Axiomatic Theory of the Reinforced Concrete Beam Undergoing a Pure Bending Moment

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Abstract

This work provides an axiomatic foundational theory of the reinforced concrete (RC) beam undergoing a pure bending moment. Measure theory in mathematics is used as the underlying axiomatic theory to describe the RC beam structure. Novel concepts related to RC beams based on measure theory are developed. These concepts include **RCS**-space, the General Stress Function (GSF), the Moment of Resistance (MoR), and some other derivative concepts. An **RCS**-space is a measure space representing the cross section of a RC beam. The GSF is an $\boldsymbol{L^2}$-function describing the stresses on the RC beam cross section including the concrete and the reinforcements stresses. As a consequence, the properties of **RCS**-space can be stated as theorems. The MoR of a RC beam is presented as a Lebesgue integral expression involving the GSF. As a validation, we demonstrate the derivation of the nominal moment equation of the conventional RC beam following from this axiomatic theory. The result indicates that this axiomatic theory serves as the general theory for RC beams in terms of bending moment.

Keywords: Measure Spaces, Lebesgue Spaces, Reinforced Concrete Beam, Structural Mechanics

1 Introduction

A reinforced concrete (RC) structure undergoing a pure bending can be considered as a reinforced concrete (RC) beam [1]. The moment of resistance (MoR) of a RC beam can be computed from the force couple principle [2]. As some bending moment M occurs on a RC beam, by the force couple principle, a pair of forces ϕ and ψ (the couple) with the same scalar quantity but of opposite direction is assumed to occur on the cross section of the RC beam separated by some distance $d > 0$ such that

$$M_r = |\phi|\, d = |\psi|\, d\,, \tag{1}$$

where M_r is the MoR of the RC beam in classical sense. As a design consideration, structural engineers suggest a factor $c \in \mathbb{R}$ satisfying $0 < c \leq 1$ to be added for a safety reason [3]. And hence the inequality

$$M < c\, M_r$$

must be satisfied.

However, this existing theory can be viewed informal. This paper will provide an axiomatic formulation of the concrete structure undergoing a pure bending moment by making use of measure theory in mathematics. Equation (1) will be the underlying principle of the axiomatic formulation of the MoR of RC beams. The concept of MoR generalises the nominal moment and other moments at all mechanical evolutionary stages of the RC beam such as the elastic moment, the cracking moment, etc.

The measure theoretic system allows us to formulate the MoR of a RC beam with any kind of reinforcement and configuration in a unified formal theory. This is due to the generality nature of theories in pure mathematics, with no exceptions for measure theory. Whilst it is not the case in the classical theory of concrete beams.

2 Preliminaries

2.1 Measure Spaces

Measure theory is a branch of mathematics concerning the concept of measure, the generalisation of real world concepts such as the length of a one dimensional object, the area of a two dimensional object and the volume of a three dimensional object. Another notions falling into the realm of “measure” are the mass of an object and the probability of an event. The readers are referred to [4] for detailed descriptions of measure theory. In addition, the mathematical writing in this paper incorporates a huge amount of the language of first-order logic. Therefore, the readers are also referred to [5]. Several concepts in measure theory will be briefly presented as follows.

(**$\boldsymbol{\sigma}$-Algebra**) The very basic fundamental notion in measure theory is “σ-algebra” [4], which is defined as a family $\mathcal{A}$ of subsets of a set X satisfying the following axioms; (A1) $X \in \mathcal{A}$, (A2) the complement of every element of $\mathcal{A}$ is an element of $\mathcal{A}$, and (A3) the union of countable [6] elements of $\mathcal{A}$ is an element of $\mathcal{A}$. And the pair $(X, \mathcal{A})$ is called a measurable space [4].

From axioms (A1), (A2) and (A3), one can deduce the basic properties of measurable space such as $\varnothing$ is an element of σ-algebras, a σ-algebra is closed under countable intersections, σ-algebra is closed under set differentiation [6], etc [4]. Another property of measurable space is presented on the following theorem.

Theorem 2.1 (Measurable Subspace) *Let* $(X, \mathcal{A})$ *be a measurable space. Let* $E \in \mathcal{A}$. *The family*

$$\mathcal{A}_E := \{A \in \mathcal{A} \mid A \subseteq E\}$$

is a σ*-algebra on* E. *Hence* $(E, \mathcal{A}_E)$ *is a measurable space on its own right which may be called a measurable subspace of* $(X, \mathcal{A})$

Proof of Theorem 2.1. Obviously $E \in \mathcal{A}_E$, hence axiom (A1) is satisfied. Let $B \in \mathcal{A}_E$. Then $B \subseteq E$ as well as $B \in \mathcal{A}$ by definition. Note that $E \setminus B \subseteq E$ [6]. Since a σ-algebra is closed under set differentiation [4], then $E \setminus B \in \mathcal{A}$. We can consider that $E \setminus B$ is the complement of B on E, and we may denote this by $B^{C_E} := E \setminus B$. Hence $B^{C_E} \in \mathcal{A}_E$, which shows that axiom (A2) is satisfied. Let $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{A}_E$. Note that $B_k \subseteq E$ and $B_k \in \mathcal{A}$, for every $k \in \mathbb{N}$. By axiom (A3), we have that $\cup_{k=1}^{\infty} B_k \in \mathcal{A}$. And since $\cup_{k=1}^{\infty} B_k \subseteq E$, then we have $\cup_{k=1}^{\infty} B_k \in \mathcal{A}_E$, which shows that axiom (A3) is satisfied. Hence we conclude that $\mathcal{A}_E$ is a σ-algebra on E. Consequently, $(E, \mathcal{A}_E)$ is a measurable space. $\square$

We encourage the readers to deeply explore more properties of measurable spaces at [4, 7, 8].

(**Measurable Functions**) A function f which maps from a measurable space $(X, \mathcal{A}_X)$ to a measurable space $(Y, \mathcal{A}_Y)$ is called a measurable function if and only if for every $A \in \mathcal{A}_Y$, we have $f^{-1}(A) \in \mathcal{A}_X$ [4]. It may be necessary to assert that f takes the form $f : X \to Y$, instead of $f : (X, \mathcal{A}_X) \to (Y, \mathcal{A}_Y)$. This definition is analogous to that of continuous function in general topology [9], of which continuous functions are defined in terms of topology whereas measurable functions are defined in terms of measurable space. In fact, these two types of functions have important relations [4]. We encourage the readers to observe the properties of measurable functions and their relations with continuous functions at [4].

(**Step Function**) A function that is important in measure theory, especially for the formulation of Lebesgue integral [4, 7, 8, 10] is the so-called "step function" [4], which is defined as a function $s : X \to \mathbb{R}$ such that the image of s is finite, for any set X [4]. Note that, in general a step function is not necessarily measurable. However, those used to formulate Lebesgue integral are of course necessarily measurable. A theorem related to step functions which is fundamental to the formulation of Lebesgue integral is called step function approximation theorem [4, 11]. The readers are required to pay attention to the theorem.

(**Measure Space**) A measure space is a triplet $(X, \mathcal{A}, \mu)$ where $(X, \mathcal{A})$ is a measurable function, and μ is a map $\mu : \mathcal{A} \to [0, \infty]$ satisfying the axioms; (M1) there is some $A \in \mathcal{A}$ such that $\mu(A) < \infty$, and (M2) for every countable and disjoint [6] family $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$, we will have $\mu(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$

[4]. We encourage the readers to explore the properties of measure spaces at [4, 7, 8].

2.2 Lebesgue Integral and Lebesgue Spaces

Lebesgue integration is an integration technique based on measure theory, introduced by Henry Lebesgue [10]. This integration technique extends the integral to a larger class of functions. It also extends the domain of integration. The formal definition of Lebesgue integral (of nonnegative measurable functions) is presented on the following definition.

