

Analytical sensitivity analysis of dynamic problems with direct differentiation of generalized- α time integration

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Abstract: In this paper, the direct differentiation of generalized- α time integration is derived, equations are introduced and results are shown. Although generalized- α time integration has found usage, the derivation and the resulting equations for the analytical sensitivity analysis via direct differentiation are missing. Thus here, the sensitivity equations of generalized- α time integration via direct differentiation are provided. Results with generalized- α are compared with Newmark- β and their sensitivities with numerical sensitivities via forward finite differencing in terms of accuracy and performance. An example is shown for each linear structural dynamics and flexible multibody dynamics.

Keywords: sensitivity analysis; time integration; structural dynamics; flexible multibody dynamics

1. Introduction

The analytical design sensitivity analysis is developed in this paper for the generalized- α family of time integration. After deriving and introducing the design sensitivity using direct differentiation, results are compared with numerical differentiation via forward differencing in terms of accuracy and computational effort.

Efficient sensitivity analysis is the centerpiece of efficient and effective gradient-based design optimization. Other uses of sensitivity analysis include uncertainty analysis and quantification. The adjoint variable method applied to the sensitivity analysis with respect to uncertain and random variables is developed in [1]. Overviews of design sensitivity analysis are provided by [2–6]. In transient problems, sensitivity analysis must be extended to time integration [7,8], which is handled here.

A number of studies have been carried out for sensitivity analysis in transient structural dynamics and multibody dynamics. Early works on sensitivity analysis with transient structural dynamics include [9], which bridges sensitivities in static and dynamic analysis via the adjoint variable method, including maximum-value objectives and constraints. In [10], sensitivity analysis with adjoint variable method and direct differentiation is introduced for time-average and time-specific constraints. The German-language monograph [11] clearly demonstrates analytic sensitivity analysis of transient linear dynamics with Newmark time integration, providing all necessary equations for implementation, which also includes the sensitivity of the duration of a dynamic operation. The adjoint variable method for transient structural dynamics with Newmark time integration is derived by [12]. The adjoint variable method and direct differentiation of nonlinear and transient problems is handled by [13] in which the crucial aspects of the nonlinear solver for sensitivity analysis are introduced.

The sensitivity analysis of multibody systems is often limited to the governing equations as for direct differentiation with the divide-and-conquer based modal superposition method in [14], for direct differentiation and the adjoint variable method with general formulations

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of differential–algebraic equations in [15], for direct differentiation and the adjoint variable method of ordered differential equations in [16] and for the adjoint variable method with the floating frame of reference formulation in [17]. The adjoint variable method of the governing equations in descriptor form and for generic optimization functions is shown by [18] for several time integration methods including backward difference formula, Newmark, Adams–Bashforth–Moulton and Diagonally Implicit Runge–Kutta. The derivation of the adjoint variable method based on generalized- α time integration and applied to the finite-element based formulation is provided by [19].

Higher-order sensitivities can also be of interest, especially in cases of low number of degrees of freedom and design variables. Second-order sensitivities via the adjoint variable method with the Runge–Kutta–Fehlberg method are shown in [20]. Second-order sensitivities for multibody systems with the adjoint variable method and differential–algebraic equations are shown in [21]. Higher-order sensitivities are used in [22,23] to develop a surrogate model for optimization. This work is limited to first-order sensitivities as needed for gradient-based optimization, though the equations provided can be readily extended to higher-order derivatives.

The generalized- α family of time integration methods offers several time integrators via a single implementation and the generalized- α method is therefore chosen here. The α -family of methods add numerical damping to the system to avoid numerical stability problems [24].

Generalized- α time integration method has found usages in both nonlinear structural dynamics and multibody dynamics. Multibody dynamics addresses the dynamic response with kinematic constraints. Rigid multibody dynamics considers the bodies rigid, while flexible multibody dynamics allows for deformation in the bodies of the system. Applications are shown including both linear structural dynamics and flexible multibody dynamics.

In the dynamic systems addressed, the transient response (response in time domain) is of interest and therefore a temporal discretization is applied. The calculation of the system responses, here position, velocity and acceleration, can be carried out with explicit or implicit time integration methods. Implicit time integration allows for larger time steps than explicit methods, which is bounded by the Courant criterion to remain stable [25].

Newmark introduced an implicit integration scheme, now referred to as Newmark- β , in 1959 with two integration parameters γ and β [26]. The Hilber–Hughes–Taylor (HHT) method was introduced in 1977 to improve upon Newmark with the additional integration parameter α_f for numerical damping [27], while the Wood–Bossak–Zienkiewicz (WBZ, also known as simply Bossak or Wood) introduced the parameter α_m [28]. Generalized- α was introduced by Chung and Hulbert in 1993 [24] to generalize all these methods with the integration parameters α_m , α_f , β and γ . These methods can be generalized via a single implementation and is therefore referred to as the generalized- α family of time integrators. In the generalized- α method, for which the family is named, all parameters can be controlled by a single value, namely the spectral radius at infinity¹ ϱ_∞ ranging from zero, being the highest amount of damping, to unity, being none [24]. The mapping of ϱ_∞ to α_m , α_f , β and γ is shown below. The choice of this value can affect the response for a given time step, see e.g. [29].

This paper is concerned with the sensitivity analysis, which is used when one or more parameters effect the system responses. The calculation of the system responses is referred to here as primal analysis to distinguish from the sensitivity analysis. This is of interest for efficient design optimization via gradient-based optimization algorithms in addition to investigating the sensitivities themselves or for uncertainty analysis. Methods for sensitivity analysis include analytical, numerical, complex step and algorithmic approaches [5].

Analytical sensitivities can be derived either manually (by hand) or via automatic differentiation of the implemented code. Automatic differentiation and manual differentiation of rigid multibody systems are compared in [30]. Although prone to human error and not

¹ It should be noted that we differentiate between variants of the Greek letter rho: ϱ is used here for the numerical damping term, while ρ is used below for density.

automatic, the former represents an exact and one of the fastest methods for sensitivity analysis [31]. The high implementation effort of the manual sensitivity analysis though results in a typically highly reusable implementation and must therefore only be carried out once. Due to its physics- and knowledge-based nature in addition to the increased transparency, hand differentiation is preferred by the authors.

This study limits itself to analytical sensitivity analysis, which is further divided into direct differentiation (also known as forward sensitivity analysis) and the adjoint variable method (also known as backward sensitivity analysis). Generally, it can be said that direct differentiation is more efficient than the adjoint variable method if the number of design variables is less than the number of functions to differentiate, i.e. in optimization the sum of the number of objective functions (one in case of single objective optimization) and the number of constraints. Due to its nature, the adjoint variable method must be performed after the time integration of the primal analysis and integrated backwards in time. Thus, the adjoint variable method, in contrast to direct differentiation, needs dedicated solver routines and cannot reuse these implementations [7,8]. Further, for nonlinear analysis the system matrices must be stored or recalculated when integrating backwards in time. Therefore, direct differentiation is chosen in this work. The reader is referred to the following literature for the adjoint variable method: [17,18,32–34].

The scope here is thus the investigation of direct differentiation of the generalized- α method. A method in this regard is introduced in [35], yet the needed equations are not derived and shown. The present work is an extension of [7,36] where Newmark- β sensitivities are developed, which are in turn based on [11,12]. In this paper, Newmark- β serves as a comparison.

This work first generally introduces generalized- α time integration and its equations. Afterwards, the sensitivity equations are introduced for structural dynamics and flexible multibody dynamics, respectively. A numerical example is shown for each and results are provided of both primal and sensitivity analysis in comparison with Newmark- β time integration. We close with a conclusion, including discussion of this work. Further, the detailed derivation of the effective system matrices is provided in the appendix.

2. Sensitivity analysis with generalized- α time integration

In this section, we introduce the equations for an acceleration-form ($\mathbf{a}$ -form) of generalized- α time integration using a predictor–corrector scheme. The acceleration-form refers to the formulation, where the acceleration is the variable for which is being solved, and is contrasted by the displacement-form ($\mathbf{d}$ -form), where the displacement is the primary variable. The predictor–corrector scheme allows for nonlinear terms in the governing equation in which the predictor values are used to determine these values in the nonlinear solver, e.g. Newton–Raphson.

In this work, we extend this well-established time integration method to include design sensitivity analysis, which is defined by the differentiation with respect to the design variables (but could include other parameters, e.g. uncertain parameters). Sensitivity analysis is carried out here by directly differentiating both the equations of the time integration as well as the governing equations. We consider both linear structural dynamics as well as flexible multibody dynamics, which is intrinsically nonlinear. After introducing the general equations for generalized- α time integration and its sensitivities, the domain specific governing equations for linear structural dynamics and flexible multibody dynamics are given. Flexible multibody dynamics is shown using the specific approach developed in [7,37,38] with an index-1 approach and the floating frame of reference formulation.

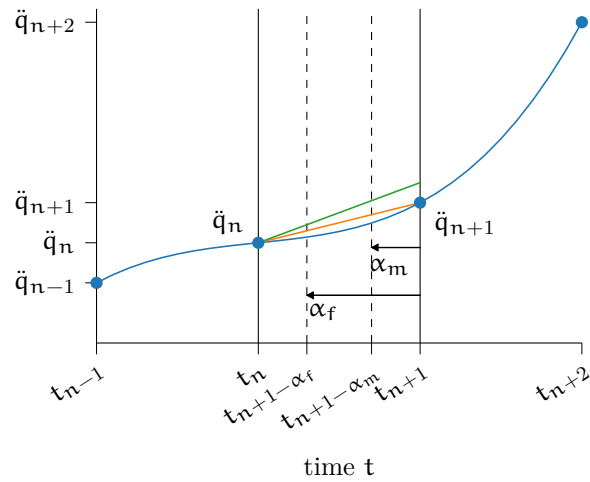


Figure 1. Time integration with — acceleration, — $\gamma = 0.5$, — $\gamma = 0.75$

2.1. Generalized- α time integration

134

The method for primal and sensitivity analysis with generalized- α time integration is introduced in the following. The basis of which are Newmark's equations for the updated position $\underline{q}$ and velocity vectors $\dot{\underline{q}}$ with the calculated value of the acceleration vector $\ddot{\underline{q}}$,

$$\underline{q}_{n+1} = \underbrace{\underline{q}_n + \Delta t \dot{\underline{q}}_n + \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{\underline{q}}_n}_{\text{predictor } \underline{q}_{\text{pred}}} + \underbrace{\beta \Delta t^2 \ddot{\underline{q}}_{n+1}}_{\text{corrector}}, \quad (1)$$

$$\dot{\underline{q}}_{n+1} = \underbrace{\dot{\underline{q}}_n + (1 - \gamma) \Delta t \ddot{\underline{q}}_n}_{\text{predictor } \dot{\underline{q}}_{\text{pred}}} + \underbrace{\gamma \Delta t \ddot{\underline{q}}_{n+1}}_{\text{corrector}}, \quad (2)$$

where γ and β are the Newmark integration parameters. Though the simulation is evaluated at the times t_n and t_{n+1} , values at intermediate points are interpolated. The position $\underline{q}$ and velocity $\dot{\underline{q}}$ use time $t_{n+1-\alpha_f}$ and the acceleration $\ddot{\underline{q}}$ uses time $t_{n+1-\alpha_m}$, giving the approximations by linear interpolation (see fig. 1),

$$\underline{q}_{n+1-\alpha_f} = (1 - \alpha_f) \underline{q}_{n+1} + \alpha_f \underline{q}_n, \quad (3)$$

$$\dot{\underline{q}}_{n+1-\alpha_f} = (1 - \alpha_f) \dot{\underline{q}}_{n+1} + \alpha_f \dot{\underline{q}}_n, \quad (4)$$

$$\ddot{\underline{q}}_{n+1-\alpha_m} = (1 - \alpha_m) \ddot{\underline{q}}_{n+1} + \alpha_m \ddot{\underline{q}}_n, \quad (5)$$

where α_m and α_f are integration parameters. A predictor–corrector scheme is used in order to accommodate nonlinear calculation and state-dependency of the system parameters, i.e. system matrices with respect to position and velocity. The predicted state variables with Newmark's equations are

$$\underline{q}_{\text{pred}} = \underline{q}_n + \Delta t \dot{\underline{q}}_n + \left(\frac{1}{2} - \beta\right) \Delta t^2 \ddot{\underline{q}}_n, \quad (6)$$

$$\dot{\underline{q}}_{\text{pred}} = \dot{\underline{q}}_n + (1 - \gamma) \Delta t \ddot{\underline{q}}_n, \quad (7)$$

and we calculate $\ddot{\underline{q}}_{n+1}$ using the effective system equations (see below for explanation of structural dynamics and flexible multibody dynamics, respectively). The predicted values are then corrected with Newmark's equations,

$$\underline{q}_{n+1} = \underline{q}_{\text{pred}} + \beta \Delta t^2 \ddot{\underline{q}}_{n+1}, \quad (8)$$

$$\dot{\underline{q}}_{n+1} = \dot{\underline{q}}_{\text{pred}} + \gamma \Delta t \ddot{\underline{q}}_{n+1}. \quad (9)$$

For the sensitivity analysis within the generalized- α time integration, we use direct differentiation of the approximations,

$$\underline{\underline{\nabla}} \mathbf{q}_{n+1-\alpha_f} = (1 - \alpha_f) \underline{\underline{\nabla}} \mathbf{q}_{n+1} + \alpha_f \underline{\underline{\nabla}} \mathbf{q}_n, \quad (10)$$

$$\underline{\underline{\nabla}} \dot{\mathbf{q}}_{n+1-\alpha_f} = (1 - \alpha_f) \underline{\underline{\nabla}} \dot{\mathbf{q}}_{n+1} + \alpha_f \underline{\underline{\nabla}} \dot{\mathbf{q}}_n, \quad (11)$$

$$\underline{\underline{\nabla}} \ddot{\mathbf{q}}_{n+1-\alpha_m} = (1 - \alpha_m) \underline{\underline{\nabla}} \ddot{\mathbf{q}}_{n+1} + \alpha_m \underline{\underline{\nabla}} \ddot{\mathbf{q}}_n, \quad (12)$$

where the nabla operator ∇ denotes the design sensitivity, i.e. total derivative with respect to the design variables, e.g. for the sensitivity scalar response with respect to the design variable vector, 135
136
137

$$\underline{\underline{\nabla}}(\cdot) = \frac{d(\cdot)}{d\mathbf{x}}. \quad (13)$$

The same direct differentiation is applied to the predictors, which are then defined by

$$\underline{\underline{\nabla}} \mathbf{q}_{\text{pred}} = \underline{\underline{\nabla}} \mathbf{q}_n + \Delta t \underline{\underline{\nabla}} \dot{\mathbf{q}}_n + \left(\frac{1}{2} - \beta\right) \Delta t^2 \underline{\underline{\nabla}} \ddot{\mathbf{q}}_n, \quad (14)$$

$$\underline{\underline{\nabla}} \dot{\mathbf{q}}_{\text{pred}} = \underline{\underline{\nabla}} \dot{\mathbf{q}}_n + (1 - \gamma) \Delta t \underline{\underline{\nabla}} \ddot{\mathbf{q}}_n. \quad (15)$$

After calculating $\underline{\underline{\nabla}} \ddot{\mathbf{q}}_{n+1}$ by use of the effective sensitivity system (see below for explanation of structural dynamics and flexible multibody dynamics, respectively), the following correctors are used:

$$\underline{\underline{\nabla}} \mathbf{q}_{n+1} = \underline{\underline{\nabla}} \mathbf{q}_{\text{pred}} + \beta \Delta t^2 \underline{\underline{\nabla}} \ddot{\mathbf{q}}_{n+1}, \quad (16)$$

$$\underline{\underline{\nabla}} \dot{\mathbf{q}}_{n+1} = \underline{\underline{\nabla}} \dot{\mathbf{q}}_{\text{pred}} + \gamma \Delta t \underline{\underline{\nabla}} \ddot{\mathbf{q}}_{n+1}. \quad (17)$$

For nonlinear analysis, a Newton solver can be integrated into the solving routine. This is explained with flexible multibody systems below in Sec. 2.3. 138
139

The time integration constants of generalized- α are α_m , α_f , β and γ and through their choice can lead to a number of established time integration methods, including the Newmark family of integrators, Hilber–Hughes–Taylor and Wood–Bossak–Zienkiewicz. Therefore, this can then be referred to as the α -family of time integrators. The equations introduced above in this section can therefore be used generally within this family. Specifically for the generalized- α time integration used here, the choice of integration constants can be tied to a single user-defined parameter, the spectral radius at infinity ϱ_∞ [24],

$$\alpha_m = \frac{2\varrho_\infty - 1}{\varrho_\infty + 1}, \quad (18)$$

$$\alpha_f = \frac{\varrho_\infty}{\varrho_\infty + 1}, \quad (19)$$

$$\gamma = \frac{1}{2} - \alpha_m + \alpha_f, \quad (20)$$

$$\beta = \frac{1}{4}(1 - \alpha_m + \alpha_f)^2. \quad (21)$$

The spectral radius at infinity is sometimes referred to as the (numerical) damping and is limited to a range of $[0, 1]$. The value $\varrho_\infty = 1$ returns Newmark- β retaining frequencies are preserved, while $\varrho_\infty = 0$ annihilates all frequencies above $\frac{\Delta t}{T}$ after one time step in which T is the period of the highest frequency of interest. The results converge to those without numerical damping when the time steps goes to zero. Thus, the level of numerical damping introduced to generalized- α time integration depends on both the step size as well as the spectral radius at infinity and the choice is critical for proper results. A parameter study with fixed time step and varying the values of the spectral radius at infinity is shown in [29]. As the present study develops a general framework for this methodology, this is not addressed here and the reader is referred to [24,39–41]. In this paper, we use a moderate 140
141
142
143
144
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146
147
148
149

value of $\varrho_\infty = 0.55$, though the sensitivity analysis methodology developed is general and independent of this parameter value.

