QUATERNION DOMAIN DISCRETE FOURIER TRANSFORM OF THE PULSES VECTOR

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***Abstract*— At present, technologies of digital representation of signals are becoming more widespread in information transmission systems. The discrete Fourier transform (DFT) is used to obtain the spectrum of such signals. The DFT is represented as a finite sum of signal samples multiplied by a discrete complex exponent.**

Modern communication systems use vector signals and many-input – many-output technologies (MIMO). These technologies can significantly increase the capacity of communication channels. As a mathematical model of a MIMO channel, multidimensional orthogonal matrices are usually used. An information vector of pulses are suppled at the input of the channel matrix. The DFT of multidimensional signals is written as a multiple sum of samples for a separate variables, that multiplied by the corresponding discrete complex exponent. The DFT of multidimensional signals in matrix form is written as a sum in the sample area, and the vector of samples is multiplied by the matrix complex exponent from the inverse diagonal matrix of the number of samples. This representation uses the same imaginary unit and, in despite of the multidimensional nature of the signals, they are considered on the same complex plane.

The purpose of the work is to present a technique for the discrete Fourier transform of a four-dimensional pulse vector, the kernel of which is a quaternion in the matrix representation. This technique considers a 4-dimensional signal in 4D quaternion space with three imaginary units and one scalar in which the MIMO signal is formed.

The exponential function from the quaternion provides a conformal mapping of 4D hypercomplex space into 4D hypercomplex space with a constant rotation radius. The exponential function of a quaternion is represented as a fundamental matrix of a quaternion carrier with discrete counts for time and frequency. Using the discrete fundamental matrices of the quaternion carrier, formulas have been obtained for the direct and inverse discrete Fourier transform of the four-dimensional impulses vector.

Formulas have been obtained for calculating the spectra of the elements of the pulse vector with the same and different cyclic shifts in time and frequency. The Parseval's identity of the quaternion discrete Fourier transform has been proven.

The calculations of vector spectra of various pulses have been presented: rectangle, sawtooth, sine, cosine with different time shifts. The trajectory of movement of quaternion scalar values in time in 3D has been shown for various cyclic shifts and projections of trajectories onto 4 orthogonal planes.

***Index Terms*— Discrete Fourier transform, Hypercomplex signals, Multiple-Input Multiple-Output – MIMO, Quaternion.**

I. INTRODUCTION

The representation of signals by samples in discrete time allows simpler implementation of devices for their generation and processing using digital devices. Digital devices work with discrete amplitude values and are accordingly less susceptible to interference. Digital devices operate with discrete amplitude values and, accordingly, are less sensitive to interference. The discrete representation of signals allows implementing fast Fourier transform (FFT), as well as fast convolution and signal filters. Discrete representation of signals makes it possible to implement FFT, as well as fast convolution and filtering of signals.

To obtain the FFT, a complex representation of harmonic oscillations is used as the sum of the values of the real and imaginary parts on the complex plane. The complex recording of multidimensional signals makes it possible to represent the FFT as a multiplication of the vector of signal samples by an orthogonal matrix. It is possible to factorize these matrices and, accordingly, use several times repeating sample values. The discrete quaternion representation of signals also makes it possible to write multidimensional signals as a multiplication of the vector of signal samples by an orthogonal matrix and, using matrix factorization and repeatability of sample values, to implement fast algorithms for the quaternion discrete Fourier transform. Discrete hypercomplex representation of signals using an orthogonal quaternion matrix can be used in systems with many inputs - many outputs (Multiple-Input Multiple-Output - MIMO). When multiplying the vector of samples of the input signal by such a channel matrix, we get the output vector, in which each element is formed from combinations of elements of the vector. It is known that the complex representation of information vectors is used in M-ary Phase Shift Keying modulation methods and makes it possible to increase the capacity of communication channels through more efficient use of the entire complex plane. The hypercomplex representation of signals allows using the entire hypervolume and, accordingly, increasing the channel capacity more significantly [1].

In connection with the above, hypercomplex signals are of great interest for increasing the efficiency of devices for discrete processing of multidimensional signals, as well as for increasing the capacity and noise immunity of information transmission channels. At present, there is a relatively small number of works on the analysis of the properties of discrete hypercomplex signals. For example, in [2], a two-dimensional quaternion discrete Fourier transform on the complex plane is presented and its properties are obtained. This transformation is commonly used for image analysis. In [3], an FFT algorithm for a complex quaternion was obtained by decomposing it into 4 complex Fourier transforms.

In [4], the MIMO scheme was represented by a discrete state-space model. With a constant channel matrix, the Kalman filter is converted into a storage recirculator (digital integrator). In [5], the Kalman filter with a constant channel matrix in the MIMO scheme was transformed into an integrator. It is shown that the optimal reception in the MIMO scheme includes multiplication of the received vector by the transposed fundamental matrix with subsequent integration. The fundamental matrix is found by solving the equation of model dynamics in the state space.

Since in the MIMO scheme the signal is represented as a vector of pulses, which is multiplied by the channel matrix, the received vector is obtained by combining all elements of the input vector. In [6], a method for calculating the Fourier transform of the impulses vector was obtained using a quaternion in the matrix representation. In [7], a method for calculating the Fourier series of the impulses vector was developed. In [8], a method for calculating the Laplace transform of the impulses vector is presented.

The purpose of the work is to present a technique for the discrete Fourier transform of a four-dimensional impulses vector whose kernel is a quaternion in a matrix representation.

II. Materials and Methods for Solving the Problem, Assumptions Made

In the matrix representation, the quaternion $q = s + xi + yj + zk$ has the form [9], p. 28:

$$\mathbf{Q} = \begin{bmatrix} s & x & y & z \\ -x & s & -z & y \\ -y & z & s & -x \\ -z & -y & x & s \end{bmatrix}, \tag{1}$$

where $s, x, y, z \in \mathbb{R}$, *i*, *j* and *k* - are imaginary units.

Accordingly, the imaginary units *i*, *j* and *k* in matrix form are written as

$$\mathbf{I} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ \mathbf{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

The basis matrices (2) are orthogonal, since $\mathbf{I}^{\mathrm{T}} = \mathbf{I}^{-1} = -\mathbf{I}$, $\mathbf{J}^{\mathrm{T}} = \mathbf{J}^{-1} = -\mathbf{J}$, $\mathbf{K}^{\mathrm{T}} = \mathbf{K}^{-1} = -\mathbf{K}$.

The quaternion in the matrix representation (1) can be written in terms of the basis matrices in the form:

$$\mathbf{Q} = \mathbf{E}s + \mathbf{I}x + \mathbf{J}y + \mathbf{K}z \,, \tag{3}$$

where $\mathbf{E} = \mathrm{diag}[1 \quad 1 \quad 1 \quad 1]$ - diagonal matrix of 1.

