Kirchhoff assumptions are employed. Small elongations and shears are assumed. Following Novozhilov, the strain expressions are derived. Then, under a small in-plane rotation assumption and using the virtual work principle, the equilibrium equations and the boundary conditions are obtained. The equations/conditions become the known von Karman ones for the case of moderate deflections. The solutions of the obtained equations may be used as benchmarks for the nonlinear structural analysis (e.g., FEM) software in the case of large deflections.

Keywords: thin plate, large deflection, geometric nonlinearity

1. Introduction

In the last decades, many researchers explored the nonlinear behavior of thin flexible plates. As a rule, they use the theory described in (Washizu 1975, Reddy 2007). The theory employs the Kirchhoff assumptions and the von Karman approach.

It is known that the latter approach describes the so-called moderate deflections (see Novozhilov 1961, 2011). The deflections are supposed to be comparable to the plate thickness. Reddy, from his side, indicates that in the von Karman approach the plate out-of-plane rotations should not be greater than 15°.

On the other hand, there are cases of considerable or even large deflections (much larger than plate thickness), e.g., considerable post-buckling shown in (Selyugin 2021).

It is known that the software for nonlinear structural analysis (like FEM) gives good results at the beginning of the nonlinear structural behavior. But in case of considerable nonlinear behavior the FEM numerical results become more different from the real structural behavior (the difference is recognized by the FEM experts).

The present paper deals with the case of large deflections. The deflections created due to the loading are much larger than the plate thickness. The virtual work principle is used to derive the equilibrium equations and the boundary conditions. The obtained equations and the conditions may be used for benchmarking the FEM software (dedicated to the nonlinear structural behavior).

2. Theoretical analysis

A thin elastic plate (homogeneous or composite) of constant thickness h is considered. The plate thickness is much smaller than the linear plate dimensions. Fig. 1 demonstrates the coordinate system and some notations. The X-Y plane coincides with the mid-plane Γ of the plate. The plate lateral surface S consists of a loaded part S_1 and the part with the prescribed displacements S_2. The mid-plane Γ intersects the lateral surface S at the smooth contour C, with the portion C_1 belonging to S_1 and the portion C_2 belonging to S_2. The normal to C is the vector $\vec{n}$ with the components l, m. The tangential direction to C is s. The n, s, z directions create the right triplet of vectors. The points of non-smoothness at the contour C may be considered in an ordinary way, this will be done in a separate paper.

The plate experiences large deflections created due to loading (they are much larger than the plate thickness).

The geometrically nonlinear elasticity theory and the Kirchhoff assumptions are employed (Novozhilov 1961, 2011). The latter assumptions are written as follows:

$$\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{1}$$

where ε is the Green strain tensor.

The deformed plate is supposed to be stable.

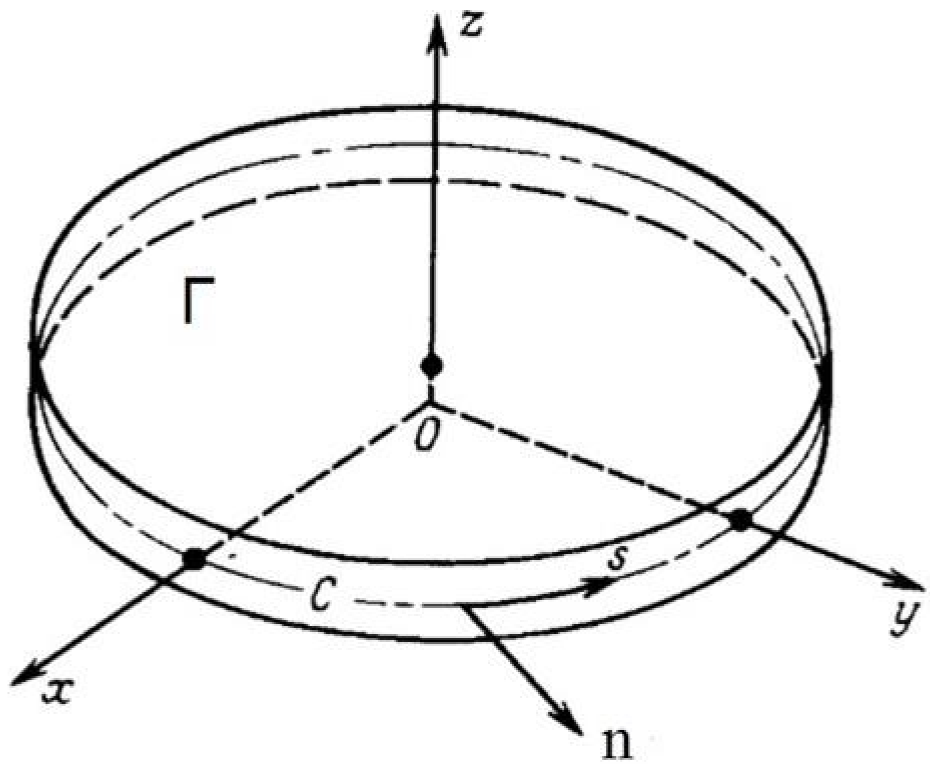

Fig. 1.

