SIMILARITIES BETWEEN INFORMATION GEOMETRY AND EUCLIDEAN GEOMETRY

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Abstract. In information geometry, some divergences are introduced as an amount which measures the discrepancy between two probability distributions. In a dually flat space, a canonical divergence is naturally introduced and the canonical divergence satisfies a relational expression called triangular relation. This can be regarded as a generalization of the law of cosines in Euclidean space. We introduce an amount corresponding to the squared distance in a dually flat space and show that more relational equations and theorems similar to Euclidean space hold in addition to the law of cosines.

Keywords: information geometry, parallelogram law, polarization identity, divergence.

1. Introduction

Given a probability distribution $p_\theta(x)$, we consider a manifold M with a parameter vector $\boldsymbol{\theta} = (\theta^1, \theta^2, \cdots, \theta^n) \in \mathbb{R}^n$ as a coordinate system. Manifolds with probability distributions as elements are called statistical manifolds, and dually flat space is an important concept of it [1]. In a dually flat space, it is possible to introduce a dual coordinate system $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_n)$. For example, it is well known that the manifold of the exponential family $p_\theta(x) = \exp(C(x) + \sum_{i=1}^n \theta^i F_i(x) - \psi(\theta))$ is a dually flat space for $\boldsymbol{\theta}$ and $\boldsymbol{\eta} = E[\mathbf{F}(x)]$, where x is a random variable, $F_i(x), C(x)$ and $\psi(\theta)$ are known functions and $E[\cdot]$ indicates expected value .

Given two probability distributions, the Kullback-Leibler divergence(KL-divergence) has long been known as an amount which measures the discrepancy. In a dually flat space, we may define an amount called a canonical divergence, and in exponential families the canonical divergence is consistent with the KL-divergence [1–4]. The canonical divergence satisfies relational expressions called triangular relations for three points P, Q and $R \in M$.This can be regarded as a generalization of the law of cosines in Euclidean space. When a curve PQ and a curve QR are "orthogonal" , the triangular relation becomes the same expression as the Pythagorean theorem [1]. This generalized Pythagorean theorem is an important role in the projection theorem.

In this paper, we study the properties similar to Euclidean space in dually flat spaces.We begin with an inner product of θ-coordinate and its dual η-coordinate .We show this inner product can be expressed by the canonical divergence and has the properties of divergence. We call this amount "affine divergence ", and study its properties. When a dually flat space is self-dual, the affine divergence is consistent with the squared Euclidean distance. The affine divergence satisfies triangular relational expression as well as the canonical divergence.

Then we study the behavior of the affine divergence on θ and η-geodesics and show that the affine divergence satisfies the same relational expression as the squared Euclidean distance.

We further show that the generalized parallelogram law and the generalized polarization identity hold like Euclidean space for the sum of vectors in θ and η-coordinate systems.

2. Canonical divergence

Let M be a dually flat space, we may define dual affine connections ∇, ∇^* and a Riemannian metric g_{ij}. There exist dual affine coordinate systems θ, η and dual convex functions $\psi(\theta), \phi(\eta)$ on M. An affine coordinate system θ corresponds to the connection ∇, and η corresponds to the connection ∇^*. $\psi(\theta)$ and $\phi(\eta)$ are in a relationship of Legendre transformation with each other.

$$\phi(\eta) = \theta^i \eta_i - \psi(\theta), \tag{1}$$

where we use Einstein notation for $i = (1, 2, \cdots, n)$. We denote $\frac{\partial}{\partial \theta^i}$ as ∂_i and $\frac{\partial}{\partial \eta_i}$ as ∂^i. Then, equations

$$\theta^i = \partial^i \phi \tag{2}$$

$$\eta_i = \partial_i \psi \tag{3}$$

hold. A relationships between the Riemannian metric and affine coordinates are

$$g_{ij} = \partial_i \eta_j \tag{4}$$

$$g^{ij} = \partial^i \theta^j. \tag{5}$$

For two points $P, Q \in M$, we may define a divergence $D(P\|Q)$ called the canonical divergence as follows [1,2].

$$D(P||Q) = \psi(\theta(P)) + \phi(\eta(Q)) - \theta^i(P)\eta_i(Q), \tag{6}$$

The canonical divergence is not symmetric with respect to P and Q.