Just like the imaginary units *i*, *j* and *k*, the corresponding matrices **I**, **J**, **K** satisfy the multiplication rules given in the following table:

$$\begin{array}{c|ccc} \times & \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \hline \mathbf{I} & -\mathbf{E} & \mathbf{K} & -\mathbf{J} \\ \mathbf{J} & -\mathbf{K} & -\mathbf{E} & \mathbf{I} \\ \mathbf{K} & \mathbf{J} & -\mathbf{I} & -\mathbf{E} \end{array}. \tag{4}$$

Since the imaginary units *i*, *j* and *k* are orthogonal and form a three-dimensional space, to obtain 1 when summing them, we divide the sum $(i + j + k)$ by the normalizing coefficient $\sqrt{3}$. As a result, the quaternion with $x = y = z$ will take the form

$$q = s + \hat{i}x \,, \tag{5}$$

where $\hat{i} = (i + j + k)/\sqrt{3}$ - is the imaginary unit of the quaternion.

Using the basis matrices (2), we represent the imaginary part of the quaternion in the form:

$$\hat{\mathbf{I}} = (\mathbf{I} + \mathbf{J} + \mathbf{K})/\sqrt{3}\,. \tag{6}$$

We will call the matrix (6) *the imaginary unit of the quaternion in the matrix notation* or *imaginary matrix unit*. The imaginary matrix unit has the properties of the imaginary unit of a complex number: $\hat{\mathbf{I}}\hat{\mathbf{I}} = -\mathbf{E}$. Since $\hat{\mathbf{I}}^{\mathrm{T}}\hat{\mathbf{I}} = \hat{\mathbf{I}}\hat{\mathbf{I}}^{\mathrm{T}} = -\hat{\mathbf{I}}\hat{\mathbf{I}} = \mathbf{E}$, the imaginary matrix unit is an orthogonal matrix.

III. Technique of Obtaining Direct Quaternion Discrete Fourier Transform

In the general case, quaternion (1) has different values for the imaginary part, i.e. $x \neq y \neq z$. Such a quaternion is written in polar form using three circular frequencies ω_1 , ω_2 , ω_3 . Here we consider a single-frequency quaternion in which $\omega_1 = \omega_2 = \omega_3 = \omega$. In the state space, the dynamics equation is represented as a homogeneous linear matrix equation with a constant state transition matrix [4]. The fundamental matrix, which is the solution of the dynamics equation, has the form [6]:

$$\mathbf{\Phi}(\omega, t) = \exp\left(\hat{\mathbf{I}}\omega t\right) = \cos\omega t\mathbf{E} + \sin\omega t\hat{\mathbf{I}}\,. \tag{7}$$

We will call it *the quaternion carrier matrix*.
As you can see, this notation uses the imaginary unit of the quaternion in the matrix notation (6). The fundamental matrix (7) satisfies the equality $\mathbf{\Phi}^{\mathrm{T}}(t) = \mathbf{\Phi}^{-1}(t)$ and, consequently, it is orthogonal, i.e. $\mathbf{\Phi}(t)\mathbf{\Phi}^{\mathrm{T}}(t) = \mathbf{\Phi}^{\mathrm{T}}(t)\mathbf{\Phi}(t) = \mathbf{E}$.

Since the time *t* is discrete, we write it as $t_k = kT_d$, where T_d - constant time interval between samples. We denote the frequency interval between discrete frequencies as $\Omega_d = 2\pi/NT_d$, and $\omega_n = n\Omega_d$ - will be a discrete frequency, (rad/s). Then $\Omega_d T_d = 2\pi/N$. Taking into account the notation made, expression (7) in discrete time and at a discrete frequency will take the following form:

$$\mathbf{\Phi}(n\Omega_d, kT_d) = \exp\left(\hat{\mathbf{I}}n\Omega_d kT_d\right) = \tag{8}$$

$$= \cos(n\Omega_d kT_d)\mathbf{E} + \sin(n\Omega_d kT_d)\hat{\mathbf{I}}\,,$$

where $k = 0,1,2,\ldots,N-1$, $n = 0,1,2,\ldots,N-1$, N – number of counts per pulse duration equal to the number of discrete frequencies.

According to the Euler formula, the expression (8) is equal to the matrix exponent $\exp\left(\hat{\mathbf{I}}n\Omega_d kT_d\right)$, with the imaginary quaternion matrix unit $\hat{\mathbf{I}}$ in degree. Obviously, the modulus of the complex number (5), written in polar form for $s = \cos\omega t$ and $x = \sin\omega t$, is equal to 1.

According to the Moivre's formula is

$$1^{1/N} = \exp\left(\hat{i}nk2\pi/N\right).$$

In matrix notations

$$\mathbf{E}^{1/N} = \exp\left(\hat{\mathbf{I}}nk2\pi/N\right). \tag{9}$$

The case $nk = 1$ is the primitive *N*th root of unity, since an integer power gives all the others. The *N* roots of unity are very important, so they are given a special notation for complex numbers:

$$W_N^k \triangleq \exp\left(i2\pi k/N\right),\ k = 0,1,2,\dots,N-1,$$

where W_N denotes the primitive N-th root of unity.

W_N is also called the generator of a mathematical group containing the N-th roots of unity and their products. N roots of unity are used in the DFT to generate sinusoids:

$$W_N^{nk} = \exp\left(i2\pi nk/N\right) \triangleq \exp\left(i\omega_n t_k\right) = \cos(\omega_n t_k) + i\sin(\omega_n t_k),$$

where $\omega_n \triangleq 2\pi n/NT_d$, $t_k \triangleq kT_d$.

Let us denote the quaternion fundamental matrices (8) in discrete time and frequency as

$$\mathbf{H}_N^{nk} = \mathrm{e}^{\hat{\mathbf{I}} nk\frac{2\pi}{N}} = \cos\left(nk\frac{2\pi}{N}\right)\mathbf{E} + \sin\left(nk\frac{2\pi}{N}\right)\hat{\mathbf{I}}. \quad (10)$$

We represent the impulses vector as a 4-dimensional vector of discrete functions of the quaternion:

$$\mathbf{q}(k) = \left[p(k) \quad u(k) \quad v(k) \quad w(k)\right]^{\mathrm{T}},$$

where $p(k)$, $u(k)$, $v(k)$, $w(k)$ - samples of the amplitudes of the input signal in the form of pulses of finite duration.

