
Optimal Design Approach for One-Dimensional Rubber-Concrete Periodic Foundations based on Analytical Approximations of Band Gaps

Qiaoyun Wu^{1,2}, Peishan Xu¹, Zhifeng Xu^{1,*}, Xuyong Chen¹

1.School of Civil Engineering and Architecture, Wuhan Institute of Technology, Wuhan, Hubei 430073

2.State Key Laboratory for Health and Safety of Bridge Structures, Wuhan, Hubei 430034

*. Corresponding author: Zhifeng Xu, 20120102@wit.edu.cn, +86-15927279709

Abstract: This research presents an optimal design approach for one-dimensional rubber-concrete periodic foundations based on the proposed analytical approximations and mapping relations of the first few band gaps. The presented design approach can search for the optimal solution in a global sense through effectively cooperating the band gaps with the superstructure's resonance frequencies. Firstly, analytical approximation formulas for the first few band gaps, mapping relations among frequency points inside the band gaps, and special approximations for the first band gap of one-dimensional rubber-concrete periodic foundations are proposed and verified. Furthermore, inspired by the proposed analytical approximations and the mapping relations, an optimal design approach for one-dimensional rubber-concrete periodic foundations is presented and applied to a practical example, whose optimality is demonstrated both theoretically and numerically.

Keywords: Periodic foundation; Band gap; Analytical approximation; Optimal design

1. INTRODUCTION

Periodic structures are obtained by repetitively arranging an identical sub-structure called the unit cell, which exhibits a unique ability to manipulate the propagation of waves according to their frequencies ¹. Theoretical studies on frequency dependent properties of periodic structures can be traced back to the 1970s, when Delph ² proposed analytical solutions for harmonic waves travelling through an ideal one-dimensional (1-D) periodic structure by using the transfer matrix method. In the paper, it is found that the dispersion spectrum of 1-D periodic structures has a wave selecting characteristic that hampers or even prohibits the transmission of waves belonging to certain frequency intervals through reflection or refraction. These frequency intervals are referred to as band gaps (BGs) or attenuation zones ⁶, while harmonic waves whose frequencies are outside BGs can travel through the periodic structure without any energy loss. On the other hand, similar conclusions can be drawn by using periodic beam models, periodic plate models, periodic shell models ³⁻⁵, etc. BGs seem to be a universal characteristic shared by many types of periodic structures of varied dimensionality ⁶⁻¹². Sigalas and Economou ¹³ demonstrated that ideal two-dimensional (2-D) cylinders embedded in matrix materials have BGs in two dimensions. Kafesaki et al. ¹⁴ showed the existence of BGs in three-dimensional (3-D) polymer matrix composites.

The above-mentioned frequency selecting property endows periodic structures with a wide application scope, such as light control ¹⁵, electronic transport manipulation ¹⁶, noise reduction ¹⁷, and seismic vibration attenuation ^{8,18-23}. Compared to conventional passive seismic control systems, periodic structures have the advantages of simple manufacturing and being able to mitigate the responses of both horizontal and vertical earthquakes simultaneously ²⁴⁻²⁶, which has continued to attract many scholars for decades. Shi et al. ⁷ proposed the idea of using periodic structures as isolators against ground motions, in which the isolator is firstly named as the periodic foundation. Later, different forms of periodic foundations have been rapidly developed, from 1-D to 2-D and 3-D, from simple unit cells to multi-material complex unit cells ^{9,10,12,27}. Xiang et al. ²⁶ verified the vibration reduction effect of BGs through a combination of numerical simulations and experimental tests. Jia et al. ²⁰ demonstrated the seismic attenuation effect of 1-D and 2-D periodic foundations through finite element analysis (FEA). Yan et al. ^{9,10} conducted experimental studies of 2-D and 3-D periodic foundations, in which the results showed that periodic foundations can have satisfactory vibration reduction performances if the periodic structures are well-designed.

Among various periodic foundations, we are mostly interested in the 1-D rubber-concrete (R-C) periodic foundations for their economy in cost, simplicity in manufacturing, and potential to alleviate both pressure wave (P-wave) and shear wave (S-wave). Reasonable design of periodic foundations should be based on a theoretical understanding of periodic structures, while knowledge of BGs is of paramount importance. Sackman et al. ²⁸ propose analytical approximation solutions for the lower bound and upper bound of the first BG of 1-D periodic structures. The solutions are accurate for calculating the upper bound of the first BG, yet the accuracy of the lower bound is not satisfactory for many practical cases. Zhifei et al. ²⁹ modified Sackman's approximation solution, which is merely a curve-fitting result that can be inaccurate for cases beyond the scope of their numerical study. Recently, Witarto et al. ³⁰ studied the significance of design parameters with respect to the first BG of 1-D periodic materials using a global sensitivity analysis, which provides clues for deriving analytical approximations for BGs.

Aiming at designing optimal periodic foundations, this research studies the frequency-related properties of 1-D periodic structures and further proposes an optimal design approach for 1-D R-C periodic foundations. Firstly, analytical approximation formulas for the first few BGs, mapping relations among frequency points inside BGs, and special approximations for the first BG of 1-D R-C periodic foundations are proposed and verified. On top of that, inspired by the proposed analytical approximations and the idea of properly distributing the superstructure's resonance frequencies into BGs, an optimal design approach for 1-D R-C periodic foundations is presented, which is further implemented in the design of a 1-D R-C periodic foundation for a 4-story steel frame. The rest of this paper is organized as follows: Section 2 presents the theoretical fundamentals of 1-D periodic structures; Section 3 proposes the analytical formulas for approximating the first few BGs, mapping relations among BGs' frequency points, and special approximations for the first BG; and Section 4 presents and validates the optimal design approach of 1-D R-C periodic foundations.

2. THEORETICAL FUNDAMENTALS ON ONE-DIMENSIONAL PERIODIC STRUCTURES

This section reviews the theoretical fundamentals of 1-D periodic structures. As shown in Fig. 1, the structure is a 1-D periodic structure that consists of n unit cells repetitively lying in the y -direction. Each unit cell contains m elastic layers, in which the thickness of the j^{th} layer is h_j , the total thickness of the unit cell is computed by $h = \sum_{j=1}^m h_j$, and the total length of the entire periodic structure is $H = nh$. The structure is ideal if $n = \infty$. The local coordinate of the j^{th} layer in the i^{th} unit cell is related to the global coordinate by $y = (i-1)h + \sum_{p=0}^{j-1} h_p + y_{ij}$, where y is the global coordinate and y_{ij} is the local coordinate of the layer. For the j^{th} layer, the Young's modulus, Poisson's ratio, and density are denoted by E_j , ν_j and ρ_j , respectively. The displacement and the stress component of the structure and the corresponding quantities of the layer are linked by $u(y, t) = u_{ij}(y_{ij}, t)$ and $\sigma(y, t) = \sigma_{ij}(y_{ij}, t)$ respectively, in which t represents time. The bottom and the top of the periodic structure are characterized by $y = 0$ and $y = H$, respectively. Incident waves with different frequencies can be input at the bottom in the x -direction (parallel to the layers) or y -direction (perpendicular to the layers), representing S-waves or P-waves, respectively. For S-waves, $u(y, t)$ is the displacement in x -direction, and $\sigma(y, t)$ is shear stress. For P-waves, $u(y, t)$ is the displacement in y -direction, and $\sigma(y, t)$ is normal stress.

The governing equation for a harmonic incident wave propagating through the j^{th} layer in the i^{th} unit cell can be described by the following wave equation³¹:

$$\frac{\partial^2 u_{ij}(y_{ij}, t)}{\partial t^2} = c_j^2 \frac{\partial^2 u_{ij}(y_{ij}, t)}{\partial y_{ij}^2} \quad (1)$$

where $c_j = \sqrt{\frac{\kappa_j}{\rho_j}}$, and κ_j is the wave modulus. For S-waves, $\kappa_j = \mu_j$, for P-waves, $\kappa_j = \lambda_j + 2\mu_j$,

$\mu_j = \frac{E_j}{2(1+\nu_j)}$ and $\lambda_j = \frac{\nu_j E_j}{(1+\nu_j)(1-2\nu_j)}$ are the Lamé constants.

The particular solution of Eq. (2) with respect to a harmonic excitation satisfies the following separable form:

$$u_{ijl}(y_{ij}, t) = \exp(i\omega_l t) u_{ijl}(y_{ij}) \quad (3)$$

$$u_{ijl}(y_{ij}) = A_{ijl} \sin(k_{jl} y_{ij}) + B_{ijl} \cos(k_{jl} y_{ij}) \quad (4)$$

where ω_l is frequency, $k_{jl} = \omega_l / c_j$ is the wave number with respect to ω_l , and i is the imaginary unit.