The ordinary notations are used hereinafter

$$\frac{\partial}{\partial x}(...) = (...),_x$$
$$\frac{\partial}{\partial y}(...) = (...),_y \qquad (2)$$

Small elongations and shears (negligible as compared to unity) are assumed. Following (Novozhilov, 1961, 2011), the displacements *u, v, w* within the plate (respectively along X, Y, Z) are supposed to be

$$\begin{aligned} u(x,y) &= \hat{u}(x,y) + z \cdot \theta(x,y) \\ \mathrm{v}(x,y) &= \hat{\mathrm{v}}(x,y) + z \cdot \psi(x,y) \\ w(x,y) &= \hat{w}(x,y) + z \cdot \chi(x,y) \end{aligned} \qquad (3)$$

where upper 'tilde' means quantities at the mid-plane. After using (3) and the definition of the Green tensor components it is obtained in (Novozhilov, 1961, 2011) that

$$\begin{aligned} \theta &= -\hat{w},_x (1 + \hat{\mathrm{v}},_y) + \hat{\mathrm{v}},_x \hat{w},_y \\ \psi &= -\hat{w},_y (1 + \hat{u},_x) + \hat{u},_y \hat{w},_x \\ \chi &= \hat{u},_x + \hat{\mathrm{v}},_y + \hat{u},_x \hat{\mathrm{v}},_y - \hat{u},_y \hat{\mathrm{v}},_x \end{aligned} \qquad (4)$$

and

$$\varepsilon_{xx} = \hat{\varepsilon}_{xx} + z\kappa_{xx} + z^2\eta_{xx}$$
$$\varepsilon_{yy} = \hat{\varepsilon}_{yy} + z\kappa_{yy} + z^2\eta_{yy} \tag{5}$$
$$\varepsilon_{xy} = \hat{\varepsilon}_{xy} + z\kappa_{xy} + z^2\eta_{xy}$$

The mid-plane strains in (5) are

$$\hat{\varepsilon}_{xx} = \hat{u},_x + \tfrac{1}{2}\left[\hat{u},_x^2 + \hat{\mathrm{v}},_x^2 + \hat{w},_x^2\right]$$
$$\hat{\varepsilon}_{yy} = \hat{\mathrm{v}},_y + \tfrac{1}{2}\left[\hat{u},_y^2 + \hat{\mathrm{v}},_y^2 + \hat{w},_y^2\right] \tag{6}$$
$$2\hat{\varepsilon}_{xy} = \hat{u},_y + \hat{\mathrm{v}},_x + \hat{u},_y \hat{u},_x + \hat{\mathrm{v}},_x \hat{\mathrm{v}},_y + \hat{w},_x \hat{w},_y$$

and other quantities in (5):

$$\kappa_{xx} = \theta,_x + \hat{u},_x \theta,_x + \hat{\mathrm{v}},_x \psi,_x + \hat{w},_x \chi,_x$$
$$\kappa_{yy} = \psi,_y + \hat{u},_y \theta,_y + \hat{\mathrm{v}},_y \psi,_y + \hat{w},_y \chi,_y \tag{7}$$
$$2\kappa_{xy} = \theta,_y + \psi,_x + \hat{u},_x \theta,_y + \hat{u},_y \theta,_x + \hat{\mathrm{v}},_x \psi,_y + \hat{\mathrm{v}},_y \psi,_x + \hat{w},_x \chi,_y + \hat{w},_y \chi,_x$$

$$\eta_{xx} = \tfrac{1}{2}\left[\theta,_x^2 + \psi,_x^2 + \chi,_x^2\right]$$
$$\eta_{yy} = \tfrac{1}{2}\left[\theta,_y^2 + \psi,_y^2 + \chi,_y^2\right] \tag{8}$$
$$2\eta_{xy} = \theta,_x \theta,_y + \psi,_x \psi,_y + \chi,_x \chi,_y$$

In (Novozhilov, 1961, 2011) it is indicated that for small elongations and shears the last (quadratic wrt to z) terms in (5) may be neglected. Then (5) is transformed to

$$\varepsilon_{xx} = \hat{\varepsilon}_{xx} + z\kappa_{xx}$$
$$\varepsilon_{yy} = \hat{\varepsilon}_{yy} + z\kappa_{yy} \tag{9}$$
$$\varepsilon_{xy} = \hat{\varepsilon}_{xy} + z\kappa_{xy}$$