Let's write the spectra vector as a 4-dimensional vector of discrete functions:

$$\mathbf{c}(n) = \left[c_p(n) \quad c_u(n) \quad c_v(n) \quad c_w(n)\right]^{\mathrm{T}},$$

where $c_p(n)$, $c_u(n)$, $c_v(n)$, $c_w(n)$ - spectra of the corresponding elements of the impulses vector.

In accordance with the notation made, we write the discrete quaternion Fourier transform (DQFT) as

$$\mathbf{c}(n) = D\left\{\mathbf{q}(k)\right\} = \sum_{k=0}^{N-1} \mathbf{H}_N^{-nk}\mathbf{q}(k). \quad (11)$$

The inverse discrete quaternion Fourier transform (IDQFT) will have the form:

$$\mathbf{q}(k) = D^{-1}\left\{\mathbf{c}(n)\right\} = \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{H}_N^{kn}\mathbf{c}(n). \quad (12)$$

Usually, the relationship between DQFT and IDQFT is denoted as $\mathbf{q}(k) \Leftrightarrow \mathbf{c}(n)$.

By definition, transformations (11), (12) have the property of linearity.

With the same pulse shape, $f(k) = p(k) = u(k) = v(k) = w(k)$, DQFT (11) can be written as

$$\mathbf{c}(n) = \sum_{k=0}^{N-1} \mathbf{H}_N^{-nk} f(k) = \sum_{k=0}^{N-1} f(k)\mathbf{H}_N^{-nk}. \quad (13)$$

Accordingly, the vector of coefficients will consist of identical impulses and IDQFT (12) will take the form:

$$\mathbf{q}(k) = \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{H}_N^{kn} c(n) = \frac{1}{N}\sum_{n=0}^{N-1} c(n)\mathbf{H}_N^{kn}. \quad (14)$$

If the pulses have different amplitudes and signs, which are determined by the amplitude vector $\mathbf{a} = \left[a_0 \quad a_1 \quad a_2 \quad a_3\right]^{\mathrm{T}}$, then the DQFT (13) can be represented as

$$\mathbf{c}(n) = \sum_{k=0}^{N-1} f(k)\mathbf{H}_N^{-nk}\mathbf{a} = \left(\sum_{k=0}^{N-1} f(k)\mathbf{H}_N^{-nk}\right)\mathbf{a}. \quad (15)$$

Let us denote the matrix in brackets of expression (15) as

$$\mathbf{C}(n) = \sum_{k=0}^{N-1} f(k)\mathbf{H}_N^{-nk}, \quad (16)$$

and call it *the matrix of spectral coefficients*.

Accordingly, due to the linearity of the transformations, the IDQFT (12) at different pulse amplitudes can be written in the form:

$$\mathbf{q}(k) = \frac{1}{N}\sum_{n=0}^{N-1} c(n)\mathbf{H}_N^{kn}\mathbf{a} = \left(\frac{1}{N}\sum_{n=0}^{N-1} c(n)\mathbf{H}_N^{kn}\right)\mathbf{a}. \quad (17)$$

The matrix in brackets of expression (17) denote as

$$\mathbf{Q}(k) = \frac{1}{N}\sum_{n=0}^{N-1} c(n)\mathbf{H}_N^{kn}, \quad (18)$$

and call it *the matrix of impulses coefficients*.

The vector of spectral coefficients for pulses of different amplitudes is obtained by multiplying the matrix of spectral coefficients by the amplitude vector:

$$\mathbf{c}(n) = \mathbf{C}(n)\mathbf{a}. \quad (19)$$

Similarly, the vector of identical in shape impulses of different amplitudes will be obtained by multiplying the matrix of impulses coefficients by the amplitude vector:

$$\mathbf{q}(k) = \mathbf{Q}(k)\mathbf{a}. \quad (20)$$

If $\mathbf{q}(k) \Leftrightarrow \mathbf{c}(n)$, then to calculate the spectrum of the impulses vector with the same cyclic shift, you need to multiply the matrix $\mathbf{H}_N^{\pm nd}$ by the spectrum vector without a shift:

$$\mathbf{q}(k \pm d) \Leftrightarrow \mathbf{H}_N^{\pm nd}\mathbf{c}(n). \quad (21)$$

A counterclockwise shift of $\mathbf{q}(k)$ samples (advance) occurs at d>0, and a clockwise shift of $\mathbf{q}(k)$ samples (delay) occurs at d<0.

From (10) and (21) we obtain a number of properties of the discrete quaternion fundamental matrix:

$$\mathbf{H}_N^N = \mathbf{E},\ \mathbf{H}_N^{kN} = \mathbf{E},\ \mathbf{H}_N^{N/2} = -\mathbf{E},\ \mathbf{H}_N^{N/4} = \hat{\mathbf{I}}, \quad (22)$$

$$\mathbf{H}_N^{3N/4} = -\hat{\mathbf{I}} = \hat{\mathbf{I}}^{\mathrm{T}},\ \mathbf{H}_{2N}^k = \mathbf{H}_N^{k/2},\ \mathbf{H}_N^{kn}\mathbf{H}_N^{km} = \mathbf{H}_N^{(n+m)k},$$

$$\mathbf{H}_N^{kn} = \mathbf{H}_N^{(k+mN)n}.$$

In the general case, cyclic shifts of impulses can be different: d_0, d_1, d_2, d_3. The matrix of different shift consists of the corresponding columns of matrices of the same shift and has the form:

$$\mathbf{D}(N,n,d_0,d_1,d_2,d_3) = \quad (23)$$

$$= \frac{1}{\sqrt{3}}\left[\begin{matrix} \sqrt{3}\cos\left(nd_0\frac{2\pi}{N}\right) & \sin\left(nd_1\frac{2\pi}{N}\right) \\ -\sin\left(nd_0\frac{2\pi}{N}\right) & \sqrt{3}\cos\left(nd_1\frac{2\pi}{N}\right) \\ -\sin\left(nd_0\frac{2\pi}{N}\right) & \sin\left(nd_1\frac{2\pi}{N}\right) \\ -\sin\left(nd_0\frac{2\pi}{N}\right) & -\sin\left(nd_1\frac{2\pi}{N}\right) \end{matrix}\right. \cdots$$

$$\begin{matrix} \sin\left(nd_2\frac{2\pi}{N}\right) & \sin\left(nd_3\frac{2\pi}{N}\right) \\ -\sin\left(nd_2\frac{2\pi}{N}\right) & \sin\left(nd_3\frac{2\pi}{N}\right) \\ \sqrt{3}\cos\left(nd_2\frac{2\pi}{N}\right) & -\sin\left(nd_3\frac{2\pi}{N}\right) \\ \sin\left(nd_2\frac{2\pi}{N}\right) & \sqrt{3}\cos\left(nd_3\frac{2\pi}{N}\right) \end{matrix} .$$