2.2. Structural dynamics

The governing equation for linear structural dynamics is defined by the following second-order equation of motion:

$$\underline{\underline{m}} \ddot{\underline{q}}(t) + \underline{\underline{d}} \dot{\underline{q}}(t) + \underline{\underline{k}} \underline{q}(t) - \underline{F}(t) = \underline{0}, \quad (22)$$

where $\underline{\underline{m}}$ is the mass matrix, $\underline{\underline{d}}$ is the damping matrix, $\underline{\underline{k}}$ is the stiffness matrix and $\underline{F}$ is the external force vector. For linear structural dynamics the system matrices are constant and independent of position and time. For the solving scheme, we add the residual force vector $\underline{R}(t)$, which is to be zero (to an allowable value) at all times, to the right-hand side. The α -family of integration methods use intermediate point approximations between $\underline{t}_n$ and $\underline{t}_{n+1}$, namely $\underline{t}_{n+1-\alpha_f}$ and $\underline{t}_{n+1-\alpha_m}$, revealing

$$\underline{R}_{n+1-\alpha_f} = \underline{\underline{m}} \ddot{\underline{q}}_{n+1-\alpha_m} + \underline{\underline{d}} \dot{\underline{q}}_{n+1-\alpha_f} + \underline{\underline{k}} \underline{q}_{n+1-\alpha_f} - \underline{F}_{n+1-\alpha_f}, \quad (23)$$

where the linear approximation of the intermediate external force is

$$\underline{F}_{n+1-\alpha_f} = (1 - \alpha_f) \underline{F}_{n+1} + \alpha_f \underline{F}_n. \quad (24)$$

The effective equation of motion is defined by

$$\underline{\underline{m}}_{\text{eff}} \ddot{\underline{q}}_{n+1} = \underline{F}_{\text{eff}}, \quad (25)$$

where

$$\underline{\underline{m}}_{\text{eff}} = (1 - \alpha_m) \underline{\underline{m}} + (1 - \alpha_f) \gamma \Delta t \underline{\underline{d}} + (1 - \alpha_f) \beta \Delta t^2 \underline{\underline{k}}, \quad (26)$$

$$\begin{aligned} \underline{F}_{\text{eff}} = & (1 - \alpha_f) \underline{F}_{n+1} + \alpha_f \underline{F}_n + \\ & - \alpha_m \underline{\underline{m}} \ddot{\underline{q}}_n - \underline{\underline{d}} \left((1 - \alpha_f) \dot{\underline{q}}_{\text{pred}} + \alpha_f \dot{\underline{q}}_n \right) - \underline{\underline{k}} \left((1 - \alpha_f) \underline{q}_{\text{pred}} + \alpha_f \underline{q}_n \right). \end{aligned} \quad (27)$$

For the derivation of the effective mass matrix $\underline{\underline{m}}_{\text{eff}}$ and effective force vector $\underline{F}_{\text{eff}}$ for generalized- α method, see Sec. A.1.

The sensitivity analysis is derived using the direct differentiation which is applied to the governing equation (23). This reveals the sensitivity of the force balance for the α -family of integration methods as

$$\begin{aligned} \underline{\underline{\nabla R}}_{n+1-\alpha_f} = & \underline{\underline{\nabla m}} \ddot{\underline{q}}_{n+1-\alpha_m} + \underline{\underline{m}} \underline{\underline{\nabla \ddot{q}}}_{n+1-\alpha_m} + \underline{\underline{\nabla d}} \dot{\underline{q}}_{n+1-\alpha_f} + \underline{\underline{d}} \underline{\underline{\nabla \dot{q}}}_{n+1-\alpha_f} + \\ & + \underline{\underline{\nabla k}} \underline{q}_{n+1-\alpha_f} + \underline{\underline{k}} \underline{\underline{\nabla q}}_{n+1-\alpha_f} - \underline{\underline{\nabla F}}_{n+1-\alpha_f}, \end{aligned} \quad (28)$$

where the intermediate external force sensitivity is found with the approximation

$$\underline{\underline{\nabla F}}_{n+1-\alpha_f} = (1 - \alpha_f) \underline{\underline{\nabla F}}_{n+1} + \alpha_f \underline{\underline{\nabla F}}_n. \quad (29)$$

The design sensitivities of the system matrices are three-dimensional matrices and therefore efficient handling is needed.

The effective equation of motion for sensitivities is defined by the differentiation of (25),

$$\underline{\underline{m}}_{\text{eff}} \underline{\underline{\nabla \ddot{q}}}_{n+1} = \underline{F}_{\text{pseudo}}, \quad (30)$$

where

$$\underline{F}_{\text{pseudo}} = \underline{\underline{\nabla F}}_{\text{eff}} - \underline{\underline{\nabla m}}_{\text{eff}} \ddot{\underline{q}}_{n+1}, \quad (31)$$

and fully expressed by

$$\begin{aligned} \underline{\underline{F}}_{\text{pseudo}} = & (1 - \alpha_f) \underline{\underline{\nabla}} \underline{\underline{F}}_{n+1} + \alpha_f \underline{\underline{\nabla}} \underline{\underline{F}}_n - \underline{\underline{\nabla}} \underline{\underline{m}} \left((1 - \alpha_m) \underline{\underline{\ddot{q}}}_{n+1} + \alpha_m \underline{\underline{\ddot{q}}}_n \right) - \alpha_m \underline{\underline{m}} \underline{\underline{\nabla}} \underline{\underline{\ddot{q}}}_n + \\ & - \underline{\underline{\nabla}} \underline{\underline{d}} \left((1 - \alpha_f) \underline{\underline{\dot{q}}}_{n+1} + \alpha_f \underline{\underline{\dot{q}}}_n \right) - \underline{\underline{d}} \left((1 - \alpha_f) \underline{\underline{\nabla}} \underline{\underline{\dot{q}}}_{\text{pred}} + \alpha_f \underline{\underline{\nabla}} \underline{\underline{\dot{q}}}_n \right) + \\ & - \underline{\underline{\nabla}} \underline{\underline{k}} \left((1 - \alpha_f) \underline{\underline{q}}_{n+1} + \alpha_f \underline{\underline{q}}_n \right) - \underline{\underline{k}} \left((1 - \alpha_f) \underline{\underline{\nabla}} \underline{\underline{q}}_{\text{pred}} + \alpha_f \underline{\underline{\nabla}} \underline{\underline{q}}_n \right). \end{aligned} \quad (32)$$

For the derivation of the effective mass matrix $\underline{\underline{m}}_{\text{eff}}$ and effective pseudo force $\underline{\underline{F}}_{\text{pseudo}}$ for generalized- α method, see Sec. A.1 and Sec. A.2.

2.3. Flexible multibody dynamics

Flexible multibody dynamics is the analysis of mechanical systems consisting of flexible bodies constrained by kinematic joints modeled with differential-algebraic equations. This work builds upon previous work of analytical sensitivity analysis with multibody dynamics, including [15,42,43]. In addition to the kinematic constraints w.r.t. structural dynamics, multibody dynamics also must integrate a nonlinear solver in order to consider dependencies of the system matrices and vectors on the state variables. This is carried out with Newton steps.

The flexible multibody system is modeled with the floating frame of reference formulation [44–46]. The generic index-3 differential-algebraic equation for a flexible multibody system with holonomic constraints is written as follows:

$$\underline{\underline{m}}(\underline{q}, t) \underline{\underline{\ddot{q}}}(t) + \underline{\underline{d}}(\underline{q}, t) \underline{\underline{\dot{q}}}(t) + \underline{\underline{k}}(\underline{q}, t) \underline{q}(t) + \underline{\underline{J}}_{\Phi}^T(\underline{q}, t) \underline{\lambda}(t) = \underline{F}_{\text{ext}}(\underline{q}, \underline{\dot{q}}, t) + \underline{F}_v(\underline{q}, \underline{\dot{q}}, t), \quad (33)$$

$$\underline{\Phi}(\underline{q}, t) = 0, \quad (34)$$

where $\underline{q}$, $\underline{\dot{q}}$ and $\underline{\ddot{q}}$ are the generalized position, velocity and acceleration vectors, $\underline{\lambda}$ is the vector of Lagrange multipliers of kinematic constraints, the matrices $\underline{\underline{m}}$, $\underline{\underline{d}}$, and $\underline{\underline{k}}$ are the mass, damping and stiffness matrices, $\underline{\underline{J}}_{\Phi}$ is the Jacobian matrix of kinematic constraints, $\underline{F}_{\text{ext}}$ is the vector of external forces, $\underline{F}_v$ is the quadratic velocity vector and $\underline{\Phi}$ is the vector of kinematic constraints at position level.

Numerical problems related to the solution of index-3 differential-algebraic equations [47] can be avoided by a number of methods including index reduction, coordinate reduction, Augmented Lagrangian formulation and preconditioning [48,49]. Although generalized- α time integration is suitable for solving index-3 or index-2 differential-algebraic equations directly, in this work we address the widely used index reduction leading to an index-1 differential-algebraic equation in which the kinematic constraint function is differentiated twice with respect to time,

$$\underline{\underline{m}} \underline{\underline{\ddot{q}}} + \underline{\underline{d}} \underline{\underline{\dot{q}}} + \underline{\underline{k}} \underline{q} + \underline{\underline{J}}_{\Phi}^T \underline{\lambda} = \underline{F}_{\text{ext}} + \underline{F}_v, \quad (35)$$

$$\underline{\underline{J}}_{\Phi} \underline{\underline{\ddot{q}}} = \underline{F}_c, \quad (36)$$

where $\underline{F}_c$ is the right-hand side of acceleration constraints. As index-1 differential-algebraic equations are susceptible to drift in the kinematic constraints, Baumgarte stabilization [50] is used. Kinematic constraint drift is stabilized in this method by adding damping and restoring terms to the acceleration constraint equations that are proportional to the velocity and position constraints. Details of its implementation, including the sensitivity analysis are described in [51,52].

The generalized- α time integration is based on

$$(\underline{\underline{m}} \underline{\underline{\ddot{q}}})_{n+1-\alpha_m} + (\underline{\underline{d}} \underline{\underline{\dot{q}}})_{n+1-\alpha_f} + (\underline{\underline{k}} \underline{q})_{n+1-\alpha_f} + (\underline{\underline{J}}_{\Phi}^T \underline{\lambda})_{n+1-\alpha_m} = \underline{F}_{\text{ext},n+1-\alpha_f} + \underline{F}_{v,n+1-\alpha_f}, \quad (37)$$

$$(\underline{\underline{J}}_{\Phi} \underline{\underline{\ddot{q}}})_{n+1-\alpha_m} = \underline{F}_{c,n+1-\alpha_f}, \quad (38)$$

with the intermediate approximations of all force terms of the equations of motion. These intermediate approximations are shown in Sec. B.1. Inserting the intermediate approximations of the force terms acting on the flexible multibody system leads to the equations of motion in residual form given by

$$\begin{aligned} \underline{R}_{1,n+1-\alpha_f} = & (1 - \alpha_m) \underline{m}_{n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{m}_n \ddot{\underline{q}}_n + (1 - \alpha_f) \underline{d}_{n+1} \dot{\underline{q}}_{n+1} + \alpha_f \underline{d}_n \dot{\underline{q}}_n + \\ & + (1 - \alpha_f) \underline{k}_{n+1} \underline{q}_{n+1} + \alpha_f \underline{k}_n \underline{q}_n + (1 - \alpha_m) \underline{J}_{\Phi,n+1}^T \underline{\lambda}_{n+1} + \alpha_m \underline{J}_{\Phi,n}^T \underline{\lambda}_n + \\ & - (1 - \alpha_f) \underline{F}_{\text{ext},n+1} - \alpha_f \underline{F}_{\text{ext},n} - (1 - \alpha_f) \underline{F}_{v,n+1} - \alpha_f \underline{F}_{v,n}, \end{aligned} \quad (39)$$

$$\underline{R}_{2,n+1-\alpha_f} = (1 - \alpha_m) \underline{J}_{\Phi,n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{J}_{\Phi,n} \ddot{\underline{q}}_n - (1 - \alpha_f) \underline{F}_{c,n+1} - \alpha_f \underline{F}_{c,n}. \quad (40)$$

In the case of a nonlinear mass matrix, as is the case with the floating frame of reference formulation, this weighted formulation of the residual equations are first-order accurate. For a formulation of second-order accuracy see e.g. [53], where the dynamic equilibrium is enforced exactly at the time steps of the simulation. The derived time integration and its differentiation with the weighted residual formulation can be extended and applied to further formulations.

The effective equation of motion is defined by

$$\begin{bmatrix} \underline{m}_{\text{eff}} & \underline{J}_{\Phi,\text{eff}}^T \\ \underline{J}_{\Phi,\text{eff}} & \underline{0} \end{bmatrix} \begin{bmatrix} \ddot{\underline{q}}_{n+1} \\ \underline{\lambda}_{n+1} \end{bmatrix} = \begin{bmatrix} \underline{F}_{a,\text{eff}} \\ \underline{F}_{c,\text{eff}} \end{bmatrix}, \quad (41)$$

where

$$\underline{m}_{\text{eff}} = (1 - \alpha_m) \underline{m}_{n+1} + (1 - \alpha_f) \gamma \Delta t \underline{d}_{n+1} + (1 - \alpha_f) \beta \Delta t^2 \underline{k}_{n+1}, \quad (42)$$

$$\underline{J}_{\Phi,\text{eff}} = (1 - \alpha_m) \underline{J}_{\Phi,n+1}, \quad (43)$$

$$\begin{aligned} \underline{F}_{a,\text{eff}} = & (1 - \alpha_f) \underline{F}_{\text{ext},n+1} + \alpha_f \underline{F}_{\text{ext},n} + (1 - \alpha_f) \underline{F}_{v,n+1} + \alpha_f \underline{F}_{v,n} + \\ & - \alpha_m \underline{m}_n \ddot{\underline{q}}_n - (1 - \alpha_f) \underline{d}_{n+1} \dot{\underline{q}}_{\text{pred}} - \alpha_f \underline{d}_n \dot{\underline{q}}_n + \\ & - (1 - \alpha_f) \underline{k}_{n+1} \underline{q}_{\text{pred}} - \alpha_f \underline{k}_n \underline{q}_n - \alpha_m \underline{J}_{\Phi,n}^T \underline{\lambda}_n, \end{aligned} \quad (44)$$

$$\underline{F}_{c,\text{eff}} = (1 - \alpha_f) \underline{F}_{c,n+1} + \alpha_f \underline{F}_{c,n} - \alpha_m \underline{J}_{\Phi,n} \ddot{\underline{q}}_n. \quad (45)$$

The derivation of the effective equation of motion and the terms $\underline{m}_{\text{eff}}$, $\underline{J}_{\Phi,\text{eff}}$, $\underline{F}_{a,\text{eff}}$ and $\underline{F}_{c,\text{eff}}$ is shown in Sec. B.1.

With flexible multibody dynamics, the system parameters (matrix- and vector-valued for $n_{\text{dof}} > 1$) including mass, damping, stiffness, constraints and forces are generally dependent on state variables. Therefore, a nonlinear solver is needed to solve the system and the Newton–Raphson method is used here of the following form:

$$\begin{bmatrix} \frac{\partial \underline{R}_{1,n+1-\alpha_f}}{\partial \ddot{\underline{q}}_{n+1}} & \frac{\partial \underline{R}_{1,n+1-\alpha_f}}{\partial \underline{\lambda}_{n+1}} \\ \frac{\partial \underline{R}_{2,n+1-\alpha_f}}{\partial \ddot{\underline{q}}_{n+1}} & \frac{\partial \underline{R}_{2,n+1-\alpha_f}}{\partial \underline{\lambda}_{n+1}} \end{bmatrix} \begin{bmatrix} \Delta \ddot{\underline{q}}_{n+1} \\ \Delta \underline{\lambda}_{n+1} \end{bmatrix} + \begin{bmatrix} \underline{R}_{1,n+1-\alpha_f} \\ \underline{R}_{2,n+1-\alpha_f} \end{bmatrix} = \underline{0}. \quad (46)$$

With the dependencies shown in (33) and (34), the terms of the Jacobian matrix (tangent operator) of the residuals with respect to $\underline{\ddot{q}}_{n+1}$ and $\underline{\lambda}_{n+1}$ are given by,

$$\begin{aligned} \frac{\partial \underline{R}_{1,n+1-\alpha_f}}{\partial \underline{\ddot{q}}_{n+1}} = & (1 - \alpha_m) \left(\beta \Delta t^2 \frac{\partial \underline{m}_{n+1}}{\partial \underline{q}_{n+1}} \underline{\ddot{q}}_{n+1} + \underline{m}_{n+1} \right) + \\ & + (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{d}_{n+1}}{\partial \underline{q}_{n+1}} \underline{\dot{q}}_{n+1} + \gamma \Delta t \underline{d}_{n+1} \right) + \\ & + (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{k}_{n+1}}{\partial \underline{q}_{n+1}} \underline{q}_{n+1} + \beta \Delta t^2 \underline{k}_{n+1} \right) + \\ & + (1 - \alpha_m) \beta \Delta t^2 \frac{\partial \underline{J}_{\Phi,n+1}^T}{\partial \underline{q}_{n+1}} \underline{\lambda}_{n+1} + \\ & - (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{F}_{\text{ext},n+1}}{\partial \underline{q}_{n+1}} + \gamma \Delta t \frac{\partial \underline{F}_{\text{ext},n+1}}{\partial \underline{\dot{q}}_{n+1}} \right) + \\ & - (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{F}_{v,n+1}}{\partial \underline{q}_{n+1}} + \gamma \Delta t \frac{\partial \underline{F}_{v,n+1}}{\partial \underline{\dot{q}}_{n+1}} \right), \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{\partial \underline{R}_{2,n+1-\alpha_f}}{\partial \underline{\ddot{q}}_{n+1}} = & (1 - \alpha_m) \left(\beta \Delta t^2 \frac{\partial \underline{J}_{\Phi,n+1}}{\partial \underline{q}_{n+1}} \underline{\ddot{q}}_{n+1} + \underline{J}_{\Phi,n+1} \right) + \\ & - (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{F}_{c,n+1}}{\partial \underline{q}_{n+1}} + \gamma \Delta t \frac{\partial \underline{F}_{c,n+1}}{\partial \underline{\dot{q}}_{n+1}} \right), \end{aligned} \quad (48)$$

$$\frac{\partial \underline{R}_{1,n+1-\alpha_f}}{\partial \underline{\lambda}_{n+1}} = (1 - \alpha_m) \underline{J}_{\Phi,n+1}^T, \quad (49)$$

$$\frac{\partial \underline{R}_{2,n+1-\alpha_f}}{\partial \underline{\lambda}_{n+1}} = \underline{0}. \quad (50)$$

The derivation of these terms is shown in Sec. B.2.

