Nilpotent Waveform Relaxation Methods for Chains of Passive Circuits

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***Abstract* — New results on the nilpotency of the waveform relaxation (WR) algorithm are presented for chains of general linear time-invariant circuits. Strictly dissipative impedance coupling is used in the WR method to decouple the cascaded parts. Three relaxation schemes: Gauss-Jacobi, Gauss-Seidel and relaxation by forward and reverse sweeping implement the WR iterations. The analysis of the operator matrices in the Fourier domain leads to the characterization of the nilpotent WR operator for the three relaxation schemes.**

***Index Terms*—Waveform relaxation, impedance coupling, transmission conditions, iteration matrix, nilpotent operators, transient simulation.**

I. Introduction

WAVEFORM RLAXATION (WR) techniques [1]–[7] were introduced in the early 1980s as an alternative to direct time-stepping algorithms used by circuit simulators for solving ordinary differential equations. The concept of WR is based on partitioning the original circuit into subcircuits that are solved independently of each other on the entire time interval of interest. Coupling effects among individual subcircuits are represented by time-domain sources that are initially assumed to be known. Circuit response is reached iteratively through repeated estimations of these coupling effects. The main steps of WR circuit computing can be found in [8].

Several techniques were proposed in the literature to improve the efficiency of WR analysis including dynamic partitioning, scheduling, and time windowing, see [9, Ch. 8, pp. 41-123] and references therein. However, the rate of convergence remains a major challenge facing WR techniques and limits their applicability. To obtain fast convergence, it is necessary that partitioning the original circuit is done in such a way that the coupling among the individual subcircuits is weak. A requirement which is difficult to fulfill in general in longitudinal partitioning schemes (LP).

The idea of optimal WR convergence was brought to circuit problems in [23] and led to the emergence of a class of WR-LP methods called optimized WR. These methods are suitable for strongly coupled serial circuits. They attain a faster and more uniform convergence through the exchange of appropriate combinations of current and voltage waveforms between adjacent subcircuits instead of just a current or a voltage, as in classical WR [23].

Such information exchange was implemented in the so-called transmission conditions (TC) equations [23, eqn. (2.5)],[30, eqns. (17),(18)] for one-node overlap and [24, eqn. (2.9)] for two-node overlap. Optimal convergence requires nonlocal operators in the TC equations and therefore is expensive [24]. To avoid this obstacle, constant and first-degree polynomial approximations of the optimal operators were calculated for RC ladder circuit in [24] and in sequel works [25]-[27], for one lumped RLCG line type circuit [28],[29], and for the PEEC circuit [30].

The WR-LP algorithms in the early and intuitive work [10],[11] and later in [12]-[16] are not classic. According to [32, Sec. II.B], they executed what was defined later as transmission conditions [23, eq. (2.5)] on the line $-\alpha = \beta = R_o{}^{-1}$ by inserting a neutral series connection of three resistances $(-R_o)$, $(2R_o)$, and $(-R_o)$, $R_o > 0$, between every channel or TL and its terminations. The algorithm [31] also executed conditions [23, eqn. (2.5)] on region $\mathbb{R}_*^- \times \mathbb{R}_*^+$: $(\alpha, \beta) = \left(-R_2{}^{-1}, R_1{}^{-1}\right)$ by replacing insertions $(-R_o)$, $(2R_o)$, and $(-R_o)$ [10]-[13] with $(-R_1)$, $(R_1 + R_2)$, and $(-R_2)$ where R_1 and R_2 are different in general. The WR-LP [31] relies on a numerical optimization step to calculate near-to-optimal values for its coupling resistances. The replacement of coupling resistances by strictly dissipative impedances produces a general and consistent WR-LP algorithm [32, Lemma D.1]. In the one-node overlap case, coupling impedances represented kernels of convolution integral operators in the companion TC equations [32, eqns. (5),(6)]. The algorithm is optimal when its kernels are equal to the driving-point impedances of the decoupled parts themselves [32, Thms. D.3,D.4]. An optimal kernel produces a null transmission gain in the corresponding direction and results in a null local convergence factor.

The optimal WR-LP algorithm is not cost-efficient, however judicious approximations of its kernels produced cost-efficient methods at suboptimal speeds of convergence [24],[25],[29]-[32]. In the same way, a nilpotent WR-LP algorithm is not expected to be cost-efficient either but clever approximations of its nilpotency condition would also lead to efficient algorithms. Like the convergence of the WR, the existence and possibly the characterization of the nilpotent algorithm depend closely on the underlying circuit problem to be solved, see [33] for instance. The characterization of the nilpotent set would be useful in designing efficient approximation schemes.

This paper concerns the application of the WR-LP [32] for the time-domain analysis of chains of general passive linear time-invariant (LTI) circuits [34]. It presents new results on the nilpotency of the WR-LP method with respect to three relaxations schemes: Gauss-Jacobi relaxation (GJ), Gauss-Seidel relaxation (GS), and relaxation by forward and reverse sweeping (S). The iteration matrices of the corresponding operators: GJ-WR, GS-WR and S-WR are derived in the Fourier domain and their nilpotency is investigated for positive real frequencies, $\omega \geq 0$. It is shown that nilpotent operators must have null local convergence factors, but the converse is not true. It is demonstrated that all nilpotent S-WR operators are optimal whereas there exist nilpotent GJ-WR and GS-WR that are not optimal. The analysis reveals that the way forward and reverse transmission gains are set to zero affects the indices of the nilpotent GJ-WR and GS-WR. Finally, optimality condition [29, Thm. 2.1],[32, Thm. D.4] is relaxed and is replaced with two new conditions; one for chains with an odd number of subcircuits and the other is for chains with an even number of subcircuits.

The rest of the paper is organized as follows: A review of impedance coupling [32] is presented in section II.A, and the implementation of GJ-WR, GS-WR and S-WR are briefly explained in section II.B. The iteration matrices of the three WR-LP operators are constructed in section III and their nilpotency is examined in section IV. Numerical experiments are presented in section V to validate the theoretical results.

II. Preliminaries

A. Strictly dissipative impedance coupling [32]

Impedance coupling applies on any two parts P_1 and P_2 in a circuit that are connected in an open-loop configuration, either directly or via a strictly dissipative element Z_o (Fig. 1). Impedance coupling is realized by inserting a neutral series connection of three impedances $(-\zeta_1)$, ζ_Σ and $(-\zeta_2)$ at the split node between P_1 and P_2. Coupling impedances ζ_1 and ζ_2 are strictly dissipative, ie $\zeta_1(s)$ and $\zeta_2(s)$ are two strictly positive-real complex functions of the complex frequency $s = \sigma + i\omega$, $(\sigma, \omega) \in \mathbb{R}_+ \times \mathbb{R}$, ω is the real frequency and $i^2 = -1$. The two-terminal circuit ζ_Σ represents the created overlap from insertion $\{(-\zeta_1), \zeta_\Sigma, (-\zeta_2)\}$. Overlap $\zeta_\Sigma = \zeta_1 + \zeta_2$ for a direct connection and $\zeta_\Sigma = \zeta_1 + \zeta_2 + Z_o$ in the presence of Z_o (Fig. 1).

Two possible circuit realizations of impedance coupling are presented in Fig. 2 and in Fig. 3. In the first interface, coupling impedances ζ_1, ζ_2 and their additive inverses $(-\zeta_1)$ and $(-\zeta_2)$ are realized according to the procedure explained in [32, Sec II.C]. The relaxation algorithm exchanges nodal voltages at the two extremities of ζ_Σ in P_1 and P_2 in order to implement the iteration regardless of the type of ζ_1 and ζ_2 (Fig. 2)

$$w_1^{1,2(k+1)} = v_2^{1,2(k)} \tag{1}$$

$$w_2^{1,2(k+1)} = v_1^{1,2(k+\upsilon)} \tag{2}$$

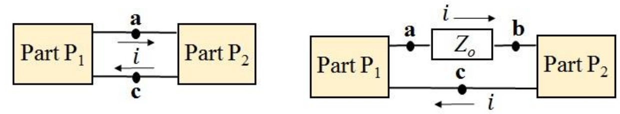

Fig. 1 Parts N_1 and N_2 are connected in an open-loop configuration. Left. Direct connection. Right. Connection via element Z_o.

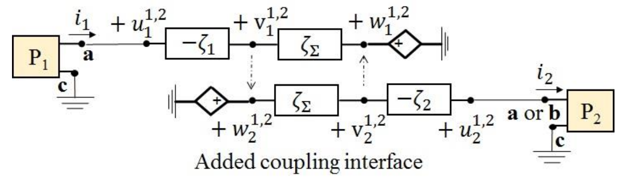

Fig. 2 First circuit representation of impedance coupling.

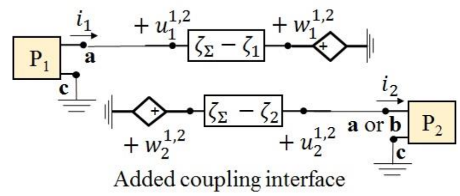

Fig. 3 Second (compact) circuit representation of impedance coupling.

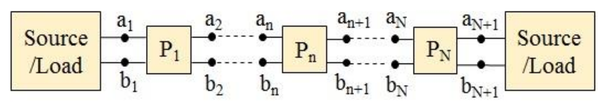

Fig. 4 Chain connection of N parts

Parameter υ =0 in the GJ relaxation and υ =0 in the GS case. Lumping ζ_Σ and $(-\zeta_1)$ together on the side of P_1 and ζ_Σ and $(-\zeta_2)$ together on the side of P_2 leads to a compact interface (Fig. 3). Nodal voltage exchange (1),(2) is no longer possible since nodes $v_2^{1,2}$ and $v_1^{1,2}$ are not available. The algorithm uses instead a more elaborate voltage exchange, expressed in the Laplace space as, see [32, eqns. (3),(4)]

$$\tilde{w}_1^{1,2(k+1)} = \tilde{u}_2^{1,2(k)}(\zeta_1 + \zeta_2)/\zeta_1 - \tilde{w}_2^{1,2(k)}\zeta_2/\zeta_1 \tag{3}$$

$$\tilde{w}_2^{1,2(k+1)} = \tilde{u}_1^{1,2(k+\upsilon)}(\zeta_1 + \zeta_2)/\zeta_2 - \tilde{w}_1^{1,2(k+\upsilon)}\zeta_2/\zeta_1 \tag{4}$$

Aside from case $\zeta_1 = \lambda\zeta_2$, $\lambda \in \mathbb{R}_+^*$, where update equations (3) and (4) use local data points at every time step

$$w_1^{1,2(k+1)} = u_2^{1,2(k)}(1 + \lambda)/\lambda - w_2^{1,2(k)}/\lambda \tag{5}$$

$$w_2^{1,2(k+1)} = u_1^{1,2(k+\upsilon)}(1 + \lambda) - w_1^{1,2(k+\upsilon)}\lambda \tag{6}$$

the source update in general requires nonlocal operators in time. When kernels ζ_1 and ζ_2 are nonproportional complex rational or irrational functions, it is possible to avoid the repetitive expensive source update by also realizing the additive inverses $(-\zeta_1)$ and $(-\zeta_2)$ in the first implementation (Fig. 2).

B. GJ-WR, GS-WR and S-WR in the chain problem

Consider a cascaded chain of N parts P_n, $1 \leq n \leq N$ and $N \geq 3$, see Fig 4. To compute the WR solution, every two consecutive parts P_n and P_{n+1}, $1 \leq n \leq N-1$, are decoupled

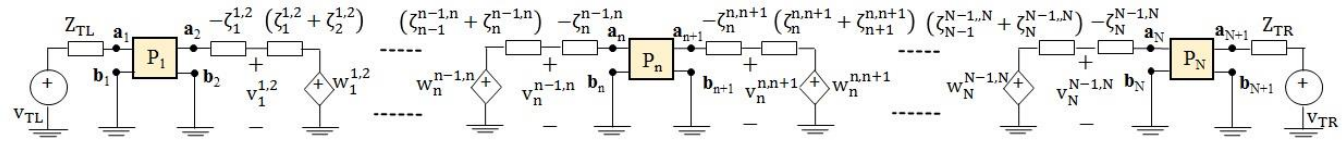

Fig. 5 Decoupled parts of the chain. First interface implementation.