By analogy with (21), we write the shift matrix for the same number of samples f of the spectra of pulses as

$$\mathbf{F}(N,f,k)=\mathbf{H}_N^{fk}=\mathrm{e}^{\hat{\mathbf{I}}fk\frac{2\pi}{N}}=\cos\left(fk\frac{2\pi}{N}\right)\mathbf{E}+\sin\left(fk\frac{2\pi}{N}\right)\hat{\mathbf{I}} .$$

The matrix of different shifts f_m, $m=0,1,2,3$, elements of the vector of spectral coefficients has the form:

$$\mathbf{F}(N,k,f_0,f_1,f_2,f_3)=\frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3}\cos\left(kf_0\frac{2\pi}{N}\right) & \sin\left(kf_1\frac{2\pi}{N}\right) \\ -\sin\left(kf_0\frac{2\pi}{N}\right) & \sqrt{3}\cos\left(kf_1\frac{2\pi}{N}\right) \\ -\sin\left(kf_0\frac{2\pi}{N}\right) & \sin\left(kf_1\frac{2\pi}{N}\right) \\ -\sin\left(kf_0\frac{2\pi}{N}\right) & -\sin\left(kf_1\frac{2\pi}{N}\right) \end{bmatrix}$$

$$\ldots \begin{matrix} \sin\left(kf_2\frac{2\pi}{N}\right) & \sin\left(kf_3\frac{2\pi}{N}\right) \\ -\sin\left(kf_2\frac{2\pi}{N}\right) & \sin\left(kf_3\frac{2\pi}{N}\right) \\ \sqrt{3}\cos\left(kf_2\frac{2\pi}{N}\right) & -\sin\left(kf_3\frac{2\pi}{N}\right) \\ \sin\left(kf_2\frac{2\pi}{N}\right) & \sqrt{3}\cos\left(kf_3\frac{2\pi}{N}\right) \end{matrix} . \qquad (24)$$

Let us show the validity of Parseval's identity for the DQFT. Multiply scalarly both IDQFT parts (12) by $\mathbf{p}^{\mathrm{T}}(k)$ and sum the terms:

$$\sum_{k=0}^{N-1}\mathbf{p}^{\mathrm{T}}(k)\mathbf{q}(k)=\sum_{k=0}^{N-1}\mathbf{p}^{\mathrm{T}}(k)\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{H}_N^{kn}\mathbf{c}(n) .$$

Because $\sum_{k=0}^{N-1}\mathbf{p}^{\mathrm{T}}(k)\mathbf{q}(k)=\sum_{k=0}^{N-1}\mathbf{q}^{\mathrm{T}}(k)\mathbf{p}(k)$, then the resulting expression can be written as

$$\sum_{k=0}^{N-1}\mathbf{p}^{\mathrm{T}}(k)\mathbf{q}(k)=\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{c}^{\mathrm{T}}(n)\mathbf{H}_N^{-kn}\sum_{k=0}^{N-1}\mathbf{p}(k) .$$

We got a double sum on the right side with one factor $\mathbf{H}_N^{-kn}$ depending on two variables n and k. First, we use $\mathbf{H}_N^{-kn}$ in the sum over k:

$$\sum_{k=0}^{N-1}\mathbf{p}^{\mathrm{T}}(k)\mathbf{q}(k)=\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{c}^{\mathrm{T}}(n)\left(\sum_{k=0}^{N-1}\mathbf{H}_N^{-kn}\mathbf{p}(k)\right).$$

In accordance with (11), we denote the vector of spectral coefficients of the vector $\mathbf{p}$ as

$$\mathbf{r}(n)=D\{\mathbf{p}(k)\}=\sum_{k=0}^{N-1}\mathbf{H}_N^{-kn}\mathbf{p}(k) .$$

After substitution, we obtain Parseval's identity:

$$\sum_{k=0}^{N-1}\mathbf{p}^{\mathrm{T}}(k)\mathbf{q}(k)=\frac{1}{N}\sum_{n=0}^{N-1}\mathbf{c}^{\mathrm{T}}(n)\mathbf{r}(n) .$$

IV. Examples of Using the Technique for Calculating the Direct Quaternion Discrete Fourier Transform

Consider the procedure for calculating the DQFT for the most common impulses.

A. Rectangle impulses

For rectangular pulses

$$p(k)=u(k)=v(k)=w(k)=\begin{cases}1 & \text{при } 0\le k\le N-1 \\ 0 & \text{in other cases}\end{cases} .$$

Therefore, expression (16) corresponds to a finite geometric progression:

$$\mathbf{C}(n)=\mathbf{E}+\left(\mathbf{H}_N^{-n}\right)^1+\left(\mathbf{H}_N^{-n}\right)^2+\cdots+\left(\mathbf{H}_N^{-n}\right)^{N-1}= \qquad (25)$$

$$=\frac{\mathbf{E}-\left(\mathbf{H}_N^{-n}\right)^N}{\mathbf{E}-\mathbf{H}_N^{-n}} .$$

For a quaternion, the Euler formulas are valid in matrix form:

$$\sin(y)\mathbf{E}=\frac{1}{2}\hat{\mathbf{I}}^{\mathrm{T}}\left(\mathrm{e}^{\hat{\mathbf{I}}y}-\mathrm{e}^{-\hat{\mathbf{I}}y}\right), \qquad (26)$$

$$\cos(y)\mathbf{E}=\frac{1}{2}\left(\mathrm{e}^{\hat{\mathbf{I}}y}+\mathrm{e}^{-\hat{\mathbf{I}}y}\right).$$

Using formula (26) for the sine, we write (25) as

$$\mathbf{C}(n)=\frac{\mathrm{e}^{-\hat{\mathbf{I}}n\pi}\left(\mathrm{e}^{\hat{\mathbf{I}}n\pi}-\mathrm{e}^{-\hat{\mathbf{I}}n\pi}\right)}{\mathrm{e}^{-\hat{\mathbf{I}}\frac{n\pi}{N}}\left(\mathrm{e}^{\hat{\mathbf{I}}\frac{n\pi}{N}}-\mathrm{e}^{-\hat{\mathbf{I}}\frac{n\pi}{N}}\right)}=$$

$$=\frac{\sin(n\pi)}{\sin\left(n\frac{\pi}{N}\right)}\mathrm{e}^{-\hat{\mathbf{I}}n\pi\left(\frac{N-1}{N}\right)}=N\frac{\operatorname{sinc}(n\pi)}{\operatorname{sinc}\left(n\frac{\pi}{N}\right)}\mathrm{e}^{-\hat{\mathbf{I}}n\pi\left(\frac{N-1}{N}\right)} .$$

For n=0 we get that

$$\lim_{n\to 0}\frac{\sin(n\pi)}{\sin\left(n\frac{\pi}{N}\right)}=N ,\ \mathrm{e}^{-\hat{\mathbf{I}}n\pi\left(\frac{N-1}{N}\right)}=\mathbf{E} .$$

For $n\neq 0$, $\lim_{n\to\neq 0}\frac{\sin(n\pi)}{\sin\left(n\frac{\pi}{N}\right)}=0$.