Substituting Eq. (3) into Hooke's law yields the following equations for the stress component

$$\sigma_{ijl}(y_{ij}, t) = \exp(i\omega_l t) \sigma_{ijl}(y_{ij}) \quad (5)$$

$$\sigma_{ijl}(y_{ij}) = \kappa_j k_{jl} [A_{ijl} \sin(k_{jl} y_{ij}) - B_{ijl} \cos(k_{jl} y_{ij})] \quad (6)$$

The general solution of Eq. (1) with respect to an arbitrary excitation can be written as the superposition of each particular solution, i.e., $u_{ij}(y_{ij}, t) = \int s(\omega_l) \exp(i\omega_l t) \mu_{ijl}(y_{ij}) d\omega_l$, in which $s(\omega_l)$ is the Fourier component that is determined by matching the initial and boundary conditions. It is noted that the general solution degenerates into the particular solution described by Eq. (3) if the excitation is a harmonic wave.

Now we focus on the cases of harmonic incident waves. By using the transfer matrix method³², Eq. (4) and Eq. (6) can be rearranged into the following matrix multiplication form:

$$\mathbf{W}_{ijl}(y_{ij}) = \mathbf{H}_{ijl}(y_{ij}) \mathbf{\Psi}_{ijl} \quad (7)$$

where $\mathbf{W}_{ijl}(y_{ij}) = \begin{bmatrix} \mu_{ijl}(y_{ij}) \\ \sigma_{ijl}(y_{ij}) \end{bmatrix}$ is the state vector with respect to ω_l , $\mathbf{\Psi}_{ijl} = \begin{bmatrix} A_{ijl} \\ B_{ijl} \end{bmatrix}$ is a constant vector, and

$$\mathbf{H}_{ijl}(y_{ij}) = \begin{bmatrix} \sin(k_{jl}y_{ij}) & \cos(k_{jl}y_{ij}) \\ \kappa_j k_{jl} \cos(k_{jl}y_{ij}) & -\kappa_j k_{jl} \sin(k_{jl}y_{ij}) \end{bmatrix} \quad (8)$$

Substituting $y_{ij} = 0$ and $y_{ij} = h_j$ into Eq. (7) yields

$$\mathbf{W}_{ijl}(h_j) = \mathbf{T}_{jl}(h_j) \mathbf{W}_{ijl}(0) \quad (9)$$

in which the following transfer matrix of the layer is defined by

$$\mathbf{T}_{jl}(h_j) = \mathbf{H}_{ijl}(h_j) [\mathbf{H}_{ijl}(0)]^{-1} = \begin{bmatrix} \cos(k_{jl}h_j) & \frac{1}{\kappa_j k_{jl}} \sin(k_{jl}h_j) \\ -\kappa_j k_{jl} \sin(k_{jl}h_j) & \cos(k_{jl}h_j) \end{bmatrix} \quad (10)$$

whose determinant $|\mathbf{T}_{jl}(h_j)| = 1$.

By using the continuity condition between layers, i.e., $\mathbf{W}_{i(j+1)l}(0) = \mathbf{W}_{(j+1)l}(h_j)$, the relationship between the top and bottom of the i^{th} unit cell is achieved by the following recursive expression:

$$\mathbf{W}_{il}(h) = \mathbf{W}_{iml}(h_m) = \mathbf{T}_{ml}(h_j) \mathbf{W}_{iml}(0) = \dots = \mathbf{T}(h) \mathbf{W}_{il}(0) \quad (11)$$

where $\mathbf{W}_{il}(h)$ and $\mathbf{W}_{il}(0)$ are the state vectors of the top and bottom of the i^{th} unit cell with respect to ω_l , $\mathbf{T}_l(h) = \prod_{j=1}^m \mathbf{T}_{jl}(h_j)$ is the transfer matrix of the unit cell with respect to ω_l . Since $|\mathbf{T}_{jl}(h_j)| = 1$, it can be obtained that the determinant $|\mathbf{T}_l(h_j)| = 1$.

Similarly, the translation relation between the top and bottom of the entire periodic structure is

$$\mathbf{W}(H) = \mathbf{W}_{nl}(h) = \mathbf{T}_l(h) \mathbf{W}_{nl}(0) = \dots = [\mathbf{T}_l(h)]^n \mathbf{W}_{nl}(0) = \mathbf{T}_l^n \mathbf{W}(0) \quad (12)$$

where $\mathbf{T}_l^n = [\mathbf{T}_l(h)]^n$ is the transfer matrix of the entire structure with respect to ω_l .

Based on the Floquet-Bloch theory³³, the state vectors of the top and bottom of a unit cell have the following relationship:

$$\mathbf{W}_{il}^\Lambda(h) = \exp(ikh) \mathbf{W}_{il}^\Lambda(0) \quad (13)$$

in which k is the wave number of the unit cell, $\mathbf{W}_{il}^\Lambda(h)$ and $\mathbf{W}_{il}^\Lambda(0)$ are in the complex eigen-modes.

Substituting Eq. (13) into Eq. (11), the following eigen-form equation is obtained:

$$[\mathbf{T}_l(h) - \exp(ikh)\mathbf{I}] \mathbf{W}_{il}^\Lambda(0) = \mathbf{0} \quad (14)$$

in which $\mathbf{I}$ is the identity matrix, and the two eigen-values $\Lambda_1 = \exp(ik_1h)$ and $\Lambda_2 = \exp(ik_2h)$ satisfy

$$|\mathbf{T}_l(h) - \exp(ik_{1 \text{ or } 2}h)\mathbf{I}| = 0 \quad (15)$$

Therefore, the solutions of Eq. (15) satisfy

$$\exp(ik_1h) + \exp(ik_2h) = T_{111}(h) + T_{122}(h) = \text{trace}[\mathbf{T}_l(h)] \quad (16)$$

$$\exp(ik_1h)\exp(ik_2h) = \exp[i(k_1 + k_2)h] = |\mathbf{T}_l(h)| = 1 \quad (17)$$

Consequently, the behavior of the state vectors can be classified into two cases: (a) no attenuation and (b) attenuation. In the case of no attenuation, $|\text{trace}[\mathbf{T}_l(h)]| < 2$, Λ_1 and Λ_2 form a pair of conjugates, $|\Lambda_{1 \text{ or } 2}| = 1$, k_1 and k_2 are two real numbers satisfying $k_1 + k_2 = 0$, and waves can propagate without any energy loss. In the case of attenuation, $|\text{trace}[\mathbf{T}_l(h)]| > 2$, Λ_1 and Λ_2 are two real numbers $|\Lambda_1\Lambda_2| = 1$, $|\Lambda_1| < 1 < |\Lambda_2|$, k_1 and k_2 become two complex numbers, and the wave amplitude is decayed per unit cell at a constant rate called the attenuation coefficient, i.e., $r = 1 - \exp(-\gamma)$, where $\gamma = -\ln|\Lambda_1|$ is the localization factor. Obviously, after travelling through the entire periodic structure, the wave amplitude is decayed by $r = 1 - \exp(-n\gamma)$. And frequencies in this case form the BGs. For ideal 1-D periodic structures, the amplitude of the output wave becomes $\lim_{n \rightarrow \infty} \exp(-n\gamma) = 0$, which indicates a total ban of wave propagation. Figs. 2 (a) and (b) show the dispersion curves of the benchmark unit cell defined by Table 1 undergoing S-waves or P-waves, respectively.

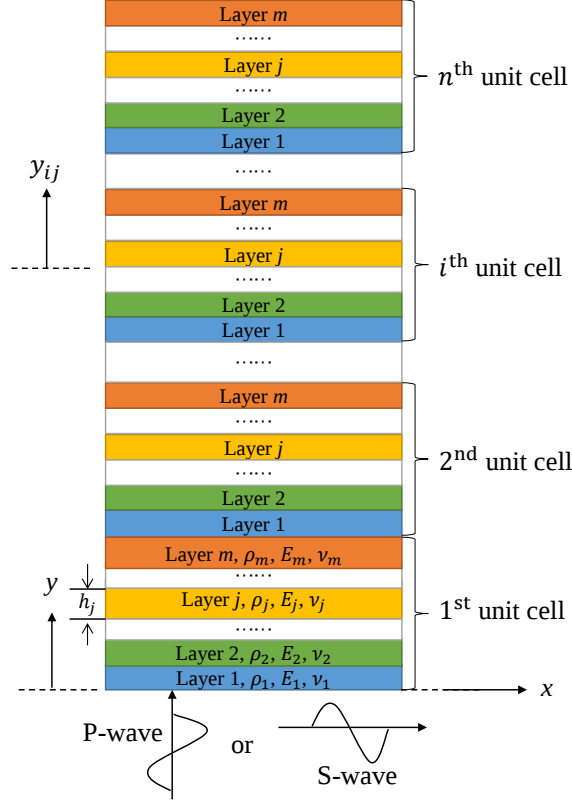


FIGURE 1 Configuration of the one-dimensional periodic structure.