ALGORITHM 1 GJ-WR

Initialize all relaxation sources
$k \leftarrow 0$: set iteration count to zero
$\varepsilon \leftarrow \epsilon$: threshold tolerance ϵ
Do
 Begin
  Solve all parts P_n using their input $w_n^{n-1,n^{(k)}}$ and $w_n^{n,n+1^{(k)}}$
  Collect $v_n^{n-1,n^{(k)}}$ for all $2 \le n \le N$ and $v_n^{n,n+1^{(k)}}$ for all $1 \le n \le N-1$
  $w_n^{n,n+1^{(k+1)}} \leftarrow v_{n+1}^{n,n+1^{(k)}}$ for all $1 \le n \le N-1$
  $w_n^{n-1,n^{(k+1)}} \leftarrow v_{n-1}^{n-1,n^{(k)}}$ for all $2 \le n \le N$
  Compute errors
  $k \leftarrow k+1$
 End
While errors $\le \varepsilon$

ALGORITHM 2 GS-WR

Initialize relaxation sources of odd All numbered parts P_{2m+1}
$k \leftarrow 0$: set iteration count to zero
$\varepsilon \leftarrow \epsilon$: threshold tolerance ϵ
Do
 Begin
  Solve all parts P_{2m-1} using inputs $w_{2m-1}^{2m-1,2m^{(k)}}$ and $w_{2m-1}^{2m-2,2m-1^{(k)}}$
  Collect all voltages $v_{2m-1}^{2m-1,2m^{(k)}}$ and $v_{2m-1}^{2m-21,2m-1^{(k)}}$
  $w_{2m}^{2m-1,2m^{(k)}} \leftarrow v_{2m-1}^{2m-1,2m^{(k)}}$ and
  $w_{2m}^{2m,2m+1^{(k)}} \leftarrow v_{2m+1}^{2m,2m+1^{(k)}}$ for all P_{2m}
  Solve all parts P_{2m} using inputs $w_{2m}^{2m-1,2m^{(k)}}$ and $w_{2m}^{2m,2m+1^{(k)}}$
  Collect all voltages $v_{2m}^{2m-1,2m^{(k)}}$ and $v_{2m}^{2m,2m+1^{(k)}}$
  $w_{2m-1}^{2m-1,2m^{(k+1)}} \leftarrow v_{2m}^{2m-1,2m^{(k)}}$ and
  $w_{2m-1}^{2m-2,2m-1^{(k+1)}} \leftarrow v_{2m-2}^{2m-2,2m-1^{(k)}}$ for all P_{2m-1}
  Compute errors
  $k \leftarrow k+1$
 End
While errors $\le \varepsilon$

by inserting a neutral series connection of three impedances $\{-\zeta_n^{n,n+1}, \zeta_n^{n,n+1}+\zeta_{n+1}^{n,n+1}, -\zeta_{n+1}^{n,n+1}\}$ at node a_{n+1} and grounding node b_{n+1}, see Fig. 5. Coupling impedances $\zeta_{n+1}^{n,n+1}$ and $\zeta_n^{n,n+1}$ represent the total additions to P_n and P_{n+1} whereas $w_n^{n,n+1}(t)$ and $w_{n+1}^{n,n+1}(t)$, $t \ge 0$, are the WR external variables and are realized as voltage sources. The iteration

ALGORITHM 3 S-WR

Initialize relaxation sources $w_n^{n,n+1}$ of all parts P_n
$k \leftarrow 0$: set double sweep count to zero
$\varepsilon \leftarrow \epsilon$: threshold tolerance ϵ
DO
 BEGIN
  $k \leftarrow k+1$
  FOR $m=1$ **TO** $N-1$ **DO**
   Solve part P_m
   Collect voltage $v_m^{m,m+1^{(k)}}$
   $w_{m+1}^{m,m+1^{(k)}} \leftarrow v_m^{m,m+1^{(k)}}$
  ENDFOR
  FOR $m=N$ **TO** 2 **DO**
   Solve part P_m
   Collect voltage $v_{m-1}^{m-1,m^{(k)}}$
   $w_{m-1}^{m-1,m^{(k)}} \leftarrow v_{m-1}^{m-1,m^{(k)}}$
  ENDFOR
  Compute errors in all relaxation sources
 END
WHILE errors $\le \varepsilon$

is sustained by the repetitive update of sources $w_n^{n-1,n}$, $2 \le n \le N$, and $w_n^{n,n+1}$, $1 \le n \le N-1$ using eqns. (1),(2). The three pseudocodes in ALGORITHMS 1, 2 and 3 summarize the main steps in executing the GJ-WR, GS-WR, and S-WR for the first implementation (Fig. 2).

III. Construction of WR Iteration Matrices

Every part P_n is represented with a LTI reciprocal two-port network. For the linear initial-value problem represented by the circuit of Fig. 5, it suffices to study the convergence of the error to the zero solution of the subsequent homogeneous problem with zero initial conditions represented by the circuit of Fig. 6. The two terminations of the chain are replaced by their respective input impedances Z_{TL} and Z_{TR}. Let $e_1^{1,2}(t)^{(k)}$,.., $e_n^{n-1,n}(t)^{(k)}$, $e_n^{n,n+1}(t)^{(k)}$,.., $e_N^{N-1,N}(t)^{(k)}$, $2 \le n \le N-1$, be the differences between the WR external variables $w_1^{1,2}(t)$,.., $w_n^{n-1,n}(t)^{(k)}$, $w_n^{n,n+1}(t)^{(k)}$,.., $w_N^{N-1,N}(t)^{(k)}$ calculated at iteration k and their final waveforms at convergence. Let $\hat{e}_1^{1,2^{(k)}}$,.., $\hat{e}_{n-1}^{n-1,n^{(k)}}$, $\hat{e}_n^{n-1,n^{(k)}}$,.., $\hat{e}_N^{N-1,N^{(k)}}$ be their respective Fourier transforms. The iteration matrices are obtained with respect to the following $2(N-1)\times 1$ error vectors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$.

$$\mathbf{e_1} = \left(\hat{e}_2^{1,2}, \hat{e}_1^{1,2}, \hat{e}_3^{2,3}, \hat{e}_2^{2,3}, \ldots, \hat{e}_n^{n-1,n}, \hat{e}_{n-1}^{n-1,n}, \ldots, \hat{e}_N^{N-1,N}, \hat{e}_{N-1}^{N-1,N}\right)^T \tag{9}$$

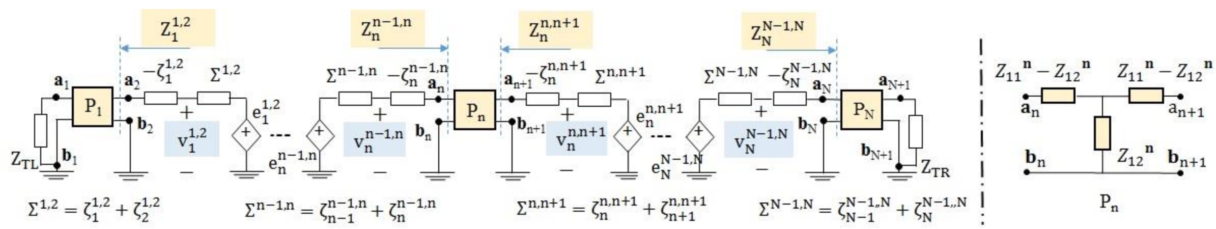

Fig. 6 Chain connection. Left. Homogeneous problem. Right. Frequency-domain representation of part P_n.

for the GJ-WR,

$$\mathbf{e_2}=\begin{pmatrix}\mathbf{e_{2,\mathit{o}}}\\ \mathbf{e_{2,\mathit{e}}}\end{pmatrix}$$

$$\mathbf{e_{2,\mathit{o}}}=\left(\hat{e}_1^{1,2},\hat{e}_3^{2,3},\hat{e}_3^{3,4},\ldots,\hat{e}_{2n+1}^{2n,2n+1},\hat{e}_{2n+1}^{2n+1,2n+2},\ldots\right)^T$$

$$\mathbf{e_{2,\mathit{e}}}=\left(\hat{e}_2^{1,2},\hat{e}_2^{2,3},\ldots,\hat{e}_{2n}^{2n-1,2n},\hat{e}_{2n}^{2n,2n+1},\ldots\right)^T \tag{10}$$

for the GS-WR, and

$$\mathbf{e_3}=\begin{pmatrix}\mathbf{e_{2,\mathit{f}}}\\ \mathbf{e_{2,\mathit{b}}}\end{pmatrix}$$

$$\mathbf{e_{2,\mathit{f}}}=\left(\hat{e}_2^{1,2},\hat{e}_3^{2,3},\ldots,\hat{e}_{n+1}^{n,n+1},\ldots,\hat{e}_{N-1}^{N-2,N-1},\hat{e}_N^{N-1,N}\right)^T$$

$$\mathbf{e_{2,\mathit{b}}}=\left(\hat{e}_1^{1,2},\hat{e}_2^{2,3},\ldots,\hat{e}_n^{n,n+1},\ldots,\hat{e}_{N-2}^{N-2,N-1},\hat{e}_{N-1}^{N-1,N}\right)^T \tag{11}$$

for the S-WR. Subscripts $\boldsymbol{o}, \boldsymbol{e}, \boldsymbol{f}$ and $\boldsymbol{b}$ stand for odd, even, forward and backward whereas T is the transpose operator. Note that it is possible to put relaxations sources $e_1^{1,2}(t)$, $e_2^{1,2}(t)$,.., $e_n^{n,n+1}(t)$, $e_{n+1}^{n,n+1}(t)$,.., $e_{N-1}^{N-1,N}(t)$, $e_N^{N-1,N}(t)$ in any order to form vectors $\boldsymbol{e_1}$ and in $\mathbf{e_{2,\mathit{o}}}$ and $\mathbf{e_{2,\mathit{e}}}$ to form $\boldsymbol{e_2}$. Different orderings in vectors $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ lead to similar iteration matrices. Ordering (9) considers the location of the primary inputs and the signal simultaneous propagations along both directions in a bidirectional chain. Ordering (10) reflects the two-time update during one GS iteration. Both GJ-WR and GS-WR update their sources simultaneously on both directions along the serial partition. The S-WR algorithm however updates its sources successively along same one direction: from P_1 to P_N (forward sweep) then from P_N back to P_1 (backward sweep), which justifies ordering (11)**.** The iteration matrix of the S-WR is not similar to those of GJ-WR and GS-WR. Finally, it is worth noticing that errors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ are used only to study convergence. In practice, the consistent algorithms retain the values of their external variables from the last two iterations. They calculate their difference and use it as an error estimate to detect convergence.

To update the error sources (1),(2), it is necessary to calculate voltages $v_1^{1,2}$ at node a_2 in P_1, $v_n^{n-1,n}$ and $v_n^{n,n+1}$ at nodes a_n and a_{n+1} in any internal part P_n, and finally $v_N^{N-1,N}$ at node a_N in P_N (Fig. 6). Basic circuit analysis of the decoupled parts in the frequency domain (Fig. 10), results in the following equations

$$\hat{v}_1^{1,2}=a_1^+\hat{e}_2^{1,2}$$

$$\hat{v}_n^{n,n+1}=b_n^+\,\hat{e}_n^{n-1,n}+a_n^+\,\hat{e}_n^{n,n+1}$$

$$\hat{v}_n^{n-1,n}=a_n^-\,\hat{e}_n^{n-1,n}+b_n^-\,\hat{e}_n^{n,n+1}\quad,(2\le n\le N-1)$$

$$\hat{v}_N^{N-1,N}=a_N^-\hat{e}_N^{N-1,N} \tag{12}$$

Coefficient a_n^+, $1\le n\le N-1$, is the forward transmission gain from part P_n to P_{n+1} whereas a_n^-, $2\le n\le N$, is the backward or reverse transmission gain from P_n back to P_{n-1}. Coefficients a_n^+, b_n^+, b_n^- and a_n^- are given as

$$a_n^+=\frac{Z_n^{n,n+1}-\zeta_n^{n,n+1}}{Z_n^{n,n+1}+\zeta_{n+1}^{n,n+1}}\quad,(1\le n\le N-1)$$

$$a_n^-=\frac{Z_n^{n-1,n}-\zeta_n^{n-1,n}}{Z_n^{n-1,n}+\zeta_{n-1}^{n-1,n}}\quad,(2\le n\le N)$$

$$b_n^+=\frac{\left(\zeta_n^{n,n+1}+\zeta_{n+1}^{n,n+1}\right)Z_{12}^{\mathbf{n}}}{\left(Z_n^{n-1,n}+\zeta_{n-1}^{n-1,n}\right)\left(Z_{22}^{\mathbf{n}}+\zeta_{n+1}^{n,n+1}\right)}$$

$$b_n^-=\frac{\left(\zeta_{n-1}^{n-1,n}+\zeta_n^{n-1n}\right)Z_{12}^{\mathbf{n}}}{\left(Z_n^{n,n+1}+\zeta_{n+1}^{n,n+1}\right)\left(Z_{11}^{\mathbf{n}}+\zeta_{n-1}^{n-1,n}\right)}\quad,(2\le n\le N-1) \tag{13}$$

where $Z_{11}^{\mathbf{n}}(i\omega)$, $Z_{12}^{\mathbf{n}}(i\omega)$, $Z_{21}^{\mathbf{n}}(i\omega)$ and $Z_{22}^{\mathbf{n}}(i\omega)$ are the impedance parameters of part P_n, $1\le n\le N$, such that $Z_{12}^{\mathbf{n}}=Z_{21}^{\mathbf{n}}$, see Fig. 6. Every interior P_n possesses two input impedances $Z_n^{n-1,n}$ and $Z_n^{n,n+1}$ taken at its nodes a_n and a_{n+1} respectively. At the extremities, $Z_1^{1,2}$ and $Z_N^{N-1,N}$ represent the input impedances of parts P_1 and P_N at nodes a_1 and a_N.

