Axiomatic Theory of the Reinforced Concrete Beam Undergoing A Pure Bending Moment

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Abstract

This work provides an axiomatic foundational theory of the reinforced concrete (RC) beam undergoing a pure bending moment. The concept of 'measure space' in measure theory is used to model the whole concrete cross section along with the area measure on the cross section. We call the measure space representing the environment an $\mathbf{RCS}$-space. Then the concrete and the reinforcements stresses are modelled as $\mathcal{L}^1$- functions. And by the additive property of $\mathcal{L}^1$-functions, these functions are generalised into a single $\mathcal{L}^1$-function coined as the general stress function. From this model, the properties of the $\mathbf{RCS}$-space can be derived as theorems. Consequently, the structural moment of resistance can be formulated following from the force couple theory in mechanics as a Lebesgue integral expression of the general stress function. This measure theoretic system allows us to formulate the moment of resistance of a concrete beam with any kind of reinforcement and configuration in a unified formal mathematical theory.

Keywords: measure space, Lebesgue space, reinforced concrete, flexural strength, structural mechanics

1 Introduction

A reinforced concrete (RC) structure undergoing a pure bending can be considered as a reinforced concrete (RC) beam. The moment of resistance of a RC beam can be computed from the force couple principle (DuBois, 1902). As some bending moment M occurs on a RC beam, by the force couple principle, a pair of forces ϕ and ψ (the couple) with the same scalar quantity but of opposite direction is assumed to occur on the cross section of the RC beam separated by some distance $d > 0$ such that

$$M_r = |\phi|\, d = |\psi|\, d\,, \tag{1}$$

where M_r is the moment of resistance of the RC beam in classical sense. As a design consideration, structural engineers suggest a factor $c \in \mathbb{R}$ satisfying $0 < c \leq 1$ to be added for a safety reason (Darwin et al., 2016). And hence the inequality

$$M < c\, M_r$$

must be satisfied.

However, this existing theory can be viewed informal. This paper will provide an axiomatic formulation of the concrete structure undergoing a pure bending moment by making use of measure theory in mathematics. Equation (1) will be the underlying principle of the axiomatic formulation of the moment of resistance of RC beams. The concept of moment of resistance generalises the nominal moment and other moments at all mechanical evolutionary stages of the RC beam such as the elastic moment, the cracking moment, etc.

The measure theoretic system allows us to formulate the moment of resistance of a concrete beam with any kind of reinforcement and configuration in a unified formal theory. This is due to the generality nature of theories in pure mathematics, with no exceptions for measure theory. Whilst it is not the case in the classical theory of concrete beams.

2 Measure Theory

2.1 Measure Spaces

Measure theory is a branch of mathematics concerning the concept of measure, the generalisation of real world concepts such as the length of a one dimensional object, the area of a two dimensional object and the volume of a three dimensional object. Another notions falling into the realm of “measure" are the mass of an object and the probability of an event. The readers are referred to (Salamon, 2016) for detailed descriptions of measure theory. In addition, the mathematical writing in this paper incorporates a huge amount of the language of first-order logic. Therefore, the readers are also referred to (Bergmann et al., 2014). A few concepts in measure theory will be covered as follows.

Definition 2.1.1 (Measurable Space). (Salamon, 2016). Let X be a set. A σ-algebra on X is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ satisfying the following axioms:

A1 $X \in \mathcal{A}$

A2 $\forall A \in \mathcal{A}\,,\ A^C \in \mathcal{A}$

A3 Given a collection $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$, then

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$$

holds.

We call every $A \in \mathcal{A}$ a measurable set. And we call the pair $(X, \mathcal{A})$ a measurable space.

Given a measurable space $(A, \mathcal{A})$, by axiom A1 and A2 we can easily deduce that $\varnothing \in \mathcal{A}$. On the other hand, by axioms A2 and A3, we can deduce that

$$\bigcap_{k=1}^{\infty} B_k \in \mathcal{A}\,,$$

with $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$. The readers are now referred to (Salamon, 2016) for the remaining basic properties of measurable spaces.

Definition 2.1.2 (Measurable Function). (Salamon, 2016). Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be measurable spaces. A function $f : X \to Y$ is called a measurable function if and only if

$$\forall A \in \mathcal{A}_Y\,,\ f^{-1}(A) \in \mathcal{A}_X\,.$$

The readers are now referred to (Salamon, 2016) for the basic properties of measurable functions. We need also to observe the notion of Borel measurability (Salamon, 2016). In this case, we need to understand the basic concepts in point-set topology (André, 2020). We will not cover these topics in this paper for simplicity, therefore the readers are referred to the respective references.

Now we move to the definition of step function, which is fundamental to the formulation of Lebesgue integral. The concept is presented on the following definition.

Definition 2.1.3 (Step Function). (Salamon, 2016). Let X be a set. A function $s : X \to \mathbb{R}$ is called a step function if and only if

$$\exists n \in \mathbb{N}\,,\ \#\mathrm{image}(s) = n\,.$$

The symbol # on definition 2.1.3 above expresses the cardinality of a countable set (Stoll, 1979). From the definition of step function as given on definition 2.1.3 we present the following proposition.

Proposition 2.1.1. Let X be a set. Let $s : X \to \mathbb{R}$ be a step function. Suppose $n \in \mathbb{N}$ such that

$$\mathrm{image}(s) = \{\alpha_1, \ldots, \alpha_n\}\,.$$

Now we define

$$\forall k \in \{1, \ldots, n\}\,,\ A_k := s^{-1}(\{\alpha_k\})\,.$$

Now let $\chi_{A_k} : X \to \{0, 1\}$ be an indicator function, that is,

$$\forall x \in X\,,\ \chi_{A_k} := \begin{cases} 1 & : x \in A_k \\ 0 & : x \notin A_k \end{cases}\,,$$

for every $k \in \{1, \dots, n\}$. Then we can express s by

$$\forall x \in X\,,\ s(x) = \sum_{k=1}^{n} \alpha_k \cdot \chi_{A_k}(x)\,.$$

The readers are now referred to (Salamon, 2016) for step function approximation theorem. Now we will oberserve the definition of measure space, which is given as follows.

Definition 2.1.4 (Measure Space)**.** (Salamon, 2016). Let $(X,\, \mathcal{A})$ be a measurable space. A map $\mu : \mathcal{A} \to [0,\, \infty]$ is called a measure on $(X,\, \mathcal{A})$ if and only if the following axioms are satisfied:

M1 $\exists A \in \mathcal{A}\,,\ \mu(A) < \infty$

M2 Given a family $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ such that

$$\forall i, j \in \mathbb{N}\big[i \neq j \implies A_i \cap A_j = \varnothing\big]\,,$$

then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

holds.

Then we call the triplet $(X,\, \mathcal{A},\, \mu)$ a measure space.