Definition 2.1 (Lebesgue Integral) [4]. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be a measurable function [4]. By step function approximation theorem [4, 11], for every $n \in \mathbb{N}$ there exists a measurable step function $s_n : X \to [0, \infty)$ such that

$$\forall x \in X\,,\ 0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq f(x) \quad \text{and} \quad \forall x \in X\,,\ \lim_{n\to\infty} s_n(x) = f(x)\,.$$

Note that we can also express s_n by

$$\forall x \in X\,,\ s_n(x) = \sum_{k=1}^{m_n} \alpha_{n,k}\chi_{A_{n,k}}(x)\,,$$

for every $n \in \mathbb{N}$ [4, 11]. Let $E \in \mathcal{A}$. (i) For every $n \in \mathbb{N}$, the Lebesgue integral of s_n over E is defined by

$$\int_E s_n\, d\mu := \sum_{k=1}^{m_n} \alpha_{n,k}\mu(E \cap A_{n,k})\,.$$

(ii) The Lebesgue integral of f over E is defined by

$$\int_E f\, d\mu := \sup_{n\in\mathbb{N}} \int_E s_n\, d\mu\,.$$

The readers are now encouraged to observe the basic properties of nonnegative Lebesgue integral, Lebesgue monotone convergence theorem, σ-additive property of Lebesgue integral and Fatou's lemma at [4].

With definition 2.1, we cannot directly present the definition of Lebesgue integral of any measurable functions including negative functions. This is due to the fact that subtractions on infinity is undefined [4, 7]. In other words, we need to ensure that the integrals of the positive and negative parts of the function [4] are bounded. For a measurable function $f : X \to \mathbb{R}$ on some measure space $(X, \mathcal{A}, \mu)$, the positive part of f, denoted by f^+, is defined by $f^+ := \max\{f, 0\}$, while the negative part of f, denoted by f^-, is defined by $f^- := \max\{-f, 0\}$. Note that $f^+, f^- \geq 0$. The functions well-defined for Lebesgue integration are called Lebesgue integrable functions. The formal definition of such functions is presented as follows.

Definition 2.2 (Lebesgue Integrable Functions) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a measurable function. Then f is said to be Lebesgue integrable if and only if

$$\int_X |f| \, d\mu < \infty$$

holds.

Now we present the Lebesgue integral of Lebesgue integrable functions as follows.

Definition 2.3 (Integral of Lebesgue Integrable Functions) Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a Lebesgue integrable function. Let $E \in \mathcal{A}$. The Lebesgue integral of f over E is defined by

$$\int_E f \, d\mu := \int_E f^+ \, d\mu - \int_E f^- \, d\mu \, .$$

The readers are now referred to [4] for the basic properties of Lebesgue integral of real-valued functions including σ-additivity of the integral, scalar multiplicity of the integral and Lebsgue dominated convergence theorem.

The family of Lebesgue integrable functions forms a vector space over $\mathbb{R}$ [4, 12], denoted by $\mathcal{L}^1$. The more general form of the vector space is the space $\mathcal{L}^p$ with $p \in [1, \infty]$ [4, 7, 13], which is formally presented on the following theorem.

Theorem 2.2 ($\mathcal{L}^p$ Space) *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p \in [0, \infty]$ A family*

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \{f : X \to \mathbb{R} \mid P_p(f) < \infty\} \, ,$$

where

$$P_p(f) := \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{1/p} & : p < \infty \\ \inf \{c \in [0, \infty] \mid |f| \leq c \text{ almost everywhere [4] } \} & : p = \infty \end{cases} ,$$

forms a vector space over $\mathbb{R}$ [4, 12].

The proof for theorem 2.2 is omitted for simplicity, since it is covered comprehensively at [4]. In addition, $\mathcal{L}^p$ is a seminormed space [4, 12], with the map $P_p : \mathcal{L}^p \to \mathbb{R}$ as presented on theorem 2.2 being the seminorm [14]. A quotient space of $\mathcal{L}^p$ with an equivalence relation defined by considering every pair $f, g \in \mathcal{L}^p$ being equivalent if $f = g$ almost everywhere [4, 15], known as an L^p space, is a normed space with the norm defined similarly to P_p [4, 13, 14]. We formally present L^p spaces on the following theorem, again without the proof.

Theorem 2.3 (L^p Spaces) *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $p \in [1, \infty]$. The family*

$$L^p(X, \mathcal{A}, \mu) := \{[f] \mid f \in \mathcal{L}^p(X, \mathcal{A}, \mu\} \, ,$$

where

$$\forall f, g \in \mathcal{L}^p(X,\, \mathcal{A},\, \mu)\big[\, g \in [f] \iff f = g \text{ almost everywhere}\big]\,,$$

is a quotient space of $\mathcal{L}^p(X,\, \mathcal{A},\, \mu)$ *[4, 13, 14]. And the map* $\|\cdot\|_p : L^p(X,\, \mathcal{A},\, \mu) \to \mathbb{R}$ *defined by*

$$\forall f \in L^p(X,\, \mathcal{A},\, \mu)\,,\ \|f\|_p := \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{1/p} & : p < \infty \\ \inf\{c \in [0,\, \infty] \mid |f| \le c \text{ almost everywhere}\} & : p = \infty \end{cases}\,,$$

is a norm on $L^p(X,\, \mathcal{A},\, \mu)$ *[4, 12]. Hence* $(L^p(X,\, \mathcal{A},\, \mu),\, \|\cdot\|_p)$ *is a normed space [4, 12, 13].*

A very important property of L^p spaces in terms of finite measure is presented on the following theorem.

Theorem 2.4 (The Inclusion $L^q \subseteq L^p$) *[16]. Let* $(X,\, \mathcal{A},\, \mu)$ *be a measure space such that* $\mu(X) < \infty$*. Let* $p, q \in [1,\, \infty)$ *such that* $p \le q$*. Theorem 2.3 asserts that* $L^p(X,\, \mathcal{A},\, \mu)$ *and* $L^q(X,\, \mathcal{A},\, \mu)$ *are normed spaces. Then*

$$L^q(X,\, \mathcal{A},\, \mu) \subseteq L^p(X,\, \mathcal{A},\, \mu)$$

holds.

The proof for theorem 2.4 can be found at [16], which is omitted in this paper. In addition, a complete L^p space is called a Banach space [4, 13]. By complete, we mean that every Cauchy sequence in the L^p space converges in the L^p space [4, 9, 13, 14, 17]. The readers are now encouraged to observe more properties of L^p spaces at [4, 13].

3 Axiomatisation of RC Beam

3.1 RCS-Space

Now we construct the axiomatic theory of RC beams in terms of pure bending moment. We use measure space to model the RC cross section with a measure representing the magnitude of the area of a given sub-cross section. The formal definition is presented as follows.

Definition 3.1 (**RCS**-Space) Given a RC beam. Let $(S,\, \mathcal{S})$ be a measurable space where S is the set of all the points in the whole cross section of the RC beam. Suppose a map $\rho : \mathcal{S} \to [0,\, \infty]$ such that $\rho(A)$ is the magnitude of the area (measure) of a region $A \in \mathcal{S}$. We call the triplet $(S,\, \mathcal{S},\, \rho)$ the **RCS**-space if and only if the following axioms are satisfied:

(R1) $\rho(S) < \infty$

(R2) Given a family $\{U_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ such that

$$\forall i, j \in \mathbb{N}\big[i \neq j \implies U_i \cap U_j = \varnothing\big]\,,$$

then

$$\rho\left(\bigcup_{k=1}^{\infty} U_k\right) = \sum_{k=1}^{\infty} \rho(U_k)$$

holds.