Using (1),(2), the interdependence between individual errors is expressed as

$$\hat{e}_2^{1,2\,(k+\nu_1)}=a_1^+\hat{e}_1^{1,2\,(k)}$$

$$\hat{e}_n^{n-1,n\,(k+\nu_1)}=b_{n-1}^+\hat{e}_{n-1}^{n-2,n-1\,(k+\nu_2)}+a_{n-1}^+\hat{e}_{n-1}^{n-1,n\,(k)}$$

$$(3\le n\le N)$$

$$\hat{e}_{n-1}^{n-1,n\,(k+\nu_1)}=a_n^-\hat{e}_n^{n-1,n\,(k+\nu_2)}+b_n^-\hat{e}_n^{n,n+1\,(k+\nu_2)}$$

$$(2\le n\le N-1)$$

$$\hat{e}_{N-1}^{N-1,N\,(k+\nu_1)}=a_N^-\hat{e}_N^{N-1,N\,(k+\nu_2)} \tag{14}$$

parameters $\upsilon_1=1$ and $\upsilon_2=0$ for GJ-WR (ALGORITHM 1), $\upsilon_1=1$ for n odd, $\upsilon_1=0$ for n even, and $\upsilon_2=0$ for GS-WR (ALGORITHM 2), and finally $\nu_1=\nu_2=1$ for S-WR (ALGORITHM 3).

A. GJ-WR

A recurrent relation over one iteration, is obtained with respect to vector $\mathbf{e_1}$

$$\mathbf{e_1}^{(k+1)}=\mathbf{J}_N\,\mathbf{e_1}^{(k)} \tag{15}$$

Matrix $\mathbf{J}_N\in\mathbb{C}^{2(N-1)\times 2(N-1)}$ is defined in eq. (16).

$$
\mathbf{J}_{\mathrm{N}} = \left[\begin{array}{cc|cc|cc|cc|cc|cc}
0 & a_1^+ & 0 & 0 & & & & & & & & \\
a_2^- & 0 & 0 & b_2^- & & & & & & & & \\
\hline
b_2^+ & 0 & 0 & a_2^+ & \ddots & \ddots & & & & & & \\
0 & 0 & a_3^- & 0 & \ddots & \ddots & & & & & & \\
\hline
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
\hline
 & & & & \ddots & \ddots & 0 & a_{\mathrm{N}-2}^+ & 0 & 0 & & \\
 & & & & \ddots & \ddots & a_{\mathrm{N}-1}^- & 0 & 0 & b_{\mathrm{N}-1}^- & & \\
\hline
 & & & & & & b_{\mathrm{N}-1}^+ & 0 & 0 & a_{\mathrm{N}-1}^+ & & \\
 & & & & & & 0 & 0 & a_{\mathrm{N}}^- & 0 & &
\end{array}\right] \tag{16}
$$

A second iteration must be performed to obtain a recurrent relation for single errors

$$
\mathbf{e_1}^{(k+2)} = (\mathbf{J}_{\mathrm{N}})^2 \, \mathbf{e_1}^{(k)} \tag{17}
$$

Matrix $(\mathbf{J}_{\mathrm{N}})^2$ in eqn. (18), represents the GJ iteration matrix over two iterations. It will be referred to as the GJ iteration matrix in the rest of the text. Its diagonal elements $a_{\mathrm{n}}^+(i\omega)a_{\mathrm{n}+1}^-(i\omega)$, $1 \le \mathrm{n} \le \mathrm{N}-1$, are the local rates of the WR for the chain partition.

A. GS-WR

A recurrent relation over one iteration with respect to vector $\mathbf{e_2}$, is produced

$$
\mathbf{e_2}^{(k+1)} = \mathbf{S}_{\mathrm{N}} \, \mathbf{e_2}^{(k)} \tag{19}
$$

$$
\mathbf{S}_{\mathrm{N}} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \tag{20}
$$

$\mathbf{S}_{\mathrm{N}} \in \mathbb{C}^{2(\mathrm{N}-1)\times 2(\mathrm{N}-1)}$ is the iteration matrix of the GS-WR. Block matrices $\mathbf{E}$, $\mathbf{F} \in \mathbb{C}^{(\mathrm{N}-1)\times(\mathrm{N}-1)}$ are given in eqns. (21),(22) for N even and in eqns. (23),(24) for N odd.

A. S-WR

A recurrent relation over two consecutive sweeps (one forward then one backward) with respect to vector $\mathbf{e_3}$, is produced

$$
\mathbf{e_3}^{(k+1)} = \mathbf{R}_{\mathrm{N}} \, \mathbf{e_3}^{(k)} \tag{25}
$$

$$
\mathbf{R}_{\mathrm{N}} = \begin{bmatrix} (\mathbf{A}^+ \circ \mathbf{B}^+)\mathbf{T}(\mathbf{A}^- \circ \mathbf{B}^-)\mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}(\mathbf{A}^- \circ \mathbf{B}^-)\mathbf{T}(\mathbf{A}^+ \circ \mathbf{B}^+) \end{bmatrix} \tag{26}
$$

$\mathbf{R}_{\mathrm{N}} \in \mathbb{C}^{2(\mathrm{N}-1)\times 2(\mathrm{N}-1)}$ is the iteration matrix of the S-WR. Operator $\circ$ refers to Hadamard product [35] and $\mathbf{T} \in \mathbb{C}^{(\mathrm{N}-1)\times(\mathrm{N}-1)}$ is the involutory backward identity matrix. Matrices $\mathbf{A}^+ := (r_{ij})$, $\mathbf{B}^+ := (s_{ij})$, $\mathbf{A}^- := (t_{ij})$ and $\mathbf{B}^- := (u_{ij})$ are lower triangular of order $(\mathrm{N}-1)$.

$$
r_{ij} = \begin{cases} 0 & , i < j \\ a_j^+ & , i \ge j \end{cases} \tag{27}
$$

$$
s_{ij} = \begin{cases} 0 & , i < j \\ \displaystyle\prod_{k=j+1}^{i} b_k^+ & , i \ge j \end{cases} \tag{28}
$$

$$
(\mathbf{J}_{\mathrm{N}})^2 = \left[\begin{array}{cc|cc|cc|cc|cc|cc|cc}
a_1^+a_2^- & 0 & 0 & a_1^+b_2^- & 0 & 0 & & & & & & & & \\
0 & a_1^+a_2^- & b_2^-a_3^- & 0 & 0 & b_2^-b_3^- & & & & & & & & \\
\hline
0 & a_1^+b_2^+ & a_2^+a_3^- & 0 & 0 & a_2^+b_3^- & \ddots & \ddots & & & & & & \\
b_2^+a_3^- & 0 & 0 & a_2^+a_3^- & b_3^-a_4^- & 0 & \ddots & \ddots & & & & & & \\
\hline
b_2^+b_3^+ & 0 & 0 & a_2^+b_3^+ & a_3^+a_4^- & 0 & \ddots & \ddots & \ddots & \ddots & & & & \\
0 & 0 & b_3^+a_4^- & 0 & 0 & a_3^+a_4^- & \ddots & \ddots & \ddots & \ddots & & & & \\
\hline
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
\hline
 & & & & \ddots & \ddots & \ddots & \ddots & a_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & 0 & 0 & a_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^- & 0 & 0 \\
 & & & & \ddots & \ddots & \ddots & \ddots & 0 & a_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & b_{\mathrm{N}-2}^-a_{\mathrm{N}-1}^- & 0 & 0 & b_{\mathrm{N}-2}^-b_{\mathrm{N}-}^- \\
\hline
 & & & & & & \ddots & \ddots & 0 & a_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^+ & a_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & 0 & 0 & a_{\mathrm{N}-2}^+b_{\mathrm{N}-}^- \\
 & & & & & & \ddots & \ddots & b_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & 0 & 0 & a_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & b_{\mathrm{N}-1}^-a_{\mathrm{N}}^- & 0 \\
\hline
 & & & & & & & & b_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^+ & 0 & 0 & a_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^+ & a_{\mathrm{N}-1}^+a_{\mathrm{N}}^- & 0 \\
 & & & & & & & & 0 & 0 & b_{\mathrm{N}-1}^+a_{\mathrm{N}}^- & 0 & 0 & a_{\mathrm{N}-1}^+a_{\mathrm{N}}^-
\end{array}\right] \tag{18}
$$

$$
\mathbf{E} = \left[\begin{array}{cc|cc|cc|cc|cc|cc|c}
a_1^+a_2^- & b_2^+a_3^- & b_2^-b_3^- & 0 & & & & & & & & & \\
a_1^+b_2^+ & a_2^+a_3^- & a_2^+b_3^- & 0 & & & & & & & & & \\
\hline
0 & b_3^+a_4^- & a_3^+a_4^- & b_4^-a_5^- & b_4^-b_5^- & 0 & & & & & & & \\
0 & b_3^+b_4^+ & a_3^+b_4^+ & a_4^+a_5^- & a_4^+b_5^- & 0 & & & & & & & \\
\hline
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
\hline
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
\hline
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\hline
 & & & & & & & & 0 & b_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & a_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & b_{\mathrm{N}-2}^-a_{\mathrm{N}-1}^- & b_{\mathrm{N}-2}^-b_{\mathrm{N}-1}^- \\
 & & & & & & & & 0 & b_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^- & a_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^+ & a_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & a_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^- \\
\hline
 & & & & & & & & & & 0 & b_{\mathrm{N}-1}^+a_{\mathrm{N}}^- & a_{\mathrm{N}-1}^+a_{\mathrm{N}}^-
\end{array}\right] \tag{21}
$$

$$\mathbf{F}=\left[\begin{array}{cc|cc|cc|cc|cc|cc|c}
a_1^+a_2^- & a_1^+b_2^- & 0 & 0 & & & & & & & & & \\
b_2^+a_3^- & a_2^+a_3^- & b_3^-a_4^- & b_3^-b_4^- & & & & & & & & & \\
\hline
b_2^+b_3^+ & a_2^+b_3^+ & a_3^+a_4^- & a_3^+b_4^- & 0 & 0 & & & & & & & \\
0 & 0 & b_4^+a_5^- & a_4^+a_5^- & b_5^-a_6^- & b_5^-b_6^- & & & & & & & \\
\hline
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
\hline
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
\hline
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\hline
 & & & & & & & & b_{\mathrm{N}-4}^+b_{\mathrm{N}-3}^+ & a_{\mathrm{N}-4}^+b_{\mathrm{N}-3}^+ & a_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & a_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^- & 0 \\
 & & & & & & & & 0 & 0 & b_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & a_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & b_{\mathrm{N}-1}^+a_{\mathrm{N}}^- \\
\hline
 & & & & & & & & & & b_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^+ & a_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^+ & a_{\mathrm{N}-1}^+a_{\mathrm{N}}^-
\end{array}\right] \tag{22}$$

$$\mathbf{E}=\left[\begin{array}{cc|cc|cc|cc|cc|cc}
a_1^+a_2^- & b_2^-a_3^- & b_2^-b_3^- & 0 & & & & & & & & \\
a_1^+b_2^+ & a_2^+a_3^- & a_2^+b_3^- & 0 & & & & & & & & \\
\hline
0 & b_3^+a_4^- & a_3^+a_4^- & b_4^-a_5^- & b_4^-b_5^- & 0 & & & & & & \\
0 & b_3^+b_4^+ & a_3^+b_4^+ & a_4^+a_5^- & a_4^+b_5^- & 0 & & & & & & \\
\hline
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
\hline
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
\hline
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\hline
 & & & & & & 0 & b_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & a_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & b_{\mathrm{N}-2}^-a_{\mathrm{N}-1}^- & b_{\mathrm{N}-3}^-b_{\mathrm{N}-2}^- & 0 \\
 & & & & & & 0 & b_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^+ & a_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^+ & a_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & a_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^- & 0 \\
\hline
 & & & & & & & & 0 & b_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & a_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & b_{\mathrm{N}-1}^-a_{\mathrm{N}}^- \\
 & & & & & & & & 0 & b_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^+ & a_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^+ & a_{\mathrm{N}-1}^+a_{\mathrm{N}}^-
\end{array}\right] \tag{23}$$

$$\mathbf{F}=\left[\begin{array}{c|cc|cc|cc|cc|cc|c}
a_1^+a_2^- & a_1^+b_2^- & 0 & & & & & & & & & \\
\hline
b_2^+a_3^- & a_2^+a_3^- & b_3^-a_4^- & b_3^-b_4^- & 0 & & & & & & & \\
b_2^+b_3^+ & a_2^+b_3^+ & a_3^+a_4^- & a_3^+b_4^- & 0 & & & & & & & \\
\hline
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
\hline
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
\hline
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\hline
 & & & & & & & 0 & b_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & a_{\mathrm{N}-3}^+a_{\mathrm{N}-2}^- & b_{\mathrm{N}-2}^-a_{\mathrm{N}-1}^- & b_{\mathrm{N}-2}^-b_{\mathrm{N}-1}^- \\
 & & & & & & & 0 & b_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^+ & a_{\mathrm{N}-3}^+b_{\mathrm{N}-2}^+ & a_{\mathrm{N}-2}^+a_{\mathrm{N}-1}^- & a_{\mathrm{N}-2}^+b_{\mathrm{N}-1}^- \\
\hline
 & & & & & & & & 0 & 0 & b_{\mathrm{N}-1}^+a_{\mathrm{N}}^- & a_{\mathrm{N}-1}^+a_{\mathrm{N}}^-
\end{array}\right] \tag{24}$$

$$t_{ij}=\begin{cases}0 & ,i<j\\ a_{\mathrm{N}+1-j}^- & ,i\ge j\end{cases} \tag{29}$$

$$u_{ij}=\begin{cases}0 & ,i<j\\ \prod_{k=\mathrm{N}-i+1}^{\mathrm{N}-j} b_k^- & ,i\ge j\end{cases} \tag{30}$$

where $1\le i,j\le \mathrm{N}-1$. Note: *In all products* $\prod_{r=P}^{Q}(.)$ *throughout the text, take* $\prod_{r=P}^{Q}(.)=1$ *when* $P>Q$. Matrix products $(\mathbf{A}^+\circ\mathbf{B}^+)\mathbf{T}(\mathbf{A}^-\circ\mathbf{B}^-)\mathbf{T}\coloneqq(v_{\boldsymbol{ij}})$ and $\mathbf{T}(\mathbf{A}^-\circ\mathbf{B}^-)\mathbf{T}(\mathbf{A}^+\circ\mathbf{B}^+)\coloneqq(w_{\boldsymbol{ij}})$ are defined as

$$v_{ij}=a_{1+j}^-\sum_{k=1}^{\min(i,j)} a_k^+\prod_{l=k+1}^{i} b_l^+\prod_{m=k+1}^{j} b_m^- \tag{31}$$

$$w_{ij}=a_j^+\sum_{k=\max(i,j)}^{\mathrm{N}-1} a_{k+1}^-\prod_{l=i+1}^{k} b_l^-\prod_{m=j+1}^{k} b_m^+ \tag{32}$$

In the next section, nilpotency is examined for the three algorithms. Matrices $(\mathbf{J}_{\mathrm{N}})^2$ (18) and $\mathbf{S}_{\mathrm{N}}$ (20) are similar. The similarity transformation matrix P is the permutation defined by $\mathbf{Pe_1}=\mathbf{e_2}$. The nilpotency analysis is therefore limited to the GJ relaxation.