Thus, the matrix of spectral coefficients will take the form:

$$\mathbf{C}(n)=N\frac{\operatorname{sinc}(n\pi)}{\operatorname{sinc}\left(n\frac{\pi}{N}\right)}\times \qquad (27)$$

$$\times\left[\cos\left(n\pi\frac{N-1}{N}\right)\mathbf{E}-\sin\left(n\pi\frac{N-1}{N}\right)\hat{\mathbf{I}}\right].$$

As can be seen, only when $n=0$ we obtain $\mathbf{C}(0)=N\mathbf{E}$, in other cases $\mathbf{C}(n)=\mathbf{0}$.

The vector of spectral coefficients at different pulse amplitudes is calculated using expression (19).

IDQFT of rectangular pulses is calculated by formula (12), as

$$\mathbf{Q}(k)=\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{e}^{\hat{\mathbf{i}}nk\frac{2\pi}{N}}N=\sum_{n=0}^{N-1}\mathrm{e}^{\hat{\mathbf{i}}0k\frac{2\pi}{N}}=\mathbf{E}.$$

The pulses vector is obtained from formula (20)

$\mathbf{q}(k)=\mathbf{Ea}=\mathbf{a}$.

Since the value of the pulse samples is the same, when the pulses are shifted, the shape of the pulse does not change in duration. The value of the spectrum differs from 0 only at n=0 and the spectrum does not change when the pulses are shifted.

B. Sawtooth impulses

For sawtooth pulses

$$p(k)=u(k)=v(k)=w(k)=\begin{cases}k & \text{при } 0\le k\le N-1\\ 0 & \text{in other cases}\end{cases}.$$

Find the matrix of spectral coefficients. By formula (16) we obtain:

$$\mathbf{C}(n)=0\mathbf{E}+1\left(\mathbf{H}_N^{-n}\right)^1+2\left(\mathbf{H}_N^{-n}\right)^2+\cdots+(N-1)\left(\mathbf{H}_N^{-n}\right)^{N-1}.$$

To obtain the sum of the series, we use the formula from [10], p. 603, 4.1.7, 2:

$$\sum_{k=1}^{n}kx^k=(1-x)^{-2}\left[x+(nx-n-1)x^{n+1}\right].$$

Substituting the corresponding values, we get

$$\sum_{k=1}^{N-1}k\left(\mathbf{H}_N^{-n}\right)^k=\frac{1}{\left(\mathbf{E}-\mathbf{H}_N^{-n}\right)^2}\times$$

$$\times\left[\mathbf{H}_N^{-n}+\left((N-1)\mathbf{H}_N^{-n}-N\right)\left(\mathbf{H}_N^{-n}\right)^N\right].$$

We use properties (22) and transform the expression to the form:

$$\mathbf{C}(n)=\frac{1}{\left(\mathbf{E}-\mathbf{H}_N^{-n}\right)^2}N\left[\mathbf{H}_N^{-n}-\mathbf{E}\right]=-\frac{N\mathbf{E}}{\mathbf{E}-\mathbf{H}_N^{-n}}.$$

We represent the resulting expression in the form of exponents and using the formula for the sine from (26), we obtain the matrix of spectral coefficients in the form:

$$\mathbf{C}(n)=-\frac{N}{2\sin\left(\frac{n\pi}{N}\right)}\left(\sin\left(\frac{n\pi}{N}\right)\mathbf{E}-\cos\left(\frac{n\pi}{N}\right)\hat{\mathbf{I}}\right)=$$

$$=\frac{N}{2\sin\left(\frac{n\pi}{N}\right)}\hat{\mathbf{I}}e^{\hat{\mathbf{i}}\frac{n\pi}{N}}. \qquad (28)$$

The vector of spectral coefficients for sawtooth pulses of different amplitudes will be obtained from the formula (19).

For $n=0$ we get $\mathbf{c}(0)=N(N-1)/2$.

To obtain the same cyclic shifts of sawtooth pulses, it is necessary to use (21). To obtain different cyclic shifts, matrix (23) must be used. This requires the following order of multiplication:

$$\mathbf{C}(n)\mathbf{D}(N,n,d_0,d_1,d_2,d_3). \qquad (29)$$

C. Sine impulses

For pulses in the form of half a sine period

$$p(k)=u(k)=v(k)=w(k)=\begin{cases}\sin\left(k\frac{\pi}{N}\right) & \text{при } 0\le k\le N-1\\ 0 & \text{in other cases}\end{cases}.$$

Let us find the matrix of spectral coefficients. Substituting these values into DQFT (16) and we get

$$\mathbf{C}(n)=\sin\left(1\frac{\pi}{N}\right)\left(\mathbf{H}_N^{-n}\right)^1+\sin\left(2\frac{\pi}{N}\right)\left(\mathbf{H}_N^{-n}\right)^2+\cdots$$

$$+\sin\left((N-1)\frac{\pi}{N}\right)\left(\mathbf{H}_N^{-n}\right)^{N-1}.$$

To obtain the sum of the series, we use the formula from [11], P. 38, 1.353, AD(361.12)a.:

$$\sum_{k=1}^{n-1}p^k\sin kx=\frac{p\sin x-p^n\sin nx+p^{n-1}\sin(n-1)x}{1-2p\cos x+p^2}.$$

We substitute into the formula $x=\pi/N$, $p=\mathbf{H}_N^{-n}$ and use the properties (22). As a result, we obtain a matrix of spectral coefficients in the form:

$$\mathbf{C}(n)=\frac{2\mathbf{H}_N^{-n}\sin\left(\frac{\pi}{N}\right)}{\mathbf{E}-2\mathbf{H}_N^{-n}\cos\left(\frac{\pi}{N}\right)+\left(\mathbf{H}_N^{-n}\right)^2}.$$

We represent the fundamental matrix in the form of an exponential and, using the Euler formulas (26), we obtain the matrix of spectral coefficients in the form:

$$\mathbf{C}(n)=\mathbf{E}\frac{\sin\left(\frac{\pi}{N}\right)}{\cos\left(n\frac{2\pi}{N}\right)-\cos\left(\frac{\pi}{N}\right)}. \qquad (30)$$

The vector of spectral coefficients for sinusoidal pulses is obtained by formula (19).