202

The direct differentiation of the equations of motion (39) and (40) leads to the governing equation of the sensitivity analysis,

$$\begin{aligned} \underline{\nabla} \underline{R}_{1,n+1-\alpha_f} = & (1 - \alpha_m) \underline{\nabla} \underline{m}_{n+1} \underline{\ddot{q}}_{n+1} + \alpha_m \underline{\nabla} \underline{m}_n \underline{\ddot{q}}_n + \\ & + (1 - \alpha_m) \underline{m}_{n+1} \underline{\nabla} \underline{\ddot{q}}_{n+1} + \alpha_m \underline{m}_n \underline{\nabla} \underline{\ddot{q}}_n + \\ & + (1 - \alpha_f) \underline{\nabla} \underline{d}_{n+1} \underline{\dot{q}}_{n+1} + \alpha_f \underline{\nabla} \underline{d}_n \underline{\dot{q}}_n + \\ & + (1 - \alpha_f) \underline{d}_{n+1} \underline{\nabla} \underline{\dot{q}}_{n+1} + \alpha_f \underline{d}_n \underline{\nabla} \underline{\dot{q}}_n + \\ & + (1 - \alpha_f) \underline{\nabla} \underline{k}_{n+1} \underline{q}_{n+1} + \alpha_f \underline{\nabla} \underline{k}_n \underline{q}_n + \\ & + (1 - \alpha_f) \underline{k}_{n+1} \underline{\nabla} \underline{q}_{n+1} + \alpha_f \underline{k}_n \underline{\nabla} \underline{q}_n + \\ & + (1 - \alpha_m) \underline{\nabla} \underline{J}_{\Phi,n+1}^T \underline{\lambda}_{n+1} + \alpha_m \underline{\nabla} \underline{J}_{\Phi,n}^T \underline{\lambda}_n + \\ & + (1 - \alpha_m) \underline{J}_{\Phi,n+1}^T \underline{\nabla} \underline{\lambda}_{n+1} + \alpha_m \underline{J}_{\Phi,n}^T \underline{\nabla} \underline{\lambda}_n + \\ & - (1 - \alpha_f) \underline{\nabla} \underline{F}_{\text{ext},n+1} - \alpha_f \underline{\nabla} \underline{F}_{\text{ext},n} - (1 - \alpha_f) \underline{\nabla} \underline{F}_{v,n+1} - \alpha_f \underline{\nabla} \underline{F}_{v,n}, \end{aligned} \quad (51)$$

$$\begin{aligned} \underline{\nabla} \underline{R}_{2,n+1-\alpha_f} = & (1 - \alpha_m) \underline{\nabla} \underline{J}_{\Phi,n+1} \underline{\ddot{q}}_{n+1} + \alpha_m \underline{\nabla} \underline{J}_{\Phi,n} \underline{\ddot{q}}_n + \\ & + (1 - \alpha_m) \underline{J}_{\Phi,n+1} \underline{\nabla} \underline{\ddot{q}}_{n+1} + \alpha_m \underline{J}_{\Phi,n} \underline{\nabla} \underline{\ddot{q}}_n + \\ & - (1 - \alpha_f) \underline{\nabla} \underline{F}_{c,n+1} - \alpha_f \underline{\nabla} \underline{F}_{c,n}. \end{aligned} \quad (52)$$

The governing sensitivity equation of the effective system is defined by the differentiation of (41),

$$\begin{bmatrix} \underline{\underline{\mathbf{m}}}_{\text{eff}} & \underline{\underline{\mathbf{J}}}_{\Phi, \text{eff}}^T \\ \underline{\underline{\mathbf{J}}}_{\Phi, \text{eff}} & \underline{\underline{\mathbf{0}}} \end{bmatrix} \begin{bmatrix} \underline{\underline{\nabla \ddot{\mathbf{q}}}}_{n+1} \\ \underline{\underline{\nabla \lambda}}_{n+1} \end{bmatrix} = \begin{bmatrix} \underline{\underline{\mathbf{F}}}_{\text{a,pseudo}} \\ \underline{\underline{\mathbf{F}}}_{\text{c,pseudo}} \end{bmatrix}, \quad (53)$$

where

$$\begin{bmatrix} \underline{\underline{\mathbf{F}}}_{\text{a,pseudo}} \\ \underline{\underline{\mathbf{F}}}_{\text{c,pseudo}} \end{bmatrix} = \begin{bmatrix} \underline{\underline{\nabla \mathbf{F}}}_{\text{a,eff}} \\ \underline{\underline{\nabla \mathbf{F}}}_{\text{c,eff}} \end{bmatrix} - \begin{bmatrix} \underline{\underline{\nabla \mathbf{m}}}_{\text{eff}} & \underline{\underline{\nabla \mathbf{J}}}_{\Phi, \text{eff}}^T \\ \underline{\underline{\nabla \mathbf{J}}}_{\Phi, \text{eff}} & \underline{\underline{\mathbf{0}}} \end{bmatrix} \begin{bmatrix} \underline{\underline{\ddot{\mathbf{q}}}}_{n+1} \\ \underline{\underline{\lambda}}_{n+1} \end{bmatrix}, \quad (54)$$

leading to

$$\begin{aligned} \underline{\underline{\mathbf{F}}}_{\text{a,pseudo}} = & (1 - \alpha_f) \underline{\underline{\nabla \mathbf{F}}}_{\text{ext}, n+1} + \alpha_f \underline{\underline{\nabla \mathbf{F}}}_{\text{ext}, n} + (1 - \alpha_f) \underline{\underline{\nabla \mathbf{F}}}_{\text{v}, n+1} + \alpha_f \underline{\underline{\nabla \mathbf{F}}}_{\text{v}, n} + \\ & - (1 - \alpha_m) \underline{\underline{\nabla \mathbf{m}}}_{n+1} \underline{\underline{\ddot{\mathbf{q}}}}_{n+1} - \alpha_m \underline{\underline{\nabla \mathbf{m}}}_n \underline{\underline{\ddot{\mathbf{q}}}}_n - \alpha_m \underline{\underline{\mathbf{m}}}_n \underline{\underline{\nabla \ddot{\mathbf{q}}}}_n + \\ & - (1 - \alpha_f) \underline{\underline{\nabla \mathbf{d}}}_{n+1} \underline{\underline{\dot{\mathbf{q}}}}_{n+1} - \alpha_f \underline{\underline{\nabla \mathbf{d}}}_n \underline{\underline{\dot{\mathbf{q}}}}_n - (1 - \alpha_f) \underline{\underline{\mathbf{d}}}_{n+1} \underline{\underline{\nabla \dot{\mathbf{q}}}}_{\text{pred}} - \alpha_f \underline{\underline{\mathbf{d}}}_n \underline{\underline{\nabla \dot{\mathbf{q}}}}_n + \\ & - (1 - \alpha_f) \underline{\underline{\nabla \mathbf{k}}}_{n+1} \underline{\underline{\mathbf{q}}}_{n+1} - \alpha_f \underline{\underline{\nabla \mathbf{k}}}_n \underline{\underline{\mathbf{q}}}_n - (1 - \alpha_f) \underline{\underline{\mathbf{k}}}_{n+1} \underline{\underline{\nabla \mathbf{q}}}_{\text{pred}} - \alpha_f \underline{\underline{\mathbf{k}}}_n \underline{\underline{\nabla \mathbf{q}}}_n + \\ & - (1 - \alpha_m) \underline{\underline{\nabla \mathbf{J}}}_{\Phi, n+1}^T \underline{\underline{\lambda}}_{n+1} - \alpha_m \underline{\underline{\nabla \mathbf{J}}}_{\Phi, n}^T \underline{\underline{\lambda}}_n - \alpha_m \underline{\underline{\mathbf{J}}}_{\Phi, n} \underline{\underline{\nabla \lambda}}_n, \end{aligned} \quad (55)$$

$$\begin{aligned} \underline{\underline{\mathbf{F}}}_{\text{c,pseudo}} = & (1 - \alpha_f) \underline{\underline{\nabla \mathbf{F}}}_{\text{c}, n+1} + \alpha_f \underline{\underline{\nabla \mathbf{F}}}_{\text{c}, n} + \\ & - (1 - \alpha_m) \underline{\underline{\nabla \mathbf{J}}}_{\Phi, n+1} \underline{\underline{\ddot{\mathbf{q}}}}_{n+1} - \alpha_m \underline{\underline{\nabla \mathbf{J}}}_{\Phi, n} \underline{\underline{\ddot{\mathbf{q}}}}_n - \alpha_m \underline{\underline{\mathbf{J}}}_{\Phi, n} \underline{\underline{\nabla \ddot{\mathbf{q}}}}_n. \end{aligned} \quad (56)$$

The derivation of the effective mass matrix and the effective constraint Jacobian matrix is shown in Sec. B.1 and the derivation of the effective pseudo force terms is shown in Sec. B.3.

The Newton–Raphson step for the sensitivity analysis is given by

$$\begin{bmatrix} \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \lambda}}_{n+1}} \\ \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \lambda}}_{n+1}} \end{bmatrix} \begin{bmatrix} \underline{\underline{\Delta \nabla \ddot{\mathbf{q}}}}_{n+1} \\ \underline{\underline{\Delta \nabla \lambda}}_{n+1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{\nabla \mathbf{R}}}_{1, n+1-\alpha_f} \\ \underline{\underline{\nabla \mathbf{R}}}_{2, n+1-\alpha_f} \end{bmatrix} = \underline{\underline{\mathbf{0}}}. \quad (57)$$

As shown in [7] the Jacobian of the residual sensitivity with respect to $\underline{\underline{\nabla \ddot{\mathbf{q}}}}_{n+1}$ and $\underline{\underline{\nabla \lambda}}_{n+1}$ reduces to

$$\begin{bmatrix} \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \lambda}}_{n+1}} \\ \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\nabla \mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\nabla \lambda}}_{n+1}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \underline{\underline{\mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\lambda}}_{n+1}} \\ \frac{\partial \underline{\underline{\mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\lambda}}_{n+1}} \end{bmatrix} \underline{\underline{\mathbf{e}}}, \quad (58)$$

wherefore a conventional solver can be used for the solution of the following:

$$\begin{bmatrix} \frac{\partial \underline{\underline{\mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\mathbf{R}}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\lambda}}_{n+1}} \\ \frac{\partial \underline{\underline{\mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\ddot{\mathbf{q}}}}_{n+1}} & \frac{\partial \underline{\underline{\mathbf{R}}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\lambda}}_{n+1}} \end{bmatrix} \begin{bmatrix} \underline{\underline{\Delta \nabla \ddot{\mathbf{q}}}}_{n+1} \\ \underline{\underline{\Delta \nabla \lambda}}_{n+1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{\nabla \mathbf{R}}}_{1, n+1-\alpha_f} \\ \underline{\underline{\nabla \mathbf{R}}}_{2, n+1-\alpha_f} \end{bmatrix} = \underline{\underline{\mathbf{0}}}. \quad (59)$$

3. Numerical results

In this section, numerical results are shown comparing the introduced method for the sensitivity analysis for generalized- α time integration (primal analysis introduced in Sec. 2) with that of Newmark- β . The methods are compared with applications to both structural dynamics as well as flexible multibody dynamics.

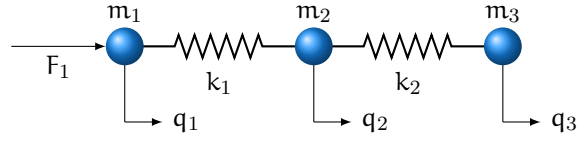


Figure 2. Structural dynamics example

3.1. Structural dynamics

The linear structural dynamic problem used here as a demonstration example is based on [54], which has found usage as a benchmark in [55–61], and shown schematically in fig. 2. This problem is defined by

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{\underline{\underline{m}}} \underbrace{\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}}_{\underline{\underline{\ddot{q}}}} + \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}_{\underline{\underline{k}}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}}_{\underline{\underline{q}}} - \underbrace{\begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{\underline{F}}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{\underline{R}}}, \quad (60)$$

with the prescribed displacement,

$$q_1 = \sin \omega t, \quad (61)$$

and the unitless values

$$\begin{aligned} m_1 &= 0, \\ m_2 &= 1, \\ m_3 &= 1, \\ k_1 &= 10^7, \\ k_2 &= 1, \\ \omega &= 1.2. \end{aligned}$$

The masses m_2 and m_3 as well as the stiffnesses k_1 and k_2 are considered design variables with respect to which the design sensitivities are calculated. The problem consists of stiff (k_1) and flexible (k_2) parts typical of complex, practical structural-mechanical systems [54].

Considering the prescribed displacement, this problem can be reduced to two degrees of freedom,

$$\underbrace{\begin{bmatrix} m_2 & 0 \\ 0 & m_3 \end{bmatrix}}_{\underline{\underline{m}}} \underbrace{\begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix}}_{\underline{\underline{\ddot{q}}}} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\underline{\underline{k}}} \underbrace{\begin{bmatrix} q_2 \\ q_3 \end{bmatrix}}_{\underline{\underline{q}}} - \underbrace{\begin{bmatrix} k_1 q_1 \\ 0 \end{bmatrix}}_{\underline{\underline{F}}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\underline{\underline{R}}}, \quad (62)$$

which we implement in our in-house, Python-based solver framework SIMULI in the time $t = [0, 10]$ with the initial conditions

$$\begin{aligned} q_2 &= 0, \\ q_3 &= 0, \\ \dot{q}_2 &= 0, \\ \dot{q}_3 &= 0. \end{aligned}$$

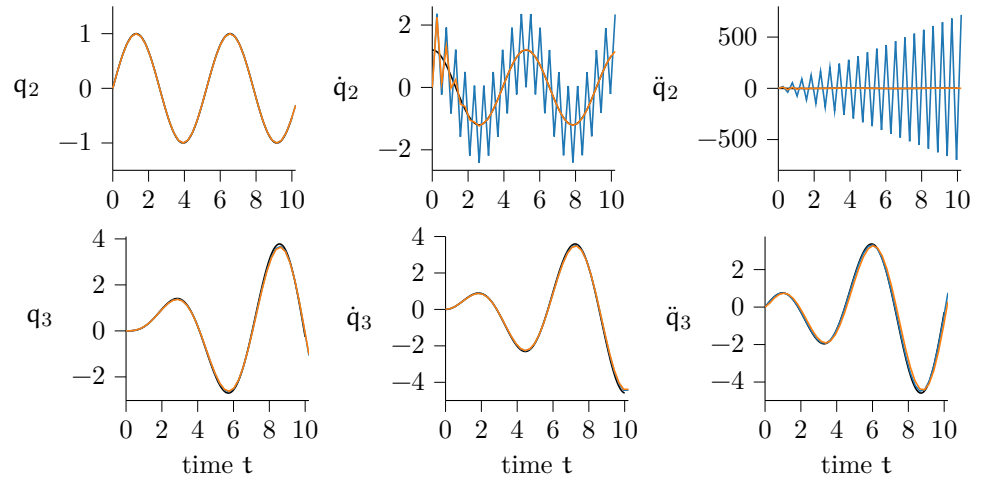


Figure 3. Responses of the structural dynamic example calculated with — Newmark- β and — generalized- α compared with — exact solution (undermost line)

As reported in [54], time steps of $\Delta t = 0.2618$ are used, giving the following values for the ratio step size to fundamental periods of the system:

$$\begin{aligned}\frac{\Delta t}{T_F} &= 0.05, \\ \frac{\Delta t}{T_1} &= 0.041667, \\ \frac{\Delta t}{T_2} &= 131.76,\end{aligned}$$

where T_F is the fundamental period of the forcing function and $T_{1,2}$ for the first and second modes, $T_1 = 6.283$ and $T_2 = 0.001987$.

The results for both Newmark- β and generalized- α time integration methods are shown in fig. 3 and compared with the exact solution. The exact solution is calculated via mode superposition per [62]. The numerical damping parameter of generalized- α time integration is set as $\rho_\infty = 0.55$. As expected with generalized- α time integration, the unphysical, high-frequency oscillatory behavior of the response can be avoided. As such, the beneficial effects of numerical damping with generalized- α time integration can be clearly seen in the velocity and acceleration of node 2, while maintaining the integrity of the other responses.

The sensitivity analysis developed above is applied and these results can be seen in fig. 4–6. Analytical sensitivity analysis with Newmark- β and generalized- α time integration methods are compared and also with their numerical sensitivity implementations via finite differencing using a relative perturbation of 1×10^{-6} .

In cases where Newmark- β time integration shows instability or noise, the numerical damping allows for usable sensitivities. In some cases, noise — albeit at a much lower level — persists in the first iterations before being damped out: after approximately 2.5 s (or approximately 10 iterations). Attention is drawn to those sensitivities susceptible to stability issues, namely:

- sensitivities of position 2 w.r.t. stiffness 1 and mass 2 (fig. 4),
- sensitivities of velocity 2 w.r.t. to stiffness 1 and mass 2 (fig. 5),
- sensitivities of acceleration 2 w.r.t. stiffness 1, stiffness 2 and mass 2 (fig. 6),
- sensitivities of acceleration 3 w.r.t. stiffness 1 and mass 2 (fig. 6).

Further, generalized- α time integration eliminates the noise in those sensitivities present when using Newmark- β . This is seen with the following sensitivity responses:

- sensitivities of velocity 2 w.r.t. stiffness 2 (fig. 5),
- sensitivities of velocity 3 w.r.t. stiffness 1 and mass 2 (fig. 5).

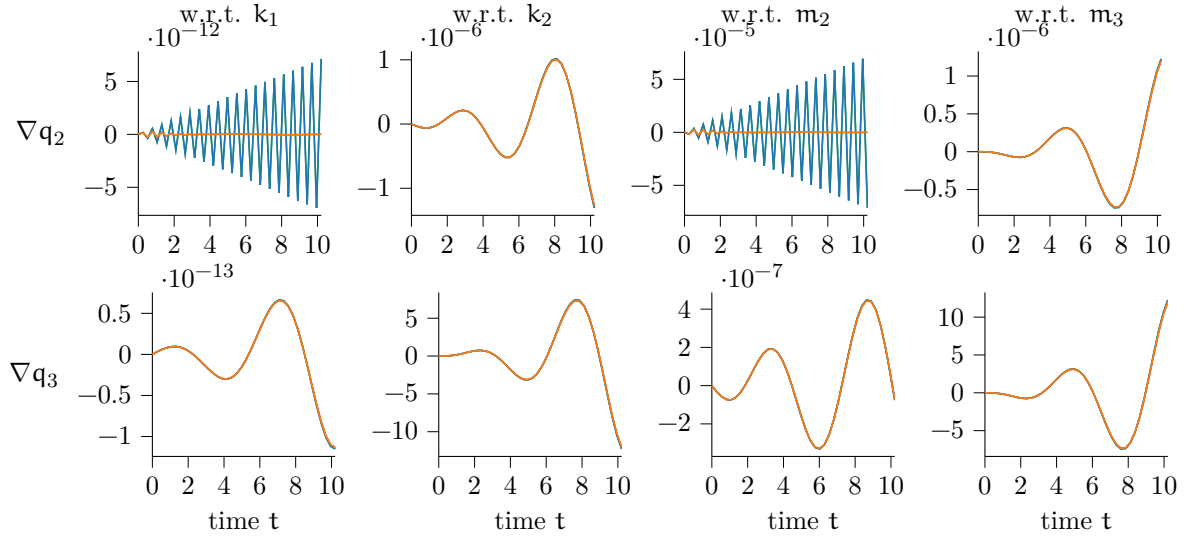


Figure 4. Position sensitivities ∇q of the structural dynamic example w.r.t. k_1 , k_2 , m_2 and m_3 calculated analytically with — Newmark- β and — generalized- α compared with those calculated numerically via finite differencing with — Newmark- β and — generalized- α (undermost lines). Note: no visible difference between finite differencing and analytical sensitivities.

The analytical and numerical results are in the most plots nearly exact. There are cases though where the negative consequences of the numerical instability with Newmark- β are increased. The readers attention is called to especially

- sensitivity of velocity 2 w.r.t. stiffness 2 (fig. 5),
- sensitivity of velocity 2 w.r.t. mass 3 (fig. 5),
- sensitivity of acceleration 2 w.r.t. stiffness 2 (fig. 6),
- sensitivity of acceleration 2 w.r.t. mass 3 (fig. 6).

It can be seen that both analytical and numerical sensitivities with generalized- α time integration are not negatively effected by numerical instabilities as the case with Newmark- β .

The number of time steps needed for the damping to take effect can be reduced by reducing the numerical damping parameter ϱ_∞ towards zero (i.e. more damping), though also effecting the physical response to a greater degree. As time continues, the instability grows with Newmark- β time integration and therefore a clear advantage can be seen with generalized- α . It can be clearly seen how gradient-based solvers would struggle to use the gradient information provided with Newmark- β time integration.

