A graduate student's perspectives on the article "Nonlocal elasticity and related variational principles" by Castrenze Polizzotto

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Abstract - This is a short note intended to highlight the contribution of the seminal article "Nonlocal elasticity and related variational principles" by Prof. Dr. Castrenze Polizzotto that appeared in the Volume 38 of International Journal of Solids and Structures, to the field of nonlocal elasticity. The considered article was instrumental in shaping the present author's concepts, understanding, and approach to integral nonlocal elasticity; henceforth, simply referred to as nonlocal elasticity.

Keywords - Nonlocal elasticity | Variational principles | Convex models | Nonlocal FEM | Castrenze Polizzotto

The present author was exposed to nonlocal elasticity following the recommendation of his doctoral advisor Prof. Dr. Fabio Semperlotti in 2018; possibly a prime time in regards to the growing acknowledgment and understanding that the notion of 'nonlocality' transcends spatio-temporal scales in both nonliving and living matter [1–3]. Nonetheless, nonlocal elasticity is a field where the reader is more likely to encounter an erroneous application; see, for example, comments in [4]. To this end, the very well-articulated article from Prof. Polizzotto [5] serves as a guiding light, and a scripture in its own right. While the concept of nonlocal elasticity can be traced back to works from Kröner [6] and Eringen [7], the considered article [5] espoused and formulated theoretical mechanisms for casting the concept in a variational framework. This had major practical implications which also inspired the present author and defined his approach to nonlocality. While the present author had to make several (using the wise words of his advisor) 'nonlinear' passes to gain clarity and some grasp of the concept of nonlocal elasticity, in retrospect, they appear very clearly formulated and organized in [5].

In the following, for the benefit of young and fresh graduate students who intend to explore the field of nonlocal mechanics, the present author, as he graduates, attempts to highlight the most important concepts that he crystallized from [5]:

- *Analysis of convexity*. The study [5] formulated a mechanism wherein the convexity of the strain energy, obtained following the Eringen's nonlocal model, could be achieved; see specifically the eigenvalue integral equation corresponding to the attenuation function in Eq. (11) of [5] and the analysis thereafter. From a broad perspective, [5] provided a foundation for the detection and analysis of inconsistencies (also called 'paradoxes') that accompanied the use of (an adapted differential version of) Eringen's nonlocal stress-stress constitutive relationship in (non-convex) modeling of nonlocal beams and other slender structural elements.

- *Analysis of thermodynamic consistency*. The implicit multiscale nature of integral nonlocal theories, wherein the effect of the microstructural is accounted in the continuum level response via nonlocal constitutive parameters, requires a a careful consideration of thermodynamic balance

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principles across the scales so that the laws of thermodynamics can be enforced in a strong form (as they should be); see also [8] from Prof. Polizzotto. This was first pointed out in [5] where thermodynamic consistency was envisaged as a firm basis for the construction of nonlocality.

- *Formulation of finite element principles*. The variational principles and theoretical basis for nonlocal finite element modeling formulated in [5] served as a basis for the finite element simulation of integral elasticity in [9] (possibly for the first time) and then in [10] (possibly for the second time; first when considering displacement-driven fractional calculus-based modeling with singular kernels). To this end, the author is happy to note that all these studies appeared in the same journal, International Journal of Solids and Structures; indeed an interesting coincidence.

Notably, all the above concepts were consistently realized in the 'displacement-driven approach to nonlocal elasticity' developed by the present author along with his co-workers, Dr. Sai Sidhardh and Prof. Dr. Semperlotti [11]. This later reformulation, when using power-law kernels available from fractional calculus, also enabled the application of nonlocal elasticity to the modeling of porous solids [12]. In the retrospect, it also appears that the application to porous solids is closely related to an advanced concept envisioned in the considered study [5], discussed here below.

The considered study [5] also put forth the concept of the 'geodetical distance', rather than the Euclidean distance, in being a key parameter in determining the diffusion of nonlocal effects via the nonlocal attenuation kernel. This reformulation was aimed to refine Eringen's model in [7] for application to nonlocal solids with discontinuities such as holes and cracks. To this end, [5] wondered the extent to which this concept would be correct and practically realizable, and indicated that the concept is a matter of future research. From a broad perspective, this concept is analogous to the recent development in [12] where, fractional-calculus based displacement-driven nonlocal elasticity was utilized to analyze the role of microstructure in the deformation of porous solids, and to accurately and efficiently model the linear and nonlinear deformation behaviour of porous solids. More specifically, analogous to the concept of geodetical distance which ultimately would lead to a heterogeneous attenuation kernel, variable-order fractional calculus was utilized in [12] to model position-dependent nonlocal effects in porous solids very naturally.

The present author continues to be fascinated by [5] and on several occasions refers back to the same article for further insights on nonlocal elasticity, and to even generate ideas for future research. To this end, very recently the author had the opportunity to come across on ResearchGate, a fairly recent article from Prof. Polizzotto (and his coworkers) [13], where they highlighted limitations of classical nonlocal elasticity theories succinctly. This occasion allowed the author a very brief exchange with Prof. Polizzotto via comments, to the end of which, Prof. Polizzotto commented (and the author takes the liberty to quote here) "Thanks for your comment. I hope this article may be source of inspiration for fu(r)ther developments. Castrenze". The author cherishes the comment.

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Competing Interests

The author declares that there are no competing interests. Except for the very brief exchange highlighted in the main text, the present author has had no previous formal or private communication with the Prof. Castrenze Polizzotto, the author of the considered study [5].

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