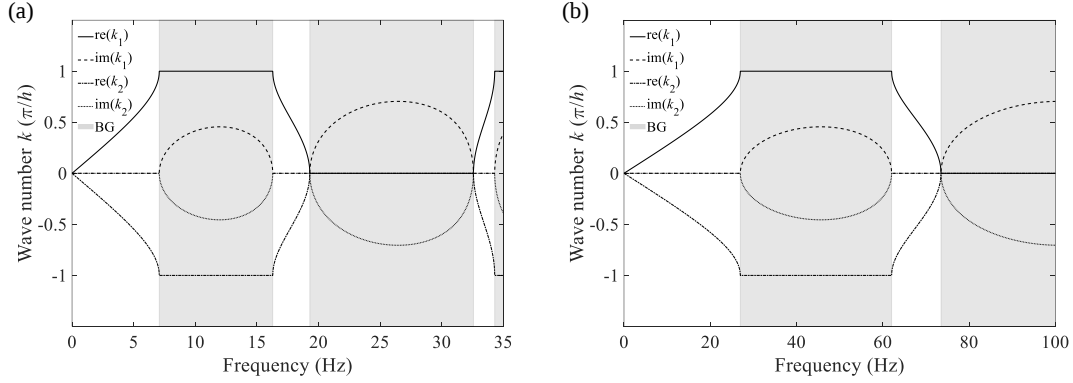


FIGURE 2 Dispersion curves of the benchmark unit cell: (a) S-wave, (b) P-wave.

TABLE 1 Configuration of the benchmark unit cell

layer label	material	Young's modulus (MPa)	density (kg/m ³)	Poisson's ratio	layer thickness (m)
1	rubber	0.1586	1277	0.463	0.2
2	concrete	40,000	2300	0.2	0.2

3. APPROXIMATIONS AND MAPPING RELATIONS FOR THE FIRST FEW BAND GAPS OF ONE-DIMENSIONAL RUBBER-CONCRETE PERIODIC FOUNDATIONS

In this section, the analytical formulas for approximating the first few BGs of 1-D R-C periodic foundations are proposed and verified, in which the 1-D R-C periodic foundation is a 1-D periodic structure whose unit cells consist of a rubber layer and a concrete layer. Besides, one-to-one mapping relations among the frequency points in different BGs are established. Besides, special approximations for the lower bound and upper bound of the 1st BG are proposed. At this point, it is defined that layer 1 is the rubber layer and layer 2 is the concrete layer.

3.1 Analytical approximations of the first few band gaps

For a 1-D R-C periodic foundation, Eq. (16) can be rewritten as:

$$\text{trace} = \Lambda_1 + \frac{1}{\Lambda_1} = 2 \cos\left(\frac{h_1}{c_1} \omega_l\right) \cos\left(\frac{h_2}{c_2} \omega_l\right) - 2D \sin\left(\frac{h_1}{c_1} \omega_l\right) \sin\left(\frac{h_2}{c_2} \omega_l\right) \quad (18)$$

where $D = \frac{1}{2} \left(\frac{c_1 \kappa_2}{c_2 \kappa_1} + \frac{c_2 \kappa_1}{c_1 \kappa_2} \right)$. It should be noted that the symmetry of Eq. (18) with respect to the superscripts indicates that the order of the two layers is trivial in the computation of BGs.

The boundaries of BGs are characterized by $\Lambda_1 = \pm 1$. Meanwhile, if $\frac{h_2}{c_2} \ll \frac{h_1}{c_1}$ and $\frac{h_2}{c_2} \omega_l$ is small, which is nearly universal for 1-D R-C periodic foundations, it is guaranteed that $\cos\left(\frac{h_2}{c_2} \omega_l\right) \approx 1$, $\sin\left(\frac{h_2}{c_2} \omega_l\right) \approx \frac{h_2}{c_2} \omega_l$, then Eq. (18) becomes

$$\cos\left(\frac{h_1}{c_1} \omega_l\right) - D \frac{h_2}{c_2} \omega_l \sin\left(\frac{h_1}{c_1} \omega_l\right) = \pm 1 \quad (19)$$

The solution of Eq. (19) provides approximations of the lower bounds and upper bounds of the first few BGs, which can be further expressed as the following form:

$$\cos\left(\frac{h_1}{c_1} \omega_l + \theta\right) \approx \cos(\pm \theta + p\pi) \quad (20)$$

where $\theta = \arctan\left(\frac{D\omega_l h_2}{c_2}\right)$, and p is a non-negative integer.

Consequently, $h_1 \omega_l / c_1 + \theta \approx \theta + p\pi$ marks the upper bounds of the first few BGs, i.e.:

$$\omega_{p-\text{end}} \approx p\pi \frac{c_1}{h_1} \quad (21)$$

It is noted that, for the 1st BG, $\omega_{1-\text{end}} \approx \frac{\pi c_1}{h_1}$ is accordance with Sackman's approximation²⁸.

On the other hand, the lower bounds are marked by $\frac{h_1}{c_1} \omega_l + \theta \approx -\theta + p\pi$, i.e., $\frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi \approx -\theta$ or $\frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi \approx \frac{1}{2} \pi - \theta$. Since $\tan(-\theta) \approx -D \frac{h_2}{c_2} \omega_l$ and $\cot(\frac{\pi}{2} - \theta) \approx D \frac{h_2}{c_2} \omega_l$ hold, by using $\cot(-\theta) \approx -\frac{c_2}{D h_2 \omega_l}$ and $\tan(\frac{\pi}{2} - \theta) \approx \frac{c_2}{D h_2 \omega_l}$, the following two equations are obtained as the implicit approximations for the lower bounds of BGs:

$$\cot\left(\frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi\right) + \frac{c_2}{D h_2 \omega_l} \approx 0 \quad (22)$$

$$\tan\left(\frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi\right) - \frac{c_2}{D h_2 \omega_l} \approx 0 \quad (23)$$

To further obtain an explicit approximation for the lower bounds of the first few BGs, substituting $\tan\left(\frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi\right) \approx \frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi$ and $\cot\left(\frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi\right) \approx \frac{\pi}{2} - \frac{1}{2} \frac{h_1}{c_1} \omega_l + p\pi$ into Eq. (22) and Eq. (23) respectively and combining the outcomes yield

$$\omega_{p-\text{start}} = \frac{c_1}{h_1} \left[\frac{p-1}{2} \pi + \sqrt{\left(\frac{p-1}{2} \pi\right)^2 + \frac{2c_2 h_1}{D c_1 h_2}} \right] \quad (24)$$

In order to verify the proposed approximation formulas, the exact values of BGs of the benchmark unit cell computed by Eq. (18) and the corresponding approximations using Eq. (21) to (24) are compared in Figs. 3 and Table 2 in terms of frequency f . At this junction, the following convention between frequency f and angular frequency ω is made: $\omega = 2\pi f$. It is seen that, except for the relatively conservative approximation using Eq. (24) on the 1st BG, all the rest of the approximations have sufficient accuracy.

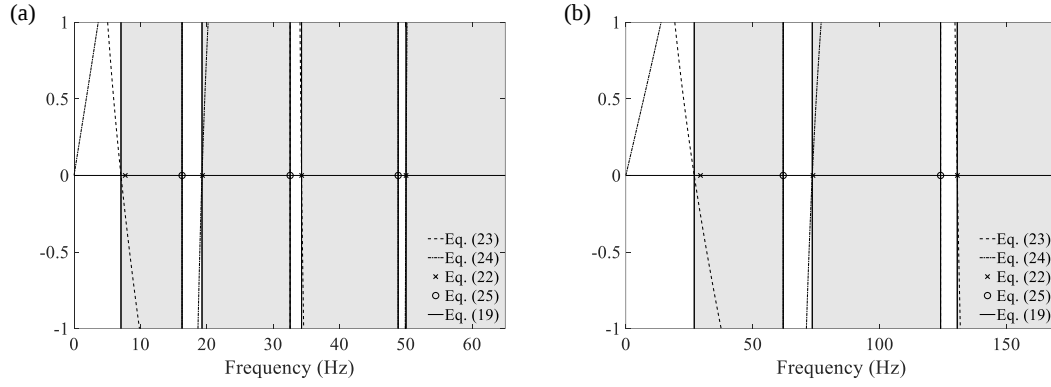


FIGURE 3 Validation of the approximations for the first few BGs: (a) S-wave, (b) P-wave.