IV. Nilpotency analysis

Let us define a necessary condition for the existence of nilpotent algorithms. First, for the GJ-WR and GS-WR.

Theorem IV.1 Nilpotent GJ-WR and GS-WR algorithms possess null local convergence factors

$$a_j^+ a_{j+1}^- = 0, \quad \text{for all } 1 \le j \le N-1 \tag{33}$$

Proof. If complex matrix $(\mathbf{J}_N)^2$ (18) is nilpotent, then its trace $\sum_{j=1}^{N-1} a_j^+ a_{j+1}^- = 0$ at every frequency point $\omega \ge 0$ [36]. A requirement met when $a_j^+ a_{j+1}^- = 0$ for all $1 \le j \le N-1$. Same condition applies to $\mathbf{S}_N$ (20) since trace is invariant under a similarity transformation [36]. ■

Next, a similar condition is presented for the S-WR. This necessary condition is also sufficient according to the following result

Theorem IV.2 A S-WR algorithm is nilpotent if and only if all its transmission gains satisfy

$$a_p^+ a_q^- = 0, \quad \text{for all } 1 \le p < q \le N \tag{34}$$

Moreover, all nilpotent S-WR algorithms converge exactly in two rounds independently of initial waveforms. One round is one forward sweep followed by one backward sweep.

Proof. Products $(\mathbf{A}^+ \circ \mathbf{B}^+)\mathbf{T}(\mathbf{A}^- \circ \mathbf{B}^-)\mathbf{T}$ (31) and $\mathbf{T}(\mathbf{A}^- \circ \mathbf{B}^-)\mathbf{T}(\mathbf{A}^+ \circ \mathbf{B}^+)$ (32) have same trace $a_2^- a_1^+ + \sum_{k=3}^{N} a_k^- \left(a_{k-1}^+ + \sum_{l=1}^{k-2} a_l^+ \prod_{m=l+1}^{k-1} b_m^- b_m^+\right)$, which must be zero if matrix $\mathbf{R}_N$ (26) is nilpotent. A requirement met when condition (34) is satisfied. Conversely, a close look at expressions (31) and (32) shows that all entries $v_{ij} = 0$ and $w_{ij} = 0$ if condition (34) is satisfied. Hence $\mathbf{R}_N = \mathbf{0}$ is the only nilpotent iteration matrix. At the end of round one, all relaxation sources attain zero DC waveforms together for the first time. One additional round will bring all system variables to the zero solution of the homogeneous problem with zero initial conditions. ■

The nilpotent S-WR algorithms of Thm. IV.2 are optimal in the sense that faster convergence is not possible. Unlike condition (34) which characterizes the nilpotent S-WR, condition (33) is not sufficient. There exists GJ-WR and GS-WR algorithms with null local convergence factors, yet they are not nilpotent according to the following result.

Theorem IV.3

(a) Let $a_2^- = a_2^+ = \cdots = a_{2k}^- = a_{2k}^+ = \cdots = 0$ while a_1^+, a_3^-, a_3^+, $a_5^-, \ldots$, a_{2k+1}^-, a_{2k+1}^+ are not equal to zero. The resulting GJ-WR and GS-WR algorithms are not nilpotent for any chain of at least three subcircuits, $N \ge 3$.

(b) Let $a_1^+ = a_3^- = a_3^+ = \cdots = a_{2k+1}^- = a_{2k+1}^+ = \cdots = 0$ while a_2^-, a_2^+, a_4^-, a_4^+, ..., a_{2k}^-, a_{2k}^+,... are not equal to zero. The resulting GJ-WR and GS-WR algorithms are not nilpotent for any chain of at least four subcircuits, $N \ge 4$.

Proof. Cases (a) and (b) are demonstrated by inspection. A proof for case (a) is presented. A similar approach is used for (b).

Let us start with matrix $\mathbf{J}_3$ in eqn. (35) where $a_2^- = a_2^+ = 0$. The column vectors of $\mathbf{J}_3$ are linearly independent in general. Matrix $\mathbf{J}_3$ is nonsingular and cannot be nilpotent. Next, let us examine matrix $\mathbf{J}_4$ in eqn. (36) where $a_2^- = a_2^+ = a_4^- = 0$. If $P_{\mathbf{J}_3}$ and $P_{\mathbf{J}_4}$ denote the characteristic polynomials of matrices $\mathbf{J}_3$ and $\mathbf{J}_4$, then $P_{\mathbf{J}_4}(x) = x^2\, P_{\mathbf{J}_3}(x)$, $x \in \mathbb{C}$. Matrix $\mathbf{J}_4$

$$\mathbf{J}_3 = \begin{bmatrix} 0 & a_1^+ & 0 & 0 \\ 0 & 0 & 0 & b_2^- \\ b_2^+ & 0 & 0 & 0 \\ 0 & 0 & a_3^- & 0 \end{bmatrix} \tag{35}$$

$$\mathbf{J}_4 = \begin{bmatrix} 0 & a_1^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2^- & 0 & 0 \\ b_2^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3^- & 0 & 0 & b_3^- \\ 0 & 0 & b_3^+ & 0 & 0 & a_3^+ \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

$$\mathbf{J}_5 = \begin{bmatrix} 0 & a_1^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2^- & & 0 & & \\ b_2^+ & 0 & 0 & 0 & \vdots & 0 & & \vdots \\ 0 & & a_3^- & & & b_3^- & \vdots & \\ & \vdots & b_3^+ & \vdots & & a_3^+ & & 0 \\ \vdots & & 0 & & 0 & 0 & & b_4^- \\ & & 0 & & b_4^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_5^- & 0 \end{bmatrix} \tag{37}$$

$$\mathbf{J}_6 = \begin{bmatrix} 0 & a_1^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2^- & & 0 & & & & \\ b_2^+ & & 0 & 0 & \vdots & 0 & \vdots & \vdots & & \vdots \\ 0 & & a_3^- & & \vdots & b_3^- & \vdots & & & \vdots \\ & \vdots & b_3^+ & & & a_3^+ & & 0 & \vdots & \\ \vdots & \vdots & 0 & \vdots & 0 & 0 & & b_4^- & \vdots & \\ \vdots & & & \vdots & b_4^+ & & 0 & 0 & & 0 \\ & & \vdots & & 0 & \vdots & a_5^- & \vdots & & b_5^- \\ & & \vdots & & 0 & & b_5^+ & & & a_5^+ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

shares the same non-zero eigenvalues of $\mathbf{J}_3$ and therefore is not nilpotent. In the same way, a close look at matrices $\mathbf{J}_5$ in eqn. (37) where $a_2^- = a_2^+ = a_4^- = a_4^+ = 0$ and $\mathbf{J}_6$ in eqn. (38) where $a_2^- = a_2^+ = a_4^- = a_4^+ = a_6^- = 0$ shows that the column vector of $\mathbf{J}_5$ are linearly independent since $a_3^- a_3^+ - b_3^+ b_3^- \ne 0$. Matrix $\mathbf{J}_5$ is therefore nonsingular. As with $\mathbf{J}_3$ and $\mathbf{J}_4$, the characteristic polynomials $P_{\mathbf{J}_5}$ and $P_{\mathbf{J}_6}$ of $\mathbf{J}_5$ and $\mathbf{J}_6$ satisfy $P_{\mathbf{J}_6}(x) = x^2\, P_{\mathbf{J}_5}(x)$. Matrix $\mathbf{J}_6$ cannot be nilpotent.

In general, any two successive matrices $\mathbf{J}_{2m+1}$ where $a_2^- = a_2^+ = \cdots = a_{2m}^- = a_{2m}^+ = 0$ and $\mathbf{J}_{2m+2}$ where $a_2^- = a_2^+ = \cdots = a_{2m}^- = a_{2m}^+ = a_{2m+2}^- = 0$ with characteristic polynomials $P_{\mathbf{J}_{2m+1}}$ and $P_{\mathbf{J}_{2m+2}}$, satisfy the following: 1) The column vectors of the $4m \times 4m$ matrix $\mathbf{J}_{2m+1}$ are linearly independent since $a_l^- a_l^+ - b_l^+ b_l^- \ne 0$, $3 \le l < 2m+1$. 2) Matrices $\mathbf{J}_{2m+1}$ and $\mathbf{J}_{2m+2}$ have the same non-zero eigenvalues since $P_{\mathbf{J}_{2m+2}}(x) = x^2\, P_{\mathbf{J}_{2m+1}}(x)$. Both $\mathbf{J}_{2m+1}$ and $\mathbf{J}_{2m+2}$ are not nilpotent. ■

The conditions of Thm. VI.3 do not satisfy condition (33) and hence produce non-nilpotent S-WR. These algorithms are not unique. In general, setting gains to zero according to the following alternating fashions

$$\begin{aligned} & a_1^+ = a_2^+ = \cdots = a_{k_1}^+ = 0, \\ & a_{k_2}^- = a_{k_2-1}^- = \cdots = a_{k_1+2}^- = 0, \\ & a_{k_2}^+ = a_{k_2+2}^+ = \cdots = a_{k_3}^+ = 0, \\ & \qquad \vdots \\ & a_{k_{r-2}}^+ = a_{k_{r-2}+1}^+ = \cdots = a_{k_{r-1}}^+ = 0, \\ & a_{k_r}^- = a_{k_{r-1}}^- = \cdots = a_{k_{r-1}+2}^- = 0 \end{aligned} \tag{39}$$

$$\vdots$$

such that $1 \le k_1 \le k_2 \le \cdots \le k_r \le \cdots \le \mathrm{N}$ and $r \ge 3$, lead to non-nilpotent GJ-WR, GS-WR and S-WR methods. Similar patterns to (39) are also constructed starting from reverse gain a_{N}^-. It is key to notice that patterns (39) satisfy (33) but not the much stronger condition (34). Therefore, the search for nilpotent GJ-WR and GS-WR algorithms will be based on two non-alternating patterns. In the first one, either all forward gains $a_{\mathrm{j}}^+ = 0$ or all reverse gains $a_{\mathrm{j}+1}^- = 0$. In the second one, the first m forward gains $a_1^+ = \cdots = a_{\mathrm{m}}^+ = 0$ and the first m' backward gains $a_{\mathrm{N}}^- = \cdots = a_{\mathrm{N}-m'+1}^- = 0$ where $m + m' \ge \mathrm{N} - 1$. A strict equality ensures all local $a_{\mathrm{j}}^+ a_{\mathrm{j}+1}^- = 0$ by making either $a_{\mathrm{j}}^+ = 0$ or $a_{\mathrm{j}+1}^- = 0$ and not simultaneously.

First, let us start by revisiting the optimal result in [29, Thm. 2.1],[32, Thm. D.4]. The following theorem proposes a new and rigorous proof of the GJ result and introduces its GS version for the first time.

Theorem IV.4 Let $a_{\mathrm{j}}^+ = a_{\mathrm{j}+1}^- = 0,\ \ 1 \le \mathrm{j} \le \mathrm{N} - 1$. The resulting GJ-WR and GS-WR algorithms converge exactly in N iterations and in $[\mathrm{N}/2] + 1$ iterations respectively, independently of initial waveforms.

