Relative Motion as a Fundamental Characteristic of Two Degree of Freedom Systems

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September 18, 2022

Abstract

Two-degree-of-freedom (2DOF) systems play an important role in many areas of mechanical engineering. There are two groups of structural compositions of 2DOF systems. One of these groups consists of two rigid bodies (masses) connected to each other by one or two (in parallel) links (springs and/or dashpots). In this paper this group is considered as the basic 2DOF system. The other structural composition of a 2DOF system consists of a basic system, in which one or both masses are connected to non-movable supports. It should be stressed that these connecting links allow relative motion between the two masses. Each mass in a 2DOF system moves independently according to its law of motion. Therefore, in 2DOF systems the values of the forces that the connecting links exert toward the masses are proportional to the *differences* between the absolute values of the displacements and/or velocities of the masses. This is contrary to 1DOF systems, where these links exert forces proportional to the absolute values of the corresponding parameters of motion. A survey of published sources from the last seventy-five years shows that all published pairs of differential equations attempting to describe the motion of 2DOF systems actually contain a mix of mathematical terms representing the absolute values of the forces exerted by the connecting links as well as the differences of these forces. None of the surveyed sources contained any rigorous solutions of any pair of simultaneous differential equations of motion of 2DOF systems. Accounting for the relative motion between the masses of 2DOF systems, I revised the differential equations from the surveyed publications and also composed the additional pairs of differential equations for all practically relevant 2DOF systems and the possible varieties of their structures. Applying the Laplace Transform methodology to these differential equations of motion, I obtained rigorous mathematical solutions for all of them. Several examples are presented in this paper.

Introduction

Movable mechanical structures predominantly consist of one- and two-degree-of-freedom (1DOF and 2DOF) systems. The investigation of the operational processes of these systems is based on the analysis of their basic parameters of motion. These parameters can be determined by composing and solving the appropriate differential equations of motion. My book [1] addresses the entire spectrum of the solutions of second-order linear differential equations of motion of 1DOF mechanical systems. This book contains the compositions and solutions of all possible linear second-order differential equations of motion for common systems. The diversity of these versions is based on utilizing all possible combinations of mathematical terms that represent loading factors.

The solutions of these differential equations were obtained by using the Laplace Transform methodology.

Currently, I am completing a similar work for 2DOF mechanical systems. Hence, I carried out a survey of published sources in the field of composing and solving second-order linear differential equations of motion of 2DOF systems, paying special attention to the structures of mathematical terms that depend on displacements and velocities of the masses of the system. This survey includes publications that were issued during the last seventy-five years. All 2DOF systems consist of two masses connected to each other by links (elastic links such as springs and fluid links such as dashpots). This paper classifies 2DOF systems by the number of links they have. In a one-link system, the masses are only connected to each other. In surveyed publications, this is sometimes referred to as an 'unrestricted', 'unconstrained', or 'ungrounded' system. In a two-link system, the masses are connected to each other, and one of the masses is also connected to a non-movable support. In surveyed publications, this is sometimes referred to as a 'restricted', 'constrained', or 'grounded' system. Finally, in a three-link system, both masses are connected to unmovable supports, as well as to each other. In surveyed publications, this also falls in the category of 'restricted' systems. It should be noted that a link can consist of a spring, a dashpot, or both in parallel. The links between the masses allow each mass to move independently according to its particular law of motion. Therefore, in a 2DOF system the two masses constantly perform relative motion between each other. The only alternative way of motion for these two masses is to perform simultaneous motion due to having identical laws of motion. In this case, these two masses move as one body, and, therefore, the two masses constitute a 1DOF system. This allows us to emphasize that the concept of relative motion represents one of the fundamental characteristics of the kinematics of motion of the masses of any 2DOF system. Each mass of a 2DOF system can be independently assigned any initial conditions of motion and each mass can be subjected to any external forces.

The motion of a 2DOF system can be described by a pair of simultaneous differential equations (one equation for each of the two masses). By expanding the Laplace Transform methodology with a few conventional algebraic procedures I solved the vast majority of published pairs of simultaneous differential equations of motion of unrestricted (one-link) 2DOF systems. A few examples of solutions for these systems are presented in my other publication [2]. However, the aforementioned survey made it clear that the overwhelming majority of publications deal with restricted systems. Therefore, the survey of publications was focused on clarifying the basics of the kinematics of motion of the masses in restricted systems, and on the structures of the mathematical terms that constitute the second order linear differential equations of motion of these systems as well as on the methodologies of solving them.