TABLE 2 Comparison between the exact values and the approximations for the first few BGs

frequency	S-wave					P-wave				
	Eq. (18)	Eq. (21)	Eq. (22)	Eq. (23)	Eq. (24)	Eq. (18)	Eq. (21)	Eq. (22)	Eq. (23)	Eq. (24)
$f_{1\text{-start}}$ (Hz)	7.078	-	7.078	-	7.726	26.970	-	26.970	-	29.434
$f_{1\text{-end}}$ (Hz)	16.286	16.287	-	-	-	62.049	62.050	-	-	-
$f_{2\text{-start}}$ (Hz)	19.294	-	-	19.2940	19.369	73.504	-	-	73.504	73.791
$f_{2\text{-end}}$ (Hz)	32.574	32.575	-	-	-	124.098	124.101	-	-	-
$f_{3\text{-start}}$ (Hz)	34.298	-	34.298	-	34.315	130.669	-	130.669	-	130.728
$f_{3\text{-end}}$ (Hz)	48.862	48.863	-	-	-	186.148	186.151	-	-	-
$f_{4\text{-start}}$ (Hz)	50.050	-	-	50.0500	50.055	190.674	-	-	190.674	190.694
$f_{4\text{-end}}$ (Hz)	65.150	65.150	65.150	-	-	248.197	248.202	-	-	-
$f_{5\text{-start}}$ (Hz)	66.052	-	66.052	-	66.0544	251.634	-	251.634	-	251.644
$f_{5\text{-end}}$ (Hz)	81.438	81.438	-	-	-	310.246	310.252	-	-	-

3.2 Mapping relations for frequency points within band gaps

From Eq. (21), it is apparent that the upper bounds of BGs form an arithmetic sequence. At this junction, the increment of this arithmetic sequence is defined as the band interval (BI), i.e., $BI = \pi \frac{c_1}{h_1}$, which is solely determined by the properties of the rubber layer. Besides, the ratio of BI for S-waves to BI for P-waves is the following constant, which is defined as the mapping ratio (MR):

$$MR = \frac{BI_S}{BI_P} = \sqrt{\frac{\mu_1}{\lambda_1 + 2\mu_1}} = \sqrt{\frac{1-2\nu_1}{2(1-\nu_1)}} \quad (25)$$

in which the subscripts ‘S’ and ‘P’ represent S-waves and P-waves, respectively.

The occupation rate of the p^{th} BG is defined as the ratio of the BG’s width to to BI, i.e.:

$$o_p = \frac{\omega_{p\text{-end}} - \omega_{p\text{-start}}}{BI} \quad (26)$$

Consequently, the lower bound and upper bound of the p^{th} BG are $\omega_{p\text{-start}} = (p-o_p)BI$ and $\omega_{p\text{-end}} = pBI$.

Substituting Eq. (21) into Eq. (24) yields

$$o_p \approx \frac{p+1}{2} - \sqrt{\left(\frac{p-1}{2}\right)^2 + \frac{4h_1}{\pi^2 \left(\frac{\rho_2}{\rho_1} + \frac{\kappa_1}{\kappa_2}\right) h_2}} \approx \frac{p+1}{2} - \sqrt{\left(\frac{p-1}{2}\right)^2 + \frac{4h_1\rho_1}{\pi^2 h_2\rho_2}} \quad (27)$$

indicating an identical o_p for S-Wave or P-Wave. It is further obtained that: 1, $\frac{\omega_{p\text{-start-S}}}{\omega_{p\text{-start-P}}} = \frac{\omega_{p\text{-end-S}}}{\omega_{p\text{-end-P}}} = MR$;

$$2, \frac{do_p}{dp} \approx \frac{\sqrt{\left(\frac{p-1}{2}\right)^2 + \frac{4h_1\rho_1}{\pi^2 h_2\rho_2}} - \left(\frac{p-1}{2}\right)}{2\sqrt{\left(\frac{p-1}{2}\right)^2 + \frac{4h_1\rho_1}{\pi^2 h_2\rho_2}}} > 0 ; \quad 3, \lim_{p \rightarrow \infty} o_p \approx \lim_{p \rightarrow \infty} \frac{p - \frac{4h_1\rho_1}{\pi^2 h_2\rho_2}}{\frac{p+1}{2} + \sqrt{\left(\frac{p-1}{2}\right)^2 + \frac{4h_1\rho_1}{\pi^2 h_2\rho_2}}} = 1 ; \quad 4, o_p \text{ is}$$

monotonic increasing with h_2/h_1 or ρ_2/ρ_1 . Fig. 4 shows comparison between the exact value of o_p and the approximation through Eq. (27) using the benchmark unit cell, in which the close match

between them demonstrates the validity of Eq. (27).

In addition, the following one-to-one mapping relations are concluded:

1. A frequency point in the p^{th} BG can be mapped to the corresponding frequency point in the q^{th} BG by
$$\omega_{l-p} = (p - \Delta \alpha_p) \text{BI} \rightarrow \omega_{l-q} = (q - \Delta \alpha_q) \text{BI} \quad (28)$$

where Δ is the BG's local coordinate pointing to negative infinity, and $0 \leq \Delta \leq 1$.

2. A frequency point for S-waves can be mapped to the corresponding frequency point for P-waves by
$$\omega_{l-S} = \text{MR} \omega_{l-P} \quad (29)$$

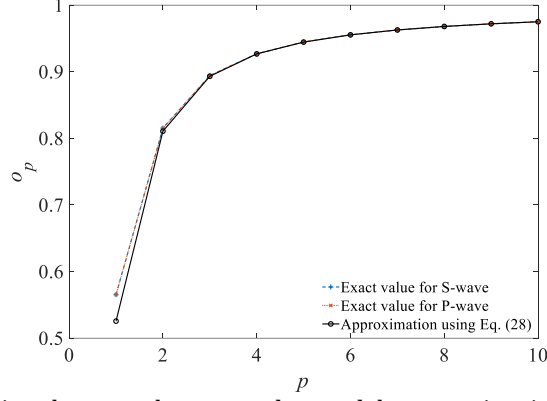


FIGURE 4 Comparison between the exact values and the approximations for occupation rates.

3.3 Special approximations of the first band gaps

From Eq. (21), the upper bound of the 1st BG, which equals the BI, can be approximated by

$$\omega_{1-\text{end-S}} = \text{BI}_S \approx \pi \sqrt{\frac{E_1}{2(1+\nu_1)\rho_1 h_1^2}} \quad (30)$$

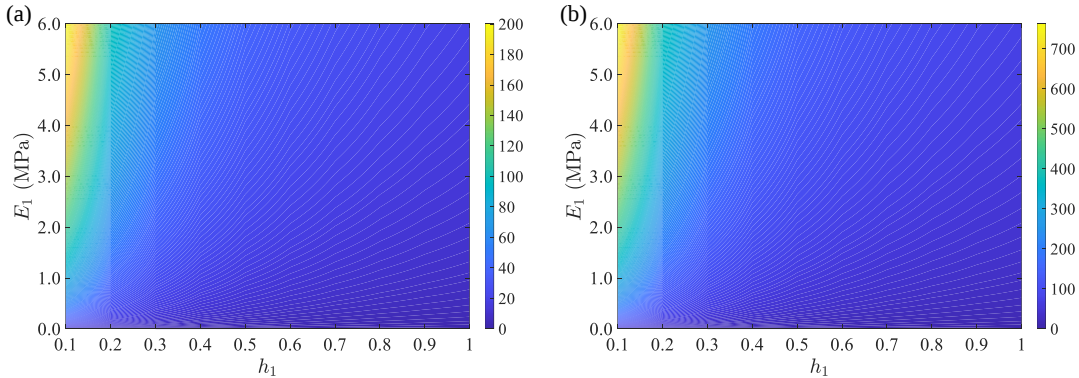
$$\omega_{1-\text{end-P}} = \text{BI}_P \approx \pi \sqrt{\frac{(1-\nu_1)E_1}{(1+\nu_1)(1-2\nu_1)\rho_1 h_1^2}} \quad (31)$$