For our modest case of four design variables, we show a speedup of two, where the speedup S is defined by the ratio of computational time for finite differencing t_{FD} to that of analytical sensitivities t_{ana} ,

$$S = \frac{t_{FD}}{t_{ana}}. \quad (63)$$

The computational time required for analytical sensitivity analysis represents approximately the ideal computational effort twice that of the primal analysis, i.e. analysis without sensitivity analysis. Computation reduction is the same for both Newmark- β and generalized- α time integration methods, see tab. 1.

3.2. Flexible slider-crank mechanism

The slider-crank mechanism as shown in fig. 7 is modeled with flexible multibody dynamics using the floating frame of reference formulation. The mechanism consists of four bodies that are connected by three revolute joints and one prismatic joint. The ground and the slider are considered to be rigid, while the crank and the rod are considered to be flexible and are modeled with Euler-Bernoulli beams. Fig. 8 shows the flexible multibody

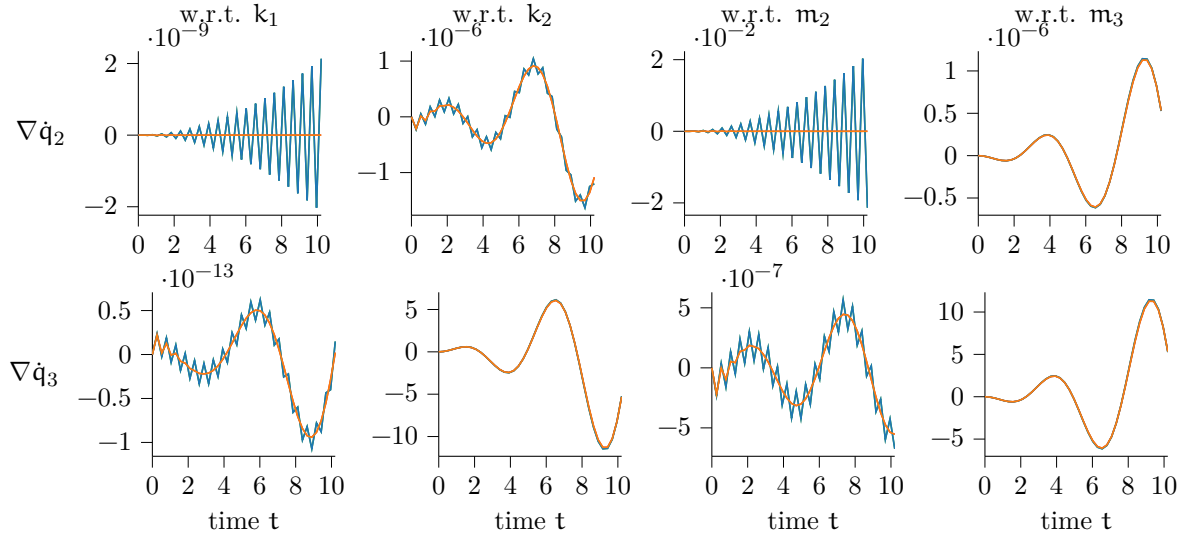


Figure 5. Velocity sensitivities $\nabla \dot{q}$ of the structural dynamic example w.r.t. k_1 , k_2 , m_2 and m_3 calculated analytically with — Newmark- β and — generalized- α compared with those calculated numerically via finite differencing with — Newmark- β and — generalized- α (undermost lines). Note: no visible difference between finite differencing and analytical sensitivities.

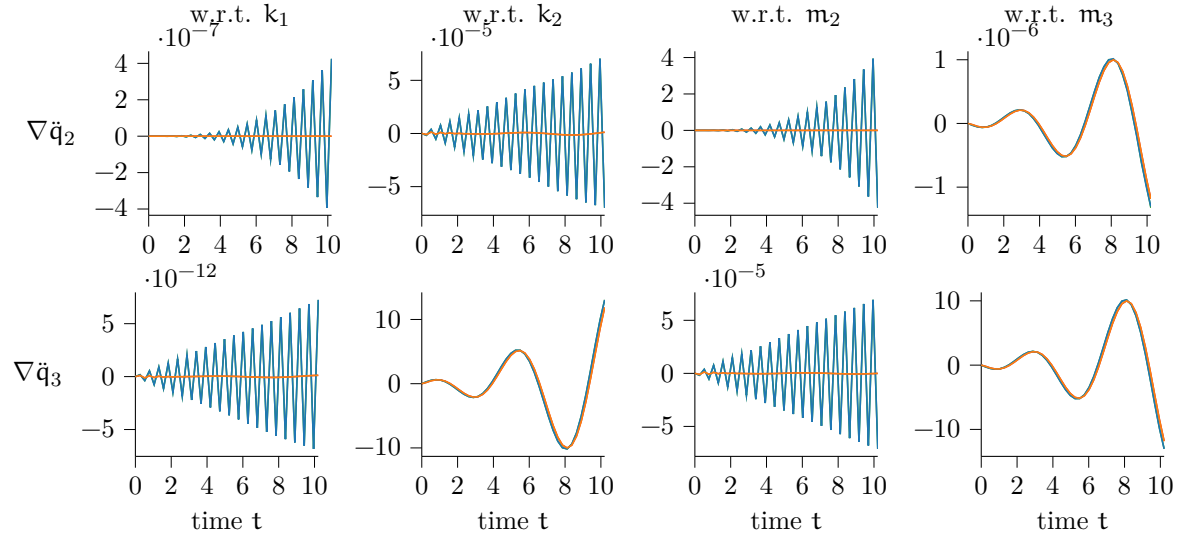


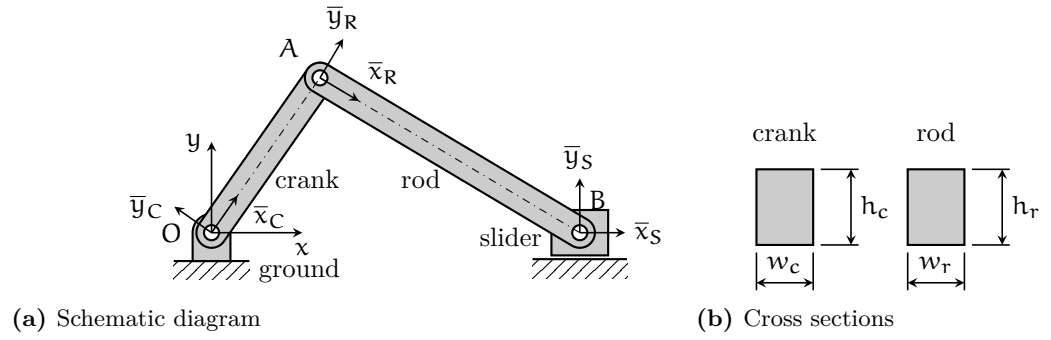
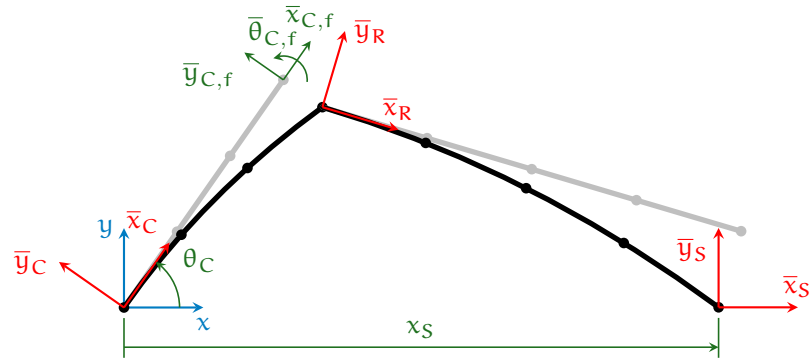
Figure 6. Acceleration sensitivities $\nabla \ddot{q}$ of the structural dynamic example w.r.t. k_1 , k_2 , m_2 and m_3 calculated analytically with — Newmark- β and — generalized- α compared with those calculated numerically via finite differencing with — Newmark- β and — generalized- α (undermost lines). Note: no visible difference between finite differencing and analytical sensitivities.

Table 1. Computational effort for the structural dynamics example

time integration	sensitivity analysis	calculation time [s] ¹	speedup ²
Newmark- β	none	0.0555	—
Newmark- β	finite differencing	0.2088	—
Newmark- β	analytical	0.1023	2.0410
generalized- α	none	0.0458	—
generalized- α	finite differencing	0.2122	—
generalized- α	analytical	0.0993	2.1370

¹Intel Core i5-7200U CPU 2.50GHz,

²speedup of analytical sensitivity analysis w.r.t. finite differencing

**Figure 7.** Slider-crank mechanism**Figure 8.** Flexible multibody system of the slider-crank mechanism with the undeformed bodies in gray, the deformed bodies in black, the inertial reference frame in blue, the body reference frames in red and the coordinates that are investigated in green

system of the mechanism. The inertial reference frame coinciding with the reference frame of the ground is shown in blue. The body reference frames of the crank, the rod and the slider are shown in red. The undeformed bodies of the crank and the rod are shown in gray and the deformed bodies are shown in black, where the points represent the nodes of the finite elements. The coordinates that are investigated in detail for the primal analysis and the sensitivity analysis are shown in green. The dimensions of the bodies are listed in tab. 2. The bodies are made of steel and tab. 3 shows the material properties. In this example, the MPa unit system is used with length in mm, mass in t, time in s, force in N, angle in rad and stress in MPa.

The generalized coordinates of a flexible multibody system modeled with the floating frame of reference formulation and Euler–Bernoulli beams are given by

$$\mathbf{q} = [\mathbf{r}^T \quad \theta \quad \bar{\mathbf{q}}_f^T]^T, \quad (64)$$

where $\mathbf{r}$ is the position of the body reference frame, θ is the orientation of the body reference frame and $\bar{\mathbf{q}}_f$ are the nodal displacements in body coordinates of each node on the flexible

Table 2. Geometry of slider-crank mechanism

property	symbol	crank	rod	slider	units
width	w	20	20	30	mm
height	h	30	30	30	mm
length	ℓ	120	180	40	mm

Table 3. Material properties

property	symbol	value	units
density	ρ	7.85×10^{-9}	$\frac{\text{t}}{\text{mm}^3}$
elastic modulus	E	210000	MPa
Poisson ratio	ν	0.3	—

body. Since no damping is considered, the equations of motion of a flexible body are given by

$$\underline{\underline{m}} \ddot{\underline{q}} + \underline{\underline{k}} \underline{q} + \underline{\underline{J}}_{\Phi}^T \underline{\lambda} = \underline{F}_{\text{ext}} + \underline{F}_{\text{v}}, \quad (65)$$

$$\underline{\underline{J}}_{\Phi} \ddot{\underline{q}} = \underline{F}_{\text{c}}, \quad (66)$$

where the system matrices of the flexible bodies are given by

$$\underline{\underline{m}} = \begin{bmatrix} \underline{\underline{m}}_{\text{tt}} & \underline{\underline{m}}_{\text{tr}} & \underline{\underline{m}}_{\text{tf}} \\ & \underline{\underline{m}}_{\text{rr}} & \underline{\underline{m}}_{\text{rf}} \\ \text{sym.} & & \underline{\underline{m}}_{\text{ff}} \end{bmatrix}, \quad (67)$$

$$\underline{\underline{k}} = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ \text{sym.} & & \underline{\underline{k}}_{\text{ff}} \end{bmatrix}, \quad (68)$$

and the system vectors are given by

$$\underline{F}_{\text{ext}} = \begin{bmatrix} \underline{F}_{\text{ext},\text{t}} \\ \underline{F}_{\text{ext},\text{r}} \\ \underline{F}_{\text{ext},\text{f}} \end{bmatrix}, \quad (69)$$

$$\underline{F}_{\text{v}} = \begin{bmatrix} \underline{F}_{\text{v},\text{t}} \\ \underline{F}_{\text{v},\text{r}} \\ \underline{F}_{\text{v},\text{f}} \end{bmatrix}, \quad (70)$$

with the index denoting translational terms t, rotational terms r and flexible terms f. 291

For the dynamic simulation of the mechanism, the above-mentioned solver framework *SiMuLi* is used. The simulation is performed in the time interval $t = [0 \text{ s}, 0.025 \text{ s}]$ with time steps $\Delta t = 1 \cdot 10^{-4} \text{ s}$. The resulting values for the ratio step size to fundamental periods of the system for the first ten non-zero eigenfrequencies are found in fig. 9. The values for ratio step size to fundamental periods range from 0.2119 for first non-zero eigenfrequency to 3.9919 for the tenth non-zero eigenfrequency and to 17.1157 for the twentieth non-zero eigenfrequency (not pictured in plot). 292
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For the initial conditions, the angular position and velocity of the crank are defined with

$$\begin{aligned} \theta_{\text{C}} &= 0 \text{ rad}, \\ \dot{\theta}_{\text{C}} &= 150\pi \frac{\text{rad}}{\text{s}}, \end{aligned}$$

and the initial conditions of the other bodies are derived through the kinematic analysis of the system. The dimensions of the cross sections of the crank h_{C} , w_{C} and the rod h_{R} , w_{R} are used as design variables in the sensitivity analysis. 299
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The results of the simulation of the coordinates that are highlighted in fig. 8 are shown in fig. 10. With Newmark- β , vibrations at high frequencies are observed. With generalized- α , vibrations are observed in the beginning of the simulation that are numerically damped with ongoing simulation time. In addition to the generalized coordinates, the maximum stresses on the upper and lower fiber of the beam elements of the flexible bodies including the 302
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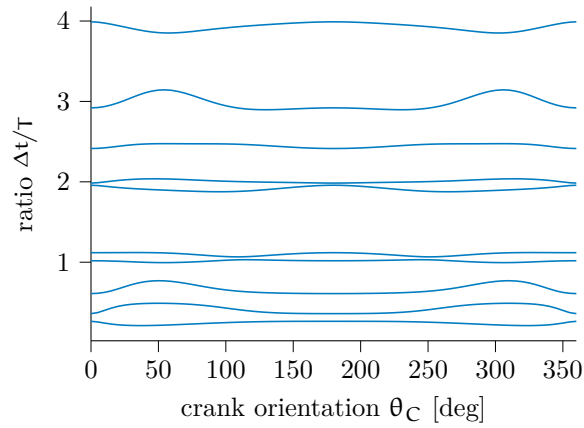


Figure 9. Ratio time step to fundamental periods for first ten non-zero eigenfrequencies of the slider–crank mechanism

crank and the rod are computed. The Kreisselmeier–Steinhauser function [63,64] is used to approximate the maximum stress value of the bodies at each time step with a differentiable function and the results are shown in fig. 11. With Newmark- β , the vibrating behavior of the system causes higher flexible deformation and therefore unrealistic high stress in the components. The numerical damping with generalized- α leads to lower maximum stress values and a more realistic stress distribution over time.

The results of the sensitivity analysis of the coordinates that are highlighted in fig. 8 are shown in figs. 12–14. The analyzed coordinates include the rotation of the floating frame of the crank θ_C , the translation in x-direction of the floating frame of the slider x_S and the three flexible coordinates of the node in A on the crank $\bar{x}_{C,f}$, $\bar{y}_{C,f}$ and $\bar{\theta}_{C,f}$. The design variables used for the sensitivity analysis are the cross-sectional dimensions of the crank h_C , w_C and the rod h_R , w_R .

Similar to above, with Newmark- β the results show a vibrating behavior that would lead to significant problems when using the sensitivities in gradient-based design optimization. With generalized- α , the numerical damping leads to usable sensitivity values after a few time steps. Fig. 12 shows that the position sensitivities of the position and orientation coordinates of the floating frames are less noisy compared to the flexible coordinates. Position sensitivities that suffer little from noise include the following:

- sensitivities of θ_C w.r.t. w_C and w_R ,
- sensitivities of x_S w.r.t. h_C , w_C , and w_R .

Position sensitivities with significant noise include the following:

- sensitivities of θ_C w.r.t. h_C and h_R ,
- sensitivities of x_S w.r.t. h_R ,
- sensitivities of $\bar{x}_{C,f}$ w.r.t. h_C , w_C , h_R and w_R ,
- sensitivities of $\bar{y}_{C,f}$ w.r.t. h_C , w_C , h_R and w_R ,
- sensitivities of $\bar{\theta}_{C,f}$ w.r.t. h_C , w_C , h_R and w_R .

The sensitivities of velocities as shown in fig. 13 and the sensitivities of accelerations as shown in fig. 14 have noisy responses with Newmark that are damped with generalized- α for all investigated coordinates and all used design variables. The same behavior also continues with the sensitivity values for the maximum stresses in the crank and the rod shown in fig. 15.