The survey shows that just a few related publications belong to the area of mathematics as, for instance, [3] - [4]. These publications, like all other surveyed publications, present the same pairs of differential equations without any attempts of getting their solutions. It should be noted that the overwhelming majority of surveyed publications belong to the theory of vibrations [5] - [16]. Therefore, the list of references in this paper is predominately limited to the well known textbooks on the theory of vibration.

In his book published in 1955, S.Timoshenko introduced a schematic of a two-link restricted 2DOF system and a related pair of simultaneous second order linear differential equations of motion. This system consists of two masses connected to each other by a spring, while one of these masses is connected by another spring to a non-movable support. In his book published in 1956 J. Den Hartog published a schematic of a three-link restricted 2DOF system and a related pair of differential equations. In this case, both masses are connected by two corresponding springs to two non-movable supports. These two pairs of differential equations supposedly describe the motion of the masses in their respective systems. It should be stressed that all surveyed published sources repeatedly describe these two schematics and these two related pairs of differential equations. All other possible structures of restricted systems are based on these two versions of 2DOF systems. The differences between them consist of the types and numbers of connections installed between the masses and the non-movable supports. It seems justifiable to consider these two systems as the basic restricted 2DOF systems.

It should be emphasized that the relative motion in 2DOF systems results in the links exerting forces toward the masses proportional to the differences between the absolute values of the parameters of motion of these masses. The mathematical terms describing these forces always are present in each pair of simultaneous differential equations of motion of a 2DOF system. Examples of the structures of these mathematical terms based on springs have the following shapes: $\Sigma K_i(x_1 - x_2)$ and $\Sigma K_i(x_2 - x_1)$, where $i = 1, 2, 3$, while K_i is the corresponding stiffness coefficient, and x_1 and x_2 are the displacements of the respective masses. It should be noted that in a 1DOF system the force exerted by the spring is proportional to the absolute value of the parameter of motion (displacement) and is expressed in the following way: Kx, where the notations are self-explanatory.

It should be emphasized, that the carried out survey did not reveal any descriptions of the considerations related to the structures of the mathematical terms characterizing the forces exerted by the connecting links, or any attempts to obtain rigorous solutions of the pairs of simultaneous differential equations of motion of restricted 2DOF systems. The survey also did not find any related expressions that could be considered as rigorous mathematical representations of functions of time describing the parameters of motion of the related masses of these systems.

Analyzing the mathematical structures of the terms included in all surveyed pairs of simultaneous differential equations of motion of restricted 2DOF systems, it became clear that all these pairs contain sums of mathematical terms such as $K_1x_1 + K_2(x_1 - x_2)$ or $K_2(x_2 - x_1) + K_3x_2$, which represent a mix of sums of expressions representing the values of forces exerted by the connecting links as proportional to the absolute value of the parameter of motion as well as to the difference between the parameters of motion, or vice-versa. This means that the surveyed differential equations do not properly describe motion and do not belong to the class of either 1DOF or 2DOF systems.

Based on considerations associated with the concept of relative motion of two masses, it became possible to revise the compositions of all surveyed pairs of differential equations of motion of the restricted 2DOF systems. Applying to them the Laplace Transform methodology, I obtained rigorous mathematical solutions of all mentioned pairs of differential equations of motion of the surveyed 2DOF mechanical systems.

Considering all this, it is possible to formulate the basic rule for composing the pairs of simultaneous second order differential equations of motion of 2DOF systems. According to this rule, the forces that depend on displacements and velocities of the masses should be represented in the differential equations as being proportional to the respective differences between the absolute values of these parameters of motion, while all other forces should be included in the equations according to the masses that they are applied to. The following examples demonstrate the rule.

$$m_1\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} + \Sigma C_i(\frac{\mathrm{d}x_1}{\mathrm{d}t} - \frac{\mathrm{d}x_2}{\mathrm{d}t}) + \Sigma K_j(x_1 - x_2) + \Sigma P_k = 0 \tag{0.1}$$

$$m_2\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} + \Sigma C_i(\frac{\mathrm{d}x_2}{\mathrm{d}t} - \frac{\mathrm{d}x_1}{\mathrm{d}t}) + \Sigma K_j(x_2 - x_1) + \Sigma P_{kj} = 0 \tag{0.2}$$

where C_i and K_j are respectively the corresponding coefficients of damping and stiffness, P is an external force, while i, j, k, and kj are the appropriate integers.