Corollary 2.1.1. *Let $(X,\, \mathcal{A},\, \mu)$ be a measure space. Then $\mu(\varnothing) = 0$.*

Proof. Let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}$ such that $\mu(A_1) < \infty$ and

$$\forall k \in \{2, \dots\}\,,\ A_k := \varnothing\,.$$

Then

$$\forall i, j \in \mathbb{N}\big[i \neq j \implies A_i \cap A_j = \varnothing\big]$$

holds. By axiom M2 of definition 2.1.4 we obtain

$$\mu(A_1) = \mu\left(A_1 \cup \bigcup_{k=2}^{\infty} \varnothing\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) = \mu(A_1) + \sum_{k=2}^{\infty} \mu(A_k) = \mu(A_1) + \infty \cdot \mu(\varnothing)\,,$$

which implies

$$\infty \cdot \mu(\varnothing) = 0\,.$$

The only solution to the equation above is $\mu(\varnothing) = 0$ in the sense of $\overline{\mathbb{R}}$ (Salamon, 2016). Hence we conclude that $\mu(\varnothing) = 0$. □

Theorem 2.1.1 (Measure Subspace)**.** *Let $(X,\, \mathcal{A},\, \mu)$ be a measure space. Let $E \in \mathcal{A}$ such that $E \neq \varnothing$. Then the following arguments hold:*

i. The family

$$\mathcal{A}_E := \{A \in \mathcal{A} \mid A \subseteq E\}$$

is a σ-algebra on E. Hence $(E,\, \mathcal{A}_E)$ is a measurable space on its own.

ii. The restriction map $\mu_E := \mu\big|_{\mathcal{A}_E} : \mathcal{A} \to [0,\, \infty]$ is a measure on $(E,\, \mathcal{A}_E)$. Hence $(E,\, \mathcal{A}_E,\, \mu_E)$ is a measure space called a measure subspace of $(X,\, \mathcal{A},\, \mu)$.

Proof. (*i*) Since $E \subseteq E$, then $E \in \mathcal{A}_E$, which means that axiom A1 is satisfied. From axiom A2 and A3 we can infer that a σ-algebra is closed under countable intersection (Salamon, 2016). Let $B \in \mathcal{A}_E$. Then $B \subseteq E$ and $B \in \mathcal{A}$. Clearly $E \in \mathcal{A}$ by definition. Then we obtain

$$E \setminus B = E \cap (X \setminus B) = E \cap B^C$$

By axiom A2, $B^C \in \mathcal{A}$. Then by the expression above, we have $E \setminus B \in \mathcal{A}_E$, which shows that axiom A2 is satisfied. Now let $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{A}_E$. Then we have $B_k \subseteq E$ and $B_k \in \mathcal{A}$, for every $k \in \mathbb{N}$. Then we have

$$\bigcup_{k=1}^{\infty} B_k \subseteq E\,.$$

And by axiom A3, we have

$$\bigcup_{k=1}^{\infty} B_k \in \mathcal{A}\,.$$

The two arguments above imply

$$\bigcup_{k=1}^{\infty} B_k \in \mathcal{A}_E\,,$$

which shows that axiom A3 is satisfied. We conclude that $\mathcal{A}_E$ is a σ-algebra on E, and $(E,\, \mathcal{A}_E)$ is a measurable space on its own.

(*ii*) Note that by definition

$$\forall A \in \mathcal{A}_E\,,\ \mu_E(A) = \mu(A)\,.$$

Since $\varnothing \subset E$, then $\varnothing \in \mathcal{A}_E$. And by corollary 2.1.1, we have

$$\mu_E(\varnothing) = \mu(\varnothing) = 0 < \infty\,,$$

which shows that axiom M1 is satisfied. Now let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{A}_E$. such that $A_i \cap A_j = \varnothing$ whenever $i \neq j$, for any $i, j \in \mathbb{N}$. Then by definition $A_k \in \mathcal{A}$, for every $k \in \mathbb{N}$. Then by axiom M2 we obtain

$$\mu_E\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) = \sum_{k=1}^{\infty} \mu_E(A_k)\,,$$

which shows that axiom M2 is satisfied. Hence μ_E is a measure on $(E,\, \mathcal{A}_E)$, and $(E,\, \mathcal{A}_E, \mu_E)$ is a measure space. $\square$

The readers are now referred to (Salamon, 2016) for the basic properties of measure spaces.

2.2 Lebesgue Integral and Lebesgue Spaces

Lebesgue integration is an integration technique based on measure theory, introduced by Henry Lebesgue (Lebesgue, 1904). This integration technique extends the integral to a larger class of functions. It also extends the domain of integration. The formal definition of Lebesgue integral is given on the following definition.

Definition 2.2.1 (Lebesgue Integral)**.** (Salamon, 2016). Let $(X,\, \mathcal{A},\, \mu)$ be a measure space. Let $f : X \to [0,\, \infty]$ be a measurable function. By step function approximation theorem (Salamon, 2016), for every $n \in \mathbb{N}$ there exists a step function $s_n : X \to [0,\, \infty)$ such that

$$\forall x \in X\,,\ 0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq f(x)$$

and

$$\forall x \in X\,,\ \lim_{n\to\infty} s_n(x) = f(x)\,.$$

Then we define as follows:

i. The Lebesgue integral of s_n is defined by

$$\int_X s_n \, d\mu := \sum_{k=1}^{m_n} \alpha_{n,k} \mu(A_{n,k}) \, ,$$

for every $n \in \mathbb{N}$.

ii. The Lebesgue integral of f is defined by

$$\int_X f \, d\mu := \sup_{n \in \mathbb{N}} \int_X s_n \, d\mu \, .$$

iii. Let $E \in \mathcal{A}$. The Lebesgue integral of f over E is defined by

$$\int_E f \, d\mu := \sup_{n \in \mathbb{N}} \int_E s_n \, d\mu \, .$$

The readers are now referred to (Salamon, 2016) for the basic properties of Lebesgue integral, Lebesgue monotone convergence theorem, σ-additive property of Lebesgue integral, and Fatou's lemma. We now moved to the integration of real-valued functions.

Definition 2.2.2 (Lebesgue Integrable Function)**.** (Salamon, 2016). Let $(X, \, \mathcal{A}, \, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a measurable function. The function f is called Lebesgue integrable if and only if

$$\int_X |f| \, d\mu < \infty \, .$$

We have seen that Lebesgue integral applies to nonnegative measurable extended real-valued functions (definition 2.2.1). We will see the essence of definition 2.2.2 on the following two definitions.

Definition 2.2.3 (Positive and Negative Parts of Functions)**.** Let X be a set. Now suppose a function $f : X \to \mathbb{R}$.

i. The positive part of f is a map $f^+ : X \to [0, \, \infty)$ defined by

$$\forall x \in X \, , \; f^+(x) := \max \{f(x), 0\} \, .$$

ii. The negative part of f is a map $f^- : X \to [0, \, \infty)$ defined by

$$\forall x \in X \, , \; f^-(x) := \max \{-f(x), 0\} \, .$$

Definition 2.2.4 (Lebesgue Integral of Real-Valued Function)**.** (Salamon, 2016). Let $(X, \, \mathcal{A}, \, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a Lebesgue integrable function. Let $E \in \mathcal{A}$. The Lebesgue integral of f over E is defined by

$$\int_E f \, d\mu := \int_E f^+ \, d\mu - \int_E f^- \, d\mu \, .$$

The readers are now referred to (Salamon, 2016) for the basic properties of Lebesgue integral of real-valued functions including σ-additivity of the integral, scalar multiplicity of the integral and Lebsgue dominated convergence theorem.

The family of Lebesgue integrable functions is a special class of functions in mathematical analysis. This family forms a vector space over $\mathbb{R}$ (Roman, 2005). We will present this notion on the following theorem. However, we omit the proof for simplicity, since the proof is covered comprehensively on (Salamon, 2016).