Substituting Eq. (30), Eq. (31), and $p=1$ into Eq. (27) yields the following approximation for the lower bound of the 1st BG

$$\omega_{1-\text{start-S}} \approx \sqrt{\frac{2E_1}{(1+\nu_1)\rho_2 h_1 h_2}} \quad (32)$$

$$\omega_{1-\text{start-P}} \approx \sqrt{\frac{4(1-\nu_1)E_1}{(1+\nu_1)(1-2\nu_1)h_1 h_2 \rho_2}} \quad (33)$$

It is evident that the upper bound of the 1st BG is determined by the density, Young's modulus, and Poisson's ratio of rubber and h_1 , while the lower bound of the 1st BG is determined by the Young's modulus and Poisson's ratio of rubber, the density of concrete, and $(h_1 h_2)^{1/2}$. Figs. 5 (a) and (b) show the upper bound of the 1st BG as a function of rubber's density and h_1 for S-wave and P-wave, respectively, and Figs. 5 (c) and (d) show the lower bound of the 1st BG as a function of concrete's density and $(h_1 h_2)^{1/2}$ for S-wave and P-wave, respectively, in which the rest of the material parameters are chosen from Table 1.



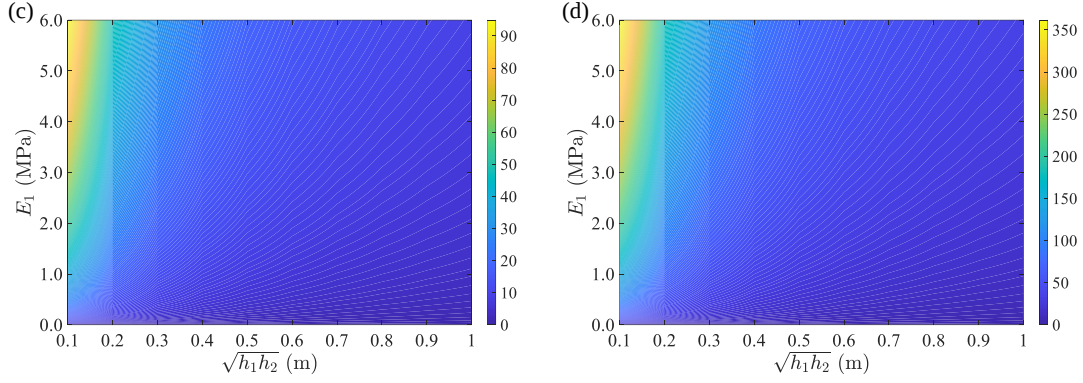


FIGURE 5 Bounds of the 1st BG: (a) $\omega_{1\text{-end-S}}$, (b) $\omega_{1\text{-end-P}}$, (c) $\omega_{1\text{-start-S}}$, (d) $\omega_{1\text{-start-P}}$.

4. OPTIMAL DESIGN APPROACH FOR ONE-DIMENSIONAL RUBBER-CONCRETE PERIODIC FOUNDATIONS

This section proposes an optimal design approach for 1-D R-C periodic foundations, in which the core idea of this design approach is to optimally fill the superstructures' resonance frequencies into the first few BGs and at the same time maximize the performance.

4.1 Problem statement

Given a 1-D R-C periodic foundation to design and a superstructure with a series of resonance zones (RZs), in which: the frequency band of seismic ground motions is $\text{FB}=(\omega_{\text{start-FB}}, \omega_{\text{end-FB}})$, and the parentheses represent an open interval; the set of RZs for S-waves is denoted by $\mathbf{RZ}_S$, in which the q^{th} element is $\text{RZ}_{q-S} = (\omega_{q\text{-start-RZs}}, \omega_{q\text{-end-RZs}})$; the set of RZs for P-waves is $\mathbf{RZ}_P$, in which the r^{th} element is $\text{RZ}_{r-P} = (\omega_{r\text{-start-RZp}}, \omega_{r\text{-end-RZp}})$; the p^{th} BG for S-waves and the s^{th} BG for P-waves are denoted by $\text{BG}_{p-S} = (\omega_{p\text{-start-S}}, \omega_{p\text{-end-S}})$ and $\text{BG}_{s-P} = (\omega_{s\text{-start-P}}, \omega_{s\text{-end-P}})$, respectively; the performance function (PF) measures the goodness of the design (e.g., $\text{PF}(h_1, h_2) = n$ since maximizing the number of unit cells would increase the attenuation effect). A good design of 1-D R-C periodic foundations should ensure the frequency responses of the superstructure are fairly alleviated for any frequency within FB, which requires every RZ to be covered by a corresponding BG. Besides, for all potential designs that meet the above requirement, the optimal one should also maximize $\text{PF}(h_1, h_2)$. Hence, as shown in Fig. 6, the optimization problem becomes the search for h_1 and h_2 maximizing $\text{PF}(h_1, h_2)$ on the condition that: 1, any RZ_{q-S} in FB should belong to a BG_{p-S} ; 2, any RZ_{r-P} in FB should belong to a BG_{s-P} ; 3, the configuration constrain (CC) is satisfied, e.g., $\text{CC}(h_1, h_2, n) = n(h_1+h_2) < H_{\text{max}}$, H_{max} is the allowed foundation height.

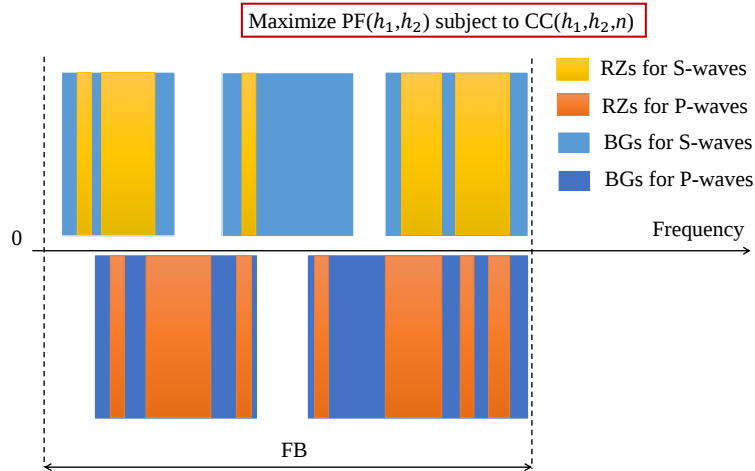


FIGURE 6 Problem statement of the optimal design.

4.2 Approach presentation

According to the optimization problem stated in subsection 4.1, a design approach is presented in this subsection based on the assumption that PF is a monotonic decreasing function of $h_1 + h_2$, since this assumption is in accordance with many different choices of PF, e.g., $\text{PF} = n$, or PF represents the least amount of used materials. As shown in Fig. 7, the design approach consists of the following steps:

1. Determine FB based on the anti-seismic requirements;
2. Determine the RZs (for both S-waves and P-waves) to attenuate according to the superstructure's frequency responses and FB, e.g., the superstructure has three RZs, namely (2.5, 4.5), (10.1, 20.1) and (40.4, 60.6). If FB = (5.0 50.0), then RZ = (2.5, 4.5) is rejected since it is outside the FB, RZ = (10.1, 20.1) is accepted since it is contained by the FB, and RZ = (40.4, 60.6) is first altered to (40.4, 50.0) and then accepted;
3. Map every RZ_{r-p} into RZ_{r-s} by $RZ_{r-s} = (MR \cdot \omega_{r-start-RZp}, MR \cdot \omega_{r-end-RZp})$ and append it to $\mathbf{RZ}_s$;
4. Merge overlapped elements in $\mathbf{RZ}_s$, e.g., $RZ_{q-s} = (3.3, 5.5)$, $RZ_{r-s} = (4.4, 6.6)$, then delete both RZ_{q-s} and RZ_{r-s} from $\mathbf{RZ}_s$ and insert $RZ_{t-s} = (3.3, 6.6)$ into $\mathbf{RZ}_s$;
5. Enumerate every RZ_{i-s} in $\mathbf{RZ}_s$ using Step 6;
6. Iterate each p from 1 to p_{max} using Steps 7 to 9 and Step 15;
7. Assume the upper bound of RZ_{i-s} is equal to the upper bound of BG_{p-s} for a given p and compute the trial BI by $BI_{i-p-s-trial} = \frac{\omega_{i-end-RZ_s}}{p}$;
8. Check the validity of $BI_{i-p-s-trial}$ by testing if none of the trial BGs' upper bounds cuts any element in $\mathbf{RZ}_s$, i.e., $BI_{S-trial}$ is valid if $\left\lceil \frac{\omega_{s-start-RZ_s}}{BI_{i-p-s-trial}} \right\rceil = \left\lceil \frac{\omega_{s-end-RZ_s}}{BI_{i-p-s-trial}} \right\rceil$ for any RZ_{s-s} in $\mathbf{RZ}_s$, where the brackets represent the operator of getting integer part;
9. For a valid $BI_{i-p-s-trial}$, perform Steps 10 to 11 to obtain the trial thickness of rubber layer $h_{1-trial}$ and the trial thickness of concrete layer $h_{2-trial}$;
10. Compute $h_{1-trial}$ according to Eq. (30), i.e., $h_{1-trial} = \frac{\pi}{BI_{i-p-s-trial}} \sqrt{\frac{E_1}{2(1+\nu_1)\rho_1}}$;
11. Obtain $h_{2-trial}$ by performing Step 12 to 14 for every RZ_{j-s} in $\mathbf{RZ}_s$;
12. Assume the lower bound of RZ_{j-s} is equal to the BG's lower bound and compute h_{2-temp} according to Eq. (24), i.e., $h_{2-temp} = \frac{2c_2 h_{1-trial}}{Dc_1 \left\{ \left(\omega_{j-start-RZ_s} \frac{h_{1-trial}}{c_1} - \frac{\pi}{2} \right)^2 - \left(\frac{\pi}{2} \left[\frac{\omega_{s-start-RZ_s}}{BI_{S-trial}} \right] \right)^2 \right\}}$;
13. Insert h_{2-temp} into the set $\mathbf{h}_{2-temp}$;
14. Obtain $h_{2-trial}$ by choosing the maximum element of $\mathbf{h}_{2-temp}$, i.e., $h_{2-trial} = \max(\mathbf{h}_{2-temp})$;
15. Compute the trial performance function $PF_{trial} = PF(h_{1-trial}, h_{2-trial})$ and insert it to the set $\mathbf{PF}_{trial}$;
16. Determine the optimal thicknesses h_{1-opt} and h_{2-opt} by $PF(h_{1-opt}, h_{2-opt}) = \max(\mathbf{PF}_{trial})$;
17. Compute the number of unit cells on the condition that $CC(h_1, h_2, n)$ is satisfied.

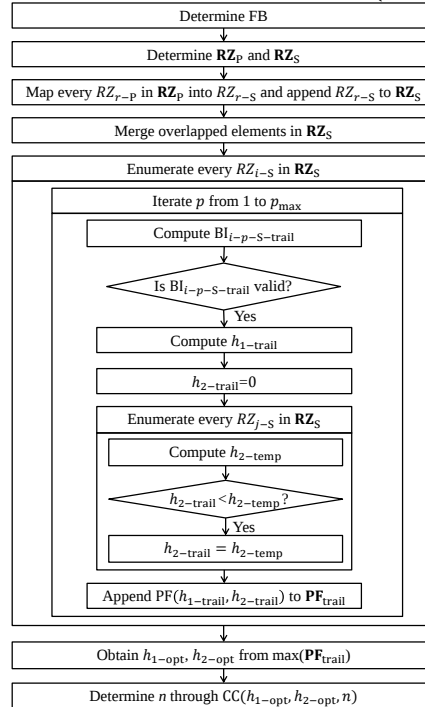


FIGURE 7 Flowchart of the optimal design approach.

4.3 Theoretical explanation

Firstly, since BGs and RZs for P-waves can be one-to-one mapped to the corresponding quantities for S-waves, it is sufficient to only consider the case of S-waves. Suppose there is a set of RZs denoted by $\mathbf{RZ}_S = \{\mathbf{RZ}_{t-S} = (\omega_{t\text{-start-RZs}}, \omega_{t\text{-end-RZs}}), 1 \leq t \leq t_{\max}\}$ and a set of BGs denoted by $\mathbf{BG}_S = \{\mathbf{BG}_{p-S} = (\omega_{p\text{-start-BGs}}, \omega_{p\text{-end-BGs}}), 1 \leq p \leq p_{\max}\}$. Each $\mathbf{RZ}_{t-S}$ is embedded in a corresponding $\mathbf{BG}_{p-S}$. According to Eq. (28), elements of $\mathbf{RZ}_S$ can be mapped to the 1st BG and therefore form a new set $\mathbf{RZ}_{\text{MP1}} = \{\mathbf{RZ}_{i\text{-MP1}}, 1 \leq i \leq i_{\max}\}$, in which overlapped elements are merged. The lower bound and upper bound of $\mathbf{RZ}_{\text{MP1}}$ are $\omega_{\text{start-MP1}}$ and $\omega_{\text{end-MP1}}$, respectively. As shown in Fig. 8, the distances between 0 to $\omega_{\text{start-MP1}}$, $\omega_{1\text{-start-S}}$ to $\omega_{\text{start-MP1}}$, 0 to $\omega_{\text{end-MP1}}$, and $\omega_{\text{end-MP1}}$ to $\omega_{1\text{-end-S}}$ are respectively Δ_1 , Δ_2 , Δ_3 and Δ_4 . By using Eq. (30) and Eq. (32), it is computed that $h_1 = \frac{\pi}{\Delta_3 + \Delta_4} \sqrt{\frac{E_1}{2(1+\nu_1)\rho_1}}$, and $h_2 = \frac{2(\Delta_3 + \Delta_4)}{\pi\rho_2(\Delta_1 - \Delta_2)^2} \sqrt{\frac{2\rho_1 E_1}{(1+\nu_1)}}$. Since it is assumed that $\text{PF}(h_1, h_2)$ is a monotonic decreasing function of $h_1 + h_2$, the optimal design is characterized by $\min[h_1 + h_2]$. Because $\frac{d(h_1 + h_2)}{d\Delta_2} = \frac{dh_2}{d\Delta_2} = \frac{4(\Delta_3 + \Delta_4)}{\pi\rho_2(\Delta_1 - \Delta_2)^3} \sqrt{\frac{2\rho_1 E_1}{(1+\nu_1)}} > 0$, $\min[h_1 + h_2]$ is reached when $\Delta_2 = 0$,

which yields $h_1 + h_2 = \frac{\pi}{\Delta_3 + \Delta_4} \sqrt{\frac{E_1}{2(1+\nu_1)\rho_1}} + \frac{2(\Delta_3 + \Delta_4)}{\pi\rho_2\Delta_1^2} \sqrt{\frac{2\rho_1 E_1}{(1+\nu_1)}}$. Similarly,

$$\frac{d(h_1 + h_2)}{d\Delta_4} = \left[\frac{4}{\pi\rho_2\Delta_1^2} - \frac{\pi}{(\Delta_3 + \Delta_4)^2\rho_1} \right] \sqrt{\frac{E_1\rho_1}{(1+\nu_1)}} > 0 \text{ if } \frac{\rho_1}{\rho_2} > \left(\frac{\pi\Delta_1}{2\Delta_3} \right)^2. \text{ Therefore, } h_1 + h_2 \text{ reaches the minimum}$$

when $\Delta_2 = \Delta_4 = 0$, which means the optimal design must ensure that the lower bound of a BG is equal to the lower bound of a RZ, and the upper bound of a BG is equal to the upper bound of a RZ. Thus, the optimal design can be obtained by exhausting all possible combinations of p , q , r and s .

The proposed design approach is literally a traversal algorithm for all the possible combinations of p , q , r and s , which guarantees the final output for the design is optimal in a global sense. To be specific, Step 5 to Step 7 collect all possible values of BI by assuming $\omega_{p\text{-end-S}} = \omega_{t\text{-end-RZs}}$. In addition, BI should separate RZs into groups without cutting any RZ, so Step 8 rejects the invalid values of BI by examining if there exist an $\omega_{p\text{-end-S}}$ locating inside a RZ. After a valid BI is determined, the next task is to find the corresponding combination of h_1 and h_2 , while h_1 is directly computed in Step 10 through Eq. (30). By assuming $\omega_{p\text{-start-S}} = \omega_{r\text{-start-RZs}}$, a $h_{2\text{-trial}}$ can be computed by using Eq. (24). From Eq. (27) it is known that ω_p is a monotonic increasing function of h_2 . This means the largest h_2 after enumerating every possible combination for $\omega_{p\text{-start-S}} = \omega_{r\text{-start-RZs}}$ is the valid value of h_2 the current BI, which is spirit of Step 11 to 14. Since the optimal combination of h_1 and h_2 is the one that maximizing PF, $h_{1\text{-opt}}$ and $h_{2\text{-opt}}$ is obtained from the max PF, which is performed by Step 15 to 16. Finally, the number of unit cells can be consequently computed by Step 17.