Hence, based on the concept of relative motion, I revised the surveyed differential equations and obtained rigorous mathematical solutions for all of them by applying the Laplace Transform methodology. Additionally, I composed equations for numerous possible structures of unrestricted and restricted 2DOF systems containing all possible combinations of elastic and fluid links. Each of these equations was also solved for all relevant loading factors and for all possible initial conditions of motion. A few relevant examples are presented in this paper.

As the carried out survey shows, it is possible to construct just two basic groups of restricted 2DOF systems: two-link systems, where only one mass is connected to a non-movable support, and three-link systems, where both are. The corresponding solutions of two- and three-link systems are presented below.

1 Analysis of Published Schematics and Related Pairs of Differential Equations of Two- and Three-Link 2DOF Systems

1.1 Two-Link 2DOF System

We begin by considering the schematic and the pair of simultaneous differential equations that are intended to describe the motion of a two-link restricted 2DOF system, as presented in the book by [5]. The schematic of this system is shown in Figure 1. In this figure K_1 and K_2 are the stiffness coefficients of the respective springs, m_1 and m_2 are the masses, and x_1 and x_2 are respectively the displacements of the masses.

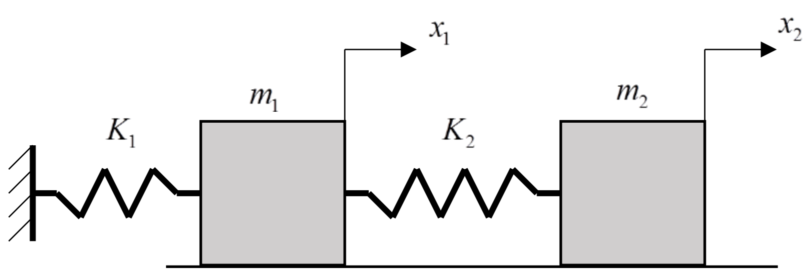

Figure 1

The mentioned above pair of simultaneous differential equations that are supposed to describe the motion of the system are presented in the mentioned above book as well as in numerous other publications. This pair of equations read:

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + K_1 x_1 + K_2(x_1 - x_2) = 0 \tag{1.1}$$

$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + (K_1 + K_2)(x_2 - x_1) = 0 \tag{1.2}$$

The analysis of the structures of the mathematical terms involved in equation (1.1) shows that the term $K_1 x_1$ characterizes the value of the force exerted by the spring as being proportional to the absolute value of the displacement of the mass m_1. However, as explained above, the structure of this mathematical term is only appropriate for 1DOF systems, not for 2DOF systems. Therefore, the pair of differential equations (1.1) and (1.2) does not appropriately describe the motion of the 2DOF system shown in Figure 1. This makes it irrelevant to try to obtain the solutions of the pair of equations (1.1) and (1.2).

1.2 A Three-Link 2DOF System

Consider now the three-link restricted 2DOF system in Figure 2, which is presented in many publications. In the book by [6] the system is shown in the vertical position, however it does not make any difference for the analysis of the structures of the mathematical terms of the related differential equations. The notations in this figure are self-explanatory.

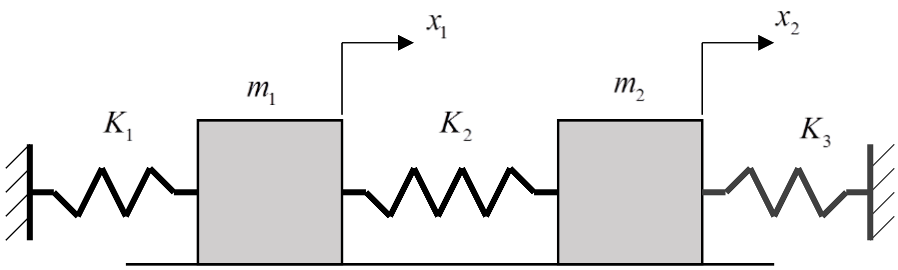

Figure 2

The corresponding pair of structurally identical simultaneous differential equations shown below are also offered in the book by [6] as well as many other publications.

