A Note on Fourier Transform Conventions Used in Wave Analyses

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July 12, 2019

Abstract

Temporal and spatial Fourier transforms are natural tools in the study of propagating waves in many applications. For example, the inverse spatial Fourier transform specifies how any wave can be built by summing plane waves. However, sign conventions necessary to describe waves are at odds with sign conventions used in spatio-temporal Fourier transforms. This note describes the problem and shows several ways that authors deal with it.

1 Introduction

There is a slightly uncomfortable relationship between how plane waves are expressed mathematically and how the Fourier transforms used to describe and analyze them are expressed. This note will discuss some of the various approaches that authors take.

Authors of "wave" books tend to describe a wave as some function $s(t,x) = p(kx - \omega t)$ or $s(t,x) = p(kx + \omega t)$ with $p(\cdot)$ a kind of base function and where the former form describes a wave propagating in the forward x direction as t increases and the latter form a wave propagating in the $-x$ direction with increasing t, both assuming that k and ω are positive. Some other authors, perhaps of the signal processing bent, insert a negative sign in the argument, as in $s(t,x) = p(\omega t \pm kx)$ with again the $-$ form propagating forward and the $+$ form propagating backward[1]. While all of these are solutions to the one-dimensional wave equation

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{c}\frac{\partial^2 s}{\partial t^2}$$

with the constraint that $k = \omega/c$ as can be verified by direct substitution, the "wave" forms reverse the base function p in time and thus if the wave s is considered to be emitted from an antenna or a loudspeaker, it will come out backwards which is probably not as intended. In three dimensions, the generic wave can be described with a spatial wave vector $\mathbf{x} = (x, y, z)'$ as $s(\mathbf{x}, t)$ and the wave equation as

$$\nabla^2 s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c}\frac{\partial^2 s}{\partial t^2}. \tag{1}$$

While any function s as described will work as a wave, it is often convenient to use a sinusoidal expression called a *monochromatic wave*. In that case, for forward traveling waves and wavenumber vector $\mathbf{k}$ described below, we have

$$s(t,\mathbf{x}) = p(\omega t - \mathbf{k}\cdot\mathbf{x}) = e^{\pm i(\omega t - \mathbf{k}\cdot\mathbf{x})} \tag{2}$$

which is used with the usual caveat that the real part is to be taken, or in the case of radar and communication systems, that quadrature processing is used in the receiver to recover both real and imaginary parts. ω is identified as the temporal radian frequency of the sinusoid in radians/s and in one dimensional cases $k = |\mathbf{k}|$ is identified as the spatial radian frequency in radians/m; we allow their cyclic frequency counterparts f and ν defined by $\omega = 2\pi f$ and $k = 2\pi\nu$ as well, and we will use either form at will, for example, expressing the same monochromatic wave above as

$$s(t,\mathbf{x}) = e^{\pm i2\pi(ft - \boldsymbol{\nu}\cdot\mathbf{x})}. \tag{3}$$

Spatial radian frequency is commonly called the *wavenumber.* The three-dimensional counterpart to k is the *wavenumber vector* $\mathbf{k} = (k_x, k_y, k_z)'$ with each component indicating the wavenumber in each direction and $|\mathbf{k}|$ is the wavenumber in the direction of propagation which is indicated by the unit vector $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$; the cyclic frequency vector $\boldsymbol{\nu}$ is defined by $\mathbf{k} = (\nu_x, \nu_y, \nu_z)' = 2\pi\boldsymbol{\nu}$. If loci of constant function value—phase for the monochromatic case—can be constructed as straight lines or planes orthogonal to $\mathbf{k}$ then the wave is called a *plane wave.*

One-dimensional Fourier transforms are traditionally defined with a "sign-i" convention as follows, first for the forward transform from time t to frequency f and then for the inverse transform (sometimes the substitution $\omega = 2\pi f$ is used but that does not concern us here). These temporal transforms are calculated at a fixed location in space, letting t and f be variable.

$$S_f(f,\mathbf{x}) = \int s(t,\mathbf{x})\, e^{-i2\pi ft} dt \tag{4}$$

*Originally published on July 27, 2018. Available at engrxiv.org. This paper has not been peer reviewed. Jerry Bauck welcomes comments and can be reached by e-mail at this address after left-shifting each alpha character by one position, e.g., f becomes e: kfssz@cbvdl.ofu.