We denote $\psi(\theta(P))$ as $\psi(P)$ and $\phi(\eta(P))$ as $\phi(P)$. For the exponential family $p_\theta(x) = \exp(C(x) + \theta^i F_i(x) - \psi(\theta))$, The Riemannian metric can be expressed as

$$g_{ij} = E[\partial_i l_\theta \partial_j l_\theta] \tag{7}$$

$$l_\theta = \log p_\theta(x). \tag{8}$$

The right hand side of (7) is Fisher information matrix. The canonical divergence is consistent with the KL-divergence. The KL-divergence is

$$D_{KL}(P\|Q) = \sum_i p_{\theta(P)}(x_i) \log \frac{p_{\theta(Q)}(x_i)}{p_{\theta(P)}(x_i)} \tag{9}$$

for discrete distribution, and

$$D_{KL}(P\|Q) = \int p_{\theta(P)}(x) \log \frac{p_{\theta(Q)}(x)}{p_{\theta(P)}(x)} dx \tag{10}$$

for continuous distribution. The canonical divergence have distance-like properties(Property 1. and 2.), and the properties similar to Euclidean space(Property 3. and 4.).

Property 1. $D(P\|Q) \geq 0$
Property 2. $D(P\|Q) = 0 \iff P = Q$
Property 3. Triangular relation

$$D(P\|Q) + D(Q\|R) - D(P\|R) = (\eta_i(R) - \eta_i(Q))(\theta^i(P) - \theta^i(Q)) \tag{11}$$

This formula plays important roles in this paper. When the dual geodesic connecting P and Q is orthogonal at Q to the dual geodesic connecting Q and R, the generalized Pythagorean Theorem holds.

$$D(P\|Q) + D(Q\|R) = D(P\|R) \tag{12}$$

Because coordinates $\boldsymbol{\theta} \equiv \{\theta^i\}$ and $\boldsymbol{\eta} \equiv \{\eta_i\}$ are affine, a curve represented in the form

$$\theta^i(Q(t)) = a^i t + \theta^i(P) \tag{13}$$

is a geodesic for ∇-connection and

$$\eta_i(Q(t)) = a_i t + \eta_i(P) \tag{14}$$

is a geodesic for ∇^*-connection, where $\{a^i\}$ and $\{a_i\}$ are constant vectors in $\mathbb{R}^n$, $t \in \mathbb{R}$ is a parameter along the geodesic.
Propertiy 4. For the point $Q(t)$ on θ-geodesic, $\theta^i(Q(t)) = a^i t + \theta^i(P)$, the following equation holds.

$$D(P||Q(T)) = a^i a^j \int_0^T t g_{ij}(\theta(Q(t)))dt \tag{15}$$

3. Affine divergence

3.1. **Definition of the affine divergence.** In this section we introduce new divergence called "affine divergence" as an inner product of dual affine coordinates.

First, we show the affine divergence satisfies three distance axioms except for the triangle inequality, and the affine divergence satisfies the properties similar to Euclidean space on geodesics.

Second, we prove the generalized parallelogram law and the generalized polarization identity.

Definition 1. A **semimetric** on M is a function $d : X \times X \to \mathbb{R}$ that satisfies

$$d(P, Q) \geq 0 \tag{16}$$

$$d(P, Q) = 0 \iff P = Q \tag{17}$$

$$d(P, Q) = d(Q, P). \tag{18}$$

Definition 2. We define the **affine divergence** $D_A : M \times M \to \mathbb{R}$ as follows.

$$D_A(P, Q) = (\eta_i(Q) - \eta_i(P))(\theta^i(Q) - \theta^i(P)) \tag{19}$$

The affine divergence is an inner product of dual affine coordinates. In a self-dual space($\theta^i = \eta_i$ for all i), the affine divergence is consistent with the squared Euclidean distance $\sum_i(\theta^i)^2$.