Axiom (R1) on definition 3.1 asserts that ρ is a finite measure. This consideration is taken into account since we do not assume infinity applicable on real world problems. Now we will develop the theory further as well as we will observe its properties.

We need to be aware that there are reinforcements installed on the cross section of a RC beam. And the cross section of these reinforcements are within the set of all points of the whole cross section. Based on this consideration, we present a formulation as presented on the following definition.

Definition 3.2 (Bipartite) Let $(S, \mathcal{S}, \rho)$ be an **RCS**-space. Let $C, R \in \mathcal{S}$ such that $C \cap R = \varnothing$. And we designate C to represent the set of all points in the concrete section, and R to represent the set of all points in the reinforcements sections. And we call the collection $\{C, R\}$ the bipartite of S.

Proposition 3.1 Given an **RCS**-space $(S, \mathcal{S}, \rho)$, and the bipartite $\{C, R\}$ of S (definition 3.2). We can consider that in a RC beam cross section, there are only concrete material and reinforcements. Hence $C \cup R = S$. Consequently, $\{C, R\}$ is a partition of S.

Follows from equation (1), then there must be some region on the cross section of a RC beam undergoing compression and some other region undergoing tension. We would like to formally define these regions. But first, we need to define the stress functions of the materials. The stress functions can be of either positive, negative or both values. By convention, negative value of a stress function indicates that the material undergoing some compression. While the positive value of a stress function indicates that the material undergoing some tension.

Definition 3.3 (Concrete and Reinforcements Stress Functions) Let $(S, \mathcal{S}, \rho)$ be an **RCS**-space. Let $\{C, R\}$ be the bipartite of S (definition 3.2). Suppose some functions $b, t : S \to \mathbb{R}$. The functions b and t are the concrete and the reinforcements stress functions respectively if the following axioms are satisfied:

(S1) $b, t \in L^2(S, \mathcal{S}, \rho)$

(S2) $\forall x \in S\,,\ b(x) \leq 0$

(S3) $\forall x \in R\,,\ b(x) = 0$

(S4) $\forall x \in C\,,\ t(x) = 0$

The designation of axiom (S2) is based on the empirical fact that the concrete has a good compressive resistance but has a very poor tensile resistance [18].

As a unification, we generalise the stress functions into a single stress function coined as the General Stress Function (GSF). It is conducted by making use of the property of L^2-functions [4].

Definition 3.4 (General Stress Function) Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $b, t \in L^2(S,\, \mathcal{S},\, \rho)$ be the concrete and the reinforcements stress functions (definition 3.3). The General Stress Function (GSF) is a function $g : S \to \mathbb{R}$ defined by

$$\forall x \in S\,,\; g(x) := b(x) + t(x)\,.$$

Proposition 3.2 Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Then there exist some functions $b, t \in L^2(S,\, \mathcal{S},\, \rho)$, which are the concrete and the reinforcement stress functions, such that $g = b + t$. Follows from theorem 2.3, $L^2(S,\, \mathcal{S},\, \rho)$ is a vector space over $\mathbb{R}$. Hence the addition of a pair of elements of $L^2(S,\, \mathcal{S},\, \rho)$ is another element of $L^2(S,\, \mathcal{S},\, \rho)$ [4, 12]. Consequently $g \in L^2(S,\, \mathcal{S},\, \rho)$.

We now present the compressive and the tensile regions of a **RCS**-space as follows.

Definition 3.5 (Compressive and Tensile Regions) Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $g \in L^2(S,\, \mathcal{S},\, \rho)$ be the GSF.

i. The compressive region of S is defined by

$$S^- := \{x \in S \mid g(x) < 0\}\,.$$

ii. The tensile region of S is defined by

$$S^+ := \{x \in S \mid g(x) > 0\}\,.$$

The first property of compressive and tensile regions is given on the following theorem.

Theorem 3.1 (The Compressive and Tensile Regions as Nonpartition) *Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let S^- and S^+ be the compressive and tensile regions of S (definition 3.5). Suppose*

$$\exists x \in S\,,\; g(x) = 0\,.$$

Then (i) $S^- \cap S^+ = \varnothing$, but (ii) $\{S^-, S^+\}$ is not a partition of S.

Proof of Thoerem 3.1. (i) Let us recall that

$$\forall x \in S\,,\; g(x) = b(x) + t(x)$$

as described by definition 3.4. And by axiom (S2), note that

$$\forall x \in S^+\,,\; g(x) = t(x)\,.$$

Then we obtain

$$\begin{aligned} S^- \cap S^+ &= \{x \in S \mid g(x) < 0\} \cap \{x \in S \mid t(x) > 0\} \\ &= \{x \in S \mid b(x) + t(x) < 0\} \cap \{x \in S \mid t(x) > 0\} \\ &= \{x \in S \mid b(x) + t(x) < 0 \wedge t(x) > 0\} . \end{aligned}$$

Now let us observe the formula within the set builder notation on the latest expression above, that is

$$b(x) + t(x) < 0 \wedge t(x) > 0 . \tag{2}$$

Supposing $b(x) = 0$, then we will have

$$b(x) + t(x) = 0 + t(x) = t(x) < 0 \ \wedge \ t(x) > 0 ,$$

which cannot exist. Then the solution to the expression (2) must be the empty set. Hence we obtain

$$S^- \cap S^+ = \{x \in S \mid b(x) + t(x) < 0 \wedge t(x) > 0\} = \varnothing ,$$

which proves statement i of the theorem.

(ii) Since there are some $x \in S$ such that $g(x) = 0$, then

$$g^{-1}(\{0\}) \neq \varnothing .$$

From definition 3.5, we can also express

$$S^- = g^{-1}((-\infty, 0)) , \quad S^+ = g^{-1}((0, \infty)) .$$

Then we obtain

$$\begin{aligned} S^- \cup S^+ &= g^{-1}((-\infty, 0)) \cup g^{-1}((0, \infty)) \\ &= g^{-1}((-\infty, 0)) \cup g^{-1}((0, \infty)) \cup \varnothing \\ &\neq g^{-1}((-\infty, 0)) \cup g^{-1}((0, \infty)) \cup g^{-1}(\{0\}) \\ &= g^{-1}((-\infty, 0) \cup \{0\} \cup (0, \infty)) \\ &= g^{-1}(\mathbb{R}) \\ &= S , \end{aligned}$$

which proves statement ii of the theorem. □

Now we may derive another property of the compressive and the tensile regions, as given on the following theorem.

Theorem 3.2 (Measurability of Compressive and Tensile Regions) *Let* $(S, \mathcal{S}, \rho)$ *be an* **RCS***-space. Let* $g : S \to \mathbb{R}$ *be the GSF (definition 3.4). Let* S^- *and* S^+ *be the compressive and tensile regions of* S *(definition 3.5). Then we have* $S^-, S^+ \in \mathcal{S}$.

Proof of Theorem 3.2. Follows from definition 3.5, we can express

$$S^- = g^{-1}((-\infty, 0)) , \quad S^+ = g^{-1}((0, \infty)) .$$

Note that both $(-\infty, 0)$ and $(0, \infty)$ are Borel sets on $\mathbb{R}$ [4, 9], which means that they are measurable sets. Proposition 3.2 asserts that $g \in L^2(S, \mathcal{S}, \rho)$, then g is measurable [4]. Consequently, by the measurability of g [4] we have

$$g^{-1}((-\infty, 0)),\ g^{-1}((0, \infty)) \in \mathcal{S} ,$$

which means $S^-, S^+ \in \mathcal{S}$. □

Follows from theorem 3.2 above, we may present the concept of measurable subspace related to the **RCS**-space as follows.