To check the correctness of the results, the sensitivity values with the introduced method are compared to numerical sensitivity analysis with forward differencing. The results are in good agreement, with only small numerical differences. In fact there are no visible differences in figs. 12–15 with the finite differencing values lying under the respective analytical values. Tab. 4 shows a comparison of the computational effort with different methods. The computational effort with generalized- α is about 30 % lower compared to

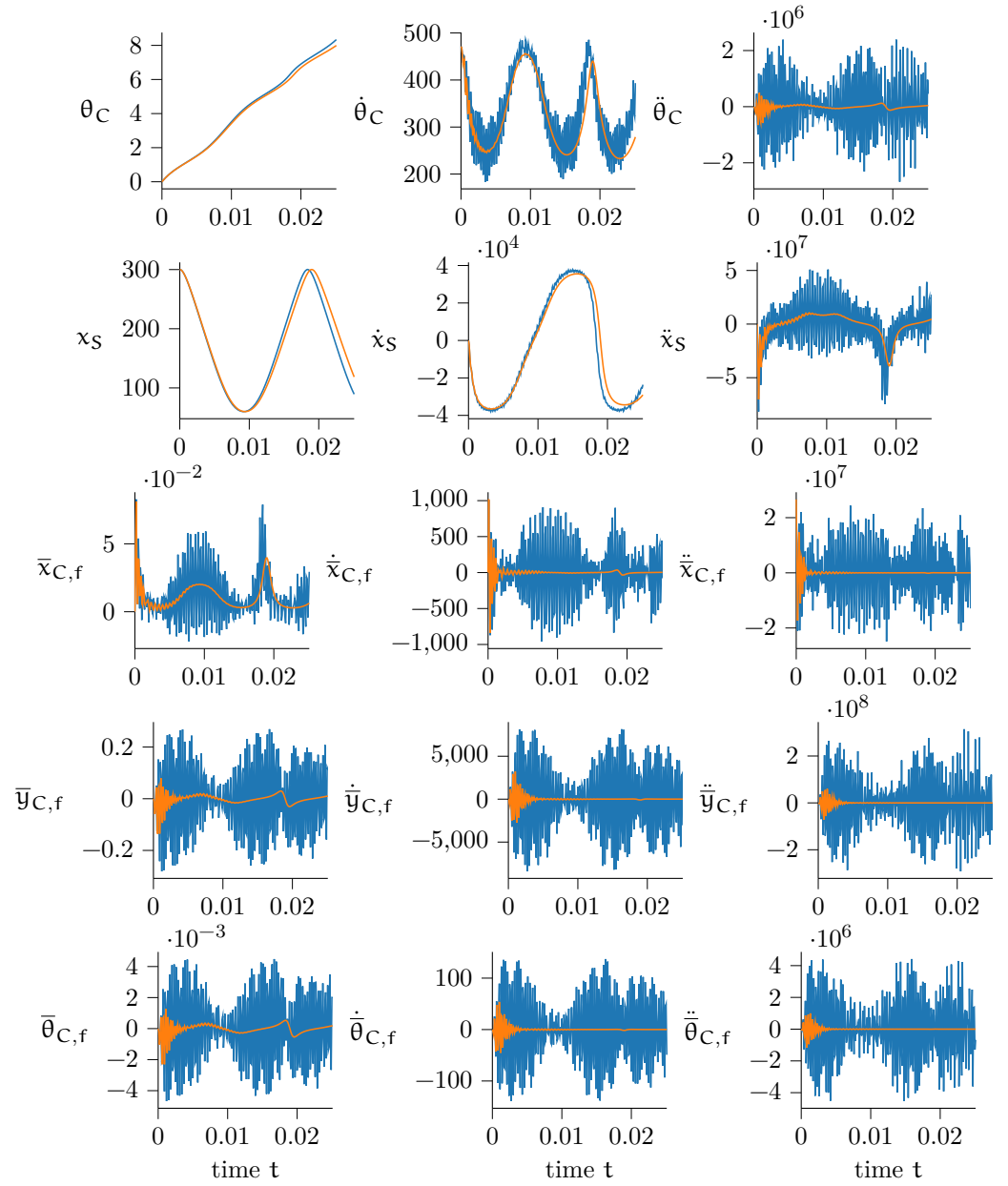


Figure 10. Responses of the slider-crank mechanism (values in MPa units, i.e. mm, $\frac{\text{mm}}{\text{s}}$, $\frac{\text{mm}}{\text{s}^2}$ for translations and rad, $\frac{\text{rad}}{\text{s}}$, $\frac{\text{rad}}{\text{s}^2}$ for rotations) calculated with — Newmark- β and — generalized- α

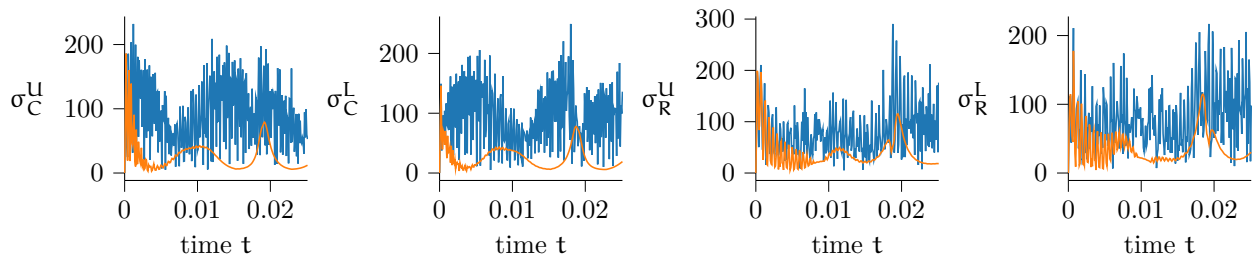


Figure 11. Maximum stress in MPa on the upper and lower fiber of the crank and the rod calculated with — Newmark- β and — generalized- α

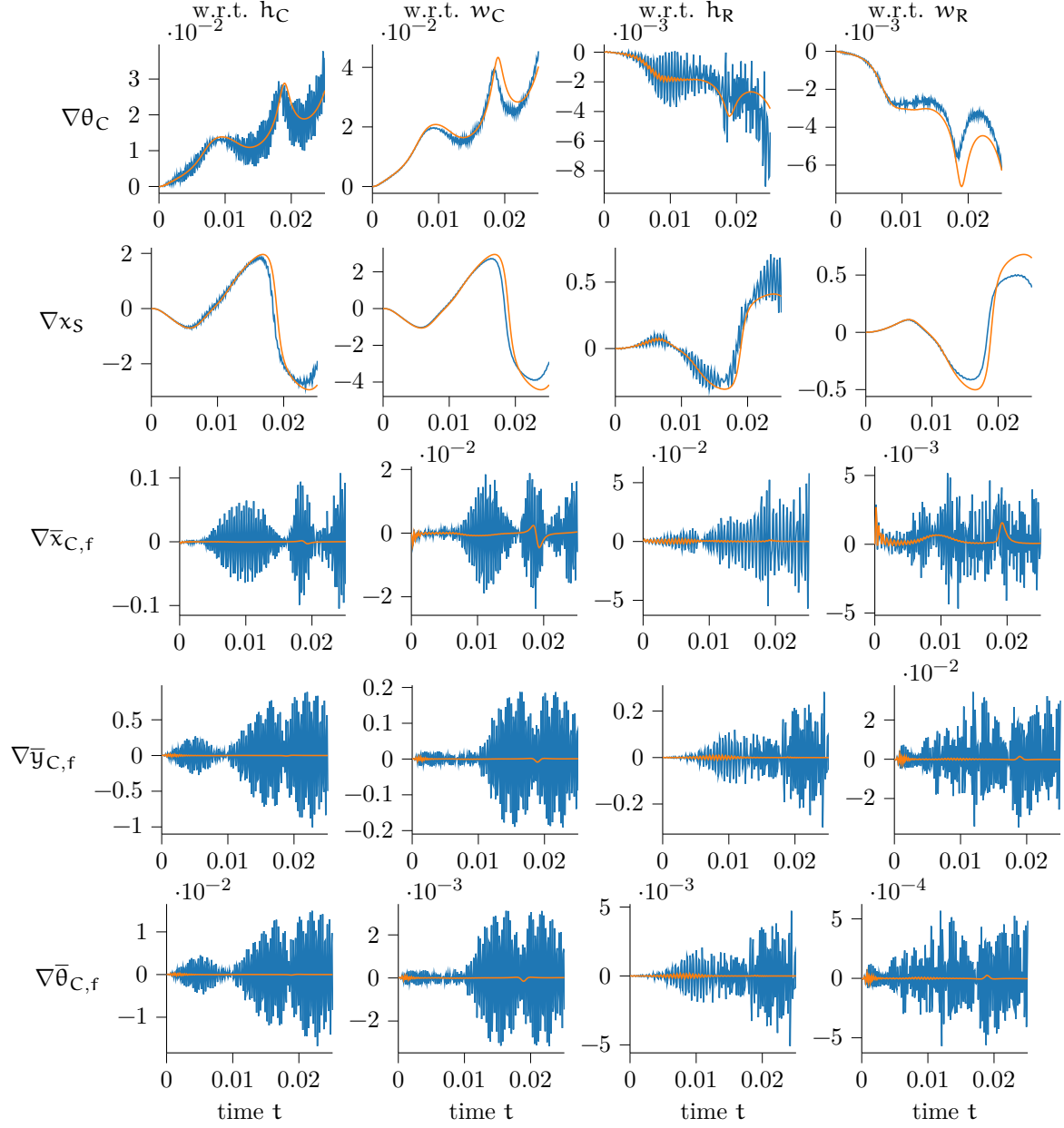


Figure 12. Position sensitivities ∇q of the slider–crank mechanism w.r.t. h_C , w_C , h_R and w_R (values in MPa system of units, i.e. mm, $\frac{\text{mm}}{\text{s}}$, $\frac{\text{mm}}{\text{s}^2}$ for translations and rad, $\frac{\text{rad}}{\text{s}}$, $\frac{\text{rad}}{\text{s}^2}$ for rotations) calculated with — Newmark- β and — generalized- α compared with those calculated numerically via finite differencing with — Newmark- β and — generalized- α (undermost lines). Note: no visible difference between finite differencing and analytical sensitivities.

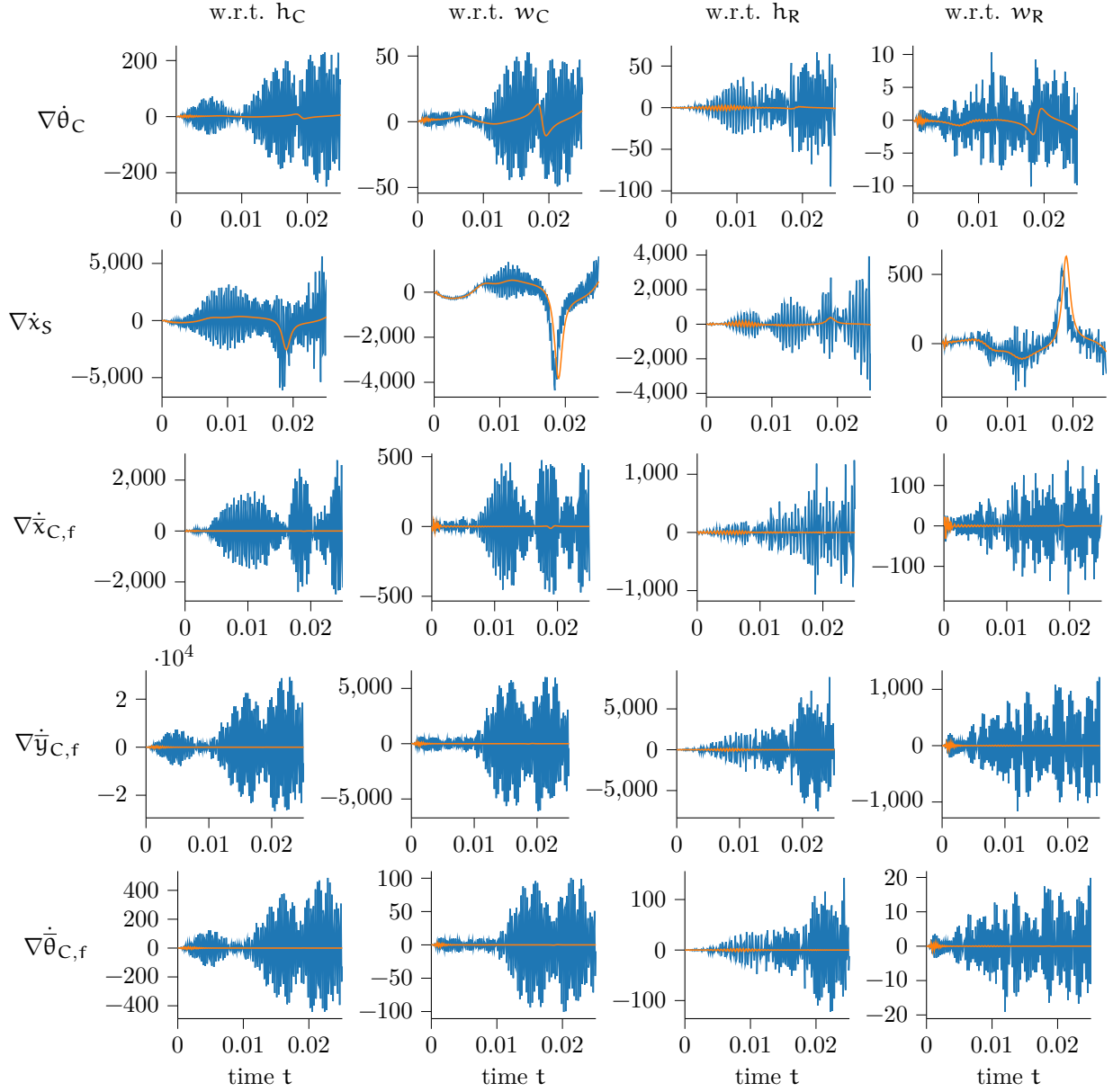


Figure 13. Velocity sensitivities $\nabla \dot{q}$ of the slider–crank mechanism w.r.t. h_C , w_C , h_R and w_R (values in MPa system of units, i.e. mm, $\frac{\text{mm}}{\text{s}}$, $\frac{\text{mm}}{\text{s}^2}$ for translations and rad, $\frac{\text{rad}}{\text{s}}$, $\frac{\text{rad}}{\text{s}^2}$ for rotations) calculated with — Newmark- β and — generalized- α compared with those calculated numerically via finite differencing with — Newmark- β and — generalized- α (undermost lines). Note: no visible difference between finite differencing and analytical sensitivities.

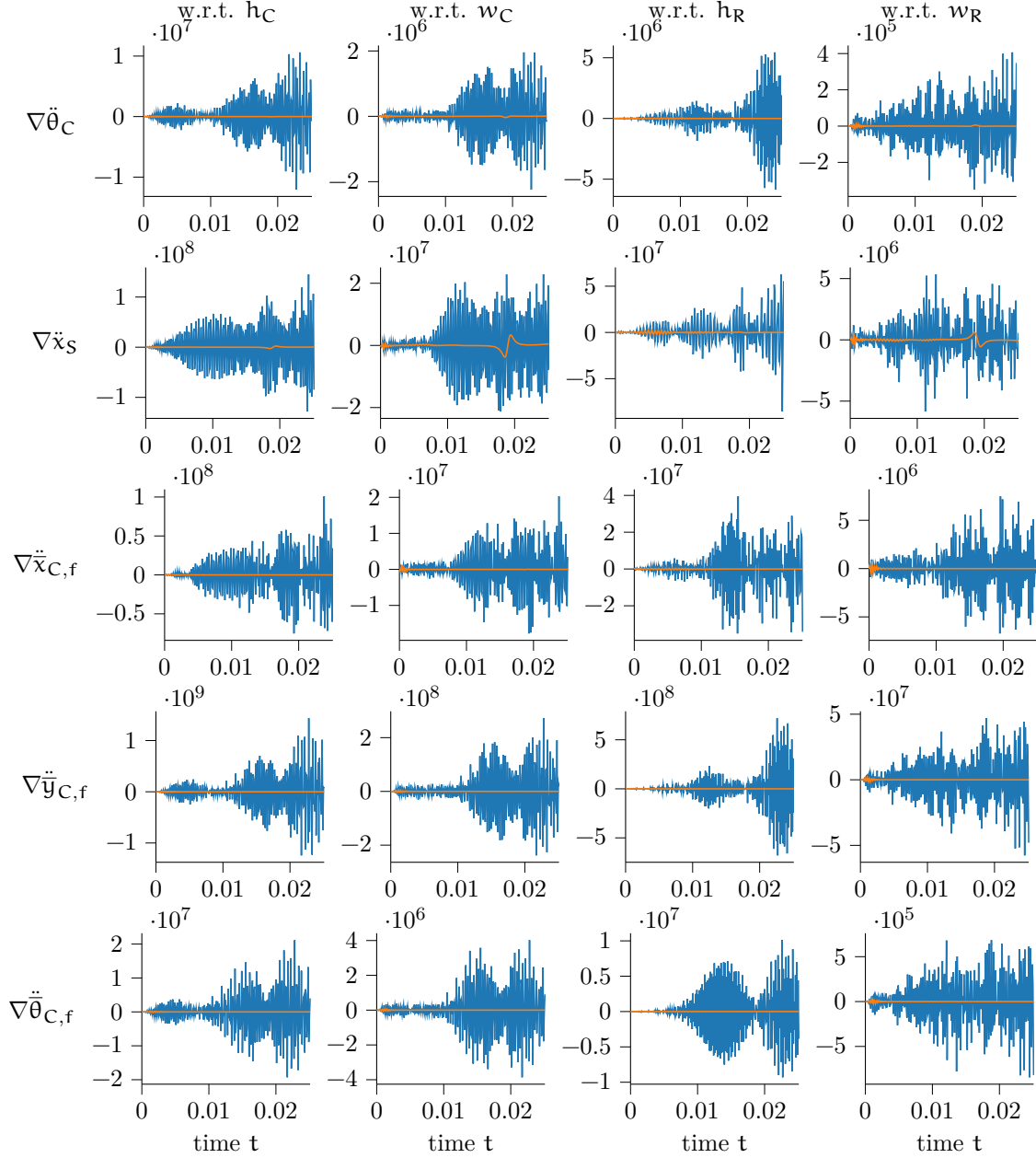


Figure 14. Acceleration sensitivities $\nabla \ddot{q}$ of the slider–crank mechanism w.r.t. h_C , w_C , h_R and w_R (values in MPa system of units, i.e. mm, $\frac{\text{mm}}{\text{s}}$, $\frac{\text{mm}}{\text{s}^2}$ for translations and rad, $\frac{\text{rad}}{\text{s}}$, $\frac{\text{rad}}{\text{s}^2}$ for rotations) calculated with — Newmark- β and — generalized- α compared with those calculated numerically via finite differencing with — Newmark- β and — generalized- α (undermost lines). Note: no visible difference between finite differencing and analytical sensitivities.

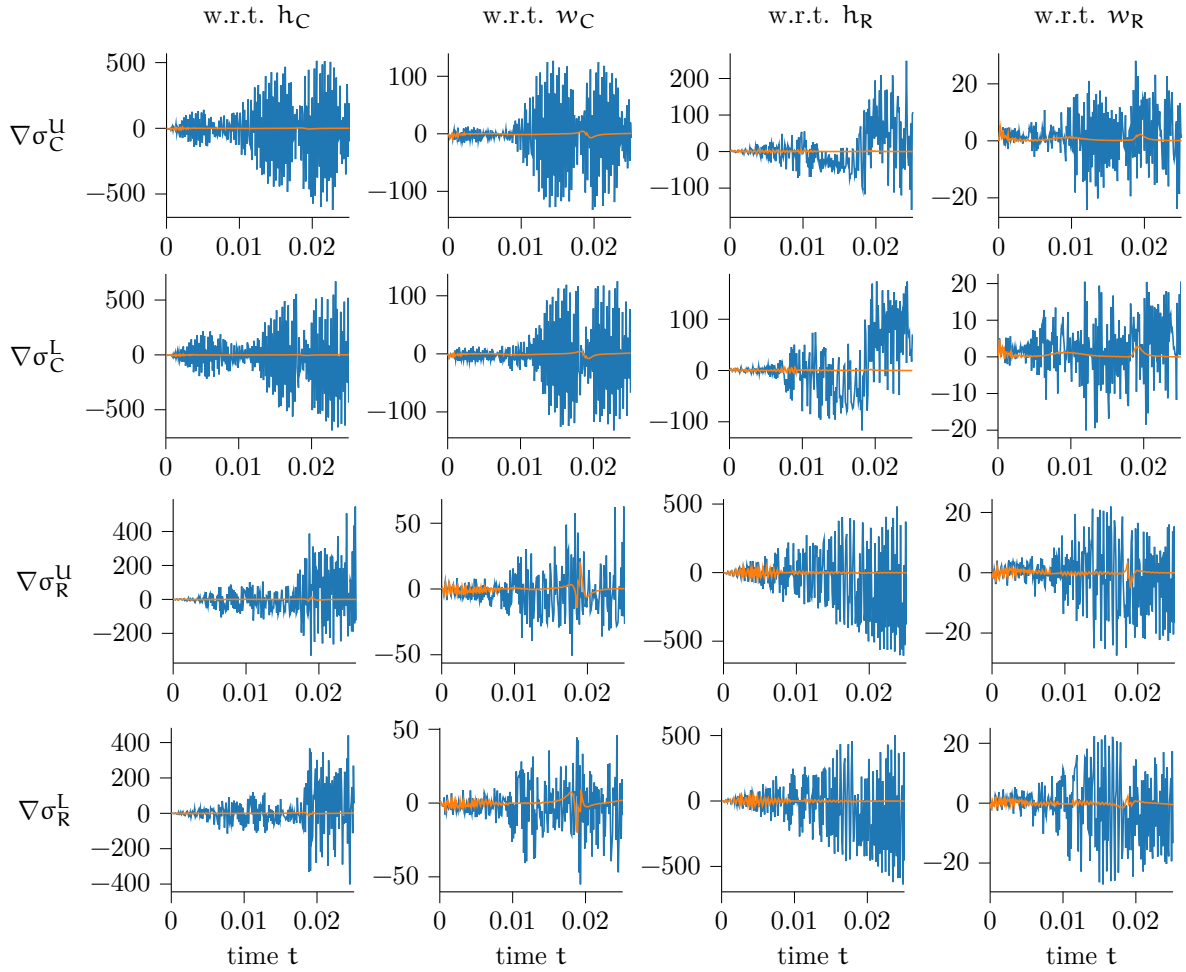


Figure 15. Maximum stress sensitivities on the upper and lower fiber of the crank and the rod (values in MPa system of units, i.e. mm, $\frac{\text{mm}}{\text{s}}$, $\frac{\text{mm}}{\text{s}^2}$ for translations and rad, $\frac{\text{rad}}{\text{s}}$, $\frac{\text{rad}}{\text{s}^2}$ for rotations) calculated with — Newmark- β and — generalized- α compared with those calculated numerically via finite differencing with — Newmark- β and — generalized- α (undermost lines). Note: no visible difference between finite differencing and analytical sensitivities.