[1] Some authors use forms such as $p[\pm(ct \pm x)]$ or $p[\pm(t \pm x/c)]$ but since these are mere scalings of the argument we shall not consider them to be essentially different than the cases presented.

$$s(t,\mathbf{x}) = \int S_f(f,\mathbf{x})\, e^{i2\pi f t} df. \quad (5)$$

Spatial Fourier transforms are also useful and we would like to maintain the same sign-i convention. Thus we hopefully write

$$S_{\boldsymbol{\nu}}(t,\boldsymbol{\nu}) = \int s(t,\mathbf{x})\, e^{-i2\pi \boldsymbol{\nu}\cdot\mathbf{x}} d\mathbf{x} \quad (6)$$

$$s(t,\mathbf{x}) = \int S_{\boldsymbol{\nu}}(t,\boldsymbol{\nu})\, e^{i2\pi \boldsymbol{\nu}\cdot\mathbf{x}} d\mathbf{k}, \quad (7)$$

now fixing time and letting the spatial and wavenumber variables become the transform variables. The latter equation is of special interest because it shows that any wave s can be expressed as an infinite summation, an integral, of plane waves over all directions and wavenumbers. This is the important concept of the *plane wave spectrum* and is the spatial corollary of composing any time function from an infinity of sinusoids as in (5) above.

Note the similarity between the exponential form describing the monochromatic wave (3) and the exponential form inside the Fourier transforms. This is important because wave fields often can be advantageously studied by using their Fourier transforms. The uncomfortable situation arises when one considers that it would be interesting to take the Fourier transform of a wave with respect to either time t or space $\mathbf{x}$, but those variables have different signs in the wave function—how do we make the inverse Fourier transforms fit our notion of a plane wave? A perusal of a few books reveals that authors deal with this in different ways.

2 Approaches

In [1] a four-dimensional transform pair is used, combining time and three spatial dimensions. A plane wave is represented by

$$s(t,\mathbf{x}) = e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \quad (8)$$

and, separating the authors' four-dimensional transform into time and space transforms, the time-frequency inverse transform is as in (5) and thus the forward transform is as in (4). (Note that we are freely mixing radian and cyclical frequency variables as promised, and focusing on the signs in the complex exponentials.) However, to accommodate the spatial variables' reversed sign, the definition of the spatial transforms is redefined as

$$S_{\boldsymbol{\nu}}(t,\boldsymbol{\nu}) = \int s(t,\mathbf{x})\, e^{i2\pi \boldsymbol{\nu}\cdot\mathbf{x}} d\mathbf{x}$$

and

$$s(t,\mathbf{x}) = \int S_{\boldsymbol{\nu}}(t,\boldsymbol{\nu})\, e^{-i2\pi \boldsymbol{\nu}\cdot\mathbf{x}} d\mathbf{k},$$

with reversed signs relative to our hoped-for forms of (6) and (7). In [2] the same convention $p(\omega t - \mathbf{k}\cdot\mathbf{x})$ is used for waves, either general or monochromatic. However, it is unclear what conventions are used for Fourier transforms.

The approach used in [3] defines a plane wave as in (8) but an alternate wavenumber vector which is the same as defined herein except with a negative sign. Thus, this alternate vector points in the direction opposite to the propagation direction. While this might seem odd, it is used in a chapter in which an antenna array is described and the reversed vector points in the direction of arriving plane waves, opposite to their direction of propagation if transmitted. It is difficult to find an actual Fourier transform definition in this multi-author book but one supposes that this negation of the wavenumber vector has the further advantage of using the traditional i-signs of (4) through (7), i.e., for both time and space transforms.

Reference [4] is an acoustics book describing radiating surfaces and defines plane waves as

$$s(t,\mathbf{x}) = e^{i(k\cdot\mathbf{x} - \omega\mathbf{t})}, \quad (9)$$

the back-ward-in time form. The i-sign problem is dealt with by keeping the traditional form of the spatial Fourier transform in (6) and (7) but defining a non-traditional, sign-reversed form for the time-frequency transforms,

$$S_f(f,\mathbf{x}) = \int s(t,\mathbf{x})\, e^{i2\pi f t} dt$$

$$s(t,\mathbf{x}) = \int S_f(f,\mathbf{x})\, e^{-i2\pi f t} df.$$

The same method is used in [5], another "wave" book. In [6] waves are initially described by both time and spatial variables as in (9) but quickly the time dependence is dropped, referring only to the "amplitude distribution" of the field; time-frequency Fourier transforms are never used and the traditional sign convention of (6) and (7) prevails.