Theorem 2.2.1 (Lebesgue Space $\mathcal{L}^1$)**.** *(Salamon, 2016). Let $(X, \, \mathcal{A}, \, \mu)$ be a measure space. We define a family*

$$\mathcal{L}^1(X, \, \mathcal{A}, \, \mu) := \left\{ f : X \to \mathbb{R} \, \middle| \, \int_X |f| \, d\mu < \infty \right\} \, .$$

The family $\mathcal{L}^1(X, \, \mathcal{A}, \, \mu)$ forms a vector space over $\mathbb{R}$, and is called a Lebesgue Space.

In fact, the more general Lebesgue space is a space $\mathcal{L}^p(X,\,\mathcal{A},\,\mu)$ with $p \in [1,\,\infty]$ (Salamon, 2016) (Bühler and Salamon, 2018). However we will only use $\mathcal{L}^1(X,\,\mathcal{A},\,\mu)$ for the axiomatisation of the concrete structure. Therefore, we do not present $\mathcal{L}^p(X,\,\mathcal{A},\,\mu)$ in this paper.

3 Axiomatisation of RC Beams

3.1 The RCS-Space

Now we build the axiomatic definition of the cross section of a RC beam by making use of the concept of measure space, as presented on the following definition.

Definition 3.1.1 ($\mathbf{RCS}$-Space)**.** Given a RC beam. Let $(S,\,\mathcal{S})$ be a measurable space where S is the set of all the points in the whole cross section of the RC beam. Suppose a map $\rho : \mathcal{S} \to [0,\,\infty)$ such that $\rho(A)$ is the magnitude of the area (measure) of a region $A \in \mathcal{S}$. We call the triplet $(S,\,\mathcal{S},\,\rho)$ the $\mathbf{RCS}$-space if and only if the following axioms are satisfied:

R1 $\rho(S) < \infty$

R2 Given a family $\{U_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ such that

$$\forall i, j \in \mathbb{N}\big[i \neq j \implies U_i \cap U_j = \varnothing\big]\,,$$

then

$$\rho\left(\bigcup_{k=1}^{\infty} U_k\right) = \sum_{k=1}^{\infty} \rho(U_k)$$

holds.

If we compare definition 3.1.1 with definition 2.1.4, we will notice that an $\mathbf{RCS}$-space is just a measure space. The only difference is that, the area measure of $\mathbf{RCS}$-space is a finite measure, while the measure given on definition 2.1.4 is not necessarily finite.

Now we need to be aware that there are reinforcements installed on the cross section of a RC beam. And the cross section of these reinforcements are within the set of all points of the whole cross section. Let us now present the following definition.

Definition 3.1.2 (Bipartite)**.** Let $(S,\,\mathcal{S},\,\rho)$ be an $\mathbf{RCS}$-space. Let $C, R \in \mathcal{S}$ such that $C \cap R = \varnothing$. And we designate C to represent the set of all points in the concrete section, and R to represent the set of all points in the reinforcements sections. And we may call the collection $\{C, R\}$ the bipartite of S.

Proposition 3.1.1. Given an $\mathbf{RCS}$-space $(S,\,\mathcal{S},\,\rho)$, and the bipartite $\{C, R\}$ of S (definition 3.1.2). We can consider that in a RC beam cross section, there are only concrete material and reinforcements. Hence $C \cup R = S$. Consequently, $\{C, R\}$ is a partition of S.

Follows from equation (1), then there must be some region on the cross section of an RC beam undergoing compression and some other region undergoing tension. We would like to formally define these regions. But first, we need to define the stress functions of the materials. The stress functions can be of either positive, negative or both values. By convention, negative value of a stress function indicates that the material undergoing some compression. While the positive value of a stress function indicates that the material undergoing some tension.

Definition 3.1.3 (Concrete and Reinforcements Stress Functions)**.** Let $(S,\,\mathcal{S},\,\rho)$ be an $\mathbf{RCS}$-space. Let $\{C, R\}$ be the bipartite of S (definition 3.1.2). Suppose some functions $b, t : S \to \mathbb{R}$. The functions b and t are the concrete and the reinforcements stress functions respectively if the following axioms are satisfied:

S1 $b, t \in \mathcal{L}^1(S,\,\mathcal{S},\,\rho)$

S2 $\forall x \in S\,,\; b(x) \leq 0$

S3 $\forall x \in R\,,\; b(x) = 0$

S4 $\forall x \in C\,,\; t(x) = 0$

The designation of axiom S2 is based on the empirical fact that the concrete has a good compressive resistance but has a very poor tensile resistance (Weidmann et al., 1990).

Now we may generalise the stress functions into a single stress function coined as the general stress function. It is conducted by making use of the property of $\mathcal{L}^1$-functions (Salamon, 2016).

Definition 3.1.4 (General Stress Function). Let $(S,\,\mathcal{S},\,\rho)$ be an $\mathbf{RCS}$-space. Let $b, t \in \mathcal{L}^1(S,\,\mathcal{S},\,\rho)$ be the concrete and the reinforcements stress functions (definition 3.1.3). The general stress function is a function $g : S \to \mathbb{R}$ defined by

$$\forall x \in S\,,\;\; g(x) := b(x) + t(x)\,.$$

Proposition 3.1.2. Let $(S,\,\mathcal{S},\,\rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Then there exist functions $b, t \in \mathcal{L}^1(S,\,\mathcal{S},\,\rho)$, which are the concrete and the reinforcement stress functions, the such that $g = b + t$. By theorem 2.2.1, $\mathcal{L}^1(S,\,\mathcal{S},\,\rho)$ is a vector space over $\mathbb{R}$. Hence the addition of a pair of elements of $\mathcal{L}^1(S,\,\mathcal{S},\,\rho)$ is another element of $\mathcal{L}^1(S,\,\mathcal{S},\,\rho)$ (Roman, 2005). Consequently $g \in \mathcal{L}^1(S,\,\mathcal{S},\,\rho)$.

We now present the compressive and the tensile regions of S as follows.

Definition 3.1.5 (Compressive and Tensile Regions). Let $(S,\,\mathcal{S},\,\rho)$ be an $\mathbf{RCS}$-space. Let $t, g \in \mathcal{L}^1(S,\,\mathcal{S},\,\rho)$ be the reinforcements and the general stress functions respectively.

i. The compressive region of S is defined by

$$S^- := \{x \in S \mid g(x) < 0\}\,.$$

ii. The tensile region of S is defined by

$$S^+ := \{x \in S \mid g(x) > 0\}\,.$$

The first property of compressive and tensile regions is given on the following theorem.

Theorem 3.1.1 (The Compressive and Tensile Regions as Nonpartition). *Let $(S,\,\mathcal{S},\,\rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let S^- and S^+ be the compressive and tensile regions of S (definition 3.1.5). Suppose*

$$\exists x \in S\,,\;\; g(x) = 0\,.$$

Then (i) $S^- \cap S^+ = \varnothing$, but (ii) $\{S^-, S^+\}$ is not a partition of S.