Proof. Reasoning by induction is used to demonstrate the result. When all coefficients $a_1^+ = a_2^- = a_2^+ = a_3^- = \cdots a_{\mathrm{N}}^- = 0$, $\mathrm{N} \ge 3$, it is possible to express the $2\mathrm{N} \times 2\mathrm{N}$ matrix $\mathbf{J}_{\mathrm{N}+1}$ in terms of $\mathbf{J}_{\mathrm{N}}$, see eqn. (40), and show that its m th power, m $\in$ $\mathbb{N}$ and m ≥ 1, satisfies eqn. (41).

$$\mathbf{J}_{\mathrm{N}+1} = \left[\begin{array}{cccc:cc} & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & \mathbf{J}_{\mathrm{N}} & & & 0 & 0 \\ & & & & 0 & b_{\mathrm{N}}^- \\ \hdashline 0 & \cdots & b_{\mathrm{N}}^+ & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right] \tag{40}$$

$$(\mathbf{J}_{\mathrm{N}+1})^{\mathrm{m}} = \left[\begin{array}{ccc:c:c} & (\mathbf{J}_{\mathrm{N}})^{\mathrm{m}} & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \mathbf{U}_{\mathrm{N,m}} \\ \hdashline & \mathbf{V}_{\mathrm{N,m}} & & 0 & 0 \\ \hdashline 0 & \cdots & 0 & 0 & 0 \end{array}\right] \tag{41}$$

Vectors $\mathbf{U}_{\mathrm{N,m}} \in \mathbb{C}^{2(\mathrm{N}-1)\times 1}$ and $\mathbf{V}_{\mathrm{N,m}} \in \mathbb{C}^{1\times 2(\mathrm{N}-1)}$ are given by

$$\begin{aligned} \mathbf{U}_{\mathrm{N,m}} &= b_{\mathrm{N}}^- (\mathbf{J}_{\mathrm{N}})^{\mathrm{m}-1} \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-2} \\ \mathbf{V}_{\mathrm{N,m}} &= b_{\mathrm{N}}^+ \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-3\,\boldsymbol{T}} (\mathbf{J}_{\mathrm{N}})^{\mathrm{m}-1} \end{aligned} \tag{42}$$

Where $\mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-3} = (0,0,\ldots,0,1,0)^T$ and $\mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-2} = (0,0,\ldots,0,1)^T$ are the $(2\mathrm{N}-3)^{\mathrm{th}}$ and $(2\mathrm{N}-2)^{\mathrm{th}}$ unit vectors in the canonical basis of the $\mathbb{C}$-space $\mathbb{C}^{2(\mathrm{N}-1)}$. A direct computation shows that $(\mathbf{J}_3)^2 = \mathbf{0}$ for $a_1^+ = a_2^- = a_2^+ = a_3^- = 0$. According to (41),(42), if $\mathbf{J}_{\mathrm{N}}$ is nilpotent of index $(\mathrm{N}-1)$, then $\mathbf{J}_{\mathrm{N}+1}$ will also be nilpotent of index N. For if there exists an integer N_0 such that $\mathbf{J}_{\mathrm{N}_0}$ were nilpotent of index less than $(\mathrm{N}_0 - 1)$, then this would have meant that $\mathbf{J}_3 = \mathbf{0}$ when counting backward; a result which is not correct. Matrices $\mathbf{J}_{\mathrm{N}}$ and $\mathbf{S}_{\mathrm{N}}$ are nilpotent of indices $(\mathrm{N}-1)$ and $[\mathrm{N}/2]$. $[.]$ denotes the integer part of a real number. One iteration is still needed to reach the zero solution, as explained in the proof of Thm. IV. 2 ■

Optimal convergence requires all gains be zero. A stringent condition which is relaxed in Cors. IV.7 and IV.9.

The next result explores the effect of zeroing all factors along one and same direction.

Theorem IV.5.

(a) Let $a_1^+ = a_2^+ = \cdots = a_{\mathrm{N}-1}^+ = 0$ or $a_{\mathrm{N}}^- = a_{\mathrm{N}-1}^- = \cdots = a_2^- = 0$. The resulting GJ-WR and GS-WR algorithms converge exactly in $(2\mathrm{N}-1)$ iterations and in N iterations respectively, independently of initial waveforms.

(b) Let $a_1^+ = a_2^+ = \cdots = a_{\mathrm{N}-1}^+ = 0$ and $a_{\mathrm{k}_1}^- = a_{\mathrm{k}_2}^- = \cdots = a_{\mathrm{k}_l}^- = 0$ where $1 \le l \le \mathrm{N}-1$ and $2 \le \mathrm{k}_l < \mathrm{k}_{l-1} < \cdots < \mathrm{k}_1 \le \mathrm{N}-1$. Or, let $a_{\mathrm{N}}^- = a_{\mathrm{N}-1}^- = \cdots = a_2^+ = 0$ and $a_{\mathrm{k}_1}^+ = a_{\mathrm{k}_2}^+ = \cdots = a_{\mathrm{k}_m}^+ = 0$ where $1 \le m \le \mathrm{N}-1$ and $2 \le \mathrm{k}_1 < \mathrm{k}_2 < \cdots < \mathrm{k}_m \le \mathrm{N}-1$. The resulting GJ-WR and GS-WR algorithms still converge exactly in $(2\mathrm{N}-1)$ iterations and in N iterations respectively, independently of initial waveforms.

Proof. The demonstration of both results (a) and (b), focuses on the case where entries $a_{\mathrm{N}}^- = \cdots = a_2^- = 0$ in matrix $\mathbf{J}_{\mathrm{N}}$ (16). A similar approach is used for $a_1^+ = \cdots = a_{\mathrm{N}-1}^+ = 0$.

Let $\mathbf{J'}_{\mathrm{N}}$ be the matrix constructed from $\mathbf{J}_{\mathrm{N}}$ by replacing value $a_{\mathrm{N}-1}^+$ of its $(2\mathrm{N}-3, 2\mathrm{N}-2)^{\mathrm{th}}$ entry with zero. Let $\mathbf{Q}_{\mathrm{N}} = a_{\mathrm{N}-1}^+ \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-3} \otimes \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-2}$, operator $\otimes$ is the outer product [35]. Matrices $\mathbf{J}_{\mathrm{N}}$, $\mathbf{Q}_{\mathrm{N}}$ and $\mathbf{J'}_{\mathrm{N}}$ satisfy $\mathbf{J}_{\mathrm{N}} = \mathbf{J'}_{\mathrm{N}} + \mathbf{Q}_{\mathrm{N}}$, $\mathbf{J'}_{\mathrm{N}}\mathbf{Q}_{\mathrm{N}} = \mathbf{Q}_{\mathrm{N}}\mathbf{J'}_{\mathrm{N}} = \mathbf{Q}_{\mathrm{N}}^{\;2} = \mathbf{0}$. Hence $\mathbf{J}_{\mathrm{N}}^{\;\mathrm{m}} = \mathbf{J'}_{\mathrm{N}}^{\;\mathrm{m}}$ for all m ≥ 2 and $\mathrm{N} \ge 3$. Reasoning by recurrence shows that

$$(\mathbf{J'}_{\mathrm{N}+1})^{2\mathrm{m}} = \left[\begin{array}{ccc:c:c} & (\mathbf{J'}_{\mathrm{N}})^{2\mathrm{m}} & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \mathbf{U'}_{\mathrm{N,2m}} \\ \hdashline & \mathbf{V'}_{\mathrm{N,2m}} & & 0 & 0 \\ \hdashline 0 & \cdots & 0 & 0 & 0 \end{array}\right] \tag{43}$$

Vectors $\mathbf{U'}_{\mathrm{N,2m}} \in \mathbb{C}^{2(\mathrm{N}-1)\times 1}$ and $\mathbf{V'}_{\mathrm{N,2m}} \in \mathbb{C}^{1\times 2(\mathrm{N}-1)}$ are given by

$$\begin{aligned} \mathbf{U'}_{\mathrm{N,2m}} &= b_{\mathrm{N}}^- (\mathbf{J'}_{\mathrm{N}})^{2\mathrm{m}-1} \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-2} \\ \mathbf{V'}_{\mathrm{N,2m}} &= b_{\mathrm{N}}^+ \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-3\,\boldsymbol{T}} (\mathbf{J'}_{\mathrm{N}})^{2\mathrm{m}-1} \end{aligned} \tag{44}$$

A direct computation shows that $(\mathbf{J'}_3)^4 = \mathbf{0}$. Hence $(\mathbf{J'}_4)^6 = \mathbf{0}$ according to eqns. (43),(44) and more generally $(\mathbf{J'}_{\mathrm{N}})^{2\mathrm{N}-2} = (\mathbf{J}_{\mathrm{N}})^{2\mathrm{N}-2} = \mathbf{0}$. Matrix $\mathbf{S}_{\mathrm{N}}$ is also nilpotent of index $(\mathrm{N}-1)$, which concludes the proof of result (a).

Next, it is shown by recurrence that $\mathbf{J'}_{\mathrm{N}}^{\;2\mathrm{N}-3} = \left(a_1^+ \prod_{k=2}^{\mathrm{N}-1} b_k^- b_k^+\right) \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-3} \otimes \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-2}$. Result (b) follows from the fact that $\mathbf{J}_{\mathrm{N}}^{\;2\mathrm{N}-3} = \mathbf{J'}_{\mathrm{N}}^{\;2\mathrm{N}-3} \ne \mathbf{0}$ if and only if $a_1^+ \ne 0$. First, $\mathbf{J'}_3^{\;3} = a_1^+ b_2^- b_2^+ \mathbf{u}_4^3 \otimes \mathbf{u}_4^4$ by direct computation. Next, matrix $(\mathbf{J'}_{\mathrm{N}+1})^{2\mathrm{N}-2}$ is calculated using eqns. (43),(44) for m $= \mathrm{N}-1$. Its bloc matrix $(\mathbf{J'}_{\mathrm{N}})^{2\mathrm{N}-2} = \mathbf{0}$ (Thm. IV.5(a)) and its vectors $\mathbf{U'}_{\mathrm{N},2\mathrm{N}-2} = b_{\mathrm{N}}^- a_1^+ \prod_{k=2}^{\mathrm{N}-1} b_k^- b_k^+ \, \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-3}$, $\mathbf{V'}_{\mathrm{N},2\mathrm{N}-2} = b_{\mathrm{N}}^+ a_1^+ \prod_{k=2}^{\mathrm{N}-1} b_k^- b_k^+ \, \mathbf{u}_{2\mathrm{N}-2}^{2\mathrm{N}-2\,\boldsymbol{T}}$ (recurrence hypothesis). Hence, matrix $(\mathbf{J}_{\mathrm{N}+1})^{2\mathrm{N}-2} = a_1^+ \prod_{k=2}^{\mathrm{N}-1} b_k^- b_k^+ \left(b_{\mathrm{N}}^+ \mathbf{u}_{2\mathrm{N}}^{2\mathrm{N}-1} \otimes \mathbf{u}_{2\mathrm{N}}^{2\mathrm{N}-2} + b_{\mathrm{N}}^- \mathbf{u}_{2\mathrm{N}}^{2\mathrm{N}-3} \otimes \mathbf{u}_{2\mathrm{N}}^{2\mathrm{N}}\right)$. Finally, result $\mathbf{J}_{\mathrm{N}+1}^{\;\;2\mathrm{N}-1} = \left(a_1^+ \prod_{k=2}^{\mathrm{N}} b_k^- b_k^+\right) \mathbf{u}_{2\mathrm{N}}^{2\mathrm{N}-1} \otimes \mathbf{u}_{2\mathrm{N}}^{2\mathrm{N}}$ follows from the multiplication of $(\mathbf{J}'_{\mathrm{N}+1})^{2\mathrm{N}-2}$ by $\mathbf{J}'_{\mathrm{N}+1}$. ■

Iteration matrices in Thm. IV.5 have maximum index $2(\mathrm{N}-1)$ [35]. For a chain of length N, there exist no nilpotent GJ-WR algorithm that converges in more than $2\mathrm{N}-1$ iterations nor there is a nilpotent GS-WR that takes more than N iterations to converge. The maximum index is still attained for any additional zeroing of some or all reverse (forward) gains at the exception of a_{N}^{-} (a_1^{+}).

The following result explores the effect of setting the first m local convergence rates $a_{\mathrm{j}}^{+}a_{\mathrm{j+1}}^{-}=0$ along the forward direction and the remaining $(\mathrm{N}-m-1)$ ones along the reverse direction.

Theorem IV.6 Let $a_1^{+}=a_2^{+}=\cdots=a_m^{+}=0$, and $a_{\mathrm{N}}^{-}=a_{\mathrm{N}-1}^{-}=\cdots=a_{m+2}^{-}=0$, $1\le m\le \mathrm{N}-2$. The resulting GJ-WR and GS-WR algorithms converge exactly in $1+2\max(m,\mathrm{N}-m-1)$ iterations and in $1+\max(m,\mathrm{N}-m-1)$ iterations respectively, independently of initial waveforms.