Table 4. Computational effort for the slider–crank mechanism

time integration	sensitivity analysis	calculation time [mm : ss] ¹	speedup ²
Newmark- β	none	03:44	—
Newmark- β	finite differencing	18:12	—
Newmark- β	analytical	04:29	4.0595
generalized- α	none	02:42	—
generalized- α	finite differencing	13:22	—
generalized- α	analytical	03:30	3.8190

¹Intel Core i7-8700 CPU 3.20GHz²speedup of analytical sensitivity analysis w.r.t. finite differencing

the computational effort with Newmark- β . The main reason for that is the continuing vibrating behavior with Newmark- β , requiring more iteration steps of the nonlinear solver. The comparison of the sensitivity methods with numerical finite differencing and analytical direct differentiation shows the high efficiency of the analytical method. With the four shown design variables, a speedup as shown in (63) of about $S \approx 4$ of the analytical method compared to the numerical method is obtained. The additional computational effort of analytical sensitivity analysis is just a fraction of the computational effort of the primal analysis. This is because of the simplification shown in (59), which allows to use the Jacobian matrix of the residual built in the primal analysis also for the sensitivity analysis. This allows for a further reduced computational effort.

4. Conclusion

Analytical sensitivity analysis is crucial for efficient design optimization, parameter fitting and uncertainty analysis. In this work, the direct differentiation of the generalized- α time integration is derived and implemented for linear structural dynamics and flexible multibody dynamics, the latter of which requiring a nonlinear solve routine. All equations needed for implementation are provided and extendable to further governing equations. The beneficial effects of the step-size dependent numerical damping provided by the generalized- α method can be clearly seen in the comparison of the results with Newmark- β .

The objective of this work is to develop the analytical sensitivity analysis for the α -family of time integration, which has a single implementation for several common time integration schemes. This thus has consequences for gradient-based optimization frameworks for dynamic systems.

If the functions of interest are nonsmooth, discontinuous and bifurcated, non-gradient-based methods can be successfully used, e.g. design cases with crash analysis. Such cases can be solved with gradient-free optimization algorithms (e.g. [65–68]) and approximation methods (e.g. [69–71]).

The system responses with respect to the design variables are indeed often smooth, continuous and unbifurcated and thus enabling gradient-based methods where efficient and accurate sensitivity analysis is critical for both effectiveness and efficiency. This is shown above with linear structural dynamics and flexible multibody dynamics. The method and equations developed are of general nature and are extendable to further formulations and applicable to larger problems, both of larger number of degrees of freedom and number of design variables.

With the above study and introduction of the direct differentiation of the α -family of time integrators, the authors hope to provide the readers as design engineers a tool in the pursuit of optimal structures and mechanisms under dynamic loading.

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preparation, E.W. and V.G.; writing—review and editing, E.W. and V.G.; visualization, E.W. and V.G.; supervision, E.W.; project administration, E.W.; funding acquisition, E.W. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest: The authors declare no conflict of interest

Appendix A. Derivation of effective equations of motion for structural dynamics

Appendix A.1. Primal analysis

The base for the α -family of integration methods is the equation of motion which for linear structural dynamics is given by

$$\underline{\underline{m}} \ddot{\underline{q}}_{n+1-\alpha_m} + \underline{\underline{d}} \dot{\underline{q}}_{n+1-\alpha_f} + \underline{\underline{k}} \underline{q}_{n+1-\alpha_f} = \underline{F}_{n+1-\alpha_f}, \quad (\text{A1})$$

and the intermediate values shown in eqs. (3)–(5). Inserting the intermediate values into the equation of motion leads to

$$\underline{\underline{m}} \left((1 - \alpha_m) \ddot{\underline{q}}_{n+1} + \alpha_m \ddot{\underline{q}}_n \right) + \underline{\underline{d}} \left((1 - \alpha_f) \dot{\underline{q}}_{n+1} + \alpha_f \dot{\underline{q}}_n \right) + \underline{\underline{k}} \left((1 - \alpha_f) \underline{q}_{n+1} + \alpha_f \underline{q}_n \right) = (1 - \alpha_f) \underline{F}_{n+1} + \alpha_f \underline{F}_n. \quad (\text{A2})$$

With Newmark's eqs, (1) and (2), this becomes

$$\begin{aligned} \underline{\underline{m}} \left((1 - \alpha_m) \ddot{\underline{q}}_{n+1} + \alpha_m \ddot{\underline{q}}_n \right) + \underline{\underline{d}} \left((1 - \alpha_f) \left(\dot{\underline{q}}_{\text{pred}} + \gamma \Delta t \ddot{\underline{q}}_{n+1} \right) + \alpha_f \dot{\underline{q}}_n \right) + \\ + \underline{\underline{k}} \left((1 - \alpha_f) \left(\underline{q}_{\text{pred}} + \beta \Delta t^2 \ddot{\underline{q}}_{n+1} \right) + \alpha_f \underline{q}_n \right) = (1 - \alpha_f) \underline{F}_{n+1} + \alpha_f \underline{F}_n. \end{aligned} \quad (\text{A3})$$

This can be rearranged to the following form

$$\begin{aligned} ((1 - \alpha_m) \underline{\underline{m}} + (1 - \alpha_f) \gamma \Delta t \underline{\underline{d}} + (1 - \alpha_f) \beta \Delta t^2 \underline{\underline{k}}) \ddot{\underline{q}}_{n+1} = \\ (1 - \alpha_f) \underline{F}_{n+1} + \alpha_f \underline{F}_n - \alpha_m \underline{\underline{m}} \ddot{\underline{q}}_n - \underline{\underline{d}} \left((1 - \alpha_f) \dot{\underline{q}}_{\text{pred}} + \alpha_f \dot{\underline{q}}_n \right) - \underline{\underline{k}} \left((1 - \alpha_f) \underline{q}_{\text{pred}} + \alpha_f \underline{q}_n \right). \end{aligned} \quad (\text{A4})$$

Therefore, the effective equation of motion is defined by

$$\underline{\underline{m}}_{\text{eff}} \ddot{\underline{q}}_{n+1} = \underline{F}_{\text{eff}}, \quad (\text{A5})$$

where

$$\underline{\underline{m}}_{\text{eff}} = (1 - \alpha_m) \underline{\underline{m}} + (1 - \alpha_f) \gamma \Delta t \underline{\underline{d}} + (1 - \alpha_f) \beta \Delta t^2 \underline{\underline{k}}, \quad (\text{A6})$$

$$\underline{F}_{\text{eff}} = (1 - \alpha_f) \underline{F}_{n+1} + \alpha_f \underline{F}_n - \alpha_m \underline{\underline{m}} \ddot{\underline{q}}_n - \underline{\underline{d}} \left((1 - \alpha_f) \dot{\underline{q}}_{\text{pred}} + \alpha_f \dot{\underline{q}}_n \right) - \underline{\underline{k}} \left((1 - \alpha_f) \underline{q}_{\text{pred}} + \alpha_f \underline{q}_n \right). \quad (\text{A7})$$

Appendix A.2. Sensitivity analysis

The governing equation for sensitivities is the first derivative of the equation of motion,

$$\begin{aligned} \underline{\underline{\nabla m}} \ddot{\underline{q}}_{n+1-\alpha_m} + \underline{\underline{m}} \underline{\underline{\nabla \ddot{q}}}_{n+1-\alpha_m} + \underline{\underline{\nabla d}} \dot{\underline{q}}_{n+1-\alpha_f} + \underline{\underline{d}} \underline{\underline{\nabla \dot{q}}}_{n+1-\alpha_f} + \underline{\underline{\nabla k}} \underline{q}_{n+1-\alpha_f} + \underline{\underline{k}} \underline{\underline{\nabla q}}_{n+1-\alpha_f} = \\ \underline{\underline{\nabla F}}_{n+1-\alpha_f}, \end{aligned} \quad (\text{A8})$$

where we introduce the intermediate sensitivity values (10)–(12),

$$\begin{aligned} & \underline{\underline{\nabla m}} \left((1 - \alpha_m) \underline{\underline{\ddot{q}}}_{n+1} + \alpha_m \underline{\underline{\ddot{q}}}_n \right) + \underline{\underline{m}} \left((1 - \alpha_m) \underline{\underline{\nabla \ddot{q}}}_{n+1} + \alpha_m \underline{\underline{\nabla \ddot{q}}}_n \right) + \\ & + \underline{\underline{\nabla d}} \left((1 - \alpha_f) \underline{\underline{\dot{q}}}_{n+1} + \alpha_f \underline{\underline{\dot{q}}}_n \right) + \underline{\underline{d}} \left((1 - \alpha_f) \underline{\underline{\nabla \dot{q}}}_{n+1} + \alpha_f \underline{\underline{\nabla \dot{q}}}_n \right) + \\ & + \underline{\underline{\nabla k}} \left((1 - \alpha_f) \underline{\underline{q}}_{n+1} + \alpha_f \underline{\underline{q}}_n \right) + \underline{\underline{k}} \left((1 - \alpha_f) \underline{\underline{\nabla q}}_{n+1} + \alpha_f \underline{\underline{\nabla q}}_n \right) = \\ & (1 - \alpha_f) \underline{\underline{\nabla F}}_{n+1} + \alpha_f \underline{\underline{\nabla F}}_n, \end{aligned} \quad (A9)$$

and which we update with Newmark's sensitivity eqs. (16) and (17)

$$\begin{aligned} & \underline{\underline{\nabla m}} \left((1 - \alpha_m) \underline{\underline{\ddot{q}}}_{n+1} + \alpha_m \underline{\underline{\ddot{q}}}_n \right) + \underline{\underline{m}} \left((1 - \alpha_m) \underline{\underline{\nabla \ddot{q}}}_{n+1} + \alpha_m \underline{\underline{\nabla \ddot{q}}}_n \right) + \\ & + \underline{\underline{\nabla d}} \left((1 - \alpha_f) \underline{\underline{\dot{q}}}_{n+1} + \alpha_f \underline{\underline{\dot{q}}}_n \right) + \underline{\underline{d}} \left((1 - \alpha_f) \left(\underline{\underline{\nabla \dot{q}}}_{\text{pred}} + \gamma \Delta t \underline{\underline{\nabla \ddot{q}}}_{n+1} \right) + \alpha_f \underline{\underline{\nabla \dot{q}}}_n \right) + \\ & + \underline{\underline{\nabla k}} \left((1 - \alpha_f) \underline{\underline{q}}_{n+1} + \alpha_f \underline{\underline{q}}_n \right) + \underline{\underline{k}} \left((1 - \alpha_f) \left(\underline{\underline{\nabla q}}_{\text{pred}} + \beta \Delta t^2 \underline{\underline{\nabla \ddot{q}}}_{n+1} \right) + \alpha_f \underline{\underline{\nabla q}}_n \right) = \\ & (1 - \alpha_f) \underline{\underline{\nabla F}}_{n+1} + \alpha_f \underline{\underline{\nabla F}}_n. \end{aligned} \quad (A10)$$

This equation can be rearranged to the following form,

$$\begin{aligned} & ((1 - \alpha_m) \underline{\underline{m}} + (1 - \alpha_f) \gamma \Delta t \underline{\underline{d}} + (1 - \alpha_f) \beta \Delta t^2 \underline{\underline{k}}) \underline{\underline{\nabla \ddot{q}}}_{n+1} = \\ & (1 - \alpha_f) \underline{\underline{\nabla F}}_{n+1} + \alpha_f \underline{\underline{\nabla F}}_n - \underline{\underline{\nabla m}} \left((1 - \alpha_m) \underline{\underline{\ddot{q}}}_{n+1} + \alpha_m \underline{\underline{\ddot{q}}}_n \right) - \alpha_m \underline{\underline{m}} \underline{\underline{\nabla \ddot{q}}}_n + \\ & - \underline{\underline{\nabla d}} \left((1 - \alpha_f) \underline{\underline{\dot{q}}}_{n+1} + \alpha_f \underline{\underline{\dot{q}}}_n \right) - \underline{\underline{d}} \left((1 - \alpha_f) \underline{\underline{\nabla \dot{q}}}_{\text{pred}} + \alpha_f \underline{\underline{\nabla \dot{q}}}_n \right) + \\ & - \underline{\underline{\nabla k}} \left((1 - \alpha_f) \underline{\underline{q}}_{n+1} + \alpha_f \underline{\underline{q}}_n \right) - \underline{\underline{k}} \left((1 - \alpha_f) \underline{\underline{\nabla q}}_{\text{pred}} + \alpha_f \underline{\underline{\nabla q}}_n \right). \end{aligned} \quad (A11)$$

Therefore, the effective equation of motion for sensitivities is defined by

$$\underline{\underline{m}}_{\text{eff}} \underline{\underline{\nabla \ddot{q}}}_{n+1} = \underline{\underline{F}}_{\text{pseudo}}, \quad (A12)$$

where $\underline{\underline{m}}_{\text{eff}}$ is given by eq. (A6) and the pseudo load is defined by

$$\begin{aligned} \underline{\underline{F}}_{\text{pseudo}} = & (1 - \alpha_f) \underline{\underline{\nabla F}}_{n+1} + \alpha_f \underline{\underline{\nabla F}}_n - \underline{\underline{\nabla m}} \left((1 - \alpha_m) \underline{\underline{\ddot{q}}}_{n+1} + \alpha_m \underline{\underline{\ddot{q}}}_n \right) - \alpha_m \underline{\underline{m}} \underline{\underline{\nabla \ddot{q}}}_n + \\ & - \underline{\underline{\nabla d}} \left((1 - \alpha_f) \underline{\underline{\dot{q}}}_{n+1} + \alpha_f \underline{\underline{\dot{q}}}_n \right) - \underline{\underline{d}} \left((1 - \alpha_f) \underline{\underline{\nabla \dot{q}}}_{\text{pred}} + \alpha_f \underline{\underline{\nabla \dot{q}}}_n \right) + \\ & - \underline{\underline{\nabla k}} \left((1 - \alpha_f) \underline{\underline{q}}_{n+1} + \alpha_f \underline{\underline{q}}_n \right) - \underline{\underline{k}} \left((1 - \alpha_f) \underline{\underline{\nabla q}}_{\text{pred}} + \alpha_f \underline{\underline{\nabla q}}_n \right). \end{aligned} \quad (A13)$$

Appendix B. Derivation of effective equations of motion for flexible multibody dynamics

Appendix B.1. Primal analysis

The base of the generalized- α method for index-1 differential algebraic equations of a flexible multibody system is given by

$$(\underline{\underline{m}} \underline{\underline{\ddot{q}}})_{n+1-\alpha_m} + (\underline{\underline{d}} \underline{\underline{\dot{q}}})_{n+1-\alpha_f} + (\underline{\underline{k}} \underline{\underline{q}})_{n+1-\alpha_f} + \left(\underline{\underline{J}}_{\Phi}^T \underline{\underline{\lambda}} \right)_{n+1-\alpha_m} = \underline{\underline{F}}_{\text{ext},n+1-\alpha_f} + \underline{\underline{F}}_{v,n+1-\alpha_f}, \quad (A14)$$

$$(\underline{\underline{J}}_{\Phi} \underline{\underline{\ddot{q}}})_{n+1-\alpha_m} = \underline{\underline{F}}_{c,n+1-\alpha_f}. \quad (A15)$$

The intermediate point approximations of the force terms acting on the flexible multibody system are given by

$$(\underline{\underline{m}} \ddot{\underline{q}})_{n+1-\alpha_m} = (1 - \alpha_m) \underline{\underline{m}}_{n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{\underline{m}}_n \ddot{\underline{q}}_n, \quad (\text{A16})$$

$$(\underline{\underline{d}} \dot{\underline{q}})_{n+1-\alpha_f} = (1 - \alpha_f) \underline{\underline{d}}_{n+1} \dot{\underline{q}}_{n+1} + \alpha_f \underline{\underline{d}}_n \dot{\underline{q}}_n, \quad (\text{A17})$$

$$(\underline{\underline{k}} \underline{q})_{n+1-\alpha_f} = (1 - \alpha_f) \underline{\underline{k}}_{n+1} \underline{q}_{n+1} + \alpha_f \underline{\underline{k}}_n \underline{q}_n, \quad (\text{A18})$$

$$(\underline{\underline{J}}_{\Phi}^T \underline{\underline{\lambda}})_{n+1-\alpha_m} = (1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1}^T \underline{\underline{\lambda}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n}^T \underline{\underline{\lambda}}_n, \quad (\text{A19})$$

$$(\underline{\underline{J}}_{\Phi} \ddot{\underline{q}})_{n+1-\alpha_m} = (1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n} \ddot{\underline{q}}_n, \quad (\text{A20})$$

$$\underline{F}_{\text{ext}, n+1-\alpha_f} = (1 - \alpha_f) \underline{F}_{\text{ext}, n+1} + \alpha_f \underline{F}_{\text{ext}, n}, \quad (\text{A21})$$

$$\underline{F}_{v, n+1-\alpha_f} = (1 - \alpha_f) \underline{F}_{v, n+1} + \alpha_f \underline{F}_{v, n}, \quad (\text{A22})$$

$$\underline{F}_{c, n+1-\alpha_f} = (1 - \alpha_f) \underline{F}_{c, n+1} + \alpha_f \underline{F}_{c, n}. \quad (\text{A23})$$