Proof. (*i*) Let us recall that

$$\forall x \in S\,,\;\; g(x) = b(x) + t(x)$$

as described by definition 3.1.4. And by axiom S2, note that

$$\forall x \in S^+\,,\;\; g(x) = t(x)\,.$$

Then we obtain

$$\begin{aligned} S^- \cap S^+ &= \{x \in S \mid g(x) < 0\} \cap \{x \in S \mid t(x) > 0\} \\ &= \{x \in S \mid b(x) + t(x) < 0\} \cap \{x \in S \mid t(x) > 0\} \\ &= \{x \in S \mid b(x) + t(x) < 0 \wedge t(x) > 0\}\,. \end{aligned}$$

Now let us observe the formula within the set builder notation on the latest expression above, that is

$$b(x) + t(x) < 0 \wedge t(x) > 0\,. \tag{2}$$

Supposing $b(x) = 0$, then we will have

$$b(x) + t(x) = 0 + t(x) = t(x) < 0 \;\wedge\; t(x) > 0\,,$$

which cannot exist. Then the solution to the expression (2) must be the empty set. Hence we obtain

$$S^- \cap S^+ = \{x \in S \mid b(x) + t(x) < 0 \wedge t(x) > 0\} = \varnothing\,,$$

which proves statement *i* of the theorem.

(*ii*) Since there are some $x \in X$ such that $g(x) = 0$, then

$$g^{-1}(\{0\}) \neq \varnothing\,.$$

From definition 3.1.5, we can also express

$$S^- = g^{-1}((-\infty,\, 0))\,, \quad S^+ = g^{-1}((0,\, \infty))\,.$$

Then we obtain

$$\begin{aligned} S^- \cup S^+ &= g^{-1}((-\infty,\, 0)) \cup g^{-1}((0,\, \infty)) \\ &= g^{-1}((-\infty,\, 0)) \cup g^{-1}((0,\, \infty)) \cup \varnothing \\ &\neq g^{-1}((-\infty,\, 0)) \cup g^{-1}((0,\, \infty)) \cup g^{-1}(\{0\}) \\ &= g^{-1}((-\infty,\, 0) \cup \{0\} \cup (0,\, \infty)) \\ &= g^{-1}(\mathbb{R}) \\ &= S\,, \end{aligned}$$

which proves statement *ii* of the theorem. □

Now we may derive another property of the compressive and the tensile regions, as given on the following theorem.

Theorem 3.1.2 (Measurability of Compressive and Tensile Regions)**.** *Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let S^- and S^+ be the compressive and tensile regions of S (definition 3.1.5). Then we have $S^-, S^+ \in \mathcal{S}$.*

Proof. Follows from definition 3.1.5, we can express

$$S^- = g^{-1}((-\infty,\, 0))\,, \quad S^+ = g^{-1}((0,\, \infty))\,.$$

Note that both $(-\infty,\, 0)$ and $(0,\, \infty)$ are Borel sets on $\mathbb{R}$ (Salamon, 2016) (André, 2020), which means that they are measurable sets. Proposition 3.1.2 asserts that $g \in \mathcal{L}^1(S,\, \mathcal{S},\, \rho)$, then g is measurable. Consequently, by definition 2.1.2 we have

$$g^{-1}((-\infty,\, 0)),\; g^{-1}((0,\, \infty)) \in \mathcal{S}\,,$$

which means $S^-, S^+ \in \mathcal{S}$. □

Follows from theorem 3.1.2 above, we may present a concept as given on the following definition.

Definition 3.1.6 ($\mathbf{RCS}^-$ and $\mathbf{RCS}^+$ Measurable Spaces)**.** Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress functions (definition 3.1.4). Let S^- and S^+ be the compressive and tensile regions of S (definition 3.1.5). Theorem 3.1.2 asserts that $S^-, S^+ \in \mathcal{S}$.

i. We define

$$\mathcal{S}^- := \{A \in \mathcal{S} \mid A \subseteq S^-\}\,.$$

By part i theorem of 2.1.1, $(S^-,\, \mathcal{S}^-)$ is a measurable space which we call the $\mathbf{RCS}^-$ measurable space.

ii. We define

$$\mathcal{S}^+ := \{A \in \mathcal{S} \mid A \subseteq S^+\}\,.$$

By part i of theorem 2.1.1, $(S^+,\, \mathcal{S}^+)$ is a measurable subspace which we call the $\mathbf{RCS}^+$ measurable space.

3.2 Forces on the RCS-Space

Our intention in building a measure theoretic model for the RC beam is to compute the forces on the section of the RC beam, and then with the forces we can compute the moment of resistance of the RC beam. Now we will introduce a concept called the force map, as given on the following definition.

Definition 3.2.1 (Force Map)**.** Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). We call a map $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ defined by

$$\forall A \in \mathcal{S}\,,\; \mathrm{F}(A) := \int_A g\, d\rho$$

the force map on $(S,\, \mathcal{S},\, \rho)$.

Form definition 3.2.1 above, we can derive several properties of the force map as given on the following theorem.

Theorem 3.2.1 (Negative and Positive Forces)**.** *Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ be the force map (definition 3.2.1). Let $(S^-,\, \mathcal{S}^-)$ and $(S^+,\, \mathcal{S}^+)$ be the $\mathbf{RCS}^-$ and $\mathbf{RCS}^+$ measurable spaces respectively (definition 3.1.6). Then the following statements hold:*

i. $\forall A \in \mathcal{S}^-\,,\; \mathrm{F}(A) < 0$

ii. $\forall A \in \mathcal{S}^+\,,\; \mathrm{F}(A) > 0$

Proof. Recalling definition 3.1.6, then we have $\mathcal{S}^-, \mathcal{S}^+ \subset \mathcal{S}$. Then the Lebesgue integral of g over some element of either $\mathcal{S}^-$ or $\mathcal{S}^+$ with respect to ρ is well defined. Let us recall the bipartite $\{C, R\}$ (definition 3.1.2). Let $X \in \mathcal{S}^-$ and $Y \in \mathcal{S}^+$. And suppose

$$X_b := X \cap C\,, \quad X_t := X \cap R$$

and

$$Y_b := Y \cap C\,, \quad Y_t := Y \cap R\,.$$

Hence $\{X_b, X_t\}$ is a partition of X and $\{Y_b, Y_t\}$ is a partition of Y since $\{C, R\}$ is a partition of S. And let $b, t \in \mathcal{L}^1(S,\, \mathcal{S},\, \rho)$ be the concrete and the reinforcements stress functions (definition 3.1.3). Then by the properties of Lebesgue integral (Salamon, 2016), we obtain