Substituting (4) in (7), after cumbersome transformations we obtain

$$\kappa_{xx} = -\theta(x,y)\hat{u},_{xx} - \psi(x,y)\hat{\mathrm{v}},_{xx} - (1+\chi(x,y))\hat{w},_{xx}$$
$$\kappa_{yy} = -\theta(x,y)\hat{u},_{yy} - \psi(x,y)\hat{\mathrm{v}},_{yy} - (1+\chi(x,y))\hat{w},_{yy} \tag{10}$$
$$\kappa_{xy} = -\theta(x,y)\hat{u},_{xy} - \psi(x,y)\hat{\mathrm{v}},_{xy} - (1+\chi(x,y))\hat{w},_{xy}$$

Further the in-plane rotations are supposed to small. We have (see Novozhilov 1961, 2011)

$$e_{zz} = w,_{zz} \sim \varphi_z^2 \approx \omega_z^2 \tag{11}$$

where the in-plane rotation is

$$\omega_z = \tfrac{1}{2}(\hat{u},_y - \hat{\mathrm{v}},_x) \tag{12}$$

and e_{zz} is the zz component of the infinitesimal strain. Hence

$$\chi \sim \varphi_z^2 \tag{13}$$

and in the case of small in-plane rotations (negligible as compared to unity)

$$|\chi| << 1 \tag{14}$$

This means that (10) may be rewritten as

$$
\begin{aligned}
\kappa_{xx} &= -\theta(x,y)\hat{u},_{xx} - \psi(x,y)\hat{\text{v}},_{xx} - \hat{w},_{xx} \\
\kappa_{yy} &= -\theta(x,y)\hat{u},_{yy} - \psi(x,y)\hat{\text{v}},_{yy} - \hat{w},_{yy} \\
\kappa_{xy} &= -\theta(x,y)\hat{u},_{xy} - \psi(x,y)\hat{\text{v}},_{xy} - \hat{w},_{xy}
\end{aligned} \tag{15}
$$

The next step of our analysis is to use the virtual work principle (Washizu 1975). It is written as follows

$$
\int_V dV\left[\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy}\right] - \int_\Gamma q(x,y)\delta w d\Gamma - \int_{S_1}\left[f_x\delta u + f_y\delta v + f_z\delta w\right]dS = 0 \tag{16}
$$

where $q, f_x, f_x, f_x, \delta, V$ respectively are the transverse external force per unit area applied at the surface *Γ* of the plate, the X-Y-Z forces per unit area applied at the part S_1 of the plate lateral surface *S*, the variation symbol, the volume of the plate.

We introduce the following notations:

$$
\begin{aligned}
N_x &= \int_{-h/2}^{h/2}\sigma_{xx}dz, & M_x &= \int_{-h/2}^{h/2}z\sigma_{xx}dz, & F_x &= \int_{-h/2}^{h/2}f_x dz, & \widetilde{M}_x &= \int_{-h/2}^{h/2}zf_x dz \\
N_y &= \int_{-h/2}^{h/2}\sigma_{yy}dz, & M_y &= \int_{-h/2}^{h/2}z\sigma_{yy}dz, & F_y &= \int_{-h/2}^{h/2}f_y dz, & \widetilde{M}_y &= \int_{-h/2}^{h/2}zf_y dz \\
N_{xy} &= \int_{-h/2}^{h/2}\sigma_{xy}dz, & M_{xy} &= \int_{-h/2}^{h/2}z\sigma_{xy}dz & F_z &= \int_{-h/2}^{h/2}f_z dz & \widetilde{F}_z &= \int_{-h/2}^{h/2}zf_z dz
\end{aligned} \tag{17}
$$

Introducing (4), (6), (7), (9), (15), (17) in (16) and making necessary transformations, we obtain

$$
\int_\Gamma d\Gamma\left[\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + K_u\delta\hat{u} + K_v\delta\hat{v} + K_w\delta\hat{w}\right] - \int_\Gamma q(x,y)\delta w d\Gamma - \int_{S_1}\left[f_x\delta u + f_y\delta v + f_z\delta w\right]dS = 0 \tag{18}
$$

