Thermal Buckling Analysis of Laminated Composite Plates using Isogeometric Analysis

Presented in honor of Prof. Romesh C. Batra’s 70th Birthday

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Outline

- Isogeometric Analysis
- Bezier Extraction: Motivation
- Bezier Extraction: Theory
- Plate Problem
  - Numerical Investigation
- Analysis Procedure
- Results
Unifies the computer aided design (CAD) and the computer aided engineering (CAE) \(^1\).

Significant decrease in computational cost as the meshes are generated within the CAD.

Higher accuracy results because of the smoothness and the higher order continuity between elements.
B-spline basis functions are expressed by a set of non-decreasing values called the knot vector

$$\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$$  \hspace{1cm} (1)

B-spline basis functions are defined by the following recursive form

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \text{ for } p = 0 \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (2)

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \text{ for } p \geq 1$$  \hspace{1cm} (3)

where $\xi_1$ is the $i$th knot, $n$ is the number of basis functions and $p$ is the polynomial order.
IGA is implemented with the presence of $C^0$ continuous Bézier elements.

The Bezier extraction operator decomposes the Non Uniform Rational B-Splines (NURBS) into $C^0$-continuous Bezier elements using linear combinations of Bernstein polynomials.

Bezier elements bear a closer resemblance to the Lagrange elements used in finite element methods.

Allows easy incorporation into existing finite element codes without adding many changes as IGA.
Beziers Extraction: Theory

Bézier decomposition \(^2\) is accomplished by the insertion of new knots into the B-spline curve until \(k = p\). Where \(k\) is the multiplicity of the knot

\[
\bar{P}_i = \begin{cases} 
  P_1 & i = 1 \\
  a_i P_i + (1-a_i)P_{i-1} & 1 < i < n + 1 \\
  P_n & i = n + 1 
\end{cases}
\]

\[
a_i = \begin{cases} 
  1 & 1 \leq i \leq k-p \\
  \frac{\xi - \xi_i}{\xi_{i+1} - \xi_i} & k-p+1 \leq i \leq k \\
  0 & i \geq k + 1 
\end{cases}
\]

The new control points are expressed in matrix form as,

\[
\mathbf{P}^b = \mathbf{C}^T \mathbf{P}
\]

---

Knot Insertion: Visual Description

(a) (b)

(c) (d)

(e) (f)
The 2D NURBS basis functions which are written as

$$R_{i,j}(\xi,\eta) = \frac{N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}}{\sum_{i=1}^{n}\sum_{j=1}^{m}N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}}.$$  \hspace{1cm} (7)

are now written in terms of Bernstein polynomials as

$$R(\xi,\eta) = \frac{1}{W(\xi,\eta)}WN(\xi,\eta) = \frac{1}{W(\xi,\eta)}WCB(\xi,\eta).$$ \hspace{1cm} (8)

The new control points are expressed as,

$$P^b = (W^b)^{-1}C^TWP,$$ \hspace{1cm} (9)

Where $W^b$ and $W$ are derived from the weights of the control points.
Three-dimensional displacement field \((u, v, w)\) of a four-layered unit square are expressed in terms of five unknown variables as:

\[
\begin{align*}
    u(x, y, z) &= u_0(x, y) + z\varphi_x(x, y) \\
    v(x, y, z) &= v_0(x, y) + z\varphi_y(x, y) \\
    w(x, y, z) &= w_0(x, y)
\end{align*}
\]

where, \(\frac{-h}{2} \leq z \leq \frac{h}{2}\)
Material Property

\[ \frac{E_L}{E_T} = 15, \frac{G}{E_T} = 0.5, \frac{G_{TT}}{E_T} = 0.3356, \]
\[ \nu_{LT} = 0.3, \nu_{TT} = 0.49, \frac{\alpha_L}{\alpha_0} = 0.015, \frac{\alpha_T}{\alpha_0} = 1 \] (10)

Boundary Condition

**Clamped:**
\[ u_0 = v_0 = w_0 = \varphi_y = \varphi_x = 0 \text{ on } x = 0, L \]
\[ u_0 = v_0 = w_0 = \varphi_x = \varphi_y = 0 \text{ on } y = 0, W \] (11)

Loads: A unit temperature rise of 1 unit is applied on all the control points.
For the symmetric laminated plate subjected to a uniform temperature rise, only membrane forces are generated.