$$\begin{aligned}
\mathrm{F}(X) &= \int_X g\, d\rho \\
&= \int_{X_b \cup X_t} g\, d\rho \\
&= \int_{X_b} b + t\, d\rho + \int_{X_t} b + t\, d\rho \\
&= \int_S (b+t)\chi_{X_b}\, d\rho + \int_S (b+t)\chi_{X_t}\, d\rho \\
&= \int_S b\chi_{X_b}\, d\rho + \int_S t\chi_{X_b}\, d\rho + \int_S b\chi_{X_t}\, d\rho + \int_S t\chi_{X_t}\, d\rho \\
&= \int_S b\chi_{X_b}\, d\rho + 0 + 0 + \int_S t\chi_{X_t}\, d\rho \\
&= \left(\int_S b^+\chi_{X_b}\, d\rho - \int_S b^-\chi_{X_b}\, d\rho\right) + \left(\int_S t^+\chi_{X_t}\, d\rho - \int_S t^-\chi_{X_t}\, d\rho\right) \\
&= \left(0 - \int_S b^-\chi_{X_b}\, d\rho\right) + \left(0 - \int_S t^-\chi_{X_t}\, d\rho\right) \\
&= -\left(\int_S b^-\chi_{X_b}\, d\rho + \int_S t^-\chi_{X_t}\, d\rho\right) \\
&< 0
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{F}(Y) &= \int_Y g\, d\rho \\
&= \int_{Y_b \cup Y_t} g\, d\rho \\
&= \int_{Y_b} b + t\, d\rho + \int_{Y_t} b + t\, d\rho \\
&= \int_S (b+t)\chi_{Y_b}\, d\rho + \int_S (b+t)\chi_{Y_t}\, d\rho \\
&= \int_S b\chi_{Y_b}\, d\rho + \int_S t\chi_{Y_b}\, d\rho + \int_S b\chi_{Y_t}\, d\rho + \int_S t\chi_{Y_t}\, d\rho \\
&= \int_S b\chi_{Y_b}\, d\rho + 0 + 0 + \int_S t\chi_{Y_t}\, d\rho \\
&= \left(\int_S b^+\chi_{Y_b}\, d\rho - \int_S b^-\chi_{Y_b}\, d\rho\right) + \left(\int_S t^+\chi_{Y_t}\, d\rho - \int_S t^-\chi_{Y_t}\, d\rho\right) \\
&= (0-0) + \left(\int_S t^+\chi_{Y_t}\, d\rho - 0\right) \\
&= \int_S t^+\chi_{Y_t}\, d\rho \\
&> 0
\end{aligned}
$$

which prove statements *i* and *ii* respectively. □

We have designated for this theory that the RC beam undergoes a pure bending moment, which consequently require the force resultant in the system to be zero. Now we will present this principle as a postulate, which is very fundamental for this theory.

Postulate 3.2.1 (Force Equilibrium)**.** *Let* $(S,\, \mathcal{S},\, \rho)$ *be an* $\mathbf{RCS}$*-space. Let* $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ *be the force map (definition 3.2.1). Then*

$$\mathrm{F}(S) = 0$$

must hold.

A consequence of postulate 3.2.1 is given on the following theorem.

Theorem 3.2.2 (Compressive and Tensile Regions Equilibrium)**.** *Let* $(S,\, \mathcal{S},\, \rho)$ *be an* $\mathbf{RCS}$*-space. Let* S^- *and* S^+ *be the compressive and tensile regions of* S *(definition 3.1.5). Let* $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ *be the force map (definition 3.2.1). Then we have*

$$\mathrm{F}(S^-) = -\mathrm{F}(S^+)\,.$$

Proof. Theorem 3.1.1 asserts that $\{S^-, S^+\}$ is not a partition of S even though the sets are disjoint. Then let us define

$$N := S \setminus (S^- \cup S^+)\,.$$

Consequently, $\{S^-, S^+, N\}$ is a partition of S. Note that by axiom A2, $N \in \mathcal{S}$. By definition 3.1.5 and theorem 3.1.1 we will obtain

$$N = \{x \in S \mid g(x) = 0\}\,.$$

Hence we have

$$\mathrm{F}(N) = \int_N g\, d\rho = \int_S g\chi_N\, d\rho = 0\,.$$

By postulate 3.2.1 and σ-additive property of Lebesgue integral (Salamon, 2016), we obtain

$$
\begin{aligned}
0 &= \mathrm{F}(S) \\
&= \mathrm{F}(S^- \cup S^+ \cup N)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{S^- \cup S^+ \cup N} g \, d\rho \\
&= \int_{S^-} g \, d\rho + \int_{S^+} g \, d\rho + \int_N g \, d\rho \\
&= \int_{S^-} g \, d\rho + \int_{S^+} g \, d\rho + 0 \\
&= \mathrm{F}(S^-) + \mathrm{F}(S^+) \, .
\end{aligned}
$$

Theorem 3.2.1 asserts that $\mathrm{F}(S^-) < 0$ and $\mathrm{F}(S^+) > 0$, which means that both forces cannot be 0. Hence the expression above implies

$$
\mathrm{F}(S^-) = -\mathrm{F}(S^+) \, ,
$$

which proves the theorem. □

3.3 Central of Gravity of Sets and UM-Distance

One of important components in the theory of flexural bending is the distance between the centroids of the compressive and tensile regions. By the force couple principle (Du-Bois, 1902), we require the distance separating the forces in the couple to compute the equivalent moment by multiplying one of the absolute value of the forces with the distance. This distance is unidirectional to the rotation of the occurring moment. We call the distance the $\mathbf{UM}$-distance. We use the concept of pseudometric space (Kurepa, 1934) to formally define the $\mathbf{UM}$-distance on an $\mathbf{RCS}$-space.

Definition 3.3.1 ($\mathbf{UM}$-Distance)**.** Let $(S, \, \mathcal{S}, \, \rho)$ be an $\mathbf{RCS}$-space. Let $\mathcal{U} \subseteq \mathcal{S}$. A map

$$
d_{um} : \mathcal{U} \to [0, \, \infty)
$$

is called a $\mathbf{UM}$-distance on $\mathcal{U}$ if and only if the following axioms are satisfied:

U1 $\forall U \in \mathcal{U} \, , \; d_{um}(U, U) = 0$

U2 $\forall U, V \in \mathcal{U} \, , \; d_{um}(U, V) = d_{um}(U, V)$

U3 $\forall U, V, W \in \mathcal{U} \, , \; d_{um}(U, V) \leq d_{um}(U, W) + d_{um}(W, V)$

Now our task is to construct a $\mathbf{UM}$-distance on an $\mathbf{RCS}$-space. However, we will first present a new measure space concept on the $\mathbf{RCS}$-space as given by the following theorem.

Theorem 3.3.1 (τ-Measure)**.** *Let $(S, \, \mathcal{S}, \, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let $A \in \mathcal{S}$. By theorem 2.1.1, a family*

$$
\mathcal{S}_A := \{ B \in \mathcal{S} \mid B \subseteq A \}
$$

is a σ-algebra on A. Hence $(A, \, \mathcal{S}_A)$ is a measurable space. A map $\tau_A : \mathcal{S}_A \to [0, \, 1]$ defined by

$$
\forall B \in \mathcal{S}_A \, , \; \tau_A(B) := \frac{1}{\int_A |g| \, d\rho} \int_B |g| \, d\rho
$$

is a measure on $(A, \, \mathcal{S}_A)$. Consequently $(A, \, \mathcal{S}_A, \, \tau_A)$ is a measure space.

Proof. Obviously, $\varnothing \in \mathcal{S}_A$. Then we obtain

$$
\tau_A(\varnothing) = \frac{1}{\int_A |g| \, d\rho} \int_\varnothing |g| \, d\rho = \frac{1}{\int_A |g| \, d\rho} \cdot 0 = 0 < \infty \, ,
$$

which shows that axiom M1 is satisfied. Now let $\{B_k\}_{k=1}^{\infty} \subseteq \mathcal{S}_A$ such that $B_i \cap B_j = \varnothing$ whenever $i \neq j$. Then by σ-additive property of Lebesgue integral (Salamon, 2016), we obtain

$$
\tau_A \left(\bigcup_{k=1}^{\infty} B_k \right) = \frac{1}{\int_A |g| \, d\rho} \int_{\bigcup_{k=1}^{\infty} B_k} |g| \, d\rho
$$

$$= \frac{1}{\int_A |g|\, d\rho} \sum_{k=1}^{\infty} \int_{B_k} |g|\, d\rho$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\int_A |g|\, d\rho} \int_{B_k} |g|\, d\rho$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \tau_A(B_k)$$

$$= \sum_{k=1}^{\infty} \tau_A(B_k)\,,$$

which shows that axiom M2 is satisfied. Hence τ_A is a measure on $(A,\, \mathcal{S}_A)$. □

Now we introduce a function called the unidirectional function as given on the following definition.