Proposition 1. The affine divergence can be expressed as the sum of the canonical divergence.

$$D_A(P, Q) = D(P\|Q) + D(Q\|P) \tag{20}$$

Proof. The result follows by substituting $P = R$ in the triangular relation(11) and using (19).

For the exponential family $p_\theta(x) = \exp(C(x) + \theta^i F_i(x) - \psi(\theta))$, the affine divergence can be expressed as follows.

$$D_A(P, Q) = D_{KL}(P\|Q) + D_{KL}(Q\|P) \tag{21}$$

The right hand side of this equation is the Jeffreys divergence [5, 6].

Proposition 2. The affine divergence satisfies semimetric axioms.
Proof. By using Property1., Property2. of the canonical divergence and the definition of the affine divergence(20), equation (16) and (17) follow. Equation (18) is trivial by (20).

3.2. **Properties along geodesics.** We show that the canonical divergence and the affine divergence monotonically increase along θ and η-geodesics. For three points $P, Q, R \in M$ on the same geodesic, we show that the same relational expression as Euclidean space holds.

Proposition 3. Let $\{a^i\}$ and $\{a_i\}$ be constant vectors in $\mathbb{R}^n$ and t be $t \in \mathbb{R}$. When points $P, Q \in M$ are on θ-geodesic

$$\theta^i(Q(t)) = a^i t + \theta^i(P), \tag{22}$$

the affine divergence can be expressed as

$$D_A(P, Q(T)) = T a^i a^j \int_0^T g_{ij}(\theta(t))dt. \tag{23}$$

When $P, Q \in M$ are on η-geodesic

$$\eta_i(Q(t)) = a_i t + \eta_i(P), \tag{24}$$

the affine divergence can be expressed as

$$D_A(P, Q(T)) = T a_i a_j \int_0^T g^{ij}(\eta(t))dt. \tag{25}$$

Because g_{ij} and g^{ij} are positive definite, the affine divergence monotonically increases with T.

Proof. We prove for θ.The same is true of η.
By combining

$$\frac{d\eta_i(\theta(t))}{dt} = a^j \partial_j \eta_i(\theta(t)) \tag{26}$$

and (4) yields

$$\eta_i(Q(T)) - \eta_i(P) = a^j \int_0^T g_{ij}(\theta(t))dt. \tag{27}$$

By substituting (27) to (19), we have the result. Equation (25) corresponds to Property 4. of the canonical divergence.

Corollary 1. Let $\{a_i\}$ be a constant vector in $\mathbb{R}^n$ and t be $t \in \mathbb{R}$. When points $P, Q \in M$ are on η-geodesic

$$\eta_i(Q(t)) = a_i t + \eta_i(P), \tag{28}$$

the canonical divergence can be expressed

$$D(P \| Q(T)) = a_i a_j \int_0^T dt \int_0^t dt' g^{ij}(\eta(t')). \tag{29}$$

Because g_{ij} is positive definite, the canonical divergence $D(P\|Q(T))$ monotonically increases with T.

Proof. Denoting ∂_Q^i as $\frac{\partial}{\partial \eta_i(Q)}$ and combining the canonical divergence definition(6) and (2) yields

$$\partial_Q^i D(P\|Q(t)) = \theta(Q(t))^i - \theta(P)^i. \tag{30}$$

Hence, we have

$$\frac{dD(P\|Q(t))}{dt} = \frac{1}{t}(\eta(Q(t))_i - \eta(P)_i)(\theta(Q(t))^i - \theta(P)^i) = \frac{1}{t} D_A(P, Q(t)). \tag{31}$$

By substituting (25) to this equation and integrating with respect to t, we have the result.