Definition 3.6 (**RCS**$^-$ and **RCS**$^+$ Measurable Spaces) Let $(S, \mathcal{S}, \rho)$ be an **RCS**-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let S^- and S^+ be the compressive and tensile regions of S (definition 3.5). Theorem 3.2 asserts that $S^-, S^+ \in \mathcal{S}$.

i. We define
$$\mathcal{S}^- := \{A \in \mathcal{S} \mid A \subseteq S^-\}\,.$$
By theorem 2.1, $(S^-, \mathcal{S}^-)$ is a measurable space which we call the **RCS**$^-$ measurable space.

ii. We define
$$\mathcal{S}^+ := \{A \in \mathcal{S} \mid A \subseteq S^+\}\,.$$
By theorem 2.1, $(S^+, \mathcal{S}^+)$ is a measurable subspace which we call the **RCS**$^+$ measurable space.

3.2 Forces on the RCS-Space

Our intention in building a measure theoretic model for the RC beam is to compute the forces on the section of the RC beam, and then with the forces we can compute the MoR of the RC beam. The force can be given as the Lebesgue integral of the GSF. Since GSF is an L^2 function and as **RCS**-space has a finite measure, then by theorem 2.4 GSF is also an L^1 function. Consequently, its Lebesgue integration is well-defined. We may present this integration process as a map from the σ-algebra of the **RCS**-space to $\mathbb{R}$, of which we call the map the "force map". We formally define the force map as follows.

Definition 3.7 (Force Map) Let $(S, \mathcal{S}, \rho)$ be an **RCS**-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Proposition 3.2 asserts that $g \in L^2(S, \mathcal{S}, \rho)$. Then by theorem 2.4, we have $g \in L^1(S, \mathcal{S}, \rho)$. Hence g has a well-defined Lebesgue integral. A map $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ defined by
$$\forall A \in \mathcal{S}\,,\ \mathrm{F}(A) := \int_A g\, d\rho$$
is called the force map on $(S, \mathcal{S}, \rho)$.

From definition 3.7 above, we can derive several properties of the force map as given on the following theorem.

Theorem 3.3 (Negative and Positive Forces) *Let $(S, \mathcal{S}, \rho)$ be an* **RCS**-*space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ be the force map (definition 3.7). Let $(S^-, \mathcal{S}^-)$ and $(S^+, \mathcal{S}^+)$ be the* **RCS**$^-$ *and* **RCS**$^+$ *measurable spaces respectively (definition 3.6). Then the following statements hold:*

i. $\forall A \in \mathcal{S}^-\,,\ \mathrm{F}(A) < 0$

ii. $\forall A \in \mathcal{S}^+\,,\ \mathrm{F}(A) > 0$

Proof of Theorem 3.3. Recalling definition 3.6, we have $\mathcal{S}^-, \mathcal{S}^+ \subset \mathcal{S}$. Then the Lebesgue integral of the GSF over some element of either $\mathcal{S}^-$ or $\mathcal{S}^+$ with respect to ρ is well defined. Let us recall the bipartite $\{C, R\}$ (definition 3.2). Let $V \in \mathcal{S}^-$ and $W \in \mathcal{S}^+$. And suppose

$$V_b := V \cap C\ , \quad V_t := V \cap R$$

and

$$W_b := W \cap C\ , \quad W_t := W \cap R\,.$$

Hence $\{V_b, V_t\}$ is a partition of V and $\{W_b, W_t\}$ is a partition of W since $\{C, R\}$ is a partition of S. And let $b, t \in L^2(S, \mathcal{S}, \rho)$ be the concrete and the reinforcements stress functions (definition 3.3). Then by the properties of Lebesgue integral [4], we obtain

$$\begin{aligned}
\mathrm{F}(V) &= \int_V g\,d\rho = \int_{V_b \cup V_t} g\,d\rho = \int_{V_b} b + t\,d\rho + \int_{V_t} b + t\,d\rho \\
&= \int_S (b+t)\chi_{V_b}\,d\rho + \int_S (b+t)\chi_{V_t}\,d\rho \\
&= \int_S b\chi_{V_b}\,d\rho + \int_S t\chi_{V_b}\,d\rho + \int_S b\chi_{V_t}\,d\rho + \int_S t\chi_{V_t}\,d\rho \\
&= \int_S b\chi_{V_b}\,d\rho + 0 + 0 + \int_S t\chi_{V_t}\,d\rho \\
&= \left(\int_S b^+\chi_{V_b}\,d\rho - \int_S b^-\chi_{V_b}\,d\rho\right) + \left(\int_S t^+\chi_{V_t}\,d\rho - \int_S t^-\chi_{V_t}\,d\rho\right) \\
&= \left(0 - \int_S b^-\chi_{V_b}\,d\rho\right) + \left(0 - \int_S t^-\chi_{V_t}\,d\rho\right) \\
&= -\left(\int_S b^-\chi_{V_b}\,d\rho + \int_S t^-\chi_{V_t}\,d\rho\right) \\
&< 0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{F}(W) &= \int_W g\,d\rho = \int_{W_b \cup W_t} g\,d\rho = \int_{W_b} b + t\,d\rho + \int_{W_t} b + t\,d\rho \\
&= \int_S (b+t)\chi_{W_b}\,d\rho + \int_S (b+t)\chi_{W_t}\,d\rho \\
&= \int_S b\chi_{W_b}\,d\rho + \int_S t\chi_{W_b}\,d\rho + \int_S b\chi_{W_t}\,d\rho + \int_S t\chi_{W_t}\,d\rho \\
&= \int_S b\chi_{W_b}\,d\rho + 0 + 0 + \int_S t\chi_{W_t}\,d\rho \\
&= \left(\int_S b^+\chi_{W_b}\,d\rho - \int_S b^-\chi_{W_b}\,d\rho\right) + \left(\int_S t^+\chi_{W_t}\,d\rho - \int_S t^-\chi_{W_t}\,d\rho\right) \\
&= (0 - 0) + \left(\int_S t^+\chi_{W_t}\,d\rho - 0\right) \\
&= \int_S t^+\chi_{W_t}\,d\rho \\
&> 0
\end{aligned}$$

which prove statements i and ii respectively. □

We have designated for this theory that the RC beam undergoes a pure bending moment, which consequently requires the force resultant in the system to be zero. Now we will present this principle on the following postulate, which is very fundamental for this theory.

Postulate 3.1 (Force Equilibrium) *Let* $(S, \mathcal{S}, \rho)$ *be an* **RCS**-*space. Let* $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ *be the force map (definition 3.7). Then*

$$\mathrm{F}(S) = 0$$

must hold.

A consequence of postulate 3.1 is given on the following theorem.

Theorem 3.4 (Compressive and Tensile Regions Equilibrium) *Let* $(S, \mathcal{S}, \rho)$ *be an* **RCS**-*space. Let* S^- *and* S^+ *be the compressive and tensile regions of* S *(definition 3.5). Let* $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ *be the force map (definition 3.7). Then we have*

$$\mathrm{F}(S^-) = -\mathrm{F}(S^+)\,.$$

Proof of Theorem 3.4. Theorem 3.1 asserts that $\{S^-, S^+\}$ is not a partition of S even though the sets are disjoint. Then let us define

$$N := S \setminus (S^- \cup S^+)\,.$$

Consequently, $\{S^-, S^+, N\}$ is a partition of S. Since a σ-algebra is closed under set differentiation [4], then $N \in \mathcal{S}$. By definition 3.5 and theorem 3.1 we will obtain

$$N = \{x \in S \mid g(x) = 0\}\,.$$

Follows from the property of Lebesgue integral [4], we obtain

$$\mathrm{F}(N) = \int_N g\, d\rho = \int_S g\chi_N\, d\rho = 0\,.$$

By postulate 3.1 and σ-additive property of Lebesgue integral [4], we obtain

$$\begin{aligned}
0 &= \mathrm{F}(S) = \mathrm{F}(S^- \cup S^+ \cup N) \\
&= \int_{S^- \cup S^+ \cup N} g\, d\rho = \int_{S^-} g\, d\rho + \int_{S^+} g\, d\rho + \int_N g\, d\rho = \int_{S^-} g\, d\rho + \int_{S^+} g\, d\rho + 0 \\
&= \mathrm{F}(S^-) + \mathrm{F}(S^+)\,.
\end{aligned}$$