Definition 3.3.2 (Unidirectional Function)**.** Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let $r \in S$ be a point called a reference point. Suppose a map $Y_r : S \to \mathbb{R}$ such that for every $x \in S$, $|Y_r(x)|$ is the distance between r and x unidirectional to the moment on the RC beam, and for every $A \in \mathcal{S}$, $Y_r\big|_A \in \mathcal{L}^1(A,\, \mathcal{S}_A,\, \tau_A)$ holds where $(A,\, \mathcal{S}_A,\, \tau_A)$ is a measure space as given on theorem 3.3.1. And we call Y_r the unidirectional function. By convention, Y_r can be of either positive or negative value. If r is closer to the extreme fibre than x then $Y_r(x)$ is positive and vice versa, for every $x \in S$. And the signed distance between r and the centroid of A is given by

$$\int_A Y_r\big|_A\, d\tau_A\,.$$

An important property of the unidirectional function is given on the following theorem.

Theorem 3.3.2. *Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let $r \in S$ be a reference point. Let $Y_r : S \to \mathbb{R}$ be the unidirectional function (definition 3.3.2). Let $A \in \mathcal{S}$. By theorem 3.3.1, $(A,\, \mathcal{S}_A,\, \tau_A)$ is a measure space. The expression*

$$\int_A Y_r\big|_A\, d\tau_A = \frac{1}{\int_A |g|\, d\rho} \int_A Y_r |g|\, d\rho$$

holds.

Proof. Definition 3.3.2 asserts that $Y_r\big|_A \in \mathcal{L}^1(A,\, \mathcal{S}_A,\, \tau_A)$. Hence

$$\int_A \Big| Y_r\big|_A \Big|\, d\tau_A < \infty\,.$$

By definition, we have that $Y_r = Y_r\big|_A$ everywhere on A (Salamon, 2016). Then by the property of Lebesgue integral (Salamon, 2016) and follows from theorem 3.3.1, we obtain

$$\int_A Y_r\big|_A\, d\tau_A = \frac{1}{\int_A |g|\, d\rho} \int_A Y_r\big|_A |g|\, d\rho = \frac{1}{\int_A |g|\, d\rho} \int_A Y_r |g|\, d\rho\,,$$

which proves the theorem. □

Now we present the $\mathbf{UM}$-distance on an $\mathbf{RCS}$-space on the following theorem.

Theorem 3.3.3 ($\mathbf{UM}$-Distance)**.** *Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let $(S^-,\, \mathcal{S}^-)$ and $(S^+,\, \mathcal{S}^+)$ be the $\mathbf{RCS}^-$ and $\mathbf{RCS}^+$ measurable spaces (definition 3.1.6). Let $r \in S$ be a reference point and let $Y_r : S \to \mathbb{R}$ be the unidirectional function (definition 3.3.2). A map $d_{um} : \mathcal{S}^- \cup \mathcal{S}^+ \to [0,\, \infty)$ defined by*

$$\forall A, B \in \mathcal{S}^- \cup \mathcal{S}^+\,,\ d_{um}(A, B) := \left| \int_A Y_r\big|_A\, d\tau_A - \int_B Y_r\big|_B\, d\tau_B \right|$$

is a $\mathbf{UM}$-distance on $\mathcal{S}^- \cup \mathcal{S}^+$, where $(U,\, \mathcal{S}_U,\, \tau_U)$ is a measure space with respect to theorem 3.3.1, for every $U \in \mathcal{S}^- \cup \mathcal{S}^+$.

Proof. Let $U, V, W \in \mathcal{S}^- \cup \mathcal{S}^+$. By the Lebesgue integral property (Salamon, 2016), we have

$$\begin{aligned} d_{um}(U,U) &= \left| \int_U Y_r\big|_U \, d\tau_U - \int_U Y_r\big|_U \, d\tau_U \right| \\ &= \left| \int_U Y_r\big|_U - Y_r\big|_U \, d\tau_U \right| \\ &= \left| \int_U 0 \, d\tau_U \right| \\ &= 0 \,, \end{aligned}$$

which shows that axiom U1 is satisfied. We also have

$$\begin{aligned} d_{um}(U,V) &= \left| \int_U Y_r\big|_U \, d\tau_U - \int_V Y_r\big|_V \, d\tau_V \right| \\ &= \left| (-1) \left(\int_V Y_r\big|_V \, d\tau_V - \int_U Y_r\big|_U \, d\tau_U \right) \right| \\ &= |-1| \left| \int_V Y_r\big|_V \, d\tau_V - \int_U Y_r\big|_U \, d\tau_U \right| \\ &= \left| \int_V Y_r\big|_V \, d\tau_V - \int_U Y_r\big|_U \, d\tau_U \right| \\ &= d_{um}(V,U) \,, \end{aligned}$$

which shows that axiom U2 is satisfied. And again, we obtain

$$\begin{aligned} d_{um}(U,V) &= \left| \int_U Y_r\big|_U \, d\tau_U - \int_V Y_r\big|_V \, d\tau_V \right| \\ &= \left| \int_U Y_r\big|_U \, d\tau_U - \int_W Y_r\big|_W \, d\tau_W + \int_W Y_r\big|_W \, d\tau_W - \int_V Y_r\big|_V \, d\tau_V \right| \\ &\leq \left| \int_U Y_r\big|_U \, d\tau_U - \int_W Y_r\big|_W \, d\tau_W \right| + \left| \int_W Y_r\big|_W \, d\tau_W - \int_V Y_r\big|_V \, d\tau_V \right| \\ &= d_{um}(U,W) + d_{um}(W,V) \,, \end{aligned}$$

which shows that axiom U3 is satisfied. Hence we conclude that d_{um} is a $\mathbf{UM}$-distance on $\mathcal{S}^- \cup \mathcal{S}^+$. □

The formal definition of the $\mathbf{UM}$-space is presented on the following definition.

Definition 3.3.3 ($\mathbf{UM}$-Space)**.** Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Let $(S^-,\, \mathcal{S}^-)$ and $(S^+,\, \mathcal{S}^+)$ be the $\mathbf{RCS}^-$ and $\mathbf{RCS}^+$ measurable spaces (definition 3.1.6). Let $r \in S$ be a reference point and let $Y_r : S \to \mathbb{R}$ be the unidirectional function (definition 3.3.2). By theorem 3.3.3, a map $d_{um} : \mathcal{S}^- \cup \mathcal{S}^+ \to [0,\, \infty)$ defined by

$$\forall A, B \in \mathcal{S}^- \cup \mathcal{S}^+ \,,\; d_{um}(A,B) := \left| \int_A Y_r\big|_A \, d\tau_A - \int_B Y_r\big|_B \, d\tau_B \right|$$

is a $\mathbf{UM}$-distance on $\mathcal{S}^- \cup \mathcal{S}^+$. And we call the pair $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{um})$ the $\mathbf{UM}$-space.