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + K_1 x_1 + K_2(x_1 - x_2) = 0 \tag{1.3}$$

$$m_2 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + K_2(x_2 - x_1) + K_3 x_2 = 0 \tag{1.4}$$

Equations (1.3) and (1.4) respectively show that the mathematical terms K_1x_1 and K_3x_2 describe forces exerted by the springs, the values of which are proportional to the absolute values of the displacements of the related masses. As discussed earlier, these mathematical terms are appropriate only for describing 1DOF systems, but not 2DOF systems. Therefore, the pair of equations (1.3) and (1.4) do not describe the motion of a 2DOF system and it is not justifiable to proceed with finding solutions.

The schematics shown in Figures 1 and 2 are based on the same visualization principles as are accepted for 1DOF systems. These schematics provide a complete understanding of the structures of two-link and three-link systems. However, a schematic does not determine the mathematical structures of the terms that constitute the differential equations of motion. For instance, a schematic does not determine the mathematical structure of the force of inertia.

It should be stressed that, according to the concept of relative motion, in a 2DOF system the motion of each mass is influenced by all links of the system. This should be reflected in the compositions of the pars of simultaneous differential equations of motion. Examples of these equations and their solutions are presented below.

2 Example of the Composition and the Solution of a Pair of Simultaneous Differential Equations of Motion of a Two-Link 2DOF System.

We begin by presenting the solution for a two-link system, the first basic restricted system.

Accounting for the concept of relative motion, I revised the pair of simultaneous differential equations (1.1) and (1.2), which are based on the schematic shown in Figure 1. The revised version of these equations is presented below. Note that all notations of parameters defined in Section 2 only apply to this section.

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + (K_1 + K_2)(x_1 - x_2) = 0 \tag{2.1}$$

$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + (K_1 + K_2)(x_2 - x_1) = 0 \tag{2.2}$$

The initial conditions of motion are:
for $t = 0$

$$x_1(0) = 0, \quad \frac{\mathrm{d}x_1(0)}{\mathrm{d}t} = V_1, \quad x_2(0) = 0, \quad \frac{\mathrm{d}x_2(0)}{\mathrm{d}t} = 0 \tag{2.3}$$

Dividing equation (2.1) by m_1 and equation (2.2) by m_2, and applying to them Laplace Transform pairs 3, 1, and 2 presented in the Appendix, we convert the differential equations of motion (2.1) and (2.2) with the initial conditions of motion according to expression (2.3) from the time domain into the corresponding system of simultaneous algebraic equations in the Laplace domain.

$$s^2 x_1(s) - sV_1 + \omega_{11}^2 x_1(s) - \omega_{11}^2 x_2(s) + \omega_{21}^2 x_1(s) - \omega_{21}^2 x_2(s) = 0 \tag{2.4}$$

$$s^2 x_2(s) + \omega_{12}^2 x_2(s) - \omega_{12}^2 x_1(s) + \omega_{22}^2 x_2(s) - \omega_{22}^2 x_1(s) = 0 \tag{2.5}$$

where

$$\omega_{11}^2 = \frac{K_1}{m_1}; \quad \omega_{21}^2 = \frac{K_2}{m_1}; \quad \omega_{12}^2 = \frac{K_1}{m_2}; \quad \omega_{22}^2 = \frac{K_2}{m_2}; \tag{2.6}$$

Each of the mentioned algebraic equations contains two unknowns $x_1(s)$ and $x_2(s)$ that respectively represent the displacements of the masses m_1 and m_2 in the Laplace domain.

Applying to the equations (2.4) and (2.5) the method of substitutions, we eliminate from each of these equations the corresponding extra unknown and obtain a pair of simultaneous equations with one unknown in each equation. These equations are presented below.

$$x_1(s) = \frac{sV_1}{s^2 + \omega^2} + \frac{V_1 \omega_2^2}{s(s^2 + \omega^2)} \tag{2.7}$$

.

$$x_2(s) = \frac{V_1 \omega_2^2}{s(s^2 + \omega^2)} \tag{2.8}$$

where

$$\omega_1^2 = \omega_{11}^2 + \omega_{21}^2; \quad \omega_2^2 = \omega_{12}^2 + \omega_{22}^2 \tag{2.9}$$

while

$$\omega^2 = \omega_1^2 + \omega_2^2 \tag{2.10}$$

Applying to equations (2.7) and (2.8) Laplace Transform pairs 2, 4, 1, 1, and 5, we invert these equations from the Laplace domain into the time domain and obtain the solutions of the differential equations (2.1) and (2.2) for the initial conditions of motion (2.3).