where

$$
\begin{aligned}
&K_u = -N_{x,x} - N_{xy,y} - \left(N_x \hat{u},_x + N_{xy} \hat{u},_y\right),_x - \left(N_y \hat{u},_y + N_{xy} \hat{u},_x\right),_y - \left(M_x \theta\right),_{xx} - \left(M_y \theta\right),_{yy} - \\
&-2\left(M_{xy} \theta\right),_{xy} + \Omega,_x \hat{w},_y - \Omega,_y \hat{w},_x \\
&K_v = -N_{y,y} - N_{xy,x} - \left(N_x \hat{v},_x + N_{xy} \hat{v},_y\right),_x - \left(N_y \hat{v},_y + N_{xy} \hat{v},_x\right),_y - \left(M_x \psi\right),_{xx} - \left(M_y \psi\right),_{yy} - \\
&-2\left(M_{xy} \psi\right),_{xy} + R,_y \hat{w},_x - R,_x \hat{w},_y \\
&K_w = -M_{x,xx} - M_{y,yy} - 2M_{xy,xy} - \left(N_x \hat{w},_x + N_{xy} \hat{w},_y\right),_x - \left(N_y \hat{w},_y + N_{xy} \hat{w},_x\right),_y + R,_x \left(1 + \hat{u},_x\right) - \\
&-R,_y \hat{v},_x + \Omega,_y \left(1 + \hat{u},_x\right) - \Omega,_x \hat{u},_y \\
&P_x = \left[N_x \left(1 + \hat{u},_x\right) + N_{xy} \hat{u},_y + \left(M_x \theta\right),_x + \left(M_{xy} \theta\right),_y - \Omega \hat{w},_y\right] \delta\hat{u} + \\
&+\left[N_{xy} \left(1 + \hat{v},_y\right) + N_x \hat{v},_x + \left(M_x \psi\right),_x + \left(M_{xy} \psi\right),_y + R \hat{w},_y\right] \delta\hat{v} + \\
&+\left[N_{xy} \hat{w},_y + N_x \hat{w},_x + M_{x,x} + M_{xy,y} - R\left(1 + \hat{v},_y\right) + \Omega \hat{u},_y\right] \delta\hat{w} - \\
&-\left(M_x \theta\right) \delta\hat{u},_x - \left(M_x \psi\right) \delta\hat{v},_x - M_x \delta\hat{w},_x - M_{xy} \theta \delta\hat{u},_y - M_{xy} \psi \delta\hat{v},_y - M_{xy} \delta\hat{w},_y \\
&P_y = \left[N_{xy} \left(1 + \hat{u},_x\right) + N_y \hat{u},_y + \left(M_{xy} \theta\right),_x + \left(M_y \theta\right)_y + \Omega \hat{w},_x\right] \delta\hat{u} + \\
&+\left[N_y \left(1 + \hat{v},_y\right) + N_{xy} \hat{v},_x + \left(M_{xy} \psi\right),_x + \left(M_y \psi\right),_y - R \hat{w},_x\right] \delta\hat{v} + \\
&+\left[N_y \hat{w},_y + N_{xy} \hat{w},_x + M_{y,y} + M_{xy,x} + R \hat{v},_x - \Omega\left(1 + u,_x\right)\right] \delta\hat{w} - \\
&-M_{xy} \theta \delta\hat{u},_x - M_{xy} \psi \delta\hat{v},_x - M_{xy} \delta\hat{w},_x - M_y \theta \delta\hat{u},_y - M_y \psi \delta\hat{v},_y - M_y \delta\hat{w},_y \\
&\Omega = -M_x \hat{v},_{xx} - M_y \hat{v},_{yy} - 2M_{xy} \hat{v},_{xy} \qquad (19) \\
&R = -M_x \hat{u},_{xx} - M_y \hat{u},_{yy} - 2M_{xy} \hat{u},_{xy}
\end{aligned}
$$

Equilibrium equations for *x, y, z* are respectively written as

$$
\begin{aligned}
K_u &= 0 \\
K_v &= 0 \qquad (20) \\
K_w - q &= 0
\end{aligned}
$$

The boundary conditions follow from the Gauss theorem:

$$
\int_\Gamma \left(P_{x,x} + P_{y,y}\right) d\Gamma = \int_C \left(\begin{pmatrix} P_x \\ P_y \end{pmatrix}^T , \vec{n} \right) dC \qquad (21)
$$

where T means transposition. Equating to zero the multipliers to the variations of the independent variables we obtain the required stationarity conditions for the virtual work.

We indicate some relations to be used for the contour terms (*a, b, d* are some functions of *x,y*):

$$
\begin{aligned}
&\frac{\partial}{\partial x} = l \frac{\partial}{\partial n} - m \frac{\partial}{\partial s} \\
&\frac{\partial}{\partial y} = m \frac{\partial}{\partial n} + l \frac{\partial}{\partial s} \qquad (22) \\
&a d,_x + b d,_y = \left(al + bm\right) d,_n + \left(-am + bl\right) d,_s
\end{aligned}
$$