Proof. The idea is to demonstrate that iteration matrix $\mathbf{J}_{\mathrm{N}}$ (16), is nilpotent of index $2\max(m,\mathrm{N}-m-1)$.

$$\mathbf{J}_{\mathrm{N}}=\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix} \tag{45}$$

It is shown that its bloc matrices $\mathbf{M}_{11}\in\mathbb{C}^{2m\times 2m}$, $\mathbf{M}_{12}\in\mathbb{C}^{2m\times 2(\mathrm{N}-m-1)}$, $\mathbf{M}_{21}\in\mathbb{C}^{2(\mathrm{N}-m-1)\times 2m}$ and $\mathbf{M}_{22}\in\mathbb{C}^{2(\mathrm{N}-m-1)\times 2(\mathrm{N}-m-1)}$ satisfy

$$\mathbf{M}_{21}(\mathbf{M}_{11})^{p}\mathbf{M}_{12}=\mathbf{0}\qquad 0\le p<m \tag{46}$$

$$\begin{matrix}(\mathbf{M}_{11})^{p}\mathbf{M}_{12}=\mathbf{0}\\ \mathbf{M}_{21}(\mathbf{M}_{11})^{p}=\mathbf{0}\end{matrix}\qquad p\ge m \tag{47}$$

$$\mathbf{M}_{12}(\mathbf{M}_{22})^{p}\mathbf{M}_{21}=\mathbf{0}\qquad 0\le p<\mathrm{N}-m-1 \tag{48}$$

$$\begin{matrix}(\mathbf{M}_{22})^{p}\mathbf{M}_{21}=\mathbf{0}\\ \mathbf{M}_{12}(\mathbf{M}_{22})^{p}=\mathbf{0}\end{matrix}\qquad p\ge \mathrm{N}-m-1 \tag{49}$$

Using eqns. (46)-(49), it is shown by recurrence that matrix $(\mathbf{J}_{\mathrm{N}})^{p}$ is given by the following relation

$$(\mathbf{J}_{\mathrm{N}})^{p}=\begin{bmatrix}(\mathbf{M}_{11})^{p} & \sum_{k=0}^{p-1}(\mathbf{M}_{11})^{k}\mathbf{M}_{12}(\mathbf{M}_{22})^{p-1-k}\\ \sum_{k=0}^{p-1}(\mathbf{M}_{22})^{k}\mathbf{M}_{21}(\mathbf{M}_{11})^{p-1-k} & (\mathbf{M}_{22})^{p}\end{bmatrix}\quad (p\ge 1) \tag{50}$$

Matrices $(\mathbf{M}_{11})^{2m}=\mathbf{0}$ and $(\mathbf{M}_{22})^{2(\mathrm{N}-m-1)}=\mathbf{0}$ (Thm. IV.5(a)). In addition, $\sum_{k=0}^{p-1}(\mathbf{M}_{11})^{k}\mathbf{M}_{12}(\mathbf{M}_{22})^{p-1-k}=\mathbf{0}$ and $\sum_{k=0}^{p-1}(\mathbf{M}_{22})^{k}\mathbf{M}_{21}(\mathbf{M}_{11})^{p-1-k}=\mathbf{0}$ only when $p\ge 2\max(m,\mathrm{N}-m-1)$, see eqns. (46)-(49). ■

When length N is odd, the optimality condition in Thm. IV.4 is relaxed according to the following result.

Corollary IV.7 The GJ-WR and GS-WR algorithms are optimal for any odd number N, $\mathrm{N}\ge 3$, of serial parts if and only if $a_1^{+}=a_2^{+}=\cdots=a_{(\mathrm{N}-1)/2}^{+}=0$ and $a_{\mathrm{N}}^{-}=a_{\mathrm{N}-1}^{-}=\cdots=a_{(\mathrm{N}+3)/2}^{-}=0$.

Proof. The necessity is clear from Thm. IV.1. The sufficiency follows from applying Thm. IV.6 for $m=(\mathrm{N}-1)/2$. ■

After all forward or all reverse gains are zero (Thm. IV5(a)), it is possible to produce faster nilpotent algorithms if zeroing of remaining gains in the opposite direction starts from first part, that is P_1 in the forward direction and P_{N} in the backward one.

Corollary IV.8

(a) Let $a_{\mathrm{N}}^{-}=a_{\mathrm{N}-1}^{-}\ldots=a_2^{-}=0$ and $a_1^{+}=a_2^{+}=\cdots=a_{\mathrm{m}}^{+}=0$ such as $1\le m\le -1+\mathrm{N}/2$ for all even $\mathrm{N}\ge 4$ and $1\le m\le -1+(\mathrm{N}-1)/2$ for all odd $\mathrm{N}\ge 5$. The resulting nilpotent GJ-WR and GS-WR algorithms converge exactly in $(2\mathrm{N}-2m-1)$ iterations and in $(\mathrm{N}-m)$ iterations respectively, independently of initial waveforms.

(b) Let $a_1^{+}=a_2^{+}=a_3^{+}=\cdots=a_{\mathrm{N}-1}^{+}=0$ and $a_{\mathrm{N}}^{-}=\cdots=a_{\mathrm{N}-m}^{-}=0$ such as $0\le m\le -1+(\mathrm{N}-3)/2$ for all odd $\mathrm{N}\ge 5$, and $0\le m\le -1+(\mathrm{N}-2)/2$ for all even $\mathrm{N}\ge 4$. The resulting nilpotent algorithms GJ-WR and GS-WR converge exactly in $(2\mathrm{N}-2m-3)$ and in $(\mathrm{N}-m-1)$ iterations respectively, independently of initial waveforms.

Proof. Part (a) follows directly from the application of Thm. IV.6. Part (b) follows from part (a) after taking $m'=\mathrm{N}-m-2$. ■

Next, the optimality condition in Thm. IV.4 is relaxed for even numbers N of parts.

Corollary IV.9 The GJ-WR and GS-WR algorithms are optimal for any even number $\mathrm{N}\ge 4$ of serial parts if and only if the first $\mathrm{N}/2$ forward transmission gains $a_1^{+}=a_2^{+}=\cdots=a_{\mathrm{N}/2}^{+}=0$ and the first $\mathrm{N}/2$ reverse transmission gains $a_{\mathrm{N}}^{-}=a_{\mathrm{N}-1}^{-}=\cdots=a_{1+\mathrm{N}/2}^{-}=0$.

Proof. The main steps in the proof of Thm. IV.6 are again used here to demonstrate that ${\mathbf{J}_{\mathrm{N}}}^{\mathrm{N}-1}=\mathbf{0}$ after taking $m=\mathrm{N}/2$. Bloc matrices $\mathbf{M}_{11}\in\mathbb{C}^{\mathrm{N}\times\mathrm{N}}$, $\mathbf{M}_{12}\in\mathbb{C}^{\mathrm{N}\times(\mathrm{N}-2)}$, $\mathbf{M}_{21}\in\mathbb{C}^{(\mathrm{N}-2)\times\mathrm{N}}$ and $\mathbf{M}_{22}\in\mathbb{C}^{(\mathrm{N}-2)\times(\mathrm{N}-2)}$ of $\mathbf{J}_{\mathrm{N}}$, see eqn. (45), satisfy eqns. (46)-(49) and $(\mathbf{J}_{\mathrm{N}})^{p}$ is also given by eqn. (50). Matrix ${\mathbf{M}_{11}}^{\mathrm{N}-2}=\mathbf{0}$ (Cor. IV.8) and ${\mathbf{M}_{22}}^{\mathrm{N}-2}=\mathbf{0}$ (Thm. IV.4), whereas $\sum_{k=0}^{p-1}(\mathbf{M}_{11})^{k}\mathbf{M}_{12}(\mathbf{M}_{22})^{p-1-k}=\mathbf{0}$ and $\sum_{k=0}^{p-1}(\mathbf{M}_{22})^{k}\mathbf{M}_{21}(\mathbf{M}_{11})^{p-1-k}=\mathbf{0}$ when $p\ge \mathrm{N}-1$. ■

Discussion

Zeroing all transmission gains in an alternate fashion does not produce nilpotent GJ-WR, GS-WR and S-WR algorithms despite having zero valued local rates. For a chain of N parts, the nilpotent GJ-WR and GS-WR algorithms converge exactly in $(2\mathrm{N}-1)$ iterations and in N iterations respectively when all forward or all reverse gains are set to zero (Thm. IV.5). It is possible to further reduce the index of the nilpotent operator by successively zeroing the remaining gains. Starting from a_{N}^{-} if all direct gains are

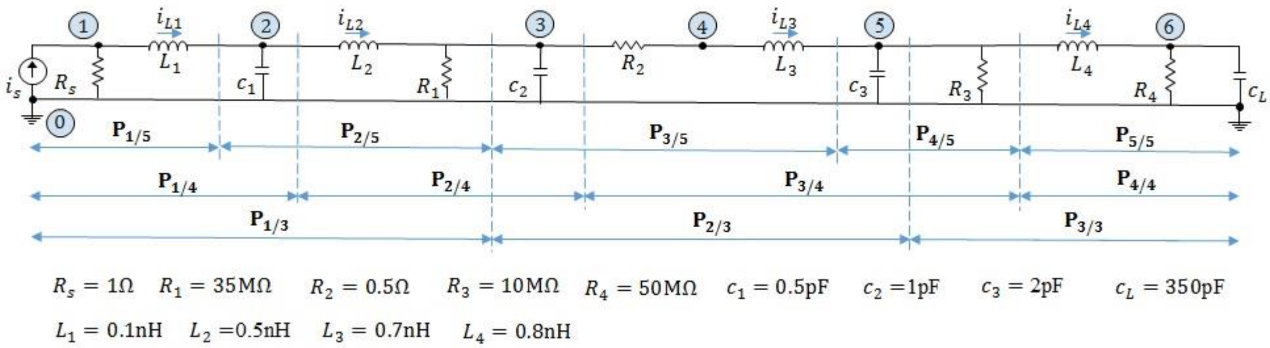

Fig. 7 RLC circuit.

already zero and from a_1^+ if all reverse gains are already zero. The number of GJ iterations decreases from $(2\text{N}-3)$ to $(\text{N}+1)$ for N even and to $(\text{N}+2)$ for N odd (Cor. IV.8). Convergence is optimal at $m = (\text{N}-1)/2$ for N odd (Cor. IV.7) and at $m = \text{N}/2$ for N even (Cor. IV.9). This means that in part (a) of Cor. IV.8 for instance, it is not necessary to set remaining forward gains $a^+_{(\text{N}+1)/2} = \cdots = a^+_{\text{N}-1} = 0$ since the number of GJ iterations will plateau at N. To produce nilpotent algorithms, it is not necessary to set every local rate $a_\text{n}^+ a_{\text{n}+1}^- = 0$ by having $a_\text{n}^+ = a_{\text{n}+1}^- = 0$ (Thm. IV. 4 and Cor. IV.8). The best way would be to zero local convergence rates along two directions without alternating (Thm. IV.6, Cors. IV.7 and IV.9).

The index of the nilpotent GJ-WR decreases by steps of two iterations until it plateaus at N. One exception occurs for even values of N where the last step is one iteration only (Thm. IV.5 and Cor. IV.8). This observation agrees with the fact that two GJ iterations are *equivalent* to one GS iteration. The nilpotent set is therefore completely characterized. Hence, condition (34) is necessary and sufficient for the existence of all three nilpotent algorithms. It produces one index for all nilpotent S-WR and $1 + [\text{N}/2]$ distinct indices for all nilpotent GJ-WR (or GS-WR).

The circuit realization of the optimal conditions in Thm. IV.4 and in Cors. IV.7 and IV.9 produce enlarged partitions of different sizes. Due to the adjacency pattern in the chain, making $a_\text{n}^+ = 0$, see eqn. (13), requires corresponding kernel $\zeta_\text{n}^{\text{n},\text{n}+1}$ be exactly equal to the driving-point impedance of the whole segment $\{\text{P}_\text{n}, \text{P}_{\text{n}-1}, \dots, \text{P}_1\}$ while $a_{\text{n}+1}^- = 0$, see eqn. (13), requires $\zeta_{\text{n}+1}^{\text{n},\text{n}+1}$ be exactly equal to the driving-point impedance of the entire second segment $\{\text{P}_{\text{n}+1}, \text{P}_{\text{n}+2}, \dots, \text{P}_\text{N}\}$. The implementation of the optimal condition in Thm. IV.4 makes all enlarged part $\text{P}_1, \dots, \text{P}_\text{N}$ duplicates of the original circuit itself, that is the entire chain against only one part $\text{P}_{(\text{N}+1)/2}$ at the middle of the chain for the first relaxed condition (Cor. IV.7) and against only two parts $\text{P}_{\text{N}/2}$ and $\text{P}_{1+\text{N}/2}$ also at the middle of the chain for the second relaxed condition (Cor. IV.9). Since cost-efficiency is attained at suboptimal speeds of convergence, the approximation of the relaxed conditions reveals more attractive. They require the approximation of $(\text{N}-1)$ or N optimal kernels for N odd or even against $2(\text{N}-1)$ optimal kernels according to Thm. IV.4. Moreover, the remaining $(\text{N}-1)$ or $(\text{N}-2)$ kernels can for instance be set apriori to decrease the complexity of the approximation problem and keep the size of the enlarged parts minimal. This way, decreasing the costs of both the approximation step and WR iteration.

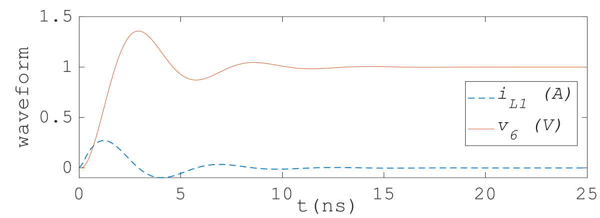

Fig. 8 Voltage at node 6 and current though L_1.