Lemma 1. Let t be $t \in \mathbb{R}$ and P, R be points on a dually flat space M. For a point $\boldsymbol{\theta}(Q) = (1-t)\boldsymbol{\theta}(P) + t\boldsymbol{\theta}(R)$ on θ-geodesic,

$$D(P\|R) = D(P\|Q) + D(Q\|R) + \frac{t}{1-t} D_A(R, Q) \tag{32}$$

holds. For a point $\boldsymbol{\eta}(Q) = (1-t)\boldsymbol{\eta}(P) + t\boldsymbol{\eta}(R)$ on η-geodesic,

$$D(P\|R) = D(P\|Q) + D(Q\|R) + \frac{1-t}{t} D_A(P, Q) \tag{33}$$

holds.

Proof. We prove for θ-geodesic.The same is true of η-geodesic.
By triangular relation (11), we have

$$D(P\|R) = D(P\|Q) + D(Q\|R) + (\eta_i(R) - \eta_i(Q))(\theta^i(Q) - \theta^i(P)). \tag{34}$$

By assumption,

$$\theta^i(Q) - \theta^i(P) = \frac{t}{1-t}(\theta^i(R) - \theta^i(Q)) \tag{35}$$

holds. By substituting (35) to (34), the result follows.
When $t \in (0, 1)$, taking into account $D_A(R, Q) \geq 0$, we have

$$D(P\|R) \geq D(P\|Q) + D(Q\|R). \tag{36}$$

Corollary 2. Let t be $t \in \mathbb{R}$ and P, R be points on a dually flat space M. For a point $\boldsymbol{\theta}(Q) = (1-t)\boldsymbol{\theta}(P) + t\boldsymbol{\theta}(R)$ on θ-geodesic,

$$D(R\|P) = D(Q\|P) + D(R\|Q) + \frac{1-t}{t} D_A(Q, P) \tag{37}$$

holds.
For a point $\boldsymbol{\eta}(Q) = (1-t)\boldsymbol{\eta}(P) + t\boldsymbol{\eta}(R)$ on η-geodesic,

$$D(R\|P) = D(Q\|P) + D(R\|Q) + \frac{t}{1-t} D_A(R, Q) \tag{38}$$

holds.

Proof. We exchange P and R in (34), we can prove corollary2. in the same way as Lemma1.

Theorem 1. Let t be $t \in \mathbb{R}$ and P, R be points on a dually flat space M. For a point $\boldsymbol{\theta}(Q) = (1-t)\boldsymbol{\theta}(P) + t\boldsymbol{\theta}(R)$ on θ-geodesic or $\boldsymbol{\eta}(Q) = (1-t)\boldsymbol{\eta}(P) + t\boldsymbol{\eta}(R)$ on η-geodesic,

$$D_A(P, R) = \frac{1}{t} D_A(P, Q) + \frac{1}{1-t} D_A(Q, R) \tag{39}$$

holds.

Proof. Taking the sum of (32) and (37), the result follows.
Theorem1. holds for the points P, Q, R on the same straight line and the squared Euclidean distance.

3.3. **Generalized expansion formula for the sum of vectors, parallelogram law, polarization identity.** For the affine divergence, we show that the generalized law of cosines holds as well as the canonical divergence. We consider about the sum of vectors $\boldsymbol{\theta}$ or $\boldsymbol{\eta}$, we show that the generalized expansion formula for the sum of vectors, the parallelogram law and the polarization identity hold for the affine divergence.

Definition 3. We define $\langle , \rangle : M \times M \to \mathbb{R}$ as

$$\langle Q, R\rangle_P \equiv \frac{1}{2}(\theta^i(Q) - \theta^i(P))(\eta_i(R) - \eta_i(P)) + \{Q \leftrightarrow R\}. \tag{40}$$

Symbol $\{Q \leftrightarrow R\}$ means replacement of R and Q.

When $Q = R$,

$$\langle Q, Q\rangle_P = D_A(P, Q) \tag{41}$$

holds.For a self-dual space($\theta^i = \eta_i$ for all i), this is consistent with a dot product.