Continuing (22), we obtain:

$$\int_{\Gamma}\left(P_{x,x}+P_{y,y}\right)d\Gamma=\int_{C}\left(lP_x+mP_y\right)dC=\int_{C}dC\{\delta\hat{u}\left[\left(1+\hat{u},_x\right)\left(N_x l+N_{xy}m\right)+\hat{u},_y\left(N_{xy}l+N_y m\right)+\right.$$
$$+\Omega\left(-l\hat{w},_y+m\hat{w},_x\right)+l\left(\left(M_x\theta\right),_x+\left(M_{xy}\theta\right),_y\right)+m\left(\left(M_{xy}\theta\right),_x+\left(M_y\theta\right),_y\right)\Big]+\delta\hat{v}\Big[\hat{v},_x\left(lN_x+mN_{xy}\right)+$$
$$+\left(1+\hat{v},_y\right)\left(lN_{xy}+mN_y\right)+R\left(l\hat{w},_y-m\hat{w},_x\right)+l\left(\left(M_x\psi\right),_x+\left(M_{xy}\psi\right),_y\right)+m\left(\left(M_{xy}\psi\right),_x+\left(M_y\psi\right),_y\right)\Big]+$$
$$+\delta\hat{w}\Big[l\left(N_x\hat{w},_x+N_{xy}\hat{w},_y+M_{x,x}+M_{xy,y}-R\left(1+\hat{v},_y\right)+\Omega\hat{u},_y\right)+m\Big(N_{xy}\hat{w},_x+N_y\hat{w},_y+M_{xy,x}+M_{y,y}+$$
$$+R\hat{v},_x-\Omega\left(1+\hat{u},_x\right)\Big)\Big]-\left(l^2M_x\theta+2lmM_{xy}\theta+m^2M_y\theta\right)\delta\hat{u},_n-\left(-lm\left(M_x-M_y\right)\theta+\left(l^2-m^2\right)M_{xy}\theta\right)\delta\hat{u},_s-$$
$$-\left(l^2M_x\psi+2lmM_{xy}\psi+m^2M_y\psi\right)\delta\hat{v},_n-\left(-lm\left(M_x-M_y\right)\psi+\left(l^2-m^2\right)M_{xy}\psi\right)\delta\hat{v},_s-$$
$$-\left(l^2M_x+2lmM_{xy}+m^2M_y\right)\delta\hat{w},_n-\left(-lm\left(M_x-M_y\right)+\left(l^2-m^2\right)M_{xy}\right)\delta\hat{w},_s\}$$

(23)

The last integral in (18) may be transformed as follows:

$$\int_{S_1}\left[f_x\delta u+f_y\delta v+f_z\delta w\right]dS=\int_{S_1}dS\{f_x\delta\left(\hat{u}-z\theta\right)+f_y\delta\left(\hat{v}-z\psi\right)+f_z\delta\left(\hat{w}-z\chi\right)\}=$$
$$=\int_{C_1}dC\{F_x\delta\hat{u}+F_y\delta\hat{v}+F_z\delta\hat{w}+\widetilde{M}_x\left[-\left(1+\hat{v},_y\right)\delta\hat{w},_x-\hat{w},_x\delta\hat{v},_y+\hat{v},_x\delta\hat{w},_y+\hat{w},_y\delta\hat{v},_x\right]+$$
$$+\widetilde{M}_y\left[-\left(1+\hat{u},_x\right)\delta\hat{w},_y-\hat{w},_y\delta\hat{u},_x+\hat{u},_y\delta\hat{w},_x+\hat{w},_x\delta\hat{u},_y\right]+\widetilde{F}_z\left[\delta\hat{u},_x+\delta\hat{v},_y+\hat{u},_x\delta\hat{v},_y+\hat{v},_y\delta\hat{u},_x-\right.$$
$$\left.-\hat{u},_y\delta\hat{v},_x-\hat{v},_x\delta\hat{u},_y\right]\}=\int_{C_1}dC\{F_x\delta\hat{u}+F_y\delta\hat{v}+F_z\delta\hat{w}+\delta\hat{u},_x\left[-\widetilde{M}_y\hat{w},_y+\widetilde{F}_z\left(1+\hat{v},_y\right)\right]+$$
$$+\delta\hat{u},_y\left[\widetilde{M}_y\hat{w},_x-\widetilde{F}_z\hat{v},_x\right]+\delta\hat{v},_x\left[\widetilde{M}_x\hat{w},_y-\widetilde{F}_z\hat{u},_y\right]+\delta\hat{v},_y\left[-\widetilde{M}_x\hat{w},_x+\widetilde{F}_z\left(1+\hat{u},_x\right)\right]+$$
$$+\delta\hat{w},_x\left[-\widetilde{M}_x\left(1+\hat{v},_y\right)+\widetilde{M}_y\hat{u},_y\right]+\delta\hat{w},_y\left[\widetilde{M}_x\hat{v},_x-\widetilde{M}_y\left(1+\hat{u},_x\right)\right]=$$
$$=\int_{C_1}dC\{F_x\delta\hat{u}+F_y\delta\hat{v}+F_z\delta\hat{w}+\left[\left(-\widetilde{M}_y\hat{w},_y+\widetilde{F}_z\left(1+\hat{v},_y\right)\right)l+\left(\widetilde{M}_y\hat{w},_x-\widetilde{F}_z\hat{v},_x\right)m\right]\delta\hat{u},_n+$$
$$+\left[-\left(-\widetilde{M}_y\hat{w},_y+\widetilde{F}_z\left(1+\hat{v},_y\right)\right)m+\left(\widetilde{M}_y\hat{w},_x-\widetilde{F}_z\hat{v},_x\right)l\right]\delta\hat{u},_s+$$
$$+\left[\left(\widetilde{M}_x\hat{w},_y-\widetilde{F}_z\hat{u},_y\right)l+\left(-\widetilde{M}_x\hat{w},_x+\widetilde{F}_z\left(1+\hat{u},_x\right)\right)m\right]\delta\hat{v},_n+$$
$$+\left[-\left(\widetilde{M}_x\hat{w},_y-\widetilde{F}_z\hat{u},_y\right)m+\left(-\widetilde{M}_x\hat{w},_x+\widetilde{F}_z\left(1+\hat{u},_x\right)\right)l\right]\delta\hat{v},_s+$$
$$+\left[\left(-\widetilde{M}_x\left(1+\hat{v},_y\right)+\widetilde{M}_y\hat{u},_y\right)l+\left(\widetilde{M}_x\hat{v},_x-\widetilde{M}_y\left(1+\hat{u},_x\right)\right)m\right]\delta\hat{w},_n+$$
$$+\left[-\left(-\widetilde{M}_x\left(1+\hat{v},_y\right)+\widetilde{M}_y\hat{u},_y\right)m+\left(\widetilde{M}_x\hat{v},_x-\widetilde{M}_y\left(1+\hat{u},_x\right)\right)l\right]\delta\hat{w},_s\}$$