Theorem 3.3 asserts that $\mathrm{F}(S^-) < 0$ and $\mathrm{F}(S^+) > 0$, which means that both forces cannot be 0. Hence the expression above implies

$$\mathrm{F}(S^-) = -\mathrm{F}(S^+)\,,$$

which proves the theorem. □

3.3 Central of Gravity of Sets and UM-Distance

One of important components in the theory of bending moment is the distance between the centroids of the compressive and tensile regions. By the force couple principle [2], we require the distance separating the forces in the couple to compute the equivalent moment by multiplying one of the absolute value of the forces with the distance. This distance is unidirectional to the rotation of the occurring moment. We call the distance the **UM**-distance. We use the concept of pseudometric space [19] to formally define the **UM**-distance on an **RCS**-space, which is presented as follows.

Definition 3.8 (**UM**-Distance) Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $\mathcal{U} \subseteq \mathcal{S}$. A map

$$d : \mathcal{U} \to [0,\, \infty)$$

is called a **UM**-distance on $\mathcal{U}$ if and only if the following axioms are satisfied:

(U1) $\forall U \in \mathcal{U}\,,\; d(U, U) = 0$

(U2) $\forall U, V \in \mathcal{U}\,,\; d(U, V) = d(U, V)$

(U3) $\forall U, V, W \in \mathcal{U}\,,\; d(U, V) \leq d(U, W) + d(W, V)$

Axioms (U1), (U2) and (U3) are known as the axioms of pseudometric [19]. Thus, a **UM**-distance is in fact a pseudometric on a family of measurable sets of a **RCS**-space. Now our task is to construct a **UM**-distance on an **RCS**-space. However, we will need a new measure space concept on the **RCS**-space which is presented on the following theorem.

Theorem 3.5 (τ-Measure) *Let $(S,\, \mathcal{S},\, \rho)$ be an* **RCS***-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $A \in \mathcal{S}$. By theorem 2.1, a family*

$$\mathcal{S}_A := \{B \in \mathcal{S} \mid B \subseteq A\}$$

is a σ-algebra on A. Hence $(A,\, \mathcal{S}_A)$ is a measurable space. A map $\tau_A : \mathcal{S}_A \to [0,\, 1]$ defined by

$$\forall B \in \mathcal{S}_A\,,\; \tau_A(B) := \frac{1}{\int_A |g|\, d\rho} \int_B |g|\, d\rho$$

is a measure on $(A,\, \mathcal{S}_A)$. Consequently $(A,\, \mathcal{S}_A,\, \tau_A)$ is a measure space.

Proof of Theorem 3.5. Obviously axiom (M1) must be satisfied since τ_A is finite, as shown by

$$\tau_A(A) = \frac{1}{\int_A |g|\, d\rho} \int_A |g|\, d\rho = 1 < \infty\,.$$

Now let $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{S}_A$ such that $B_i \cap B_j = \varnothing$ whenever $i \neq j$. Then by σ-additive property of Lebesgue integral [4], we obtain

$$\begin{aligned}\tau_A\left(\bigcup_{k=1}^{\infty} B_k\right) &= \frac{1}{\int_A |g|\, d\rho} \int_{\bigcup_{k=1}^{\infty} B_k} |g|\, d\rho = \frac{1}{\int_A |g|\, d\rho} \sum_{k=1}^{\infty} \int_{B_k} |g|\, d\rho \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\int_A |g|\, d\rho} \int_{B_k} |g|\, d\rho = \lim_{n\to\infty} \sum_{k=1}^{n} \tau_A(B_k)\end{aligned}$$

$$= \sum_{k=1}^{\infty} \tau_A(B_k)\,,$$

which shows that axiom (M2) is satisfied. Hence τ_A is a measure on $(A,\, \mathcal{S}_A)$. □

Now we introduce a function called the unidirectional function (UF), as given on the following definition.

Definition 3.9 (Unidirectional Function) Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $r \in S$ be a point called a reference point. Suppose a map $Y_r : S \to \mathbb{R}$ such that for every $x \in S$, $|Y_r(x)|$ is the distance between r and x unidirectional to the moment on the RC beam, and for every $A \in \mathcal{S}$, $Y_r\big|_A \in L^2(A,\, \mathcal{S}_A,\, \tau_A)$ holds where $(A,\, \mathcal{S}_A,\, \tau_A)$ is a measure space following from theorem 3.5. And we call Y_r the unidirectional function (UF). By convention, Y_r can be of either positive or negative value. If r is closer to the extreme compression fibre than x then $Y_r(x)$ is positive and vice versa, for every $x \in S$. Follows from theorem 2.4, $Y_r\big|_A \in L^1(S,\, \mathcal{S},\, \rho)$, which means that its Lebesgue integral is well-defined. And

$$\int_A Y_r\big|_A \, d\tau_A$$

is the signed distance between r and the centroid of A.

Now let us observe the properties of UF, as given on the following two theorems.

Theorem 3.6 *Let $(S,\, \mathcal{S},\, \rho)$ be an* **RCS**-*space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $r \in S$ be a reference point, and let $Y_r : S \to \mathbb{R}$ be the UF (definition 3.9). Let $A \in \mathcal{S}$. By theorem 3.5, $(A,\, \mathcal{S}_A,\, \tau_A)$ is a measure space. The expression*

$$\int_A Y_r\big|_A \, d\tau_A = \frac{1}{\int_A |g| \, d\rho} \int_A Y_r |g| \, d\rho$$

holds.

Proof of Theorem 3.6. Note that $Y_r\big|_A = Y_r$ everywhere on A [4]. Note that we can decompose $Y_r\big|_A$ into its positive and negative parts [4], that is

$$Y_r\big|_A = Y_r\big|_A^+ - Y_r\big|_A^- \,.$$

And by step function approximation theorem [4, 11], there exist step functions $y_m, z_n : A \to [0,\, \infty)$ satisfying

$$\forall x \in A \left[0 \le y_1(x) \le y_2(x) \le \cdots \le Y_r\big|_A^+(x)\right] \;\wedge\; \forall x \in A \left[\lim_{m\to\infty} y_m(x) = Y_r\big|_A^+(x)\right]$$

and

$$\forall x \in A \left[0 \le z_1(x) \le z_2(x) \le \cdots \le Y_r\big|_A^-(x)\right] \;\wedge\; \forall x \in A \left[\lim_{n\to\infty} z_n(x) = Y_r\big|_A^-(x)\right]$$

respectively, for every $m, n \in \mathbb{N}$. Now suppose

$$f := \frac{|g|}{\int_A |g| \, d\rho} \,.$$

Hence f is measurable [4] and satisfies $0 \leq f < \infty$. Follows from the construction of Lebesgue integral (definition 2.1 and [4]) as well as the properties of Lebesgue integral including Lebesgue monotone convergence theorem [4], we obtain