- ABD matrix is calculated using the material properties.
- A linear static analysis is then solved to determine the thermal in-plane resultants.
- The stress resultants are then used to compute the initial stress stiffness matrix.
- The stiffness matrix and the geometric stiffness matrix are used to solve the eigenvalue problem.
Results

The critical buckling temperature is normalized as \( T_{cr} = a_0 T_{cr} \)

<table>
<thead>
<tr>
<th>L/h</th>
<th>FSDT(^3)</th>
<th>HSDT(^3)</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.3348</td>
<td>0.3352</td>
<td>0.3349</td>
</tr>
<tr>
<td>10</td>
<td>0.1655</td>
<td>0.1601</td>
<td>0.1656</td>
</tr>
</tbody>
</table>

Table – critical buckling temperature of a symmetric four-layer [0/90/90/0] laminated plate. 10x10 mesh and \( p = 2 \)

Thermal buckling mode shapes for various L/h ratios are presented.

Plate with a Central Circular Cutout

Figure – $[0^\circ/90^\circ/90^\circ/0^\circ]$  
Figure – $[45^\circ/-45^\circ/-45^\circ/45^\circ]$  
Figure – $[90^\circ/0^\circ/0^\circ/90^\circ]$  
Figure – $[60^\circ/-60^\circ/-60^\circ/60^\circ]$
Numerical Investigation: Hybrid Composites

Material Property - I

\[
\frac{E_L}{E_T} = 15, \frac{G_{LT}}{E_T} = 0, 5, \frac{G_{TT}}{E_T} = 0.3356, \\
\nu_{LT} = 0.3, \nu_{TT} = 0.49, \alpha_L/\alpha_0 = 0.015, \alpha_T/\alpha_0 = 1
\]

Material Property - II (Isotropic)

\[
\frac{E_L}{E_T} = 1, \frac{G_{LT}}{E_T} = 0.3846, \frac{G_{TT}}{E_T} = 0.3846, \\
\nu_{LT} = 0.3, \nu_{TT} = 0.3, \alpha_L/\alpha_0 = 1, \alpha_T/\alpha_0 = 1
\]

Boundary Condition

\[
\text{Clamped:} \quad u_0 = v_0 = w_0 = \phi_y = \phi_x = 0 \quad \text{on} \quad x = 0, L \\
\quad u_0 = v_0 = w_0 = \phi_x = \phi_y = 0 \quad \text{on} \quad y = 0, W
\]

0/90/90/0 Laminate where the 1st and the 3rd layer have Material Property -I and the 2nd and 4th layer have Material Property -II
Results: Hybrid Composites

The critical buckling temperature is normalized as

\[ T_{cr} = 1000 \times \alpha_0 \times T_{cr} \]

<table>
<thead>
<tr>
<th>Mode</th>
<th>IGA</th>
<th>abaqus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.241</td>
<td>0.245</td>
</tr>
<tr>
<td>2</td>
<td>0.310</td>
<td>0.308</td>
</tr>
<tr>
<td>3</td>
<td>0.390</td>
<td>0.387</td>
</tr>
<tr>
<td>4</td>
<td>0.423</td>
<td>0.425</td>
</tr>
<tr>
<td>5</td>
<td>0.485</td>
<td>0.481</td>
</tr>
</tbody>
</table>

Table – critical buckling temperature of a symmetric four-layer [0/90/90/0] laminated plate. Radius/Width=0.15
Acknowledgement

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References