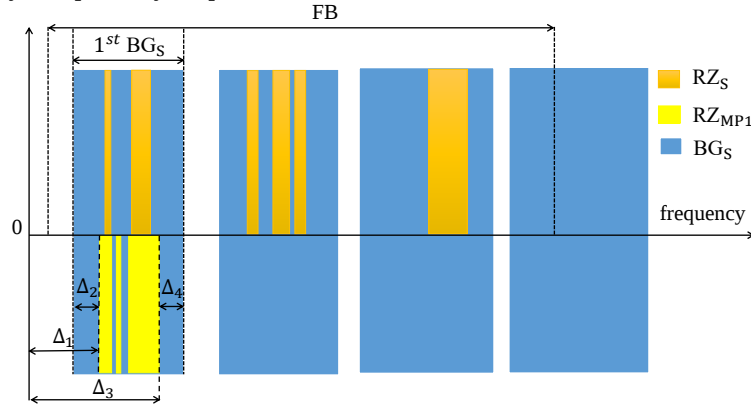


FIGURE 8 Illustration on optimality.

4.4 Design example

To further demonstrate its practicality, the proposed approach is applied to the design of a 1-D R-C periodic foundation for the steel frame structure described in Table 3, which undertakes a broad frequency range of seismic vibrations. The periodic foundation to be designed is a 10 m by 10 m square foundation with a maximum depth of 4 m, in which the materials to be used are rubber and concrete (described in Table 1). The configurations of the frame and the frame-foundation coupled system are shown in Figs. 9 (a) and (b), respectively. In the design, FB was determined to be 2.5 to 60.0 Hz. Firstly, the frequency response functions (FRFs) of the steel frame are computed through ABAQUS 6.14 using

the “SteadyStateDirectStep”, in which $FRF = 20\log_{10}(u_{out}/u_{in})$, u_{out} and u_{in} are the output displacement and the input displacement, respectively. The FRFs for S-waves and P-waves are obtained by measuring the output displacements from the nodes shown in Fig. 9 (a) after assigning the 4 bottom nodes a unit input displacement in the x -direction and the y -direction, respectively. Afterward, $\mathbf{RZ}_S$ and $\mathbf{RZ}_P$ were determined by checking if FRF_{max} (represented by the solid black curves in Figs. 10) in FB is greater than the corresponding threshold, where FRF_{max} consists of the max value of FRFs at each frequency point. The thresholds for S-waves and P-waves are 0.0 and 3.0, respectively. The reason for the threshold of P-Waves being larger than that of S-Waves is that horizontal ground motions are usually more dangerous. As shown in Figs. 10 (a) and (b), it is finally determined that: $\mathbf{RZ}_S = \{(3.104, 3.478), (6.625, 7.056), (12.120, 14.442), (23.516, 23.859), (29.233, 29.512), (30.823, 31.888), (34.855, 35.526), (36.841, 37.321), (42.446, 43.098)\}$; and $\mathbf{RZ}_P = \{(15.679, 15.947), (19.638, 21.926), (22.452, 32.595), (38.879, 60.000)\}$. The program that implemented the optimal design approach presented in subsection 5.2 was written in MATLAB and applied to the foundation design, where $PF = 4/(h_1+h_2)$. After the computation, $\mathbf{PF}_{trial}$ is obtained as the set of 14 combinations of h_1 and h_2 (shown in Table 4), from which we obtained $h_{1-opt} = 0.377$ m and $h_{2-opt} = 0.655$ m. The design results of AZs and BGs using the predicted optimal configuration are shown in Fig. 11, from which it is seen that all the AZs are well distributed in the corresponding BGs. Finally, after practical consideration of manufacturing, the thicknesses of the rubber layer and the concrete layer were determined to be 0.36 m and 0.64 m, respectively, and the number of unit cells was 4.

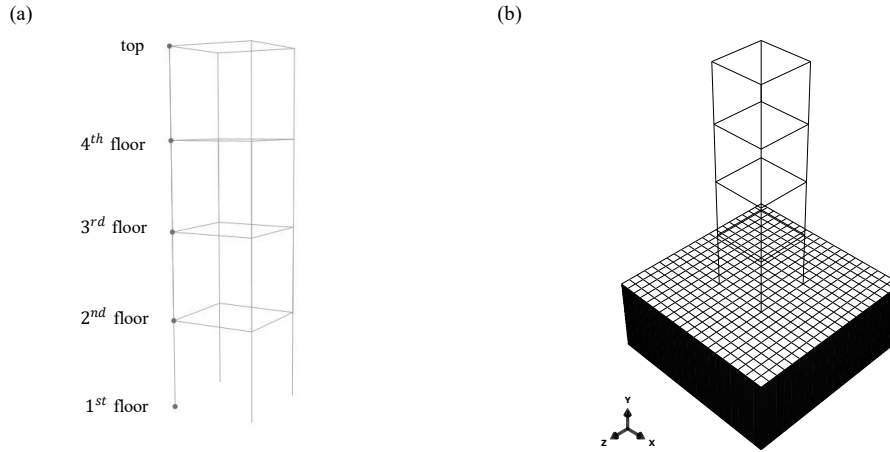


FIGURE 9 Configurations: (a) the superstructure, (b) the superstructure-foundation coupled system.

Table 3 Configuration of the steel frame

structure	number of stories	4
	length (m)	3
	width (m)	3
	story height (m)	3
	cross-section of beams (m)	0.05 by 0.05 rectangle
	cross-section of columns (m)	0.05 by 0.05 rectangle
material	Young's modulus (MPa)	200000.0
	Poisson's ratio	0.3
	density (kg/m ³)	7850

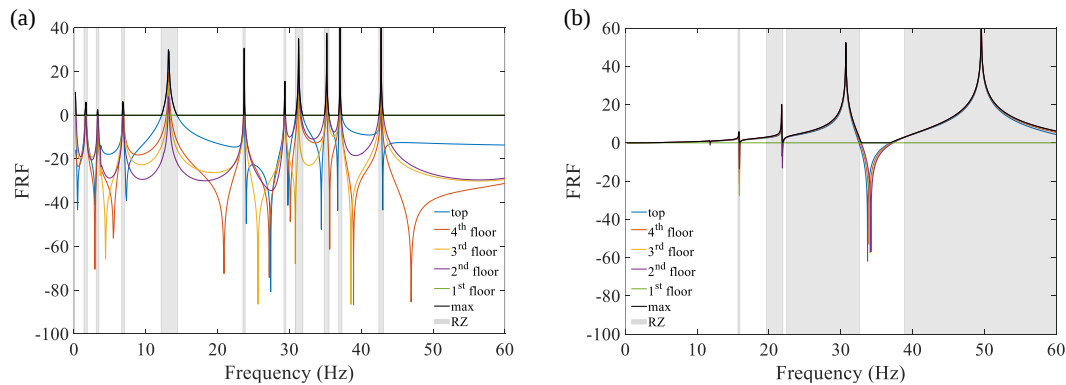


FIGURE 10 Frequency responses and resonance zones of the superstructure: (a) S-wave, (b) P-wave.