Proposition 3.3.1. Let $(S,\, \mathcal{S},\, \rho)$ be an $\mathbf{RCS}$-space. Let $g : S \to \mathbb{R}$ be the general stress function (defintion 3.1.4). Let $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{um})$ be the $\mathbf{UM}$-space (definition 3.3.3). Follows from thoerem 3.3.2, we obtain

$$\forall A, B \in \mathcal{S}^- \cup \mathcal{S}^+ \,,\; d_{um}(A,B) = \left| \frac{1}{\int_A |g| \, d\rho} \int_A Y_r |g| \, d\rho - \frac{1}{\int_B |g| \, d\rho} \int_B Y_r |g| \, d\rho \right| .$$

3.4 Moment of Resistance

Now we present the formal description of the moment of resistance of an RC beam undergoing a pure bending moment as a postulate and its derivation below.

Postulate 3.4.1 (Moment of Resistance). *Let* $(S,\, \mathcal{S},\, \rho)$ *be an* $\mathbf{RCS}$*-space. Let* $g : S \to \mathbb{R}$ *be the general stress function (definition 3.1.4). Let* $r \in S$ *be a reference point, and let* $Y_r : S \to \mathbb{R}$ *be the unidirectional function (definition 3.3.2). Let* $(S^- \cup S^+,\, d_{um})$ *be the* $\mathbf{UM}$*-space (definition 3.3.3). The moment of resistance of the* $\mathbf{RCS}$*-space is given by*

$$\mathcal{M}(S,\, \mathcal{S},\, \rho) = \left| \int_{S^*} g \, d\rho \right| \left| \frac{1}{\int_{S^-} |g| \, d\rho} \int_{S^-} Y_r |g| \, d\rho - \frac{1}{\int_{S^+} |g| \, d\rho} \int_{S^+} Y_r |g| \, d\rho \right| ,$$

for every $* \in \{-, +\}$.

Derivation. By equation (1) (as implied by the force couple principle (Du-Bois, 1902)), the moment of a RC beam is equal to the multiplication of the absolute value of the force on either compressive or tensile region with the distance separating the compressive and tensile regions. Then by our axiomatic theory, we have

$$\mathcal{M}(S,\, \mathcal{S},\, \rho) := |\mathrm{F}(S^*)| \cdot d_{um}(S^-, S^+) ,$$

such that $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ is the force map (definition 3.2.1), and $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{um})$ is the $\mathbf{UM}$-space (definition 3.3.3). Follows from definition 3.2.1 and proposition 3.3.1, we obtain

$$\begin{aligned}
\mathcal{M}(S,\, \mathcal{S},\, \rho) &= |\mathrm{F}(S^*)| \cdot d_{um}(S^-, S^+) \\
&= \left| \int_{S^*} g \, d\rho \right| \left| \frac{1}{\int_{S^-} |g| \, d\rho} \int_{S^-} Y_r |g| \, d\rho - \frac{1}{\int_{S^+} |g| \, d\rho} \int_{S^+} Y_r |g| \, d\rho \right| ,
\end{aligned}$$

hence the postulate. □

4 Validation

As a simple validation, we will present the derivation of the conventional RC beam's nominal moment equation via our axiomatic theory as presented on section 3. A conventional RC beam is a RC beam with steel rebars being the sole reinforcement.

Suppose we are given a rectangular conventional RC beam undergoing some ultimate moment $M_u > 0$. Let $b, h > 0$ be the breadth and the height of the beam section respectively. Let $f'_c, f_s, f_y > 0$ be the magnitude of the concrete ultimate compressive stress, the steel compressive stress and the magnitude of the steel yield stress respectively. Let $A_s, A'_s > 0$ be the magnitude of the area of the compressive and tensile regions of the rebars respectively. Let $d > 0$ be the distance between the extreme compressive fibre and the central of gravity of the tensile region of the rebars, and let $d' > 0$ be the distance between the extreme compressive fibre and the central of gravity of the compressive region of the rebars. Suppose $d' < d$.

Now suppose an $\mathbf{RCS}$-space $(S,\, \mathcal{S},\, \rho)$ (definition 3.1.1) representing the RC beam, where S is the set of all points on the RC beam cross section. Then we have

$$\rho(S) = bh \, .$$

Let $g : S \to \mathbb{R}$ be the general stress function (definition 3.1.4). Based on the assumptions given above as well as equivalent rectangular stress distribution principle of the concrete (Darwin et al., 2016), we designate

$$\mathrm{image}(g) = \{-0.85 f'_c, -f_s, 0, f_y\} \, .$$

Consequently, g is a step function by definition 2.1.3. For convenience, we define

$$\begin{aligned}
C^- &:= \{x \in S \mid g(x) = -0.85 f'_c\} \, , \\
R^- &:= \{x \in S \mid g(x) = -f_s\} \, , \text{ and} \\
R^+ &:= \{x \in S \mid g(x) = f_y\} \, .
\end{aligned}$$

The compressive and tensile regions of the $\mathbf{RCS}$-space (definition 3.1.5) are given by

$$\begin{aligned} S^- &= \{x \in S \mid g(x) < 0\} \\ &= \{x \in S \mid g(x) = -0.85 f'_c \vee g(x) = -f_s\} \\ &= \{x \in S \mid g(x) = -0.85 f'_c\} \cup \{x \in S \mid g(x) = -f_s\} \\ &= C^- \cup R^- \end{aligned} \tag{3}$$

and

$$S^+ = \{x \in S \mid g(x) > 0\} = \{x \in S \mid g(x) = f_y\} = R^+ \tag{4}$$

respectively. Then by Whitney stress block principle (Whitney, 1937), there exists some $a > 0$ such that $a < d$ and

$$\rho(C^-) = ab\,.$$

And we also have

$$\rho(R^-) = A'_s$$

and

$$\rho(R^+) = A_s\,.$$

Follows from equation (3) and by axiom R2, we have

$$\rho(S^-) = \rho(C^- \cup R^-) = \rho(C^-) + \rho(R^-) = ab + A'_s\,.$$

And follows from equation (4), we have

$$\rho(S^+) = \rho(R^+) = A_s\,.$$

Let $\mathrm{F} : \mathcal{S} \to \mathbb{R}$ be the force map (definition 3.2.1). Then the force on S^- is given by

$$\begin{aligned} \mathrm{F}(S^-) &= \int_{S^-} g\, d\rho \\ &= \int_{S^-} g^+\, d\rho - \int_{S^-} g^-\, d\rho \\ &= \left[f_y \cdot \rho(R^+ \cap S^-)\right] - \left[0.85 f'_c \cdot \rho(C^- \cap S^-) + f_s \cdot \rho(R^- \cap S^-)\right] \\ &= \left[f_y \cdot \rho(\varnothing)\right] - \left[0.85 f'_c \cdot \rho(C^-) + f_s \cdot \rho(R^-)\right] \\ &= 0 - \left[0.85 f'_c \cdot ab + f_s \cdot A'_s\right] \\ &= -0.85 f'_c \cdot ab - f_s \cdot A'_s\,. \end{aligned}$$