(24)

Observing (18) - (24), we obtain

$$
\begin{aligned}
&\int_\Gamma \left(P_{x,x}+P_{y,y}\right)d\Gamma-\int_{S_1}\left[f_x\delta u+f_y\delta v+f_z\delta w\right]dS=\\
&\int_C dC\{\delta\hat{u}\left[\left(1+\hat{u},_x\right)\left(N_x l+N_{xy}m\right)+\hat{u},_y\left(N_{xy}l+N_y m\right)+\Omega\left(-l\hat{w},_y+m\hat{w},_x\right)+\right.\\
&+l\left(\left(M_x\theta\right),_x+\left(M_{xy}\theta\right),_y\right)+m\left(\left(M_{xy}\theta\right),_x+\left(M_y\theta\right),_y\right)\Big]+\\
&+\delta\hat{v}\left[\hat{v},_x\left(lN_x+mN_{xy}\right)+\left(1+\hat{v},_y\right)\left(lN_{xy}+mN_y\right)+R\left(l\hat{w},_y-m\hat{w},_x\right)+l\left(\left(M_x\psi\right),_x+\left(M_{xy}\psi\right),_y\right)+\right.\\
&+m\left(\left(M_{xy}\psi\right),_x+\left(M_y\psi\right),_y\right)\Big]+\\
&+\delta\hat{w}\left[l\left(N_x\hat{w},_x+N_{xy}\hat{w},_y+M_{x,x}+M_{xy,y}-R\left(1+\hat{v},_y\right)+\Omega\hat{u},_y\right)+m\left(N_{xy}\hat{w},_x+N_y\hat{w},_y+M_{xy,x}+M_{y,y}+\right.\right.\\
&\left.\left.+R\hat{v},_x-\Omega\left(1+\hat{u},_x\right)\right)\right]-\\
&-\left(l^2M_x\theta+2lmM_{xy}\theta+m^2M_y\theta\right)\delta\hat{u},_n-\left(-lm\left(M_x-M_y\right)\theta+\left(l^2-m^2\right)M_{xy}\theta\right)\delta\hat{u},_s-\\
&-\left(l^2M_x\psi+2lmM_{xy}\psi+m^2M_y\psi\right)\delta\hat{v},_n-\left(-lm\left(M_x-M_y\right)\psi+\left(l^2-m^2\right)M_{xy}\psi\right)\delta\hat{v},_s-\\
&-\left(l^2M_x+2lmM_{xy}+m^2M_y\right)\delta\hat{w},_n-\left(-lm\left(M_x-M_y\right)+\left(l^2-m^2\right)M_{xy}\right)\delta\hat{w},_s\}-\\
&-\int_{C_1}dC\{F_x\delta\hat{u}+F_y\delta\hat{v}+F_z\delta\hat{w}+\left[\left(-\tilde{M}_y\hat{w},_y+\tilde{F}_z\left(1+\hat{v},_y\right)\right)l+\left(\tilde{M}_y\hat{w},_x-\tilde{F}_z\hat{v},_x\right)m\right]\delta\hat{u},_n+\\
&+\left[-\left(-\tilde{M}_y\hat{w},_y+\tilde{F}_z\left(1+\hat{v},_y\right)\right)m+\left(\tilde{M}_y\hat{w},_x-\tilde{F}_z\hat{v},_x\right)l\right]\delta\hat{u},_s+\\
&+\left[\left(\tilde{M}_x\hat{w},_y-\tilde{F}_z\hat{u},_y\right)l+\left(-\tilde{M}_x\hat{w},_x+\tilde{F}_z\left(1+\hat{u},_x\right)\right)m\right]\delta\hat{v},_n+\\
&+\left[-\left(\tilde{M}_x\hat{w},_y-\tilde{F}_z\hat{u},_y\right)m+\left(-\tilde{M}_x\hat{w},_x+\tilde{F}_z\left(1+\hat{u},_x\right)\right)l\right]\delta\hat{v},_s+\\
&+\left[\left(-\tilde{M}_x\left(1+\hat{v},_y\right)+\tilde{M}_y\hat{u},_y\right)l+\left(\tilde{M}_x\hat{v},_x-\tilde{M}_y\left(1+\hat{u},_x\right)\right)m\right]\delta\hat{w},_n+\\
&+\left[-\left(-\tilde{M}_x\left(1+\hat{v},_y\right)+\tilde{M}_y\hat{u},_y\right)m+\left(\tilde{M}_x\hat{v},_x-\tilde{M}_y\left(1+\hat{u},_x\right)\right)l\right]\delta\hat{w},_s\}
\end{aligned}
\tag{25}
$$