$$x_1 = \frac{V_1}{\omega}(1 - \frac{\omega_2^2}{\omega^2}) \sin \omega t - \frac{V_1 \omega_2^2 t}{\omega^2} \tag{2.11}$$

$$x_2 = \frac{V_1 \omega_2^2}{\omega^2}(t - \frac{1}{\omega} \sin \omega t) \tag{2.12}$$

Since

$$\frac{\omega_2^2}{\omega^2} < 1 \tag{2.13}$$

it becomes obvious that the two masses are performing anti-phase vibratory motion. The compound frequency of vibration is

$$\omega = \sqrt{\frac{K_1}{m_1} + \frac{K_2}{m_1} + \frac{K_1}{m_2} + \frac{K_2}{m_2}} \tag{2.14}$$

In the case where the spring rates and masses are respectively equal, we obtain:

$$\omega = 2\sqrt{\frac{K}{m}} \tag{2.15}$$

Taking the first derivatives from expressions (2.11) and (2.12), we respectively determine the velocities of the masses.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = V_1(1 - \frac{\omega_2^2}{\omega^2}) \cos \omega t + \frac{V_1 \omega_2^2}{\omega^2} \tag{2.16}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\frac{V_1 \omega_2^2}{\omega^2} \cos \omega t + \frac{V_1 \omega_2^2}{\omega^2} \tag{2.17}$$

Taking for equations (2.11), (2.12), (2.16), and (2.17) that $t = 0$, we determine that $x_1(0) = 0$ and $x_2(0) = 0$, while $\frac{\mathrm{d}x_1(0)}{\mathrm{d}t} = V_1$ as it is expected to be according to the initial conditions of motion (2.3).

The first derivatives from equations (2.16) and (2.17) yield the accelerations of the masses.

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -V_1 \omega (1 - \frac{\omega_2^2}{\omega^2}) \sin \omega t \tag{2.18}$$

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = \frac{V_1 \omega_2^2}{\omega} \sin \omega t \tag{2.19}$$

Respectively substituting into equations (2.1) and (2.2) the presented above obtained corresponding parameters of motion (2.18), (2.19), (2.11), and (2.12), we realize that the initial equations (2.1) and (2.2) turn out into zeros. This confirms the correctness of the presented mathematical solutions.

3 Example of the Composition and the Solution of a Pair of Simultaneous Differential Equations of Motion of a Three-Link 2DOF System.

Now we consider a three-link 2DOF system, which is the second basic restricted system. Applying the concept of relative motion, I revised the pair of simultaneous differential equations (1.3) and (1.4), which are based on the schematic shown in Figure 2. The revised version of the pair of the related simultaneous differential equations are presented below. Note that all notations of parameters defined in Section 3 only apply to this section.

$$m_1 \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + (K_1 + K_2 + K_3)(x_1 - x_2) = 0 \tag{3.1}$$

$$m_2 \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + (K_1 + K_2 + K_3)(x_2 - x_1) = 0 \tag{3.2}$$

The initial conditions of motion are:
for $t = 0$

$$x_1(0) = 0, \quad \frac{\mathrm{d}x_1(0)}{\mathrm{d}t} = V_1, \quad x_2(0) = 0, \quad \frac{\mathrm{d}x_2(0)}{\mathrm{d}t} = 0 \tag{3.3}$$

Dividing equation (3.1) by m_1 and equation (3.2) by m_2, and applying to them the Laplace Transform pairs 3, 1, and 2 that are presented in the Appendix, we convert these differential equations of motion (3.1) and (3.2) with the initial conditions of motion according to expression (3.3) from the time domain into the corresponding system of simultaneous algebraic equations in the Laplace domain.

$$s^2 x_1(s) - sV_1 + \omega_{11}^2 x_1(s) - \omega_{11}^2 x_2(s) + \omega_{21}^2 x_1(s) - \omega_{21}^2 x_2(s) + \omega_{3.1}^2 x_1(s) - \omega_{3.1}^2 x_2(s) = 0 \tag{3.4}$$

$$s^2 x_2(s) + \omega_{12}^2 x_2(s) - \omega_{12}^2 x_1(s) + \omega_{22}^2 x_2(s) - \omega_{22}^2 x_1(s) + \omega_{32}^2 x_1(s) - \omega_{32}^2 x_2(s) = 0 \tag{3.5}$$

where

$$\omega_{11}^2 = \frac{K_1}{m_1}; \quad \omega_{21}^2 = \frac{K_2}{m_1}; \quad \omega_{31}^2 = \frac{K_3}{m_1}; \quad \omega_{12}^2 = \frac{K_1}{m_2}; \quad \omega_{22}^2 = \frac{K_2}{m_2}; \quad \omega_{32}^2 = \frac{K_3}{m_2}; \tag{3.6}$$

Each of these two algebraic equations contains two unknowns $x_1(s)$ and $x_2(s)$ that respectively represent the displacements of the masses m_1 and m_2 in the Laplace domain.