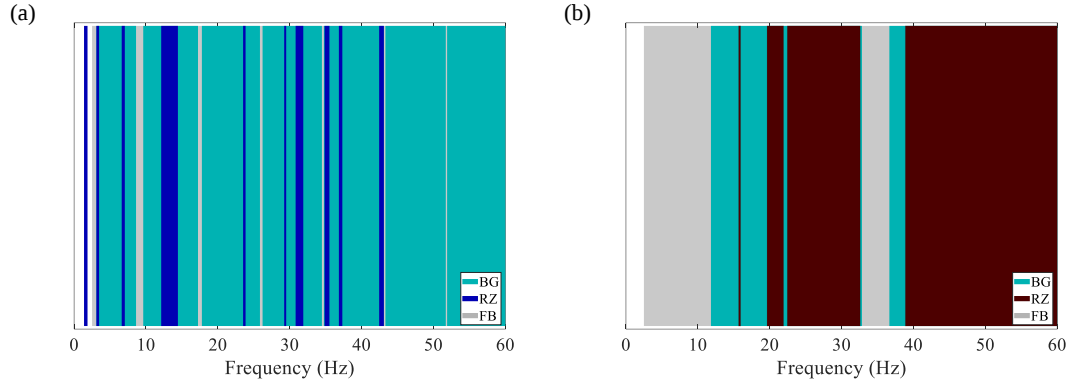


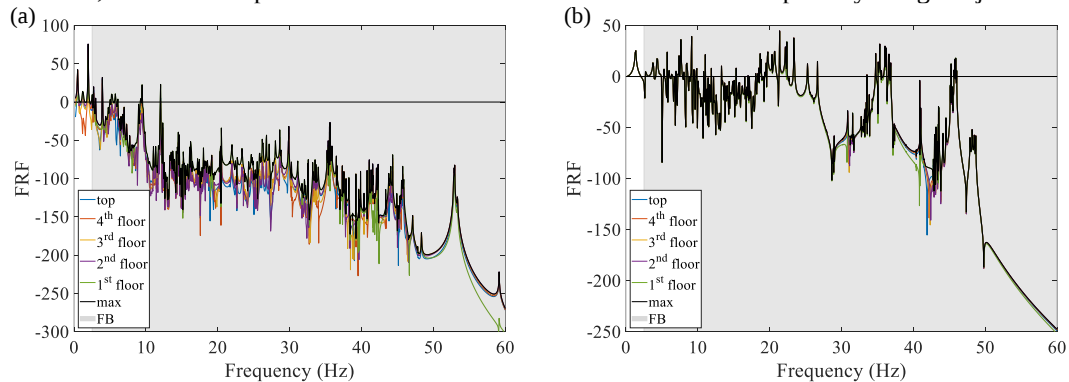
FIGURE 11 Band gaps and resonance zones of the optimal design: (a) S-wave, (b) P-wave.

Table 4 Trial combinations of the layer thicknesses

$h_{1\text{-trial}}$ (m)	0.136	0.110	0.331	0.102	0.091	0.183	0.366	0.087	0.174	0.349	0.075	0.151	0.377	0.1911
$h_{2\text{-trial}}$ (m)	1.814	2.244	1.920	2.424	2.701	1.350	0.675	2.838	1.419	0.765	3.277	1.638	0.655	1.296
PF_{trial}	2.050	1.698	1.776	1.582	1.432	2.607	3.838	1.367	2.510	3.587	1.193	2.234	3.870	2.689

4.5 Numerical validation

To verify the optimality of the optimal design, a finite element method (FEM) analysis using ABABQUS 6.14 was performed. In the analysis, three frame-foundation coupled systems were built and analyzed (shown in Fig. 9 (b)), in which the frames are identical (described by Table 3) and the three different foundations are namely the optimally designed foundation described in Subsection 4.4, the concrete foundation using the concrete described in Table 1, and the benchmark foundation that consists of 10 benchmark unit cells described by Table 1. For the foundation, the normal section was a square with a side length of 10 m. The mesh grid sizes along the transversal surface and the longitude direction were 0.5 m and 0.02 m, which is validated by a mesh convergence check. The element type for the foundation is C3D8, and the steel frame is modeled by B33 elements. All the materials in the simulations are isotropic elastic. Huang et al.³⁴ recently proposed a band-limited uniform damping model for simulating the periodic foundations. Nevertheless, no damping is included in the current simulations because the primary objective of this subsection is to verify the optimal design approach rather than perform more realistic simulations. Unit harmonic displacements of different frequencies were uniformly assigned to the bottom surface, in which displacements along the x -axis or y -axis represent S-waves or P-waves, respectively. The “SteadyStateDirectStep” is adopted to compute the FRFs of frame’s nodes (shown in Fig. 9 (a)), where the frequency ranges from 0.2 to 60.0 Hz for both S-Waves and P-waves. Figs. 12 (a)-(f) show the simulation results of the FRFs and FRF_{max} with respect to the three foundations. It is seen that the optimally designed foundation outperforms the concrete foundation and the benchmark foundation for both the case of S-waves and the case of P-waves. By comparing Fig. 12 (a) and Fig. 12 (e), we can conclude that ensuring the superstructure’s AZs are completely covered by BGs is a good strategy in designing the 1-D R-C foundations, which guarantees satisfactory vibration attenuation results. Furthermore, the vibration attenuation performance of the optimally designed foundation for S-waves is so outstanding that nearly all FRFs in FB are smaller than 0 and the majority are even below -50, while the benchmark foundation failed to attenuate the incident waves whose frequencies are inside the AZs but outside the BGs. In addition, the attenuation effect of the optimally designed foundation is expected to be even better if the configuration can strictly follow the optimal design. Yet the performance for P-waves is relatively ordinary due to the higher threshold for FRFs, which is acceptable since the attenuation of P-waves is not the primary design objective.



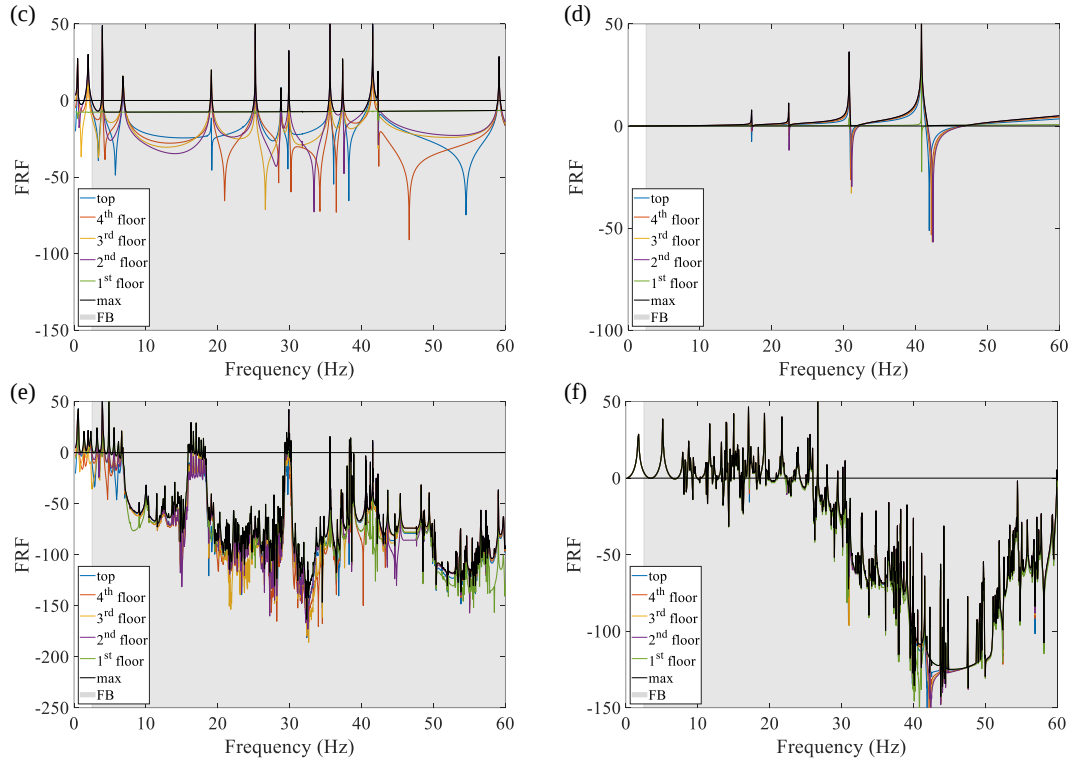


FIGURE 12 Frequency responses of the steel frame: (a) using the optimally designed foundation under S-waves, (b) using the optimally designed foundation under P-waves, (c) using the concrete foundation under S-waves, (d) using the concrete foundation under P-waves, (e) using the benchmark foundation under r S-waves, (f) using the benchmark foundation under P-waves.

5. CONCLUSION

This research studies frequency-related properties of 1-D periodic structures, from which the analytical approximations for the first few BGs, mapping relations among different band gaps' frequency points, and special approximations for the 1st BG of 1-D R-C periodic foundations are proposed. Based on the proposed analytical approximations and the mapping relations, an optimal design approach for 1-D R-C periodic foundations is proposed and validated. In addition, the following conclusions are drawn:

1. The first few BGs of 1-D R-C periodic foundations can be accurately approximated by the proposed analytical formulas in Section 3;
2. The frequency points of different BGs can be linked by the mapping relation in Section 3;
3. The design approach proposed in Section 4 is optimal in a global sense.
4. Distributing all the superstructure's RZs into the corresponding BGs can achieve satisfactory vibration attenuation results.

6. ACKNOWLEDGEMENT

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