Following the regular procedure, we should exclude the variations of the tangential derivatives $\delta\hat{u},_s, \delta\hat{v},_s, \delta\hat{w},_s$ from (25). The way of doing that is described in (Washizu 1975), (Reddy 2007). Having the contour integral like below and integrating by parts, we write

$$\int_G H\frac{\partial\delta\beta}{\partial s}ds=\left(H\delta\beta\right)\Big|_G-\int_G \delta\beta\frac{\partial H}{\partial s}ds \tag{26}$$

where *H, δβ* are some functions at the contour *G* and the first rhs term means the difference of the squared product value at the ends of the contour.

We introduce the notations:

$$A_u=\left(lm\left(M_x-M_y\right)\theta-\left(l^2-m^2\right)M_{xy}\theta\right)+\left[\left(-\tilde{M}_y\hat{w},_y+\tilde{F}_z\left(1+\hat{v},_y\right)\right)m-\left(\tilde{M}_y\hat{w},_x-\tilde{F}_z\hat{v},_x\right)l\right] \tag{27}$$

$$A_v=\left(lm\left(M_x-M_y\right)\psi-\left(l^2-m^2\right)M_{xy}\psi\right)+\left[\left(\tilde{M}_x\hat{w},_y-\tilde{F}_z\hat{u},_y\right)m-\left(-\tilde{M}_x\hat{w},_x+\tilde{F}_z\left(1+\hat{u},_x\right)\right)l\right] \tag{28}$$

$$A_w=\left(lm\left(M_x-M_y\right)-\left(l^2-m^2\right)M_{xy}\right)+\left[\left(-\tilde{M}_x\left(1+\hat{v},_y\right)+\tilde{M}_y\hat{u},_y\right)m-\left(\tilde{M}_x\hat{v},_x-\tilde{M}_y\left(1+\hat{u},_x\right)\right)l\right] \tag{30}$$

Then all terms in (25) containing variations of the tangential derivatives of $\hat{u},\hat{v},\hat{w}$ are written as follows:

$$A_u\delta\hat{u},_s+A_v\delta\hat{v},_s+A_w\delta\hat{w},_s \tag{31}$$

It should be noted that the square-bracketed terms are non-zero ones at the loaded part C_1 only.

Transforming the contour integral of (31) in view of (26) and removing the terms at the ends of the contour (the variations of the displacements at the ends of the supported contour portion C_2 are equal to zero), we obtain the under-integral term, corresponding to the tangential derivatives of the displacements $\hat{u}, \hat{v}, \hat{w}$ as:

$$\frac{\partial A_u}{\partial s}\delta\hat{u} + \frac{\partial A_v}{\partial s}\delta\hat{v} + \frac{\partial A_w}{\partial s}\delta\hat{w} \tag{32}$$