Applying to equations (3.4) and (3.5) the method of substitutions, we eliminate from each of the equations the corresponding extra unknowns and obtain two algebraic equations with one unknown in each. These equations are presented below.

$$x_1(s) = \frac{sV_1}{s^2 + \omega^2} + \frac{V_1 \omega_2^2}{s(s^2 + \omega_2^2)} \tag{3.7}$$

$$x_2(s) = \frac{V_1 \omega_2^2}{s(s^2 + \omega^2)} \tag{3.8}$$

where

$$\omega_1^2 = \omega_{11}^2 + \omega_{21}^2 + \omega_{31}^2; \quad \omega_2^2 = \omega_{12}^2 + \omega_{22}^2 + \omega_{32}^2 \tag{3.9}$$

and

$$\omega^2 = \omega_1^2 + \omega_2^2 \tag{3.10}$$

Applying to equations (3.7) and (3.8) Laplace Transform pairs 2, 4, 1, 1, and 5, we invert these equations from the Laplace domain into the time domain and obtain the solutions of the differential equations (3.1) and (3.2) for the initial conditions of motion (3.3).

$$x_1 = \frac{V_1}{\omega}(1 - \frac{\omega_2^2}{\omega^2}) \sin \omega t - \frac{V_1 \omega_2^2 t}{\omega^2} \tag{3.11}$$

$$x_2 = \frac{V_1 \omega_2^2}{\omega^2}(t - \frac{1}{\omega} \sin \omega t) \tag{3.12}$$

Since

$$\frac{\omega_2^2}{\omega^2} < 1 \tag{3.13}$$

it becomes obvious that the two masses are performing anti-phase vibratory motion. The compound frequency of vibration is

$$\omega = \sqrt{\frac{K_1}{m_1} + \frac{K_2}{m_1} + \frac{K_3}{m_1} + \frac{K_1}{m_2} + \frac{K_2}{m_2} + \frac{K_3}{m_2}} \tag{3.14}$$

In the case where the spring rates and masses are respectively equal, we obtain:

$$\omega = 2.45\sqrt{\frac{K}{m}} \tag{3.15}$$

Taking the first derivatives from expressions (3.11) and (3.12), we respectively determine the velocities of the masses.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = V_1(1 - \frac{\omega_2^2}{\omega^2}) \cos \omega t + \frac{V_1 \omega_2^2}{\omega^2} \tag{3.16}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\frac{V_1 \omega_2^2}{\omega^2} \cos \omega t + \frac{V_1 \omega_2^2}{\omega^2} \tag{3.17}$$

Taking for equations (3.11), (3.12), (3.16), and (3.17) that $t = 0$, we determine that $x_1(0) = 0$, $\frac{\mathrm{d}x_1(0)}{\mathrm{d}t} = V_1$, and $x_2(0) = 0$, while $\frac{\mathrm{d}x_2(0)}{\mathrm{d}t} = 0$ as it is expected to be according to the initial conditions of motion (3.3).

The first derivatives from equations (3.16) and (3.17) yield the accelerations of the masses.

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} = -V_1 \omega (1 - \frac{\omega_2^2}{\omega^2}) \sin \omega t \tag{3.18}$$

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = \frac{V_1 \omega_2^2}{\omega} \sin \omega t \tag{3.19}$$

Respectively substituting into equations (3.1) and (3.2) the presented above obtained corresponding parameters of motion (3.18), (3.19), (3.11), and (3.12), we realize that the initial equations (3.1) and (3.2) turn out into zeros. This confirms the correctness of the presented mathematical solutions.

Conclusions

This paper is focused on composing and solving the pairs of simultaneous second order linear differential equations of motion of 2DOF systems. In order to get familiar with the related basics, I surveyed related published sources from the last seventy-five years. This survey indicated that the vast majority of published sources are dealing with 2DOF systems in which one or both masses are respectively connected to one or two non-movable supports. Additionally, the survey found that the overwhelming majority of related publications are textbooks in the area of the theory of vibration. These textbooks present the schematics and the pairs of the simultaneous second-order linear differential equations that supposedly describe the motion of these systems. The survey did not find any rigorous mathematical solutions of these differential equations or any kinematic aspects that are specifically related to 2DOF systems.