Corollary 3. (Generalized law of cosines)
For points $P, Q, R \in M$,

$$D_A(P, Q) + D_A(Q, R) - 2\langle P, R\rangle_Q = D_A(P, R) \tag{42}$$

holds.

Proof. By exchanging P and R in triangular relation(11) and taking the sum with original triangular relation (11), we show this corollary. Equation (42) is the generalized law of cosines.

Theorem 2. (Generalized expansion formula for the sum of vectors)
When points $P, Q, R, S \in M$ satisfy $\boldsymbol{\theta}(P)+\boldsymbol{\theta}(R) = \boldsymbol{\theta}(Q)+\boldsymbol{\theta}(S)$ or $\boldsymbol{\eta}(P)+\boldsymbol{\eta}(R) = \boldsymbol{\eta}(Q)+\boldsymbol{\eta}(S)$, the following equations hold.

$$D_A(P, R) = D_A(P, Q) + D_A(P, S) + 2\langle Q, S\rangle_R \tag{43}$$

$$D_A(P, R) = D_A(R, Q) + D_A(R, S) + 2\langle Q, S\rangle_P \tag{44}$$

Proof. We prove for $\boldsymbol{\theta}(P)+\boldsymbol{\theta}(R) = \boldsymbol{\theta}(Q)+\boldsymbol{\theta}(S)$. The same is true of $\boldsymbol{\eta}(P)+\boldsymbol{\eta}(R) = \boldsymbol{\eta}(Q)+\boldsymbol{\eta}(S)$. By the definition of the affine divergence

$$D_A(P, R) = (\eta_i(R) - \eta_i(P))(\theta^i(R) - \theta^i(P)) \tag{45}$$

and the assumption $\boldsymbol{\theta}(P) + \boldsymbol{\theta}(R) = \boldsymbol{\theta}(Q) + \boldsymbol{\theta}(S)$, $D_A(P, R)$ can be expressed as

$$D_A(P, R) = (\eta_i(R) - \eta_i(P))(\theta^i(Q) - \theta^i(P) + \theta^i(S) - \theta^i(P)). \tag{46}$$

Furthermore, the following relational equation holds.

$$\begin{aligned}(\eta_i(R) - \eta_i(P))(\theta^i(Q) - \theta^i(P)) &= (\eta_i(R) - \eta_i(Q))(\theta^i(Q) - \theta^i(P)) + D_A(P, Q)\\ &= (\eta_i(R) - \eta_i(Q))(\theta^i(R) - \theta^i(S)) + D_A(P, Q)\end{aligned} \tag{47}$$

We now use the assumption $\boldsymbol{\theta}(P) + \boldsymbol{\theta}(R) = \boldsymbol{\theta}(Q) + \boldsymbol{\theta}(S)$ again. In the same way, we obtain the following relational equation.

$$\begin{aligned}(\eta_i(R) - \eta_i(P))(\theta^i(S) - \theta^i(P)) &= (\eta_i(R) - \eta_i(S))(\theta^i(S) - \theta^i(P)) + D_A(P, S)\\ &= (\eta_i(R) - \eta_i(Q))(\theta^i(R) - \theta^i(Q)) + D_A(P, S)\end{aligned} \tag{48}$$

Substituting (47) and (48) to (46), and using (40), we prove that (43) holds. Because the assumption is symmetric with respect to P and R, we exchange P and R in (43) and we prove that (44) holds.
Theorem 2. is a generalization of the expansion formula for the squared norm of the sum of vectors.

$$\|(y - x) + (z - x)\|^2 = \|y - x\|^2 + \|z - x\|^2 + 2(y - x, z - x), \tag{49}$$

where $(,)$ is a dot product and $\|\cdot\|$ is a Euclidean norm.