When $n=0$

$$\mathbf{c}(0)=\frac{\sin\left(\frac{\pi}{N}\right)}{1-\cos\left(\frac{\pi}{N}\right)}=\frac{\sin\left(\frac{\pi}{N}\right)}{2\sin^2\left(\frac{\pi}{2N}\right)}.$$

To obtain the same cyclic shifts of sine pulses, it is necessary to use formula (21), and to obtain different pulse shifts, first, matrix (30) must be multiplied by a matrix of different shifts (23), observing the order of multiplication (29).

D. Cosine impulses

For pulses in half period cosine

$$p(k)=u(k)=v(k)=w(k)=\begin{cases}\cos\left(k\frac{\pi}{N}\right) & \text{при } 0\le k\le N-1\\ 0 & \text{in other cases}\end{cases}$$

The matrix of spectral coefficients of the cosine vector has the form:

$$\mathbf{C}(n) = 1 + \cos\left(1\frac{\pi}{N}\right)\left(\mathbf{H}_N^{-n}\right)^1 + \cos\left(2\frac{\pi}{N}\right)\left(\mathbf{H}_N^{-n}\right)^2 + \cdots$$

$$+\cos\left((N-1)\frac{\pi}{N}\right)\left(\mathbf{H}_N^{-n}\right)^{N-1}.$$

Let us use the formula from [11], P. 38, 1.352, AD(361.13)ai.

$$\sum_{k=1}^{n-1} p^k \cos kx = \frac{1 - p\cos x - p^n \cos nx + p^{n-1}\cos(n-1)x}{1 - 2p\cos x + p^2}.$$

After substituting into the formula as the values $x = \pi/N$ and $p = \mathbf{H}_N^{-n}$, we simplify the expression using the properties of the fundamental matrix (22). As a result, we obtain a matrix of spectral coefficients in the form:

$$\mathbf{C}(n) = \frac{2\mathbf{E} - 2\mathbf{H}_N^{-n}\cos\left(\frac{\pi}{N}\right)}{\mathbf{E} - 2\mathbf{H}_N^{-n}\cos\left(\frac{\pi}{N}\right) + \left(\mathbf{H}_N^{-n}\right)^2}.$$

Representing the fundamental matrix as a matrix exponent and using the Euler formulas (26), we arrive at the expression:

$$\mathbf{C}(n) = \mathbf{E} + \frac{\sin\left(n\frac{2\pi}{N}\right)}{\cos\left(n\frac{2\pi}{N}\right) - \cos\left(\frac{\pi}{N}\right)}\hat{\mathbf{I}}. \qquad (31)$$

To obtain the vector of spectral coefficients, it is necessary to use the expression (19).

When $n = 0$

$$\mathbf{c}(0) = \frac{2 - 2\cos\left(\frac{\pi}{N}\right)}{3 - 2\cos\left(\frac{\pi}{N}\right)}.$$

To obtain various cyclic shifts of cosine pulses, it is necessary, as in the previous cases, to multiply the matrix of spectral coefficients (31) by matrices (21) or (23). For different shifts, the order of multiplication (29) must be observed.

E. Pulse of various shapes

Let us consider a more general case, when the pulse vector consists of pulses of different shapes, which have different cyclic shifts. Assume that the pulses in the vector are arranged in the following order: rectangular, sine, sawtooth, cosine. The corresponding matrices of spectral coefficients are represented by formulas (27), (30), (28), (31). Using the obtained matrices of spectral coefficients at various shifts, we will compose a matrix of coefficients for various impulses by choosing columns corresponding to the shape of the impulses and the cyclic shift. As a result, we obtain a combined matrix of spectral coefficients:

$$\mathbf{C}(N,n,d_0,d_1,d_2,d_3) = \qquad (32)$$

$$\frac{1}{\sqrt{3}}\left[\mathbf{C}_0(N,n,d_0) \quad \mathbf{C}_1(N,n,d_1) \quad \mathbf{C}_2(N,n,d_2) \quad \mathbf{C}_3(N,n,d_3)\right],$$

where

$$\mathbf{C}_0(N,n,d_0) = \begin{bmatrix} \sqrt{3}N\dfrac{\operatorname{sinc}(n\pi)}{\operatorname{sinc}\left(n\frac{\pi}{N}\right)}\cos\left(n\pi\dfrac{2d_0 - N + 1}{N}\right) \\ -N\dfrac{\operatorname{sinc}(n\pi)}{\operatorname{sinc}\left(n\frac{\pi}{N}\right)}\sin\left(n\pi\dfrac{2d_0 - N + 1}{N}\right) \\ -N\dfrac{\operatorname{sinc}(n\pi)}{\operatorname{sinc}\left(n\frac{\pi}{N}\right)}\sin\left(n\pi\dfrac{2d_0 - N + 1}{N}\right) \\ -N\dfrac{\operatorname{sinc}(n\pi)}{\operatorname{sinc}\left(n\frac{\pi}{N}\right)}\sin\left(n\pi\dfrac{2d_0 - N + 1}{N}\right) \end{bmatrix},$$

$$\mathbf{C}_1(N,n,d_1) = \begin{bmatrix} \dfrac{\sin\left(\frac{\pi}{N}\right)}{\cos\left(n\frac{2\pi}{N}\right) - \cos\left(\frac{\pi}{N}\right)}\sin\left(nd_1\dfrac{2\pi}{N}\right) \\ \sqrt{3}\dfrac{\sin\left(\frac{\pi}{N}\right)}{\cos\left(n\frac{2\pi}{N}\right) - \cos\left(\frac{\pi}{N}\right)}\cos\left(nd_1\dfrac{2\pi}{N}\right) \\ \dfrac{\sin\left(\frac{\pi}{N}\right)}{\cos\left(n\frac{2\pi}{N}\right) - \cos\left(\frac{\pi}{N}\right)}\sin\left(nd_1\dfrac{2\pi}{N}\right) \\ -\dfrac{\sin\left(\frac{\pi}{N}\right)}{\cos\left(n\frac{2\pi}{N}\right) - \cos\left(\frac{\pi}{N}\right)}\sin\left(nd_1\dfrac{2\pi}{N}\right) \end{bmatrix},$$