Applying these linear approximations of the generalized- α method to the equations of motion leads to

$$\begin{aligned} & (1 - \alpha_m) \underline{\underline{m}}_{n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{\underline{m}}_n \ddot{\underline{q}}_n + \\ & + (1 - \alpha_f) \underline{\underline{d}}_{n+1} \dot{\underline{q}}_{n+1} + \alpha_f \underline{\underline{d}}_n \dot{\underline{q}}_n + \\ & + (1 - \alpha_f) \underline{\underline{k}}_{n+1} \underline{q}_{n+1} + \alpha_f \underline{\underline{k}}_n \underline{q}_n + \\ & + (1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1}^T \underline{\underline{\lambda}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n}^T \underline{\underline{\lambda}}_n = (1 - \alpha_f) \underline{F}_{\text{ext}, n+1} + \alpha_f \underline{F}_{\text{ext}, n} + \\ & + (1 - \alpha_f) \underline{F}_{v, n+1} + \alpha_f \underline{F}_{v, n}, \end{aligned} \quad (\text{A24})$$

$$(1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n} \ddot{\underline{q}}_n = (1 - \alpha_f) \underline{F}_{c, n+1} + \alpha_f \underline{F}_{c, n}. \quad (\text{A25})$$

Inserting Newmark's eqs. (1) and (2) gives

$$\begin{aligned} & (1 - \alpha_m) \underline{\underline{m}}_{n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{\underline{m}}_n \ddot{\underline{q}}_n + \\ & + (1 - \alpha_f) \underline{\underline{d}}_{n+1} (\dot{\underline{q}}_{\text{pred}} + \gamma \Delta t \ddot{\underline{q}}_{n+1}) + \alpha_f \underline{\underline{d}}_n \dot{\underline{q}}_n + \\ & (1 - \alpha_f) \underline{\underline{k}}_{n+1} (\underline{q}_{\text{pred}} + \beta \Delta t^2 \ddot{\underline{q}}_{n+1}) + \alpha_f \underline{\underline{k}}_n \underline{q}_n + \\ & (1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1}^T \underline{\underline{\lambda}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n}^T \underline{\underline{\lambda}}_n = (1 - \alpha_f) \underline{F}_{\text{ext}, n+1} + \alpha_f \underline{F}_{\text{ext}, n} + \\ & + (1 - \alpha_f) \underline{F}_{v, n+1} + \alpha_f \underline{F}_{v, n}, \end{aligned} \quad (\text{A26})$$

$$(1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1} \ddot{\underline{q}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n} \ddot{\underline{q}}_n = (1 - \alpha_f) \underline{F}_{c, n+1} + \alpha_f \underline{F}_{c, n}. \quad (\text{A27})$$

which can be rewritten to

$$\begin{aligned} & ((1 - \alpha_m) \underline{\underline{m}} + (1 - \alpha_f) \gamma \Delta t \underline{\underline{d}} + (1 - \alpha_f) \beta \Delta t^2 \underline{\underline{k}}) \ddot{\underline{q}}_{n+1} + \\ & + (1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1}^T \underline{\underline{\lambda}}_{n+1} = (1 - \alpha_f) \underline{F}_{\text{ext}, n+1} + \alpha_f \underline{F}_{\text{ext}, n} + \\ & + (1 - \alpha_f) \underline{F}_{v, n+1} + \alpha_f \underline{F}_{v, n} - \alpha_m \underline{\underline{m}}_n \ddot{\underline{q}}_n + \\ & - (1 - \alpha_f) \underline{\underline{d}}_{n+1} \dot{\underline{q}}_{\text{pred}} - \alpha_f \underline{\underline{d}}_n \dot{\underline{q}}_n + \\ & - (1 - \alpha_f) \underline{\underline{k}}_{n+1} \underline{q}_{\text{pred}} - \alpha_f \underline{\underline{k}}_n \underline{q}_n + \\ & - \alpha_m \underline{\underline{J}}_{\Phi, n}^T \underline{\underline{\lambda}}_n, \end{aligned} \quad (\text{A28})$$

$$(1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1} \ddot{\underline{q}}_{n+1} = (1 - \alpha_f) \underline{F}_{c, n+1} + \alpha_f \underline{F}_{c, n} - \alpha_m \underline{\underline{J}}_{\Phi, n} \ddot{\underline{q}}_n. \quad (\text{A29})$$

The equations of motion of the effective system is given by

$$\begin{bmatrix} \underline{\underline{m}}_{\text{eff}} & \underline{\underline{J}}_{\Phi, \text{eff}}^T \\ \underline{\underline{J}}_{\Phi, \text{eff}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \ddot{\underline{q}}_{n+1} \\ \underline{\underline{\lambda}}_{n+1} \end{bmatrix} = \begin{bmatrix} \underline{F}_{a, \text{eff}} \\ \underline{F}_{c, \text{eff}} \end{bmatrix}, \quad (\text{A30})$$

where

$$\underline{\underline{m}}_{\text{eff}} = (1 - \alpha_m)\underline{\underline{m}}_{n+1} + (1 - \alpha_f)\gamma\Delta t \underline{\underline{d}}_{n+1} + (1 - \alpha_f)\beta\Delta t^2 \underline{\underline{k}}_{n+1}, \quad (\text{A31})$$

$$\underline{\underline{J}}_{\Phi, \text{eff}} = (1 - \alpha_m)\underline{\underline{J}}_{\Phi, n+1}, \quad (\text{A32})$$

$$\begin{aligned} \underline{\underline{F}}_{a, \text{eff}} = & (1 - \alpha_f)\underline{\underline{F}}_{\text{ext}, n+1} + \alpha_f \underline{\underline{F}}_{\text{ext}, n} + (1 - \alpha_f)\underline{\underline{F}}_{v, n+1} + \alpha_f \underline{\underline{F}}_{v, n} - \alpha_m \underline{\underline{m}}_n \ddot{\underline{\underline{q}}}_n + \\ & - (1 - \alpha_f)\underline{\underline{d}}_{n+1} \dot{\underline{\underline{q}}}_{\text{pred}} - \alpha_f \underline{\underline{d}}_n \dot{\underline{\underline{q}}}_n - (1 - \alpha_f)\underline{\underline{k}}_{n+1} \underline{\underline{q}}_{\text{pred}} - \alpha_f \underline{\underline{k}}_n \underline{\underline{q}}_n - \alpha_m \underline{\underline{J}}_{\Phi, n}^T \underline{\underline{\lambda}}_n, \end{aligned} \quad (\text{A33})$$

$$\underline{\underline{F}}_{c, \text{eff}} = (1 - \alpha_f)\underline{\underline{F}}_{c, n+1} + \alpha_f \underline{\underline{F}}_{c, n} - \alpha_m \underline{\underline{J}}_{\Phi, n} \ddot{\underline{\underline{q}}}_n. \quad (\text{A34})$$

Appendix B.2. Nonlinear solver

406

The base for the nonlinear solver are the residual equations of the flexible multibody system, given by

$$\begin{aligned} \underline{\underline{R}}_{1, n+1-\alpha_f} = & (1 - \alpha_m)\underline{\underline{m}}_{n+1} \ddot{\underline{\underline{q}}}_{n+1} + \alpha_m \underline{\underline{m}}_n \ddot{\underline{\underline{q}}}_n + (1 - \alpha_f)\underline{\underline{d}}_{n+1} \dot{\underline{\underline{q}}}_{n+1} + \alpha_f \underline{\underline{d}}_n \dot{\underline{\underline{q}}}_n + \\ & + (1 - \alpha_f)\underline{\underline{k}}_{n+1} \underline{\underline{q}}_{n+1} + \alpha_f \underline{\underline{k}}_n \underline{\underline{q}}_n + (1 - \alpha_m)\underline{\underline{J}}_{\Phi, n+1}^T \underline{\underline{\lambda}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n}^T \underline{\underline{\lambda}}_n + \\ & - (1 - \alpha_f)\underline{\underline{F}}_{\text{ext}, n+1} - \alpha_f \underline{\underline{F}}_{\text{ext}, n} - (1 - \alpha_f)\underline{\underline{F}}_{v, n+1} - \alpha_f \underline{\underline{F}}_{v, n}, \end{aligned} \quad (\text{A35})$$

$$\underline{\underline{R}}_{2, n+1-\alpha_f} = (1 - \alpha_m)\underline{\underline{J}}_{\Phi, n+1} \ddot{\underline{\underline{q}}}_{n+1} + \alpha_m \underline{\underline{J}}_{\Phi, n} \ddot{\underline{\underline{q}}}_n - (1 - \alpha_f)\underline{\underline{F}}_{c, n+1} - \alpha_f \underline{\underline{F}}_{c, n}. \quad (\text{A36})$$

The Newton–Raphson method is used here

407

$$\begin{bmatrix} \frac{\partial \underline{\underline{R}}_{1, n+1-\alpha_f}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} & \frac{\partial \underline{\underline{R}}_{1, n+1-\alpha_f}}{\partial \underline{\underline{\lambda}}_{n+1}} \\ \frac{\partial \underline{\underline{R}}_{2, n+1-\alpha_f}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} & \frac{\partial \underline{\underline{R}}_{2, n+1-\alpha_f}}{\partial \underline{\underline{\lambda}}_{n+1}} \end{bmatrix} \begin{bmatrix} \Delta \ddot{\underline{\underline{q}}}_{n+1} \\ \Delta \underline{\underline{\lambda}}_{n+1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{R}}_{1, n+1-\alpha_f} \\ \underline{\underline{R}}_{2, n+1-\alpha_f} \end{bmatrix} = \underline{\underline{0}}, \quad (\text{A37})$$

where the terms of the Jacobian matrix are given by

$$\begin{aligned} \frac{\partial \underline{\underline{R}}_{1, n+1-\alpha_f}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} = & (1 - \alpha_m) \frac{\partial \underline{\underline{m}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \ddot{\underline{\underline{q}}}_{n+1} + \alpha_m \frac{\partial \underline{\underline{m}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \ddot{\underline{\underline{q}}}_n + (1 - \alpha_m) \underline{\underline{m}}_{n+1} \frac{\partial \ddot{\underline{\underline{q}}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \alpha_m \underline{\underline{m}}_n \frac{\partial \ddot{\underline{\underline{q}}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \\ & + (1 - \alpha_f) \frac{\partial \underline{\underline{d}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \dot{\underline{\underline{q}}}_{n+1} + \alpha_f \frac{\partial \underline{\underline{d}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \dot{\underline{\underline{q}}}_n + (1 - \alpha_f) \underline{\underline{d}}_{n+1} \frac{\partial \dot{\underline{\underline{q}}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \alpha_f \underline{\underline{d}}_n \frac{\partial \dot{\underline{\underline{q}}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \\ & + (1 - \alpha_f) \frac{\partial \underline{\underline{k}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \underline{\underline{q}}_{n+1} + \alpha_f \frac{\partial \underline{\underline{k}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \underline{\underline{q}}_n + (1 - \alpha_f) \underline{\underline{k}}_{n+1} \frac{\partial \underline{\underline{q}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \alpha_f \underline{\underline{k}}_n \frac{\partial \underline{\underline{q}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \\ & + (1 - \alpha_m) \frac{\partial \underline{\underline{J}}_{\Phi, n+1}^T}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \underline{\underline{\lambda}}_{n+1} + \alpha_m \frac{\partial \underline{\underline{J}}_{\Phi, n}^T}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \underline{\underline{\lambda}}_n + (1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1}^T \frac{\partial \underline{\underline{\lambda}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \alpha_m \underline{\underline{J}}_{\Phi, n}^T \frac{\partial \underline{\underline{\lambda}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \\ & - (1 - \alpha_f) \frac{\partial \underline{\underline{F}}_{\text{ext}, n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} - \alpha_f \frac{\partial \underline{\underline{F}}_{\text{ext}, n}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} - (1 - \alpha_f) \frac{\partial \underline{\underline{F}}_{v, n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} - \alpha_f \frac{\partial \underline{\underline{F}}_{v, n}}{\partial \ddot{\underline{\underline{q}}}_{n+1}}, \end{aligned} \quad (\text{A38})$$

$$\begin{aligned} \frac{\partial \underline{\underline{R}}_{2, n+1-\alpha_f}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} = & (1 - \alpha_m) \frac{\partial \underline{\underline{J}}_{\Phi, n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \ddot{\underline{\underline{q}}}_{n+1} + \alpha_m \frac{\partial \underline{\underline{J}}_{\Phi, n}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} \ddot{\underline{\underline{q}}}_n + (1 - \alpha_m) \underline{\underline{J}}_{\Phi, n+1} \frac{\partial \ddot{\underline{\underline{q}}}_{n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \alpha_m \underline{\underline{J}}_{\Phi, n} \frac{\partial \ddot{\underline{\underline{q}}}_n}{\partial \ddot{\underline{\underline{q}}}_{n+1}} + \\ & - (1 - \alpha_f) \frac{\partial \underline{\underline{F}}_{c, n+1}}{\partial \ddot{\underline{\underline{q}}}_{n+1}} - \alpha_f \frac{\partial \underline{\underline{F}}_{c, n}}{\partial \ddot{\underline{\underline{q}}}_{n+1}}, \end{aligned} \quad (\text{A39})$$

$$\begin{aligned}
\frac{\partial \mathbf{R}_{1,n+1-\alpha_f}}{\partial \lambda_{n+1}} &= (1 - \alpha_m) \frac{\partial \underline{\underline{\mathbf{m}}}_{n+1}}{\partial \lambda_{n+1}} \ddot{\underline{\underline{\mathbf{q}}}}_{n+1} + \alpha_m \frac{\partial \underline{\underline{\mathbf{m}}}_n}{\partial \lambda_{n+1}} \ddot{\underline{\underline{\mathbf{q}}}}_n + (1 - \alpha_m) \underline{\underline{\mathbf{m}}}_{n+1} \frac{\partial \ddot{\underline{\underline{\mathbf{q}}}}_{n+1}}{\partial \lambda_{n+1}} + \alpha_m \underline{\underline{\mathbf{m}}}_n \frac{\partial \ddot{\underline{\underline{\mathbf{q}}}}_n}{\partial \lambda_{n+1}} + \\
&+ (1 - \alpha_f) \frac{\partial \underline{\underline{\mathbf{d}}}_{n+1}}{\partial \lambda_{n+1}} \dot{\underline{\underline{\mathbf{q}}}}_{n+1} + \alpha_f \frac{\partial \underline{\underline{\mathbf{d}}}_n}{\partial \lambda_{n+1}} \dot{\underline{\underline{\mathbf{q}}}}_n + (1 - \alpha_f) \underline{\underline{\mathbf{d}}}_{n+1} \frac{\partial \dot{\underline{\underline{\mathbf{q}}}}_{n+1}}{\partial \lambda_{n+1}} + \alpha_f \underline{\underline{\mathbf{d}}}_n \frac{\partial \dot{\underline{\underline{\mathbf{q}}}}_n}{\partial \lambda_{n+1}} + \\
&+ (1 - \alpha_f) \frac{\partial \underline{\underline{\mathbf{k}}}_{n+1}}{\partial \lambda_{n+1}} \underline{\underline{\mathbf{q}}}_{n+1} + \alpha_f \frac{\partial \underline{\underline{\mathbf{k}}}_n}{\partial \lambda_{n+1}} \underline{\underline{\mathbf{q}}}_n + (1 - \alpha_f) \underline{\underline{\mathbf{k}}}_{n+1} \frac{\partial \underline{\underline{\mathbf{q}}}_{n+1}}{\partial \lambda_{n+1}} + \alpha_f \underline{\underline{\mathbf{k}}}_n \frac{\partial \underline{\underline{\mathbf{q}}}_n}{\partial \lambda_{n+1}} + \\
&+ (1 - \alpha_m) \frac{\partial \underline{\underline{\mathbf{J}}}_{\Phi,n+1}^T}{\partial \lambda_{n+1}} \lambda_{n+1} + \alpha_m \frac{\partial \underline{\underline{\mathbf{J}}}_{\Phi,n}^T}{\partial \lambda_{n+1}} \lambda_n + (1 - \alpha_m) \underline{\underline{\mathbf{J}}}_{\Phi,n+1}^T \frac{\partial \lambda_{n+1}}{\partial \lambda_{n+1}} + \alpha_m \underline{\underline{\mathbf{J}}}_{\Phi,n}^T \frac{\partial \lambda_n}{\partial \lambda_{n+1}} + \\
&- (1 - \alpha_f) \frac{\partial \underline{\underline{\mathbf{F}}}_{\text{ext},n+1}}{\partial \lambda_{n+1}} - \alpha_f \frac{\partial \underline{\underline{\mathbf{F}}}_{\text{ext},n}}{\partial \lambda_{n+1}} - (1 - \alpha_f) \frac{\partial \underline{\underline{\mathbf{F}}}_{v,n+1}}{\partial \lambda_{n+1}} - \alpha_f \frac{\partial \underline{\underline{\mathbf{F}}}_{v,n}}{\partial \lambda_{n+1}}, \tag{A40}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{R}_{2,n+1-\alpha_f}}{\partial \lambda_{n+1}} &= (1 - \alpha_m) \frac{\partial \underline{\underline{\mathbf{J}}}_{\Phi,n+1}}{\partial \lambda_{n+1}} \ddot{\underline{\underline{\mathbf{q}}}}_{n+1} + \alpha_m \frac{\partial \underline{\underline{\mathbf{J}}}_{\Phi,n}}{\partial \lambda_{n+1}} \ddot{\underline{\underline{\mathbf{q}}}}_n + (1 - \alpha_m) \underline{\underline{\mathbf{J}}}_{\Phi,n+1} \frac{\partial \ddot{\underline{\underline{\mathbf{q}}}}_{n+1}}{\partial \lambda_{n+1}} + \alpha_m \underline{\underline{\mathbf{J}}}_{\Phi,n} \frac{\partial \ddot{\underline{\underline{\mathbf{q}}}}_n}{\partial \lambda_{n+1}} + \\
&- (1 - \alpha_f) \frac{\partial \underline{\underline{\mathbf{F}}}_{c,n+1}}{\partial \lambda_{n+1}} - \alpha_f \frac{\partial \underline{\underline{\mathbf{F}}}_{c,n}}{\partial \lambda_{n+1}}. \tag{A41}
\end{aligned}$$

The derivatives of the system parameters of the previous time step with respect to state variables of the next time step disappear,

$$\frac{\partial \underline{\underline{\mathbf{O}}}_n}{\partial \underline{\underline{\mathbf{O}}}_{n+1}} = \underline{\underline{\mathbf{0}}}, \quad \frac{\partial \underline{\underline{\mathbf{O}}}_n}{\partial \underline{\underline{\mathbf{O}}}_{n+1}} = \underline{\underline{\mathbf{0}}}. \tag{A42}$$