Theorem 3. (Generalized parallelogram law)
When points $P, Q, R, S \in M$ satisfy $\boldsymbol{\theta}(P)+\boldsymbol{\theta}(R) = \boldsymbol{\theta}(Q)+\boldsymbol{\theta}(S)$ or $\boldsymbol{\eta}(P)+\boldsymbol{\eta}(R) = \boldsymbol{\eta}(Q)+\boldsymbol{\eta}(S)$, the following equations holds.

$$D_A(P, Q) + D_A(Q, R) + D_A(R, S) + D_A(S, Q) = D_A(P, R) + D_A(Q, S) \tag{50}$$

The left hand side is the sum of four sides of rectangle $PQRS$ and the right hand side is the sum of diagonal lines of rectangle $PQRS$.
Proof. Applying Corollary3. for points Q, R and S, we have

$$2\langle Q, S\rangle_R = D_A(R, Q) + D_A(R, S) - D_A(Q, S). \tag{51}$$

Substituting (51) to (43), we have the result.
Theorem 3. is a generalization of parallelogram law,

$$2(\|y-x\|^2+\|z-x\|^2)=\|(y-x)+(z-x)\|^2+\|(z-x)-(y-x)\|^2. \tag{52}$$

Corollary 4. (Generalized polarization identity)
When points $P,Q,R,S\in M$ satisfy $\boldsymbol{\theta}(P)+\boldsymbol{\theta}(R)=\boldsymbol{\theta}(Q)+\boldsymbol{\theta}(S)$ or $\boldsymbol{\eta}(P)+\boldsymbol{\eta}(R)=\boldsymbol{\eta}(Q)+\boldsymbol{\eta}(S)$, the following equations holds.

$$2(\langle Q,S\rangle_P+\langle Q,S\rangle_R)=D_A(P,R)-D_A(Q,S). \tag{53}$$

Proof.
By using (44)and (51), the result follows.
Corollary 4 is a generalization of polarization identity,

$$4(y-x,z-x)=\|(y-x)+(z-x)\|^2-\|(z-x)-(y-x)\|^2. \tag{54}$$

Corollary 5. When points $P,Q,R,S\in M$ satisfy $\boldsymbol{\theta}(P)+\boldsymbol{\theta}(R)=\boldsymbol{\theta}(Q)+\boldsymbol{\theta}(S)$ or $\boldsymbol{\eta}(P)+\boldsymbol{\eta}(R)=\boldsymbol{\eta}(Q)+\boldsymbol{\eta}(S)$, the following equations holds.

$$\langle Q,S\rangle_P+\langle Q,S\rangle_R+\langle P,R\rangle_Q+\langle P,R\rangle_S=0 \tag{55}$$

Proof. Exchanging $(P,R)\leftrightarrow(Q,S)$ in (53) and taking the sum with (53), we have the result.
Corollary 5. is a generalization that the sum of adjacent interior angles of a parallelogram in Euclidean space is π.

4. Examples

In this section, we show concrete examples of the affine divergence.

4.1. **Normal distribution.** As representative example of continuous distribution, we think 1-dimentional normal distribution.

$$p_\theta(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-\mu)^2}{2\sigma^2}) \tag{56}$$

The relations between θ, η and σ,μ are

$$\begin{aligned}\theta^1&=-\frac{1}{2\sigma^2}\\ \theta^2&=\frac{\mu}{\sigma^2}\end{aligned} \tag{57}$$

$$\begin{aligned}\eta_1&=\sigma^2+\mu^2\\ \eta_2&=\mu.\end{aligned} \tag{58}$$

We calculate the affine divergence of normal distribution as follows.

$$D_A(P,Q)=(\eta_i(Q)-\eta_i(P))(\theta^i(Q)-\theta^i(P))=\frac{({\sigma_Q}^2-{\sigma_P}^2)^2}{2{\sigma_P}^2{\sigma_Q}^2}+(\mu_Q-\mu_P)^2(\frac{1}{2{\sigma_P}^2}+\frac{1}{2{\sigma_Q}^2}) \tag{59}$$