$$\mathbf{C}_2(N,n,d_2) = \begin{bmatrix} \dfrac{1}{2\sin\left(\frac{\pi n}{N}\right)}\cos\left(\pi n\dfrac{2d_2 + 1}{N}\right) \\ -\dfrac{1}{2\sin\left(\frac{\pi n}{N}\right)}\cos\left(\pi n\dfrac{2d_2 + 1}{N}\right) \\ -\sqrt{3}\dfrac{1}{2\sin\left(\frac{\pi n}{N}\right)}\sin\left(\pi n\dfrac{2d_2 + 1}{N}\right) \\ \dfrac{1}{2\sin\left(\frac{\pi n}{N}\right)}\cos\left(\pi n\dfrac{2d_2 + 1}{N}\right) \end{bmatrix},$$

$$\mathbf{C}_3(N,n,d_3) =$$

$$\begin{bmatrix} \dfrac{1}{\cos\left(\dfrac{\pi}{N}\right)-\cos\left(\dfrac{2\pi n}{N}\right)}\left(\sin\left(\dfrac{2\pi n d_3}{N}\right)\cos\left(\dfrac{\pi}{N}\right)-\sin\left(2\pi n\dfrac{d_3+1}{N}\right)\right) \\ \dfrac{1}{\cos\left(\dfrac{\pi}{N}\right)-\cos\left(\dfrac{2\pi n}{N}\right)}\left(\sin\left(\dfrac{2\pi n d_3}{N}\right)\cos\left(\dfrac{\pi}{N}\right)-\sin\left(2\pi n\dfrac{d_3+1}{N}\right)\right) \\ \dfrac{1}{\cos\left(\dfrac{\pi}{N}\right)-\cos\left(\dfrac{2\pi n}{N}\right)}\left(\sin\left(\dfrac{2\pi n d_3}{N}\right)\cos\left(\dfrac{\pi}{N}\right)-\sin\left(2\pi n\dfrac{d_3+1}{N}\right)\right) \\ \dfrac{1}{\cos\left(\dfrac{\pi}{N}\right)-\cos\left(\dfrac{2\pi n}{N}\right)}\left(\cos\left(\dfrac{2\pi n d_3}{N}\right)\cos\left(\dfrac{\pi}{N}\right)-\cos\left(2\pi n\dfrac{d_3+1}{N}\right)\right) \end{bmatrix}.$$

To obtain the vector of spectral coefficients, this matrix must be multiplied by the vector of amplitudes. Let the amplitude vector be $\mathbf{a}=\begin{bmatrix}3 & -2 & -1 & 2\end{bmatrix}^{\mathrm{T}}$. Quaternion vector modulus of different amplitude is $|\mathbf{q}|=\sqrt{q_0^2+q_1^2+q_2^2+q_3^2}=4,243$. Normalized quaternion is

$\mathbf{q}/|\mathbf{q}|=\begin{bmatrix}0,707 & -0,471 & -0,236 & 0,471\end{bmatrix}$.

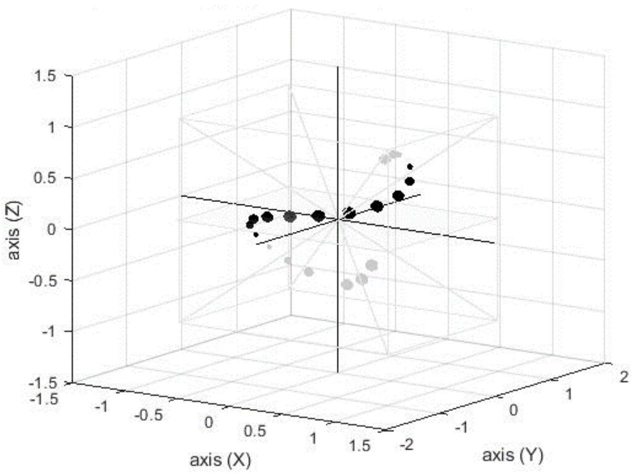
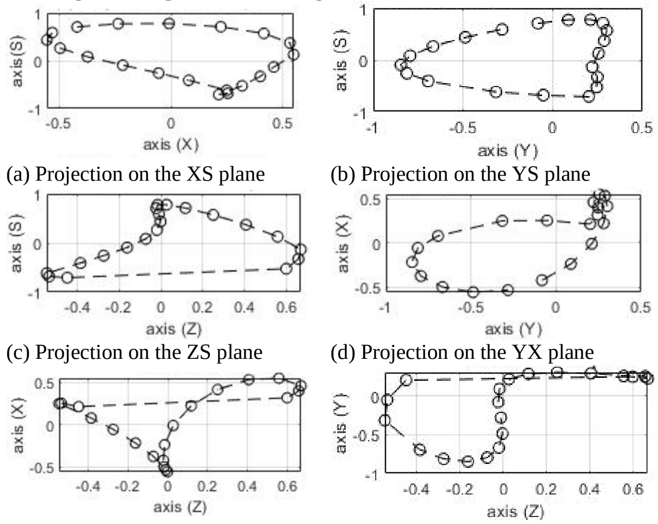


Fig. 1. The trajectory of the quaternion samples in time in 3D space.

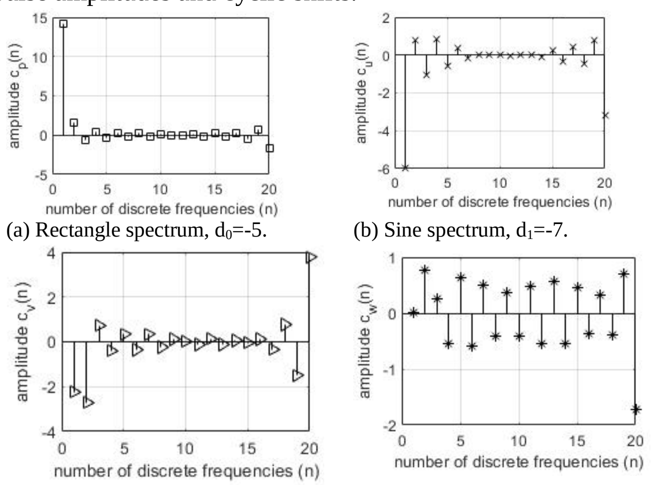
In expression (11) for the DQFT the fundamental matrix (10) is an exponent with an index in the form of an imaginary quaternion unit. Therefore, it performs an isomorphic mapping of a 4D linear quaternion space into a 4D cyclic space. Since we are considering a single-frequency quaternion, the real and three imaginary axes have the same increment in frequency when the values of the samples *n* and *k* change. At the same time, the trajectory of movement in time changes depending on the amplitudes of the pulses along different axes and the values of cyclic shifts.