The amount of work required to achieve nilpotency shows that sweeping back and forth produces the best nilpotent algorithm. Every internal part is solved four times while the two parts at the extremities are solved twice, independently of the chain's length N. In the GJ-WR and GS-WR, all parts are practically solved N times. In [9, sec. 8.3.2, pp. 58], it was observed numerically that it might be best to iterate by scheduling subcircuits alternatively in the forward and backward directions for bidirectional chains of subcircuits when primary inputs are present at the extremities. The present analysis confirms mathematically the long-standing observation.

In the following section, a numerical example is produced to check the correctness of the theoretical results in section IV.

V. Numerical Experiments

The RLC circuit of Fig. 7 is considered. The driving current signal i_s is a trapezoidal step function of rise time 0.1ns and of magnitude 1A. At the far end, load c_L is a 350pF capacitor. The model initial-value problem (IVP) of this circuit is a first-order ODE system $\mathbf{D}\dot{\boldsymbol{x}}(t) + \mathbf{G}\boldsymbol{x}(t) = \boldsymbol{u}(t), \boldsymbol{x}(0) = \mathbf{0}$ with respect to 10×1 unknowns vector $\boldsymbol{x} = (v_1, v_2, v_3, v_4, v_5, v_6, i_{L_1}, i_{L_2}, i_{L_3}, i_{L_4})^T$ where $v_1(t), \dots, v_6(t)$ are voltages at nodes 1,..,6 and $i_{L_1}(t), \dots, i_{L_4}(t)$ are currents through inductors $L_1, \dots, L_4$. Matrix $\mathbf{D} \coloneqq (\text{d}_{\text{i,j}}) \in \mathbb{R}^{10\times10}$ is diagonal and $\mathbf{G} \coloneqq (\text{g}_{\text{i,j}}) \in \mathbb{R}^{10\times10}$ is symmetric. Nonzero elements of $\mathbf{D}$ are

$$\begin{array}{llll} \text{d}_{2,2} = c_1 & \text{d}_{3,3} = c_2 & \text{d}_{5,5} = c_3 & \text{d}_{6,6} = c_L \\ \text{d}_{7,7} = -L_1 & \text{d}_{8,8} = -L_2 & \text{d}_{9,9} = -L_1 & \text{d}_{10,10} = -L_4 \end{array} \quad (51)$$

whereas upper triangular nonzero elements of $\mathbf{G}$ are

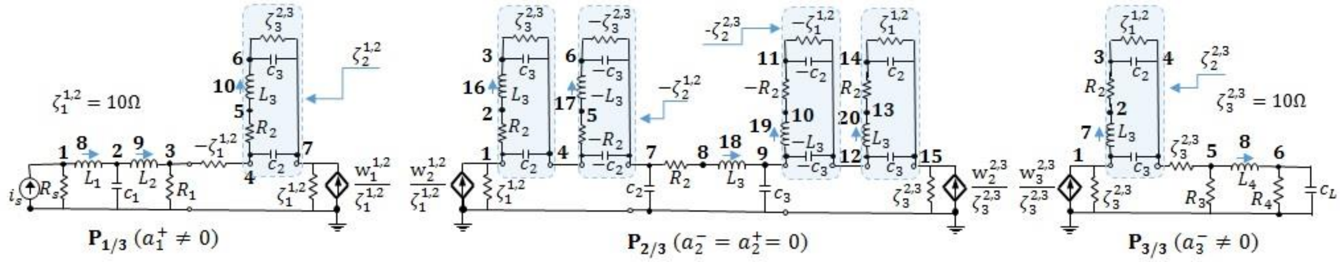

Fig. 9 Three-subcircuit partition of the non-nilpotent WR (Thm. IV.3(a))

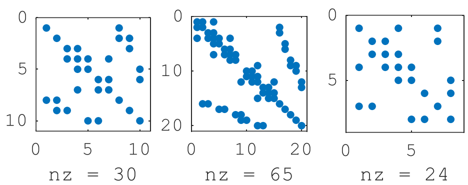

Fig. 10 Sparsity pattern of matrices $(\mathbf{G}+\mathbf{D}/\mathrm{h})$ of enlarged parts. Left. $\mathrm{P}_{1/3}$. Middle. $\mathrm{P}_{2/3}$. Right. $\mathrm{P}_{3/3}$.. Thm. IV.3(a).

$$\begin{aligned} &g_{1,1}=R_s{}^{-1} \quad g_{3,3}=R_1{}^{-1}+R_2{}^{-1} \quad g_{3,4}=-R_2{}^{-1} \quad g_{4,4}=R_2{}^{-1} \\ &g_{5,5}=R_3{}^{-1} \quad g_{6,6}=R_4{}^{-1} \quad g_{1,7}=g_{2,8}=g_{4,9}=g_{5,10}=1 \\ &g_{2,7}=g_{3,8}=g_{5,9}=g_{6,10}=-1 \end{aligned} \tag{52}$$

The IVP is solved numerically with backward Euler method on $[0,\mathrm{T}]$, $\mathrm{T}=25\mathrm{ns}$.

Its solution $\boldsymbol{x}((k+1)\mathrm{h})=(\mathbf{G}+\mathbf{D}/\mathrm{h})^{-1}\left(\boldsymbol{u}((k+1)\mathrm{h})+\mathbf{D}/\mathrm{h}\,\boldsymbol{x}(k\mathrm{h})\right)$, $k\in\mathbb{N}^*$, is calculated on points $\mathrm{t}=\mathrm{h},2\mathrm{h},3\mathrm{h},\ldots$ with stepsize $\mathrm{h}=\mathrm{T}/2^{12}$. The voltage at node 6 and current through inductor L_1 are plotted in Fig. 8.

To apply the WR algorithms of section IV to the solution of the same RLC circuit, one-node overlap longitudinal partitioning splits the circuit, first into three subcircuits: $\mathrm{P}_{1/3}$, $\mathrm{P}_{2/3}$, $\mathrm{P}_{3/3}$, then into four subcircuits: $\mathrm{P}_{1/4}$, $\mathrm{P}_{2/4}$, $\mathrm{P}_{3/4}$, $\mathrm{P}_{4/4}$, and finally into five subcircuits: $\mathrm{P}_{1/5}$, $\mathrm{P}_{2/5}$, $\mathrm{P}_{3/5}$, $\mathrm{P}_{4/5}$, $\mathrm{P}_{5/5}$, see Fig. 7. The current source is included in the first subcircuit while the load capacitor in the last one.

In every partition, subcircuits are appended with the circuit realizations of the WR kernels and their additive inverses. First, all non-optimal kernels are given a value of 10Ω to ensure corresponding gains (forward and reverse) are not zero. Next, the one-port circuit realization of every optimal kernel is constructed by duplicating the enlarged part built around the corresponding subcircuit, after setting its primary inputs and/or relaxation sources to zero. Every enlarged part is modeled with an IVP whose initial conditions are compatible with the main IVP and solved with backward Euler using same timestep $\mathrm{h}=\mathrm{T}/2^{12}$.

To monitor convergence, error is estimated by calculating the infinite norm of the difference between the successive waveforms of the relaxation sources from the last two iterations. To ensure simulation results are robust, every algorithm is run $500\times$. Each time, relaxations sources are initialized with random DC values in [0,1]. In each round of simulation, the numbers of iterations or sweeps are taken

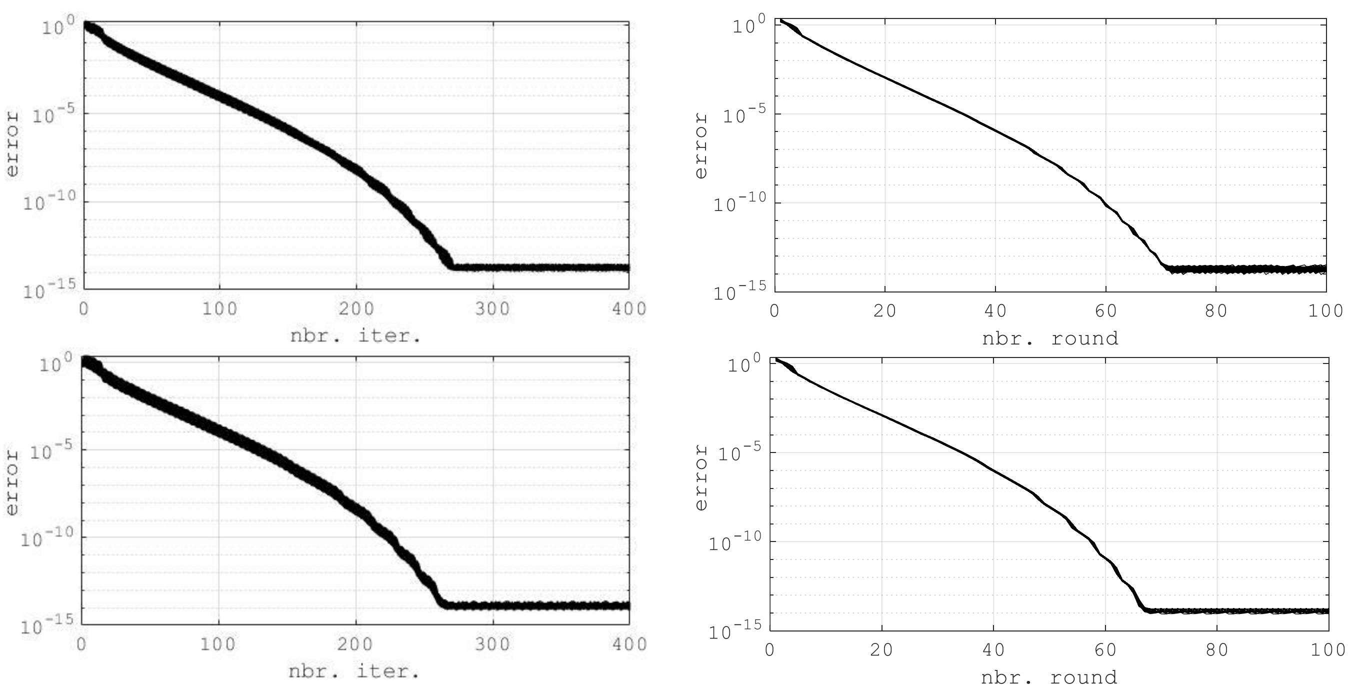

Fig. 11 Error decay. Left GJ-WR. Right. S-WR. Top. N = 3. Bottom. N = 4. (Thm. IV.3).

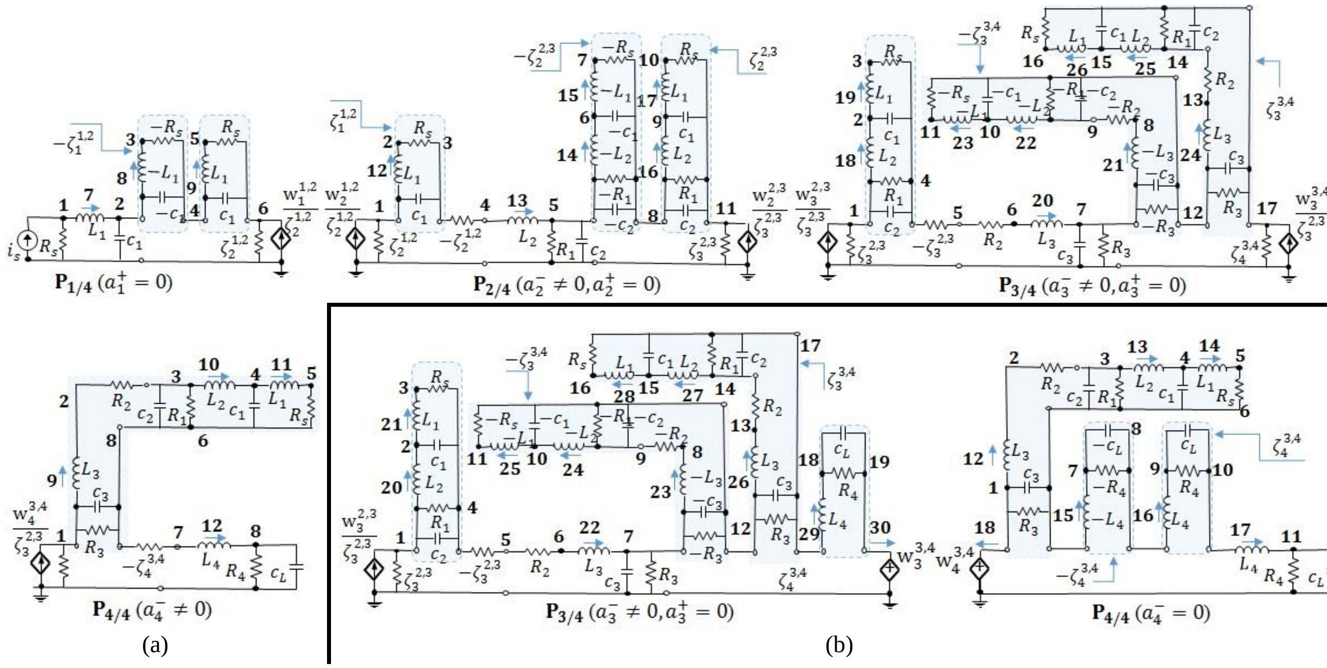

Fig. 12 Four subcircuit partition of the nilpotent algorithm. (a) $a_1^+ = a_2^+ = a_3^+ = 0$ (Thm. IV.5): (b) Replacement parts $\mathrm{P}_{3/4}$, $\mathrm{P}_{4/4}$: $a_1^+ = a_2^+ = a_3^+ = a_4^- = 0$ (Cor. IV.6 (b)).