$$\begin{aligned}
\int_A Y_r\big|_A \, d\tau_A &= \int_A Y_r\big|_A^+ \, d\tau_A - \int_A Y_r\big|_A^- \, d\tau_A \\
&= \left(\sup_{m\in\mathbb{N}} \int_A y_m \, d\tau_A\right) - \left(\sup_{n\in\mathbb{N}} \int_A z_n \, d\tau_A\right) \\
&= \left(\sup_{m\in\mathbb{N}} \sum_{i=1}^{k_m} \alpha_{m,i}\tau_A(G_{m,i})\right) - \left(\sup_{n\in\mathbb{N}} \sum_{j=1}^{l_n} \beta_{n,j}\tau_A(H_{n,j})\right) \\
&= \left(\sup_{m\in\mathbb{N}} \sum_{i=1}^{k_m} \alpha_{m,i} \frac{1}{\int_A |g| \, d\rho} \int_{G_{m,i}} |g| \, d\rho\right) \\
&\quad - \left(\sup_{n\in\mathbb{N}} \sum_{j=1}^{l_n} \beta_{n,j} \frac{1}{\int_A |g| \, d\rho} \int_{H_{n,j}} |g| \, d\rho\right) \\
&= \left(\sup_{m\in\mathbb{N}} \sum_{i=1}^{k_m} \alpha_{m,i} \int_{G_{m,i}} f \, d\rho\right) - \left(\sup_{n\in\mathbb{N}} \sum_{j=1}^{l_n} \beta_{n,j} \int_{H_{n,j}} f \, d\rho\right) \\
&= \left(\sup_{m\in\mathbb{N}} \sum_{i=1}^{k_m} \alpha_{m,i} \int_A f\chi_{G_{m,i}} \, d\rho\right) - \left(\sup_{n\in\mathbb{N}} \sum_{j=1}^{l_n} \beta_{n,j} \int_A f\chi_{H_{n,j}} \, d\rho\right) \\
&= \left(\sup_{m\in\mathbb{N}} \int_A \left(\sum_{i=1}^{k_m} \alpha_{m,i}\chi_{G_{m,i}}\right) f \, d\rho\right) \\
&\quad - \left(\sup_{n\in\mathbb{N}} \int_A \left(\sum_{j=1}^{l_n} \beta_{n,j}\chi_{H_{n,j}}\right) f \, d\rho\right) \\
&= \left(\lim_{m\to\infty} \int_A y_m f \, d\rho\right) - \left(\lim_{n\to\infty} \int_A z_n f \, d\rho\right) \\
&= \int_A \lim_{m\to\infty} y_m f \, d\rho - \int_A \lim_{n\to\infty} z_n f \, d\rho \\
&= \int_A Y_r\big|_A^+ f \, d\rho - \int_A Y_r\big|_A^- f \, d\rho \\
&= \int_A Y_r\big|_A f \, d\rho \\
&= \frac{1}{\int_A |g| \, d\rho} \int_A Y_r |g| \, d\rho \, ,
\end{aligned}$$

which proves the theorem. $\square$

Theorem 3.7 *Let $(S, \mathcal{S}, \rho)$ be an* **RCS***-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $r \in S$ be a reference point, and let $Y_r : S \to \mathbb{R}$ be the UF (definition 3.9). Let $A \in \mathcal{S}$. The expression*

$$\frac{1}{\int_A |g| \, d\rho} \int_A |Y_r g| \, d\rho < \infty$$

holds.

Proof of Theorem 3.7. By definition 3.9, we have that $Y_r\big|_A \in L^2(A,\, \mathcal{S}_A,\, \tau_A)$. By theorem 2.4, we have $Y_r\big|_A \in L^1(A,\, \mathcal{S}_A,\, \tau_A)$, which implies

$$\int_A \left|Y_r\big|_A\right| d\tau_A < \infty\,.$$

And by theorem 3.6, we obtain

$$\frac{1}{\int_A |g|\, d\rho} \int_A |Y_r g|\, d\rho = \frac{1}{\int_A |g|\, d\rho} \int_A |Y_r||g|\, d\rho = \int_A \left|Y_r\big|_A\right| d\tau_A < \infty\,,$$

which proves the theorem. □

By making use of the UF, we now establish the **UM**-distance through the following theorem.

Theorem 3.8 (**UM**-Distance) *Let $(S,\, \mathcal{S},\, \rho)$ be an* **RCS**-*space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $(S^-,\, \mathcal{S}^-)$ and $(S^+,\, \mathcal{S}^+)$ be the* **RCS**$^-$ *and* **RCS**$^+$ *measurable spaces (definition 3.6). Let $r \in S$ be a reference point and let $Y_r : S \to \mathbb{R}$ be the UF (definition 3.9). A map $d_{\mathrm{UM}} : \mathcal{S}^- \cup \mathcal{S}^+ \to [0,\, \infty)$ defined by*

$$\forall A, B \in \mathcal{S}^- \cup \mathcal{S}^+\,,\; d_{\mathrm{UM}}(A, B) := \left| \int_A Y_r\big|_A\, d\tau_A - \int_B Y_r\big|_B\, d\tau_B \right|$$

is a **UM**-*distance on $\mathcal{S}^- \cup \mathcal{S}^+$.*

Proof of Theorem 3.8. Let $U, V, W \in \mathcal{S}^- \cup \mathcal{S}^+$. Follows from theorem 3.5, we have measure spaces $(U,\, \mathcal{S}_U,\, \tau_U)$, $(V,\, \mathcal{S}_V,\, \tau_V)$ and $(W,\, \mathcal{S}_W,\, \tau_W)$. By the properties of Lebesgue integral [4], we obtain

$$d_{\mathrm{UM}}(U, U) = \left| \int_U Y_r\big|_U\, d\tau_U - \int_U Y_r\big|_U\, d\tau_U \right| = \left| \int_U Y_r\big|_U - Y_r\big|_U\, d\tau_U \right| = 0\,,$$

and

$$\begin{aligned} d_{\mathrm{UM}}(U, V) &= \left| \int_U Y_r\big|_U\, d\tau_U - \int_V Y_r\big|_V\, d\tau_V \right| \\ &= \left| (-1) \left(\int_V Y_r\big|_V\, d\tau_V - \int_U Y_r\big|_U\, d\tau_U \right) \right| \\ &= |-1| \left| \int_V Y_r\big|_V\, d\tau_V - \int_U Y_r\big|_U\, d\tau_U \right| \\ &= \left| \int_V Y_r\big|_V\, d\tau_V - \int_U Y_r\big|_U\, d\tau_U \right| \\ &= d_{\mathrm{UM}}(V, U) \end{aligned}$$

and

$$\begin{aligned} d_{\mathrm{UM}}(U, V) &= \left| \int_U Y_r\big|_U\, d\tau_U - \int_V Y_r\big|_V\, d\tau_V \right| \\ &= \left| \int_U Y_r\big|_U\, d\tau_U \left(- \int_W Y_r\big|_W\, d\tau_W + \int_W Y_r\big|_W\, d\tau_W \right) - \int_V Y_r\big|_V\, d\tau_V \right| \end{aligned}$$

$$\leq \left| \int_U Y_r\big|_U \, d\tau_U - \int_W Y_r\big|_W \, d\tau_W \right| + \left| \int_W Y_r\big|_W \, d\tau_W - \int_V Y_r\big|_V \, d\tau_V \right|$$
$$= d_{\mathrm{UM}}(U, W) + d_{\mathrm{UM}}(W, V)\,,$$

which respectively show that axioms (U1), (U2) and (U3) are satisfied. Hence d_{UM} is a **UM**-distance on $\mathcal{S}^- \cup \mathcal{S}^+$. □

Now that we have established the **UM**-distance, we will introduce a pseudometric space [19] which we call the **UM**-space.