Let us calculate the DQFT using matrix (32) with N=20. Figure 1 shows the trajectory of the quaternion samples in time in 3D space on the coordinate axes x, y, z, corresponding to the imaginary axes *i*, *j*, *k*, with cyclic shifts $d_0=-5$, $d_1=-7$, $d_2=-13$, $d_3=-10$. On the real axis, the scalar value of the quaternion is plotted, which, unlike the vector, does not depend on its location in 4D. Therefore, a scalar value is depicted as a sphere of different diameters, which indicates the magnitude of the scalar. In this case, the positive values of the scalar are shown in gray, and the negative ones in black. As can be seen from Fig. 1 the quaternion range is finite and closed in 4D.



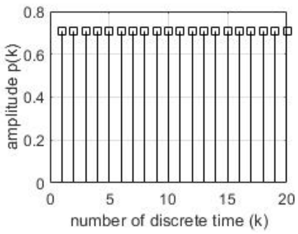
(a) Projection on the XS plane (b) Projection on the YS plane
(c) Projection on the ZS plane (d) Projection on the YX plane
(e) Projection on the ZX plane (f) Projection on the ZY plane
Fig. 2. The projections of the trajectories of quaternion samples on the plane of 4D space.

Figure 2 shows the projections of the trajectories of quaternion samples on the plane of 4D space. As can be seen from Figure 2, the values of the samples move in time cyclically. In this case, the shape of the cycles depends on the values of the pulse amplitudes and cyclic shifts.

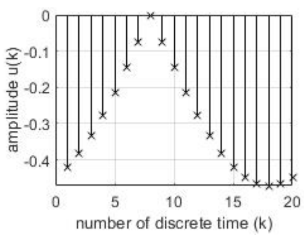


(a) Rectangle spectrum, d_0=-5. (b) Sine spectrum, d_1=-7.
(c) Sawtooth spectrum, d_2=-13. (d) Cosine spectrum, d_3=-10.
Fig. 3. The quaternion spectra of different pulses: square, sine, sawtooth, cosine.

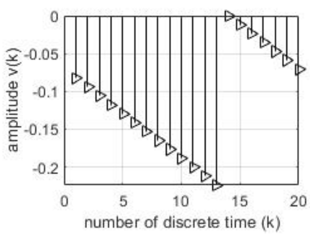
Figure 3 shows the quaternion spectra of different pulses: square, sine, sawtooth, cosine. The pulse spectra are determined by the shape of the pulses and the cyclic shift. Once again, we note that in the MIMO scheme, when the pulse vector is multiplied by the fundamental matrix, all 4 pulses participate in the formation of each pulse of the output vector.



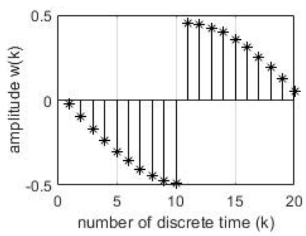
(a) Rectangle impulse, d_0=-5, a_0=3.



(b) Sine impulse, d_1=-7, a_1=-2.



(c) Rectangle impulse, d_2=-13, a_2=-1.



(d) Rectangle impulse, d_3=-10, a_3=2.

Fig. 4. Samples of various pulses: square, sine, sawtooth, cosine obtained using IDQFT.

Figure 4 shows samples of various pulses: square, sine, sawtooth, cosine obtained using IDQFT.

Since, when using IDQFT, the values of the shift matrices depend on the time samples k, then to obtain analytical expressions for the matrices of spectral coefficients with a cyclic shift of the spectrum by f_m in expressions (28), (30), (29), (31), the coefficient n must be replaced by $\left(n - f_m\right)$ in the corresponding columns m. Note that with a cyclic shift of the pulse spectra with the inverse transformation, we obtain a different shape of the pulses.

V. Conclusion

A technique for the discrete Fourier transform of a four-dimensional vector of pulses is presented, the kernel of which is a quaternion in a matrix representation. Examples of using the technique to obtain the quaternion discrete Fourier transform of a four-dimensional vector of various impulses are shown: rectangular, sawtooth, cosine and sine. The quaternion discrete Fourier transform corresponds to the MIMO transform in 4D space and can be used to calculate discrete signal spectra in MIMO channels. The resulting expressions make it possible to calculate the spectra for a vector of different pulses with different shifts in time and frequency.

References

[1]. I Emre Telatar. Capacity of Multi-antenna Gaussian Channels. Emerging Telecommunications Technologies v. 10. Issue 6, p. 569-709, 1999.

[2]. Mawardi Bahri. Discrete Quaternion Fourier Transform and Properties. Int. Journal of Math. Analysis, Vol. 7, 2013, no. 25, 1207 – 1215.

[3]. Salem Said, Nicolas Le Bihan, Stephen J. Sangwine. Fast complexified quaternion Fourier transform. February 2, 2008. arXiv:math/0603578v1 [math.NA] 24 Mar 2006.

[4]. V. Sovetov. Discrete Kalman filter in problems of optimal receiving signals. Radiotekhnika, 2016. В. Советов Дискретный Фильтр Калмана в задачах оптимального приема сигналов.// Радиотехника №3, 2016. С. 43-49.

[5]. V. Sovetov. Continuous time Kalman filter in problems of optimal receiving signals. Radiotekhnika, 2016. В. Советов. Непрерывный Фильтр Калмана в задачах оптимального приема сигналов.// Радиотехника №7, 2016. С. 111-118.

[6]. V. Sovetov. Quaternion Fourier series of periodic pulse sequence. Radiotekhnika. 2021. V. 85, № 2. P. 83-94. DOI: 10.18127/j00338486-202102-13 (In Russia). В. Советов. Кватернионное преобразование Фурье вектора импульсов. //Радиотехника том 85, №2, 2021. С. 83-94.

[7]. Quaternion Fourier series of periodic pulse sequence. Radiotekhnika. 2021. V. 85, № 3. P. 5-15. DOI: https://doi.org/10.18127/j00338486-202103-01 (In Russia). В. Советов. Кватернионный ряд Фурье периодической последовательности импульсов. // Радиотехника том 85, №3, 2021. С. 5-15.

[8]. V. Sovetov. Quaternion domain Laplace transform of the pulse vector. Electromagnetic waves and electronic systems. 2021. V. 26. № 4. P.25-35. DOI: https://doi.org/10.18127/j15604128-202104-03 (In Russia). В. Советов. Методика получения прямого кватернионного преобразования Лапласа вектора импульсов. // Электромагнитные волны и электронные системы, т. 26, №4, 2021 г., с. 25-35.

[9]. João Pedro Morais, S. Georgiev, and others. Real Quaternionic Calculus Handbook. Birkhauser. 2010. P. 222.

[10]. Integral and series Volume 1. Elementary Functions. A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev. 1986.

[11]. I. S. Gradshteyn and I. M. Ryzhik. Tables of Integrals, series, and Products. Seventh Edition. Academic press. 2007. P. 1171.