This paper highlights that in a 2DOF system, the motion of each of the two masses is characterized by its independent law of motion. This implies that the motion of each of the masses is described by different analytical expressions. Therefore, the masses of a 2DOF system constantly perform relative motion. Relative motion of the masses has a definite influence on the values of the forces that are exerted by the connecting links toward these masses. In 2DOF systems, the values of these forces are proportional to the differences between the absolute values of the corresponding parameters of motion (displacements and velocities). In contrast, in 1DOF systems, the forces exerted by the connecting links are proportional to the absolute values of the corresponding parameters of motion. Since in 2DOF systems each pair of differential equations describing the motion of the masses contains mathematical terms related to forces exerted by certain connecting links, it is necessary that the structures of these terms comply with the principle of relative motion regarding the proportionality of the exerted forces to differences between the parameters of motion.

Analyzing the compositions of all surveyed pairs of corresponding differential equations of motion, it was revealed that all these equations comprise combinations of mathematical terms that correspond to 1DOF systems and 2DOF systems. Obviously, this is not allowable in differential equations of motion, and, therefore, all surveyed differential equations of motion for 2DOF systems are not applicable to describe motion of these systems.

Accounting for the relative motion of the masses of 2DOF systems, I revised all the aforementioned surveyed differential equations of motion and based on the Laplace Transform methodology, I obtained for all of them rigorous mathematical solutions. A few typical examples are presented in this paper. In addition, I composed and solved a significant amount of pairs of simultaneous differential equations of motion of 2DOF systems addressing all possible structural compositions that are subjected to all relevant loading combinations at general initial conditions of motion.

Appendix

Table 1: Laplace Transform Pairs

#	**Time domain**	**Laplace domain**
1	$Constant$	$Constant$
2	$u(t)$ or u	$u(s)$
3	$\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}$	$s^2u(s) - s\dfrac{du(0)}{dt} - s^2u(0)$

Continued on next page

Table 1 – continued from previous page

#	Time domain	Laplace domain
4	$\sin \omega t$	$\dfrac{\omega s}{s^2+\omega^2}$
5	$\dfrac{1}{\omega^2}(t-\dfrac{1}{\omega}\sin \omega t)$	$\dfrac{1}{s(s^2+\omega^2)}$

References

[1] Spektor, M., *Applied Dynamics in Engineering*, Industrial Press, 2016

[2] Spektor, M., *Solving Engineering Problems in Dynamics*, Industrial Press, 2014

[3] Bronson, R., *2500 Solved Problems in Differential Equations*, McGraw-Hill Book Company, 1989

[4] Edwards, C., Fenney, D., *Differential Equations, Computing and Modeling*, Pearson Prentice Hall, 2019

[5] Timoshenko, S., *Vibration Problems in Engineering*, Third Edition, D. Van Nostrand Company, Inc, 1955

[6] Den Hartog, J., *Mechanical Vibrations*, Forth Edition, McGraw-Hill Book Company, Inc, 1956

[7] Thomson, W., *Theory of Vibration with Applications*, Third Edition, Prentice Hall, 1988

[8] Ogata, K., *System Dynamics*, Fourth Edition, Pearson Prentice Hall, 2004

[9] James, M., Smith, G., Wolford, J., Whaley, P., *Vibration of Mechanical and Structural Systems,* Harper & Row. Publishers, 1989

[10] Wirsching, P., Paez, T., Ortiz, K., *Random Vibrations, Theory and Practice*, Dover Books on Physics, 2006

[11] Cai, L., *Fundamentals of Mechanical Vibrations*, John Wiley & Sons, 2016

[12] Babakov, I., *Theory of Vibrations*, "Nauka", 1955 (In Russian)

[13] Shabana, A., *Theory of Vibration*, Springer-Verlag, 1991

[14] Yablonskiy, A., Noreyko, S., *Course of Theory of Vibrations*, "Vysshaya Shkola", 1966 (In Russian)

[15] Meirovitch, L. *Fundamentals of Vibration*, McGraw-Hill Book Company, 2001

[16] Kelly, S. *Mechanical Vibrations*, Cengage Learning, 2012