Definition 3.10 (**UM**-Space) Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $(S^-,\, \mathcal{S}^-)$ and $(S^+,\, \mathcal{S}^+)$ be the **RCS**$^-$ and **RCS**$^+$ measurable space (definition 3.6). Let $r \in S$ be a reference point, and let $Y_r : S \to \mathbb{R}$ be the UF (definition 3.9). By theorem 3.8, a map $d_{\mathrm{UM}} : \mathcal{S}^- \cup \mathcal{S}^+ \to [0,\, \infty)$ defined by

$$\forall A, B \in \mathcal{S}^- \cup \mathcal{S}^+\,,\; d_{\mathrm{UM}}(A, B) := \left| \int_A Y_r\big|_A \, d\tau_A - \int_B Y_r\big|_B \, d\tau_B \right|$$

is a **UM**-distance on $\mathcal{S}^- \cup \mathcal{S}^+$. The pairing $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{\mathrm{UM}})$ is called the **UM**-space.

Proposition 3.3 Let $(S,\, \mathcal{S},\, \rho)$ be an **RCS**-space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $(S^-,\, \mathcal{S}^-)$ and $(S^+,\, \mathcal{S}^+)$ be the **RCS**$^-$ and **RCS**$^+$ measurable spaces (definition 3.6). Let $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{\mathrm{UM}})$ be the **UM**-space (definition 3.10. Follows from theorem 3.6, we obtain

$$\forall A, B \in \mathcal{S}^- \cup \mathcal{S}^+\,,\; d_{\mathrm{UM}}(A, B) = \left| \frac{1}{\int_A |g|\, d\rho} \int_A Y_r |g|\, d\rho - \frac{1}{\int_B |g|\, d\rho} \int_B Y_r |g|\, d\rho \right|\,,$$

where $r \in S$ is a reference point and $Y_r : S \to \mathbb{R}$ is the UF (definition 3.9).

3.4 Moment of Resistance

Now we present the formal description of the MoR of an RC beam undergoing a pure bending moment as a postulate and its derivation below.

Postulate 3.2 (Moment of Resistance) *Let $(S,\, \mathcal{S},\, \rho)$ be an* **RCS**-*space. Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Let $r \in S$ be a reference point, and let $Y_r : S \to \mathbb{R}$ be the UF (definition 3.9). The moment of resistance (MoR) of the* **RCS**-*space is given by*

$$\mathcal{M}(S,\, \mathcal{S},\, \rho) = \left| \frac{\int_{S^*} g\, d\rho}{\int_{S^-} |g|\, d\rho} \int_{S^-} Y_r |g|\, d\rho - \frac{\int_{S^*} g\, d\rho}{\int_{S^+} |g|\, d\rho} \int_{S^+} Y_r |g|\, d\rho \right|\,,$$

for every $* \in \{-, +\}$.

Derivation By equation (1) (as implied by the force couple principle [2]), the moment of a RC beam is equal to the multiplication of the absolute value of the force on either compressive or tensile region with the distance separating the compressive and tensile regions. Then by our axiomatic theory, we have

$$\mathcal{M}(S,\, \mathcal{S},\, \rho) := |\mathrm{F}(S^*)| \cdot d_{\mathrm{UM}}(S^-, S^+)\,,$$

such that $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ is the force map (definition 3.7), and $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{\mathrm{UM}})$ is the **UM**-space (definition 3.10). Follows from definition 3.7 and proposition 3.3, we obtain

$$\begin{aligned}
\mathcal{M}(S,\, \mathcal{S},\, \rho) &= |\mathrm{F}(S^*)| \cdot d_{\mathrm{UM}}(S^-, S^+) \\
&= \left| \int_{S^*} g \, d\rho \right| \left| \frac{1}{\int_{S^-} |g| \, d\rho} \int_{S^-} Y_r |g| \, d\rho - \frac{1}{\int_{S^+} |g| \, d\rho} \int_{S^+} Y_r |g| \, d\rho \right| \\
&= \left| \frac{\int_{S^*} g \, d\rho}{\int_{S^-} |g| \, d\rho} \int_{S^-} Y_r |g| \, d\rho - \frac{\int_{S^*} g \, d\rho}{\int_{S^+} |g| \, d\rho} \int_{S^+} Y_r |g| \, d\rho \right| ,
\end{aligned}$$

hence the postulate. □

4 Deriving Conventional RC Beam Nominal Moment

As a simple validation, we will present the derivation of the conventional RC beam's nominal moment equation [3] via our axiomatic theory as presented on section 3. A conventional RC beam is a RC beam with steel reinforcement bars (rebars) being the sole reinforcement.

Suppose we are given a rectangular conventional RC beam undergoing some ultimate moment $M_u > 0$. Let $b, h > 0$ be the breadth and the height of the beam section respectively. Let $f_c', f_s, f_y > 0$ be the magnitude of the concrete ultimate compressive stress, the steel compressive stress and the magnitude of the steel yield stress respectively. Let $A_s, A_s' > 0$ be the magnitude of the area of the compressive and tensile regions of the rebars respectively. Let $d > 0$ be the distance between the extreme compressive fibre and the central of gravity of the tensile region of the rebars, and let $d' > 0$ be the distance between the extreme compressive fibre and the central of gravity of the compressive region of the rebars. Suppose $d' < d$.

Now suppose an **RCS**-space $(S,\, \mathcal{S},\, \rho)$ (definition 3.1) representing the RC beam, where S is the set of all points on the RC beam cross section. Then we have

$$\rho(S) = bh \, .$$

Let $g : S \to \mathbb{R}$ be the GSF (definition 3.4). Based on the assumptions given above as well as the equivalent rectangular stress distribution principle of the concrete [3], we designate

$$\mathrm{image}(g) = \{-0.85 f_c', -f_s, 0, f_y\} \, .$$

Consequently, g is a step function [4]. For convenience, we define

$$\begin{aligned}
C^- &:= \{x \in S \mid g(x) = -0.85 f_c'\} \, , \\
R^- &:= \{x \in S \mid g(x) = -f_s\} \, , \text{ and} \\
R^+ &:= \{x \in S \mid g(x) = f_y\} \, .
\end{aligned}$$

The compressive and tensile regions of the **RCS**-space (definition 3.5) are given by

$$\begin{aligned} S^- &= \{x \in S \mid g(x) < 0\} \\ &= \{x \in S \mid g(x) = -0.85 f_c' \vee g(x) = -f_s\} \\ &= \{x \in S \mid g(x) = -0.85 f_c'\} \cup \{x \in S \mid g(x) = -f_s\} \\ &= C^- \cup R^- \end{aligned} \tag{3}$$

and

$$S^+ = \{x \in S \mid g(x) > 0\} = \{x \in S \mid g(x) = f_y\} = R^+ \tag{4}$$

respectively. Then by Whitney stress block principle [20], there exists some $a > 0$ such that $a < d$ and

$$\rho(C^-) = ab\,.$$

And we also have

$$\rho(R^-) = A_s'$$

and

$$\rho(R^+) = A_s\,.$$

Follows from equation (3) and by axiom (R2), we have

$$\rho(S^-) = \rho(C^- \cup R^-) = \rho(C^-) + \rho(R^-) = ab + A_s'\,.$$

And follows from equation (4), we have

$$\rho(S^+) = \rho(R^+) = A_s\,.$$

Let $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ be the force map (definition 3.7). Then the force on S^- is given by

$$\begin{aligned} \mathrm{F}(S^-) &= \int_{S^-} g\,d\rho = \int_{S^-} g^+\,d\rho - \int_{S^-} g^-\,d\rho \\ &= \big[f_y \cdot \rho(R^+ \cap S^-)\big] - \big[0.85 f_c' \cdot \rho(C^- \cap S^-) + f_s \cdot \rho(R^- \cap S^-)\big] \\ &= \big[f_y \cdot \rho(\varnothing)\big] - \big[0.85 f_c' \cdot \rho(C^-) + f_s \cdot \rho(R^-)\big] \\ &= 0 - \big[0.85 f_c' \cdot ab + f_s \cdot A_s'\big] \\ &= -0.85 f_c' \cdot ab - f_s \cdot A_s'\,. \end{aligned}$$