And the force on S^+ is given by

$$\begin{aligned} \mathrm{F}(S^+) &= \int_{S^+} g\, d\rho \\ &= \int_{S^+} g^+\, d\rho - \int_{S^+} g^-\, d\rho \\ &= \left[f_y \cdot \rho(R^+ \cap S^+)\right] - \left[0.85 f'_c \cdot \rho(C^- \cap S^+) + f_s \cdot \rho(R^- \cap S^+)\right] \\ &= \left[f_y \cdot \rho(R^+)\right] - \left[0.85 f'_c \cdot \rho(\varnothing) + f_s \cdot \rho(\varnothing)\right] \\ &= \left[f_y \cdot A_s\right] - \left[0.85 f'_c \cdot 0 + f_s \cdot 0\right] \\ &= f_y \cdot A_s\,. \end{aligned}$$

And by the force equilibrium postulate (postulate 3.2.1), we have

$$f_y \cdot A_s = \mathrm{F}(S^+) = -\mathrm{F}(S^-) = 0.85 f'_c \cdot ab + f_s \cdot A'_s\,. \tag{5}$$

Now let $r \in S$ be a reference point. As a designation, suppose r is located at the extreme compressive fibre. And let $Y_r : S \to \mathbb{R}$ be the unidirectional function (definition 3.3.2). Let $(\mathcal{S}^- \cup \mathcal{S}^+,\, d_{um})$ be the $\mathbf{UM}$-space as defined by definition 3.3.3. Then we have

$$\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-} = \frac{a}{2}\,,\ \int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-} = d'\,,\ \int_{R^+} Y_r\big|_{R^+}\, d\tau_{R^+} = d\,. \tag{6}$$

Follows from theorem 3.3.2, we obtain

$$\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-} = \frac{1}{\int_{C^-}|g|\,d\rho}\int_{C^-} Y_r|g|\,d\rho$$

which implies

$$\int_{C^-} Y_r|g|\,d\rho = \left(\int_{C^-}|g|\,d\rho\right)\left(\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-}\right). \tag{7}$$

Likewise, we obtain

$$\int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-} = \frac{1}{\int_{R^-}|g|\,d\rho}\int_{R^-} Y_r|g|\,d\rho$$

which implies

$$\int_{R^-} Y_r|g|\,d\rho = \left(\int_{R^-}|g|\,d\rho\right)\left(\int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-}\right). \tag{8}$$

And by σ-additive property of Lebesgue integral (Salamon, 2016), we obtain

$$\begin{aligned}
\int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-} &= \frac{1}{\int_{S^-}|g|\,d\rho}\int_{S^-} Y_r|g|\,d\rho \\
&= \frac{1}{\int_{C^-\cup R^-}|g|\,d\rho}\int_{C^-\cup R^-} Y_r|g|\,d\rho \\
&= \frac{1}{\int_{C^-}|g|\,d\rho + \int_{R^-}|g|\,d\rho}\left(\int_{C^-} Y_r|g|\,d\rho + \int_{R^-} Y_r|g|\,d\rho\right) \\
&= \frac{1}{\int_{C^-}|g|\,d\rho + \int_{R^-}|g|\,d\rho}\Bigg[\left(\int_{C^-}|g|\,d\rho\right)\left(\int_{C^-} Y_r\big|_{C^-}\, d\tau_{C^-}\right) \\
&\qquad + \left(\int_{R^-}|g|\,d\rho\right)\left(\int_{R^-} Y_r\big|_{R^-}\, d\tau_{R^-}\right)\Bigg] \text{ (from equations (7) and (8))} \\
&= \frac{0.85f'_c\cdot ab\cdot\frac{a}{2} + f_s\cdot A'_s\cdot d'}{0.85f'_c\cdot ab + f_s\cdot A'_s} \text{ (from equation (6))}
\end{aligned} \tag{9}$$

By postulate 3.4.1, the nominal moment is given by

$$\begin{aligned}
M_n &:= \mathcal{M}(S,\,\mathcal{S},\,\rho) \\
&= \left|\int_{S^+} g\,d\rho\right|\left|\frac{1}{\int_{S^-}|g|\,d\rho}\int_{S^-} Y_r|g|\,d\rho - \frac{1}{\int_{S^+}|g|\,d\rho}\int_{S^+} Y_r|g|\,d\rho\right| \\
&= |\mathrm{F}(S^+)|\left|\int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-} - \int_{S^+} Y_r\big|_{S^+}\, d\tau_{S^+}\right| \text{ (from theorem 3.3.2)} \\
&= |\mathrm{F}(S^+)|\left|\int_{S^+} Y_r\big|_{S^+}\, d\tau_{S^+} - \int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-}\right| \\
&= |\mathrm{F}(S^+)|\left|\int_{R^+} Y_r\big|_{R^+}\, d\tau_{R^+} - \int_{S^-} Y_r\big|_{S^-}\, d\tau_{S^-}\right| \text{ (from equation (4))} \\
&= f_y\cdot A_s\left|d - \frac{0.85f'_c\cdot ab\cdot\frac{a}{2} + f_s\cdot A'_s\cdot d'}{0.85f'_c\cdot ab + f_s\cdot A'_s}\right| \text{ (from equation (9))} \\
&= f_y\cdot A_s\left(d - \frac{0.85f'_c\cdot ab\cdot\frac{a}{2} + f_s\cdot A'_s\cdot d'}{f_y\cdot A_s}\right) \text{ (from equation (3.2.1))} \\
&= f_y\cdot A_s\cdot d - \left(0.85f'_c\cdot ab\cdot\frac{a}{2} + f_s\cdot A'_s\cdot d'\right) \\
&= f_y\cdot A_s\cdot d - \left[(0.85f'_c\cdot ab + f_s\cdot A'_s)\frac{a}{2} + f_s\cdot A'_s\left(d' - \frac{a}{2}\right)\right] \\
&= f_y\cdot A_s\cdot d - \left[(f_y\cdot A_s)\frac{a}{2} - f_s\cdot A'_s\left(\frac{a}{2} - d'\right)\right] \text{ (from equation (3.2.1))} \\
&= f_y\cdot A_s\left(d - \frac{a}{2}\right) + f_s\cdot A'_s\left(\frac{a}{2} - d'\right)
\end{aligned}$$

which is the classical equation of the conventional RC beam nominal moment (Darwin et al., 2016).

5 Conclusions and Future Work

This paper has presented the axiomatic theory of the RC beam undergoing a pure bending moment by making use of measure theory in pure mathematics. We have formally defined the $\mathbf{RCS}$-space, the measure space representing the cross section of a RC beam on definition 3.1.1. The stress functions representing the stresses of the concrete and reinforcements are unified into an $\mathcal{L}^1$-function called the general stress function. The properties of $\mathbf{RCS}$-space are derived as theorems on section 3.1. The forces on the $\mathbf{RCS}$-space are formally defined as the Lebesgue integral of the general stress function on section 3.2. We have also introduced new concepts called the $\mathbf{UM}$-distance and the $\mathbf{UM}$-space to determine the central of gravity of sets within the $\mathbf{RCS}$-space as well as to determine the distances between the sets on section 3.3. Then the moment of resistance of a RC beam have been presented as a Lebesgue integral expression on postulate 3.4.1. It has also been shown that one is able to derive the classical equation for the conventional RC beam nominal moment via this axiomatic theory on section 4. It gives a clear evidence that this axiomatic theory serves as a general theory representing any kind of RC beam.

We are interested in deriving more equations for unconventional RC beams via this axiomatic theory as well as demonstrating a novel approach in the computation of RC beams' moment of resistance by treating this axiomatic theory as the foundation of the beam computation's algorithm.

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