It is clear now that (32) will give some extra terms for the boundary conditions. The developed relations lead to the boundary conditions. Namely, at C_1 the multipliers to $\delta\hat{u}, \delta\hat{v}, \delta\hat{w}, \delta\hat{u},_n, \delta\hat{v},_n, \delta\hat{w},_n$ should be equal to zero. This gives us six relations at C_1:

$$\begin{aligned}&\left[(1+\hat{u},_x)(N_x l + N_{xy} m) + \hat{u},_y (N_{xy} l + N_y m) + \Omega(-l\hat{w},_y + m\hat{w},_x) + \right.\\ &\left. + l\left((M_x\theta),_x + (M_{xy}\theta),_y\right) + m\left((M_{xy}\theta),_x + (M_y\theta),_y\right)\right] + \frac{\partial A_u}{\partial s} - F_x = 0\end{aligned} \tag{25}$$

$$\begin{aligned}&\left[\hat{v},_x (lN_x + mN_{xy}) + (1+\hat{v},_y)(lN_{xy} + mN_y) + R(l\hat{w},_y - m\hat{w},_x) + \right.\\ &\left. + l\left((M_x\psi),_x + (M_{xy}\psi),_y\right) + m\left((M_{xy}\psi),_x + (M_y\psi),_y\right)\right] + \frac{\partial A_v}{\partial s} - F_y = 0\end{aligned} \tag{26}$$

$$\begin{aligned}&\left[l\left(N_x\hat{w},_x + N_{xy}\hat{w},_y + M_{x,x} + M_{xy,y} - R(1+\hat{v},_y) + \Omega\hat{u},_y\right) - \right.\\ &\left. + m\left(N_{xy}\hat{w},_x + N_y\hat{w},_y + M_{xy,x} + M_{y,y} + R\hat{v},_x - \Omega(1+\hat{u},_x)\right)\right] + \frac{\partial A_w}{\partial s} - F_z = 0\end{aligned} \tag{27}$$

$$-\left(l^2 M_x\theta + 2lmM_{xy}\theta + m^2 M_y\theta\right) - \left[\left(-\widetilde{M}_y\hat{w},_y + \widetilde{F}_z(1+\hat{v},_y)\right)l + \left(\widetilde{M}_y\hat{w},_x - \widetilde{F}_z\hat{v},_x\right)m\right] = 0 \tag{28}$$

$$-\left(l^2 M_x\psi + 2lmM_{xy}\psi + m^2 M_y\psi\right) - \left[\left(\widetilde{M}_x\hat{w},_y - \widetilde{F}_z\hat{u},_y\right)l + \left(-\widetilde{M}_x\hat{w},_x + \widetilde{F}_z(1+\hat{u},_x)\right)m\right] = 0 \tag{29}$$

$$-\left(l^2 M_x + 2lmM_{xy} + m^2 M_y\right) - \left[\left(-\widetilde{M}_x(1+\hat{v},_y) + \widetilde{M}_y\hat{u},_y\right)l + \left(\widetilde{M}_x\hat{v},_x - \widetilde{M}_y(1+\hat{u},_x)\right)m\right] = 0 \tag{30}$$

At C_2 we obtain the clamping conditions

$$\delta\hat{u} = \delta\hat{v} = \delta\hat{w} = \delta\hat{u},_n = \delta\hat{v},_n = \delta\hat{w},_n = 0 \tag{31}$$

Other types of the boundary conditions should be considered in a similar way via analysis of the variational integral.

3. Discussion

Every equilibrium equation from (20) expressed in $\hat{u}, \hat{v}, \hat{w}$ has the order four. Hence, the order of the equilibrium equation system (20) is twelve. There are six unknown primary variables $\delta\hat{u}, \delta\hat{v}, \delta\hat{w}, \delta\hat{u},_n, \delta\hat{v},_n, \delta\hat{w},_n$ to be determined from (20). We have six boundary conditions at every contour portion, namely the natural boundary conditions (25)-(30) at C_1 and the essential boundary conditions (31) at C_2. This means that the developed system of equations is consistent.

When we consider the moderate deflections only, it is easy to see that in the quantities K_u, K_v, K_w respectively two, two and five initial terms (corresponding to the von Karman case) remain. In the boundary conditions the variables θ, ψ become equal to zero. The quantities

A_u, A_v disappear (or, in other words, equal to zero). The quantities $\hat{u},_x, \hat{u},_y, \hat{v},_x, \hat{v},_y$ are neglected and taken equal to zero. The conditions (25)-(27), (30) are transformed to the von Karman case (see Reddy 2007). The conditions (28), (29) disappear. The variables $\delta\hat{u},_n, \delta\hat{v},_n$ are removed from the list of the primary ones.

4. Conclusions

- The equilibrium equations are derived.
- The obtained system of equilibrium equations is consistent
- There are six primary variables in the system, namely $\delta\hat{u}, \delta\hat{v}, \delta\hat{w}, \delta\hat{u},_n, \delta\hat{v},_n, \delta\hat{w},_n$
- The essential and natural boundary conditions are obtained.
- For moderate deflections the equilibrium equations and the boundary conditions are transformed to ones of the von Karman case
- The solutions of the derived equations may be used as benchmarks for the nonlinear structural analysis software like FEM.

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