And the force on S^+ is given by

$$\begin{aligned} \mathrm{F}(S^+) &= \int_{S^+} g\,d\rho = \int_{S^+} g^+\,d\rho - \int_{S^+} g^-\,d\rho \\ &= \big[f_y \cdot \rho(R^+ \cap S^+)\big] - \big[0.85 f_c' \cdot \rho(C^- \cap S^+) + f_s \cdot \rho(R^- \cap S^+)\big] \\ &= \big[f_y \cdot \rho(R^+)\big] - \big[0.85 f_c' \cdot \rho(\varnothing) + f_s \cdot \rho(\varnothing)\big] \\ &= \big[f_y \cdot A_s\big] - \big[0.85 f_c' \cdot 0 + f_s \cdot 0\big] \end{aligned}$$

$$= f_y \cdot A_s\,.$$

And by the force equilibrium (postulate 3.1), we have

$$f_y \cdot A_s = \mathrm{F}(S^+) = -\mathrm{F}(S^-) = 0.85 f'_c \cdot ab + f_s \cdot A'_s\,. \tag{5}$$

Now let $r \in S$ be a reference point. As a designation, suppose r is located at the extreme compression fibre. And let $Y_r : S \to \mathbb{R}$ be the UF (definition 3.9). Let $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{um})$ be the **UM**-space (definition 3.10). Then we have

$$\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-} = \frac{a}{2}\,,\ \int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-} = d'\,,\ \int_{R^+} Y_r\big|_{R^+}\, d\tau_{R^+} = d\,. \tag{6}$$

Follows from theorem 3.6, we obtain

$$\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-} = \frac{1}{\int_{C^-} |g|\, d\rho} \int_{C^-} Y_r |g|\, d\rho$$

which implies

$$\int_{C^-} Y_r |g|\, d\rho = \left(\int_{C^-} |g|\, d\rho\right) \left(\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-}\right)\,. \tag{7}$$

Likewise, we obtain

$$\int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-} = \frac{1}{\int_{R^-} |g|\, d\rho} \int_{R^-} Y_r |g|\, d\rho$$

which implies

$$\int_{R^-} Y_r |g|\, d\rho = \left(\int_{R^-} |g|\, d\rho\right) \left(\int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-}\right)\,. \tag{8}$$

And by σ-additive property of Lebesgue integral [4], we obtain

$$
\begin{aligned}
\int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-} &= \frac{1}{\int_{S^-}|g|\,d\rho}\int_{S^-} Y_r|g|\,d\rho \\
&= \frac{1}{\int_{C^-\cup R^-}|g|\,d\rho}\int_{C^-\cup R^-} Y_r|g|\,d\rho \\
&= \frac{1}{\int_{C^-}|g|\,d\rho + \int_{R^-}|g|\,d\rho}\left(\int_{C^-} Y_r|g|\,d\rho + \int_{R^-} Y_r|g|\,d\rho\right) \\
&= \frac{1}{\int_{C^-}|g|\,d\rho + \int_{R^-}|g|\,d\rho}\Bigg[\left(\int_{C^-}|g|\,d\rho\right)\left(\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-}\right) \\
&\qquad + \left(\int_{R^-}|g|\,d\rho\right)\left(\int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-}\right)\Bigg] \quad \text{(from equations (7) and (8))} \\
&= \frac{0.85f_c'\cdot ab\cdot\frac{a}{2} + f_s\cdot A_s'\cdot d'}{0.85f_c'\cdot ab + f_s\cdot A_s'} \quad \text{(from equation (6))}
\end{aligned}
\tag{9}
$$

By postulate 3.2, the nominal moment is given by

$$
\begin{aligned}
M_n &:= \mathcal{M}(S,\,\mathcal{S},\,\rho) \\
&= \left|\frac{\int_{S^+} g\,d\rho}{\int_{S^-}|g|\,d\rho}\int_{S^-} Y_r|g|\,d\rho - \frac{\int_{S^+} g\,d\rho}{\int_{S^+}|g|\,d\rho}\int_{S^+} Y_r|g|\,d\rho\right| \\
&= |\mathrm{F}(S^+)|\left|\int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-} - \int_{S^+} Y_r\big|_{S^+}\, d\tau_{S^+}\right| \quad \text{(from theorem 3.6)} \\
&= |\mathrm{F}(S^+)|\left|\int_{S^+} Y_r\big|_{S^+}\, d\tau_{S^+} - \int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-}\right| \\
&= |\mathrm{F}(S^+)|\left|\int_{R^+} Y_r\big|_{R^+}\, d\tau_{R^+} - \int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-}\right| \quad \text{(from equation (4))} \\
&= f_y\cdot A_s\left|d - \frac{0.85f_c'\cdot ab\cdot\frac{a}{2} + f_s\cdot A_s'\cdot d'}{0.85f_c'\cdot ab + f_s\cdot A_s'}\right| \quad \text{(from equation (9))} \\
&= f_y\cdot A_s\left(d - \frac{0.85f_c'\cdot ab\cdot\frac{a}{2} + f_s\cdot A_s'\cdot d'}{f_y\cdot A_s}\right) \quad \text{(from theorem 3.4)} \\
&= f_y\cdot A_s\cdot d - \left(0.85f_c'\cdot ab\cdot\frac{a}{2} + f_s\cdot A_s'\cdot d'\right) \\
&= f_y\cdot A_s\cdot d - \left[(0.85f_c'\cdot ab + f_s\cdot A_s')\frac{a}{2} + f_s\cdot A_s'\left(d' - \frac{a}{2}\right)\right] \\
&= f_y\cdot A_s\cdot d - \left[(f_y\cdot A_s)\frac{a}{2} - f_s\cdot A_s'\left(\frac{a}{2} - d'\right)\right] \quad \text{(from theorem 3.4)} \\
&= f_y\cdot A_s\left(d - \frac{a}{2}\right) + f_s\cdot A_s'\left(\frac{a}{2} - d'\right)
\end{aligned}
$$

which is the classical equation of the conventional RC beam nominal moment [3].

This result shows that one can derive the equation of the nominal moment of the conventional RC beam from our axiomatic theory. It is an evidence that the axiomatic theory serves as the general theory of RC beam in terms of bending moment.

5 Conclusions and Future Work

This paper has presented the axiomatic theory of the RC beam undergoing a pure bending moment by making use of measure theory in mathematics. We have formally defined the **RCS**-space, the measure space representing the cross section of a RC beam on definition 3.1. The stress functions representing the stresses of the concrete and reinforcements are unified into an L^2 function called the general stress function (GSF). The properties of **RCS**-space are derived as theorems on section 3.1. The forces on the **RCS**-space are formally defined as the Lebesgue integral of the GSF on section 3.2. We have also introduced a new concept called the **UM**-space to determine the central of gravity of sets within the **RCS**-space as well as to determine the distances between the sets on section 3.3. Then the moment of resistance (MoR) of a RC beam has been presented as a Lebesgue integral expression on postulate 3.2. It has also been shown that one is able to derive the classical equation for the conventional RC beam nominal moment [3] via this axiomatic theory on section 4. It gives a clear evidence that this axiomatic theory serves as the general theory for RC beams which is believed to be compatible for any RC beams with any kind of reinforcements as well as geometric shapes.

We are interested in deriving more equations for RC beams (not necessarily conventional) via this axiomatic theory as well as demonstrating a novel approach in the computation of RC beams' MoR by treating this axiomatic theory as the foundation of the beam computation's algorithm. Another possible area of research is the study of the GSF either analytically or experimentally. It is expected that the GSF could deliver a more comprehensive analysis of RC beams compared to the classical concept in material stresses.

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