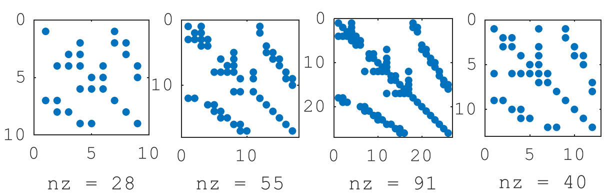

Fig. 13 Sparsity pattern of matrices $(\mathbf{G} + \mathbf{D}/\mathrm{h})$ of enlarged parts. Left to right. $\mathrm{P}_{1/4}$, $\mathrm{P}_{2/4}$, $\mathrm{P}_{3/4}$, $\mathrm{P}_{4/4}$. (Thm. IV.5).

sufficiently large in order to bring the double precision floating point value of the numerical error down to zero up to a quantity close to order 10^{-15} of the roundoff error. If the WR is nilpotent, then its index corresponds to the count value of the first iteration or round at which the error is practically zero. Starting from the index value, the numerical error should level off.

Let us start with the algorithm of Thm. IV3 for $\mathrm{N} = 3, 4$. Fig. 9 represents the three augmented parts $\mathrm{P}_{1/3}$, $\mathrm{P}_{2/3}$, $\mathrm{P}_{3/3}$ with their coupling circuitries. Kernels $\zeta_2^{1,2}$ and $\zeta_2^{2,3}$ are both optimal while $\zeta_1^{1,2} = \zeta_3^{2,3} = 10\Omega$. Matrices $(\mathbf{G} + \mathbf{D}/\mathrm{h})$ of $\mathrm{P}_{1/3}$, $\mathrm{P}_{2/3}$, $\mathrm{P}_{3/3}$ are of order 10, 20 and 8 respectively. Their

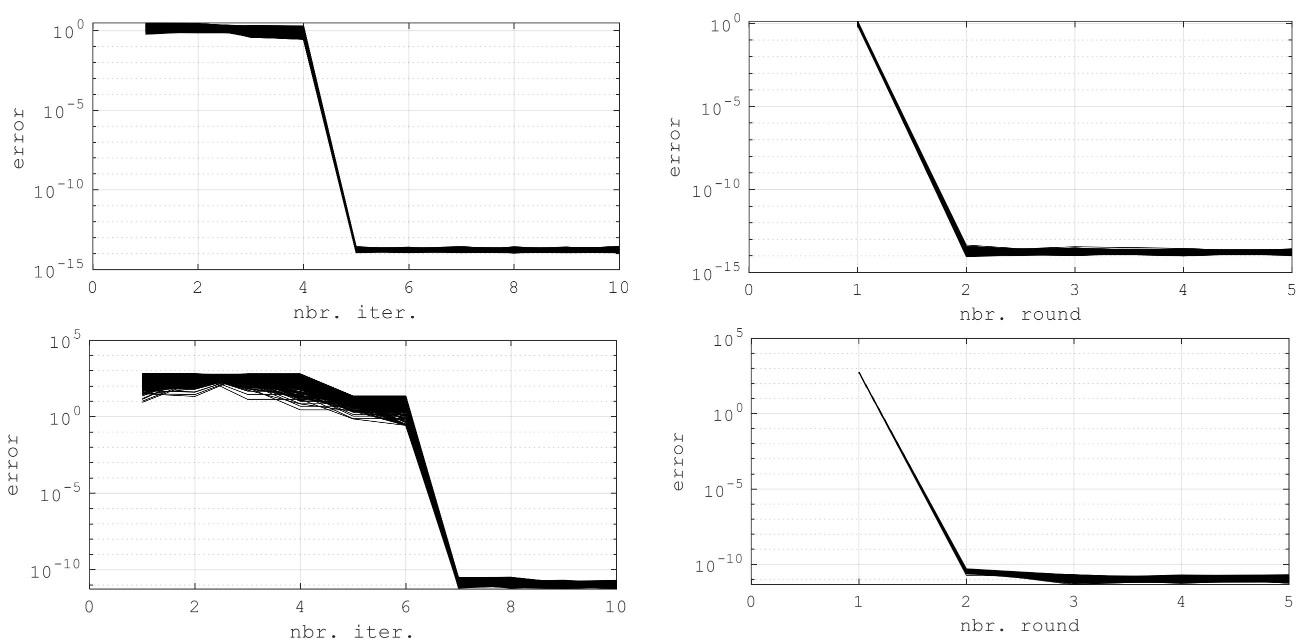

Fig. 14 Error decay. Left GJ-WR. Right S-WR. Top. $\mathrm{N} = 3$: $a_3^- = a_2^- = 0$. Bottom $\mathrm{N} = 4$: $a_1^+ = a_2^+ = a_3^+ = 0$. (Thm. IV.5).

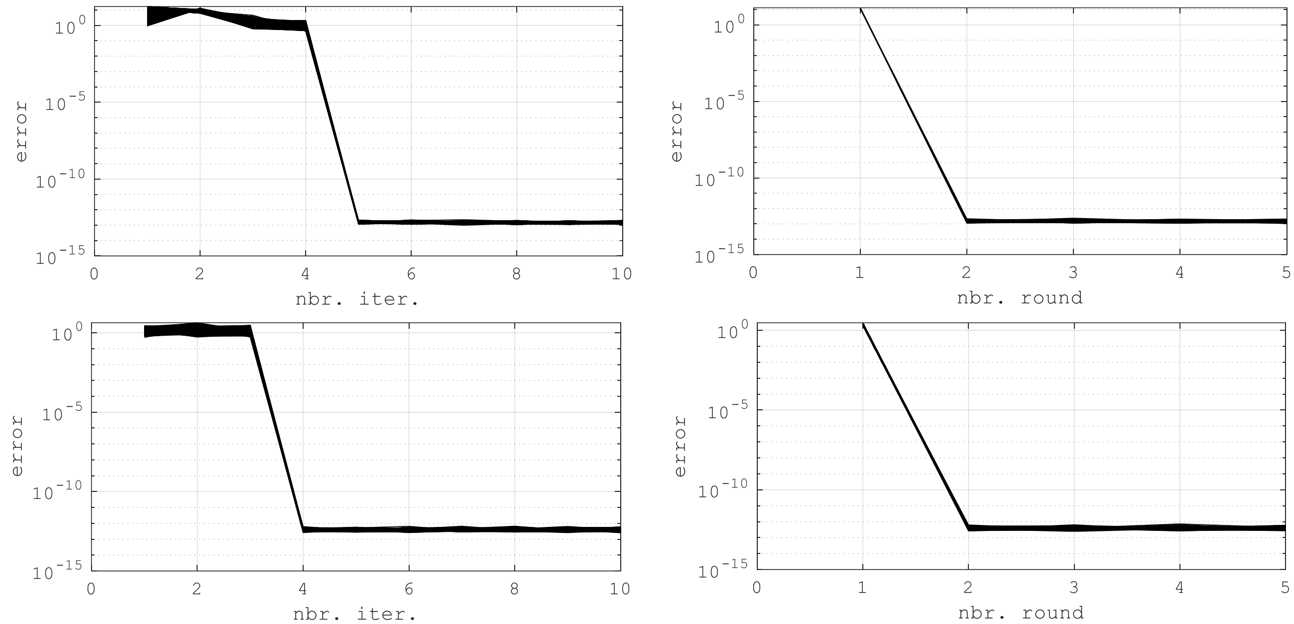

Fig. 15 Error decay. N = 4. Left. GJ-WR. Right. S-WR. Top. $a_1^+ = a_2^+ = a_3^+ = 0$ and $a_4^- = 0$. Bottom. $a_1^+ = a_2^+ = a_3^+ = 0$ and $a_4^- = a_3^- = 0$. (Cor. IV.8).

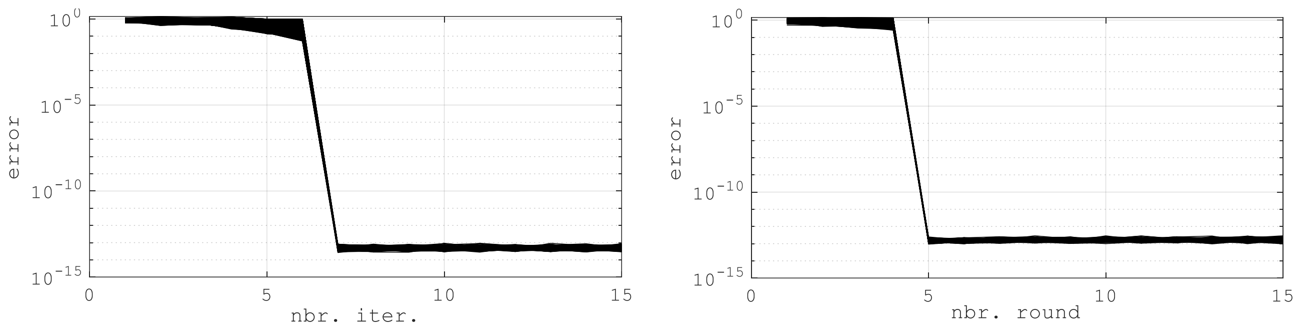

Fig. 16 Error decay. N = 5. GJ-WR. Left. $a_1^+ = a_2^+ = a_3^+ = 0$ and $a_5^- = 0$ (Thm.IV.6). Right. $a_1^+ = a_2^+ = a_3^+ = 0$ and $a_4^- = a_3^- = 0$. (Cor. IV.7).

sparsity patterns are shown in Fig. 10 in accordance with the enumeration of nodal voltages and inductor currents of Fig. 9. In the same way, parts $\mathrm{P}_{1/4}, \mathrm{P}_{2/4}$, $\mathrm{P}_{3/4}$, $\mathrm{P}_{4/4}$ are augmented with the circuit realizations of optimal kernels $\zeta_1^{1,2}$, $\zeta_3^{2,3}$ and $\zeta_3^{3,4}$ while $\zeta_2^{1,2} = \zeta_2^{2,3} = \zeta_4^{3,4} = 10\Omega$. The error evolution in Fig. 11 shows that both GJ-WR and S-WR are non-nilpotent for the two partitions.

Next, the partitions of the algorithm of Thm. IV5 are constructed by making the following kernels optimal: $\zeta_2^{1,2}$ and $\zeta_3^{2,3}$ for $\mathrm{N} = 3$, and $\zeta_1^{1,2}$, $\zeta_2^{2,3}$ and $\zeta_3^{3,4}$ for $\mathrm{N} = 4$. Fig. 12a shows the resulting partition for $\mathrm{N} = 4$. Matrices $(\mathbf{G} + \mathbf{D}/\mathrm{h})$ of parts $\mathrm{P}_{1/4}, \mathrm{P}_{2/4}$, $\mathrm{P}_{3/4}$, $\mathrm{P}_{4/4}$ are of order 9, 17, 26 and 12. Their sparsity patterns are shown on Fig. 13 in accordance with the enumeration of nodal voltages and inductor currents of Fig. 12a. The error evolution in Fig. 14 shows that S-WR converges in 2 rounds for $\mathrm{N} = 3, 4$. The GJ-WR however converges in 5 iterations for $\mathrm{N} = 3$ and in 7 iterations for $\mathrm{N} = 4$ as predicted by theory.

With all forward gains $a_1^+ = a_2^+ = a_3^+ = 0$ for $\mathrm{N} = 4$, reverse gains a_4^- and a_3^- are successively set to zero. The partition of Fig. 12a corresponds to case $a_1^+ = a_2^+ = a_3^+ = a_4^- = 0$ after parts $\mathrm{P}_{3/4}$ and $\mathrm{P}_{4/4}$ are replaced by their circuits in Fig. 12b. This time, the error decay in Fig. 15 shows that the GJ-WR converges in 5 iterations instead of 7 in accordance with the result of Cor. IV8b when $m = 0$. When $a_4^- = a_3^- = 0$, the algorithm is optimal, it converges in 4 iterations in accordance with the result of Cor. IV.9 (Fig. 15). The S-WR algorithms still converges in 2 rounds since all previous dispositions satisfy condition (34).

Finally, the partitions of the algorithm of Cor. IV.7 are constructed for $\mathrm{N} = 5$ and $m \in \{3,2\}$. The evolution of the error in Fig. 16, shows that the GJ-WR converges in 7 iterations when its gains $a_1^+ = a_2^+ = a_3^+ = a_5^- = 0$ $(m = 3)$ and in 5 iterations when $a_1^+ = a_2^+ = a_5^- = a_4^- = 0$ $(m = 2)$. The last result also confirms the relaxed optimal condition of Cor. IV.7 for $\mathrm{N} = 5$.

VI. CONCLUSION

The characterization of the sets of nilpotent GJ-WR, GS-WR and S-WR algorithms have been presented for chains of general passive circuits. It was shown that the way local convergence rates are set to zero affects the nilpotency index of the method and can even lead to non-nilpotency. It is now possible to make informed decisions on the set of optimal kernels to approximate in order to construct cost efficient methods at suboptimal speeds of convergence.

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