Many authors structure their presentations of waves so that the time dependence of the wave $s(t,\mathbf{x})$ is carefully ignored, or rather, put in the background. This can be done through the use of a version of phasors that electrical engineers typically learn in circuit and system analysis, or by simply stating that the time dependence is "understood" as in [7], in any case, carrying along only the magnitude and phase of sinusoids. In the study of waves, removing the time dependence leaves a phasor that is the complex amplitude as a function of the spatial coordinates $\mathbf{x}$. An aspect of this approach involves taking the Fourier transform with respect to time of the wave equation (1) resulting in the time-independent Helmholtz equation [8]

$$\nabla^2 s + k^2 s = 0.$$

It should be remembered that the time dependence such as $\exp(i\omega_0 t)$ can always be reinserted as needed, in either circuit or wave studies, to recover the actual signal or wave. In these texts, typically one-dimensional Fourier transformation is presented with time and frequency as the conjugate pair and two- or three-dimensional Fourier transforms are presented with $\mathbf{k}$ and $\mathbf{x}$ or $\boldsymbol{\nu}$ and $\mathbf{x}$ as the transform pairs.

The temporal and spatial analyses are not merged and so the traditional forms (4), (5), (6), and (7) suffice in these texts. The use of the phasor approach is easy to justify but restricts analyses to monochromatic fields. Many applications can't always be restricted to waves comprised of single sinusoids. For example radar where wideband signals are commonly used, or audio signals where even short segments of waves can cover much of the 10 octaves available to human hearing. In these cases the wave must be considered as at least a discrete sum of sinusoids or a continuous sum of sinusoids, thus the need for a time-frequency Fourier transform. Even so, under test conditions, these wideband systems can be excited with sinusoids and the simplified phasor approach returns with success.

In contexts such as the study of radar, the time dependence of the waves is sometimes the main concern and many authors will simply state that the time-defined one-dimensional waveform is emitted from an antenna, travels to an object r m distant were it is reflected back to the receiver. A round-trip delay is simply stated as being $2r/c$. Conventional i-signs are adequate then. However, in imaging radars, the wave nature can provide a convenient framework for understanding as well as a source of image reconstruction algorithms [2], [9]. Situations arise where the Fourier transform is performed in either the time-frequency domains or in time and one, two, or three space-wavenumber domains at the same time or in various combinations of time and space domains.

Also in the radar setting, the transmitted, outgoing, wave can be described as propagating in the increasing direction of a coordinate axis, a forward propagating wave. But the reflected, returning, wave, typically of more interest, is then usually a backward propagating wave along the same coordinate axis [10]. In this situation, both terms of the propagation exponent have the same sign, $e^{i(\omega t+\mathbf{k}\cdot\mathbf{x})}$, and both the favored forms (4) and (5) can be used for time-frequency transforms and (6) and (7) for space-wavenumber transforms.

Three other methods are available for dealing with this problem of waves versus transforms. First, simply interpret either the time or spatial axes, which ever is problematic, as being drawn backwards and reverse plots or images graphically as needed. Second, reverse the sense of the base function $p(\cdot)$ as needed, i.e., change the sign of its argument. This will swap the reversal to the other domain relative to time or space but this might be acceptable depending on the application. Third, when computing the Fourier transforms using a machine discrete Fourier transform (DFT) usually implemented as some fast Fourier transform (FFT) algorithm, simply use the opposite kind of DFT, i.e., use a forward transform when an inverse transform is indicated and vice versa, but correcting the amplitude scaling as necessary since these algorithms can have different amplitude scaling for the two kinds, forward or inverse.

3 Version History

- July 27, 2018. First published.
- July 12, 2019. Add this section. Correct several typographical and typesetting errors. Correct definition of f and ν. Expand radar comment. Minor textual fixes and improvements. Add a reference. Minor formatting changes.

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