For points $P,Q,R\in M$ which satisfies $\boldsymbol{\theta}(R)=a\boldsymbol{\theta}(P)+b\boldsymbol{\theta}(Q)$, an expected value and a variance are

$$\frac{1}{{\sigma_R}^2}=\frac{a}{{\sigma_P}^2}+\frac{b}{{\sigma_Q}^2}. \tag{60}$$

$$\mu_R=\frac{a{\sigma_Q}^2\mu_P+b{\sigma_P}^2\mu_Q}{a{\sigma_Q}^2+b{\sigma_P}^2}. \tag{61}$$

Equation(60) means weighted harmonic mean and μ_R is a point to internally divide the straight line $\overline{\mu_P\mu_Q}$ into $b{\sigma_P}^2 : a{\sigma_Q}^2$.

For the point $P, Q, R \in M$ which satisfies $\boldsymbol{\eta}(R) = a\boldsymbol{\eta}(P) + b\boldsymbol{\eta}(Q)$, an expected value and a variance are

$$\sigma_R{}^2 = a\sigma_P{}^2 + b\sigma_Q{}^2 + a(1-a)\mu_P^2 + b(1-b)\mu_Q^2 - 2ab\mu_P\mu_Q \tag{62}$$

$$\mu_R = a\mu_P + b\mu_Q. \tag{63}$$

For these quantities, if $a = t, b = 1-t, t \in \mathbb{R}$, Theorem 1. hods. In the case of $a = b = 1$, Theorem 2., the generalized parallelogram law(Theorem 3.) and the generalized polarization identity(Corollary 4.) hold.

4.2. **Binomal distribution.** As representative example of discrete distribution, we think 1-dimentional binomal distribution.

$$p_\theta(k) = \binom{n}{k} p^k (1-p)^{(n-k)}, \tag{64}$$

where n is constant and $p \in [0, 1]$. The relations between θ, η and p are

$$\theta = \ln \frac{p}{1-p} \tag{65}$$
$$\eta = np.$$

We calculate the affine divergence of binomal distribution as follows.

$$D_A(P, Q) = (\eta(Q) - \eta(P))(\theta(Q) - \theta(P)) = n(p_Q - p_P) \ln \frac{p_Q(1-p_P)}{p_P(1-p_Q)}. \tag{66}$$

For the point $P, Q, R \in M$ which satisfies $\boldsymbol{\theta}(R) = a\boldsymbol{\theta}(P) + b\boldsymbol{\theta}(Q)$, the parameter p is

$$p_R = \frac{\exp(\lambda)}{1 + \exp(\lambda)} \tag{67}$$
$$\lambda = \frac{p_P^a p_Q^b}{(1-p_P)^a (1-p_Q)^b}.$$

For the point $P, Q, R \in M$ which satisfies $\boldsymbol{\eta}(R) = a\boldsymbol{\eta}(P) + b\boldsymbol{\eta}(Q)$, the parameter p is

$$p_R = ap_P + bp_Q. \tag{68}$$

The above operation is defined only for a, b which satisfies $p_R \in [0, 1]$. For these quantities, if $a = t, b = 1-t, t \in \mathbb{R}$, Theorem 1. hods. In the case of $a = b = 1$, Theorem 2., the generalized parallelogram law(Theorem 3.) and the generalized polarization identity(Corollary4.) hold.

5. Conclusion

We have introduced the affine divergence and studied the properties similar to Euclidean space in a dually flat space. We have shown the affine divergence satisfies semimetric axioms and triangular relations as well as the canonical divergence. We also have shown that both the canonical divergence and the affine divergence increase monotonically along geodesics of affine coordinates. As a property peculiar to the affine divergence on geodesics, we have shown that the same expressions as the squared Euclidean distance hold for three points on the same geodesic.

In addition, for the sum of vectors in affine coordinate systems, we have shown that the generalized expansion formula, the generalized parallelogram law and the generalized polarization identity hold.

It is expected that dually flat spaces further have structures similar to Euclidean space.

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