The dependencies of the system parameters are shown in (33) and (34) and applying the chain rule of differentiation leads to

$$\frac{\partial \underline{\underline{\mathbf{m}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} = \frac{\partial \underline{\underline{\mathbf{m}}}}{\partial \underline{\underline{\mathbf{q}}}} \frac{\partial \underline{\underline{\mathbf{q}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}}, \tag{A43}$$

$$\frac{\partial \underline{\underline{\mathbf{d}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} = \frac{\partial \underline{\underline{\mathbf{d}}}}{\partial \underline{\underline{\mathbf{q}}}} \frac{\partial \underline{\underline{\mathbf{q}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}}, \tag{A44}$$

$$\frac{\partial \underline{\underline{\mathbf{k}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} = \frac{\partial \underline{\underline{\mathbf{k}}}}{\partial \underline{\underline{\mathbf{q}}}} \frac{\partial \underline{\underline{\mathbf{q}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}}, \tag{A45}$$

$$\frac{\partial \underline{\underline{\mathbf{J}}}_{\Phi}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} = \frac{\partial \underline{\underline{\mathbf{J}}}_{\Phi}}{\partial \underline{\underline{\mathbf{q}}}} \frac{\partial \underline{\underline{\mathbf{q}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}}, \tag{A46}$$

$$\frac{\partial \underline{\underline{\mathbf{F}}}_{\text{ext}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} = \frac{\partial \underline{\underline{\mathbf{F}}}_{\text{ext}}}{\partial \underline{\underline{\mathbf{q}}}} \frac{\partial \underline{\underline{\mathbf{q}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} + \frac{\partial \underline{\underline{\mathbf{F}}}_{\text{ext}}}{\partial \dot{\underline{\underline{\mathbf{q}}}}} \frac{\partial \dot{\underline{\underline{\mathbf{q}}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}}, \tag{A47}$$

$$\frac{\partial \underline{\underline{\mathbf{F}}}_v}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} = \frac{\partial \underline{\underline{\mathbf{F}}}_v}{\partial \underline{\underline{\mathbf{q}}}} \frac{\partial \underline{\underline{\mathbf{q}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}} + \frac{\partial \underline{\underline{\mathbf{F}}}_v}{\partial \dot{\underline{\underline{\mathbf{q}}}}} \frac{\partial \dot{\underline{\underline{\mathbf{q}}}}}{\partial \ddot{\underline{\underline{\mathbf{q}}}}}, \tag{A48}$$

$$\frac{\partial \underline{\underline{\mathbf{O}}}}{\partial \lambda} = \underline{\underline{\mathbf{0}}}. \tag{A49}$$

$$\frac{\partial \underline{\underline{\mathbf{O}}}}{\partial \lambda} = \underline{\underline{\mathbf{0}}}. \tag{A50}$$

The derivatives of the state variables with respect to $\underline{\ddot{q}}_{n+1}$ and $\underline{\lambda}_{n+1}$ are given by

$$\begin{aligned}
 \frac{\partial \underline{q}_{n+1}}{\partial \underline{\ddot{q}}_{n+1}} &= \beta \Delta t^2 \underline{\underline{e}}, & \frac{\partial \underline{q}_{n+1}}{\partial \underline{\lambda}_{n+1}} &= \underline{0}, \\
 \frac{\partial \underline{\dot{q}}_{n+1}}{\partial \underline{\ddot{q}}_{n+1}} &= \gamma \Delta t \underline{\underline{e}}, & \frac{\partial \underline{\dot{q}}_{n+1}}{\partial \underline{\lambda}_{n+1}} &= \underline{0}, \\
 \frac{\partial \underline{\ddot{q}}_{n+1}}{\partial \underline{\ddot{q}}_{n+1}} &= \underline{\underline{e}}, & \frac{\partial \underline{\ddot{q}}_{n+1}}{\partial \underline{\lambda}_{n+1}} &= \underline{0}, \\
 \frac{\partial \underline{\lambda}_{n+1}}{\partial \underline{\ddot{q}}_{n+1}} &= \underline{0}, & \frac{\partial \underline{\lambda}_{n+1}}{\partial \underline{\lambda}_{n+1}} &= \underline{\underline{e}}.
 \end{aligned} \tag{A51}$$

Applying eq. (A42), eqs. (A43)–(A50) and eq. (A51) leads to the terms of the Jacobian matrix given by

$$\begin{aligned}
 \frac{\partial \underline{R}_{1,n+1-\alpha_f}}{\partial \underline{\ddot{q}}_{n+1}} &= (1 - \alpha_m) \left(\beta \Delta t^2 \frac{\partial \underline{\underline{m}}_{n+1}}{\partial \underline{q}_{n+1}} \underline{\ddot{q}}_{n+1} + \underline{\underline{m}}_{n+1} \right) + \\
 &\quad + (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{\underline{d}}_{n+1}}{\partial \underline{q}_{n+1}} \underline{\dot{q}}_{n+1} + \gamma \Delta t \underline{\underline{d}}_{n+1} \right) + \\
 &\quad + (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{\underline{k}}_{n+1}}{\partial \underline{q}_{n+1}} \underline{q}_{n+1} + \beta \Delta t^2 \underline{\underline{k}}_{n+1} \right) + \\
 &\quad + (1 - \alpha_m) \beta \Delta t^2 \frac{\partial \underline{\underline{J}}_{\Phi,n+1}^T}{\partial \underline{q}_{n+1}} \underline{\lambda}_{n+1} + \\
 &\quad - (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{F}_{\text{ext},n+1}}{\partial \underline{q}_{n+1}} + \gamma \Delta t \frac{\partial \underline{F}_{\text{ext},n+1}}{\partial \underline{\dot{q}}_{n+1}} \right) + \\
 &\quad - (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{F}_{v,n+1}}{\partial \underline{q}_{n+1}} + \gamma \Delta t \frac{\partial \underline{F}_{v,n+1}}{\partial \underline{\dot{q}}_{n+1}} \right),
 \end{aligned} \tag{A52}$$

$$\begin{aligned}
 \frac{\partial \underline{R}_{2,n+1-\alpha_f}}{\partial \underline{\ddot{q}}_{n+1}} &= (1 - \alpha_m) \left(\beta \Delta t^2 \frac{\partial \underline{\underline{J}}_{\Phi,n+1}}{\partial \underline{q}_{n+1}} \underline{\ddot{q}}_{n+1} + \underline{\underline{J}}_{\Phi,n+1} \right) + \\
 &\quad - (1 - \alpha_f) \left(\beta \Delta t^2 \frac{\partial \underline{F}_{c,n+1}}{\partial \underline{q}_{n+1}} + \gamma \Delta t \frac{\partial \underline{F}_{c,n+1}}{\partial \underline{\dot{q}}_{n+1}} \right),
 \end{aligned} \tag{A53}$$

$$\frac{\partial \underline{R}_{1,n+1-\alpha_f}}{\partial \underline{\lambda}_{n+1}} = (1 - \alpha_m) \underline{\underline{J}}_{\Phi,n+1}^T, \tag{A54}$$

$$\frac{\partial \underline{R}_{2,n+1-\alpha_f}}{\partial \underline{\lambda}_{n+1}} = \underline{0}. \tag{A55}$$

Appendix B.3. Sensitivity analysis

410

The derivative of the equations of motion leads to the governing equation of the sensitivities given by

$$\begin{aligned}
& (1 - \alpha_m) \underline{\underline{\underline{\nabla m}}}_{n+1} \ddot{\underline{\underline{\underline{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{\nabla m}}}_n \ddot{\underline{\underline{\underline{q}}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{m}}}_{n+1} \underline{\underline{\underline{\nabla \ddot{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{m}}}_n \underline{\underline{\underline{\nabla \ddot{q}}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{\nabla d}}}_{n+1} \dot{\underline{\underline{\underline{q}}}}_{n+1} + \alpha_f \underline{\underline{\underline{\nabla d}}}_n \dot{\underline{\underline{\underline{q}}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{d}}}_{n+1} \underline{\underline{\underline{\nabla \dot{q}}}}_{n+1} + \alpha_f \underline{\underline{\underline{d}}}_n \underline{\underline{\underline{\nabla \dot{q}}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{\nabla k}}}_{n+1} \underline{\underline{\underline{q}}}_{n+1} + \alpha_f \underline{\underline{\underline{\nabla k}}}_n \underline{\underline{\underline{q}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{k}}}_{n+1} \underline{\underline{\underline{\nabla q}}}_{n+1} + \alpha_f \underline{\underline{\underline{k}}}_n \underline{\underline{\underline{\nabla q}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{\nabla J}}}_{\Phi, n+1}^T \underline{\underline{\underline{\lambda}}}_{n+1} + \alpha_m \underline{\underline{\underline{\nabla J}}}_{\Phi, n}^T \underline{\underline{\underline{\lambda}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{J}}}_{\Phi, n+1}^T \underline{\underline{\underline{\nabla \lambda}}}_{n+1} + \alpha_m \underline{\underline{\underline{J}}}_{\Phi, n}^T \underline{\underline{\underline{\nabla \lambda}}}_n = (1 - \alpha_f) \underline{\underline{\underline{\nabla F}}}_{\text{ext}, n+1} + \alpha_f \underline{\underline{\underline{\nabla F}}}_{\text{ext}, n} + \\
& + (1 - \alpha_f) \underline{\underline{\underline{\nabla F}}}_{v, n+1} + \alpha_f \underline{\underline{\underline{\nabla F}}}_{v, n}, \tag{A56}
\end{aligned}$$

and

$$\begin{aligned}
& (1 - \alpha_m) \underline{\underline{\underline{\nabla J}}}_{\Phi, n+1} \ddot{\underline{\underline{\underline{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{\nabla J}}}_{\Phi, n} \ddot{\underline{\underline{\underline{q}}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{J}}}_{\Phi, n+1} \underline{\underline{\underline{\nabla \ddot{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{J}}}_{\Phi, n} \underline{\underline{\underline{\nabla \ddot{q}}}}_n = (1 - \alpha_f) \underline{\underline{\underline{\nabla F}}}_{c, n+1} + \alpha_f \underline{\underline{\underline{\nabla F}}}_{c, n}. \tag{A57}
\end{aligned}$$

Including Newmark's sensitivity eqs. (16) and (17) leads to

$$\begin{aligned}
& (1 - \alpha_m) \underline{\underline{\underline{\nabla m}}}_{n+1} \ddot{\underline{\underline{\underline{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{\nabla m}}}_n \ddot{\underline{\underline{\underline{q}}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{m}}}_{n+1} \underline{\underline{\underline{\nabla \ddot{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{m}}}_n \underline{\underline{\underline{\nabla \ddot{q}}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{\nabla d}}}_{n+1} \dot{\underline{\underline{\underline{q}}}}_{n+1} + \alpha_f \underline{\underline{\underline{\nabla d}}}_n \dot{\underline{\underline{\underline{q}}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{d}}}_{n+1} \left(\underline{\underline{\underline{\nabla \dot{q}}}}_{\text{pred}} + \gamma \Delta t \underline{\underline{\underline{\nabla \ddot{q}}}}_{n+1} \right) + \alpha_f \underline{\underline{\underline{d}}}_n \underline{\underline{\underline{\nabla \dot{q}}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{\nabla k}}}_{n+1} \underline{\underline{\underline{q}}}_{n+1} + \alpha_f \underline{\underline{\underline{\nabla k}}}_n \underline{\underline{\underline{q}}}_n + \\
& + (1 - \alpha_f) \underline{\underline{\underline{k}}}_{n+1} \left(\underline{\underline{\underline{\nabla q}}}_{\text{pred}} + \beta \Delta t^2 \underline{\underline{\underline{\nabla \ddot{q}}}}_{n+1} \right) + \alpha_f \underline{\underline{\underline{k}}}_n \underline{\underline{\underline{\nabla q}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{\nabla J}}}_{\Phi, n+1}^T \underline{\underline{\underline{\lambda}}}_{n+1} + \alpha_m \underline{\underline{\underline{\nabla J}}}_{\Phi, n}^T \underline{\underline{\underline{\lambda}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{J}}}_{\Phi, n+1}^T \underline{\underline{\underline{\nabla \lambda}}}_{n+1} + \alpha_m \underline{\underline{\underline{J}}}_{\Phi, n}^T \underline{\underline{\underline{\nabla \lambda}}}_n = (1 - \alpha_f) \underline{\underline{\underline{\nabla F}}}_{\text{ext}, n+1} + \alpha_f \underline{\underline{\underline{\nabla F}}}_{\text{ext}, n} + \\
& + (1 - \alpha_f) \underline{\underline{\underline{\nabla F}}}_{v, n+1} + \alpha_f \underline{\underline{\underline{\nabla F}}}_{v, n}, \tag{A58}
\end{aligned}$$

and

$$\begin{aligned}
& (1 - \alpha_m) \underline{\underline{\underline{\nabla J}}}_{\Phi, n+1} \ddot{\underline{\underline{\underline{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{\nabla J}}}_{\Phi, n} \ddot{\underline{\underline{\underline{q}}}}_n + \\
& + (1 - \alpha_m) \underline{\underline{\underline{J}}}_{\Phi, n+1} \underline{\underline{\underline{\nabla \ddot{q}}}}_{n+1} + \alpha_m \underline{\underline{\underline{J}}}_{\Phi, n} \underline{\underline{\underline{\nabla \ddot{q}}}}_n = (1 - \alpha_f) \underline{\underline{\underline{\nabla F}}}_{c, n+1} + \alpha_f \underline{\underline{\underline{\nabla F}}}_{c, n}. \tag{A59}
\end{aligned}$$

which can be rewritten to

$$\begin{aligned}
 & ((1 - \alpha_m)\underline{\underline{m}} + (1 - \alpha_f)\gamma\Delta t\underline{\underline{d}} + (1 - \alpha_f)\beta\Delta t^2\underline{\underline{k}})\nabla\ddot{\underline{\underline{q}}}_{n+1} + \\
 & + (1 - \alpha_m)\underline{\underline{J}}_{\Phi,n+1}^T\nabla\lambda_{n+1} = (1 - \alpha_f)\nabla F_{\text{ext},n+1} + \alpha_f\nabla F_{\text{ext},n} + \\
 & + (1 - \alpha_f)\nabla F_{v,n+1} + \alpha_f\nabla F_{v,n} + \\
 & - (1 - \alpha_m)\nabla\underline{\underline{m}}_{n+1}\ddot{\underline{\underline{q}}}_{n+1} + \\
 & - \alpha_m\nabla\underline{\underline{m}}_n\ddot{\underline{\underline{q}}}_n - \alpha_m\underline{\underline{m}}_n\nabla\ddot{\underline{\underline{q}}}_n + \\
 & - (1 - \alpha_f)\nabla\underline{\underline{d}}_{n+1}\dot{\underline{\underline{q}}}_{n+1} - \alpha_f\nabla\underline{\underline{d}}_n\dot{\underline{\underline{q}}}_n + \\
 & - (1 - \alpha_f)\underline{\underline{d}}_{n+1}\nabla\dot{\underline{\underline{q}}}_{\text{pred}} - \alpha_f\underline{\underline{d}}_n\nabla\dot{\underline{\underline{q}}}_n + \\
 & - (1 - \alpha_f)\nabla\underline{\underline{k}}_{n+1}\underline{\underline{q}}_{n+1} - \alpha_f\nabla\underline{\underline{k}}_n\underline{\underline{q}}_n + \\
 & - (1 - \alpha_f)\underline{\underline{k}}_{n+1}\nabla\underline{\underline{q}}_{\text{pred}} - \alpha_f\underline{\underline{k}}_n\nabla\underline{\underline{q}}_n + \\
 & - (1 - \alpha_m)\nabla\underline{\underline{J}}_{\Phi,n+1}^T\lambda_{n+1} + \\
 & - \alpha_m\nabla\underline{\underline{J}}_{\Phi,n}^T\lambda_n - \alpha_m\underline{\underline{J}}_{\Phi,n}\nabla\lambda_n, \tag{A60}
 \end{aligned}$$

and

$$\begin{aligned}
 (1 - \alpha_m)\underline{\underline{J}}_{\Phi,n+1}\nabla\ddot{\underline{\underline{q}}}_{n+1} &= (1 - \alpha_f)\nabla F_{c,n+1} + \alpha_f\nabla F_{c,n} + \\
 &- (1 - \alpha_m)\nabla\underline{\underline{J}}_{\Phi,n+1}\ddot{\underline{\underline{q}}}_{n+1} - \alpha_m\nabla\underline{\underline{J}}_{\Phi,n}\ddot{\underline{\underline{q}}}_n - \alpha_m\underline{\underline{J}}_{\Phi,n}\nabla\ddot{\underline{\underline{q}}}_n. \tag{A61}
 \end{aligned}$$

The governing equation of sensitivity analysis of the effective system is given by

411

$$\begin{bmatrix} \underline{\underline{m}}_{\text{eff}} & \underline{\underline{J}}_{\Phi,\text{eff}}^T \\ \underline{\underline{J}}_{\Phi,\text{eff}} & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} \nabla\ddot{\underline{\underline{q}}}_{n+1} \\ \nabla\lambda_{n+1} \end{bmatrix} = \begin{bmatrix} \underline{\underline{F}}_{a,\text{pseudo}} \\ \underline{\underline{F}}_{c,\text{pseudo}} \end{bmatrix}, \tag{A62}$$

where

$$\begin{aligned}
 \underline{\underline{F}}_{a,\text{pseudo}} &= (1 - \alpha_f)\nabla F_{\text{ext},n+1} + \alpha_f\nabla F_{\text{ext},n} + (1 - \alpha_f)\nabla F_{v,n+1} + \alpha_f\nabla F_{v,n} + \\
 &- (1 - \alpha_m)\nabla\underline{\underline{m}}_{n+1}\ddot{\underline{\underline{q}}}_{n+1} - \alpha_m\nabla\underline{\underline{m}}_n\ddot{\underline{\underline{q}}}_n - \alpha_m\underline{\underline{m}}_n\nabla\ddot{\underline{\underline{q}}}_n + \\
 &- (1 - \alpha_f)\nabla\underline{\underline{d}}_{n+1}\dot{\underline{\underline{q}}}_{n+1} - \alpha_f\nabla\underline{\underline{d}}_n\dot{\underline{\underline{q}}}_n - (1 - \alpha_f)\underline{\underline{d}}_{n+1}\nabla\dot{\underline{\underline{q}}}_{\text{pred}} - \alpha_f\underline{\underline{d}}_n\nabla\dot{\underline{\underline{q}}}_n + \\
 &- (1 - \alpha_f)\nabla\underline{\underline{k}}_{n+1}\underline{\underline{q}}_{n+1} - \alpha_f\nabla\underline{\underline{k}}_n\underline{\underline{q}}_n - (1 - \alpha_f)\underline{\underline{k}}_{n+1}\nabla\underline{\underline{q}}_{\text{pred}} - \alpha_f\underline{\underline{k}}_n\nabla\underline{\underline{q}}_n + \\
 &- (1 - \alpha_m)\nabla\underline{\underline{J}}_{\Phi,n+1}^T\lambda_{n+1} - \alpha_m\nabla\underline{\underline{J}}_{\Phi,n}^T\lambda_n - \alpha_m\underline{\underline{J}}_{\Phi,n}\nabla\lambda_n, \tag{A63}
 \end{aligned}$$

$$\begin{aligned}
 \underline{\underline{F}}_{c,\text{pseudo}} &= (1 - \alpha_f)\nabla F_{c,n+1} + \alpha_f\nabla F_{c,n} + \\
 &- (1 - \alpha_m)\nabla\underline{\underline{J}}_{\Phi,n+1}\ddot{\underline{\underline{q}}}_{n+1} - \alpha_m\nabla\underline{\underline{J}}_{\Phi,n}\ddot{\underline{\underline{q}}}_n - \alpha_m\underline{\underline{J}}_{\Phi,n}\nabla\ddot{\underline{\underline{q}}}_n. \tag{A64}
 \end{aligned}$$

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