Generalized Integrating Factor Method Applied for One Dimensional Linear Second Order Ordinary Differential Equations

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Abstract

This manuscript introduces the concept of generalized integrating factor for one dimensional linear ordinary differential equations of order n. The procedure is used to address linear second order equations with varying and with constant coefficients, commonly found in many practical problems. The solutions are analytically derived by means of double convolutions. Analytical solutions for the constant coefficient case with different types of continuous and discontinuous excitations are discussed with examples. The concept of Heaviside series is introduced to generalize the solutions for discrete excitations.

Keywords: ordinary differential equation, second order, analytical solution, integrating factor

1. Introduction

Linear and non-homogeneous ordinary differential equations are common in many areas of applied mathematics. In special, second order ordinary differential equations appear in various practical problems in mathematics, physics and engineering, like for example the dynamic equilibrium of mechanical systems. Their solution is well known for constant coefficients and periodic forces [17], but the solution for general problems usually depends on special transformations.

Integrating factors are used as a tool to modify a general ordinary differential equation to an exact form. Once in this form, its solution is just a matter of integration. Its use is well known for first order equations [5] but its use for second order equations is not so well established in the literature [7, 1, 15, 8, 14, 2, 4, 3, 12].

We propose and discuss a simple but powerful variant of the traditional integrating factor that presents many benefits. The proposed method can be used for general orders and results in integrating factors function of the independent variable only, disregarding the coefficients of the ordinary differential equation.

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The general concept is shown in details and applied to a third-order ODE with varying coefficients to exemplify the procedure. The method is then used to solve second-order ODEs with varying coefficients and with constant coefficients. The latter case is used as starting point to derive many important analytical solutions for different types of excitations (non-homogeneous term of the ODE). Finally, we discuss an approach to use the proposed method to solve discrete forms of excitation by means of Heaviside series.

2. The generalized integrating factor method

Consider a linear first order ordinary differential equation

$$a_1(t)\dot{y}(t) + a_0(t)y(t) = f(t), \tag{1}$$

where y is the dependent variable, t is the independent variable and $\dot{y}(t)$ is the derivative of y with respect to t.

The integrating factor for first order differential equations, introduced by Leibniz, relies on the relation

$$p(t)\dot{y}(t) + \dot{p}(t)y(t) = (p(t)\dot{y}(t)), \tag{2}$$

where $(\dot{\ })$ means the derivative, with respect to t, of all the expression inside the parenthesis.

An integrating factor $\mu(t)$ is multiplied to Eq. 1 to force the appearance of Eq. 2,

$$\underbrace{\mu(t)a_1(t)}_{p(t)}\dot{y}(t) + \underbrace{\mu(t)a_0(t)}_{\dot{p}(t)}y(t) = (p(t)\dot{y}(t)). \tag{3}$$

It immediately follows that

$$\dot{p}(t) = \mu(t)a_0(t) = \big(\mu(t)\dot{a}_1(t)\big) = \dot{\mu}(t)a_1(t) + \mu(t)\dot{a}_1(t) \tag{4}$$

such that

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{a_0(t) - \dot{a}_1(t)}{a_1(t)}, \tag{5}$$

with solution

$$\mu(t) = \exp\left(\int \frac{a_0(t) - \dot{a}_1(t)}{a_1(t)}\,\mathrm{d}t\right), \tag{6}$$

allowing the use direct integration to solve

$$(\mu(t)a_1\dot{(}t)y(t)) = \mu(t)f(t), \tag{7}$$

such that

$$y(t) = \frac{1}{\mu(t)a_1(t)}\int \mu(t)f(t)\,\mathrm{d}t + C_1, \tag{8}$$

where C_1 is an integration constant. Relation given by Eq. 2 can be generalized to higher orders

$$p_j(t)\overset{(j)}{y}(t) + \dot{p}_j(t)\overset{(j-1)}{y}(t) = (p_j(t)\dot{\overset{(j-1)}{y}}(t)), \tag{9}$$

where $\overset{(j)}{y}(t)$ is the j-th derivative of y with respect to t (used for $j > 3$).

Now consider a linear and non-homogeneous ordinary differential equation of order n

$$a_n(t)\overset{(n)}{y}(t) + a_{n-1}(t)\overset{(n-1)}{y}(t) + \dots + a_2(t)\ddot{y}(t) + a_1(t)\dot{y}(t) + a_0(t)y(t) = f(t). \tag{10}$$

It can be rewritten in pairs of derivatives of $y(t)$ such that one element is a derivative higher than the other. This is achieved by partitioning the coefficients multiplying the intermediate derivatives into two terms

$$\begin{aligned}
&\underbrace{a_n(t)\overset{(n)}{y}(t) + f_{n,n-1,1}(t)\overset{(n-1)}{y}(t)}_{\pi_{n,n}} + \underbrace{f_{n,n-1,2}(t)\overset{(n-1)}{y}(t) + f_{n,n-2,1}(t)\overset{(n-2)}{y}(t)}_{\pi_{n,n-1}} + \dots \\
&\qquad + \underbrace{f_{n,3,2}(t)\ddot{y}(t) + f_{n,2,1}(t)\ddot{y}(t)}_{\pi_{n,3}} + \underbrace{f_{n,2,2}(t)\ddot{y}(t) + f_{n,1,1}(t)\dot{y}(t)}_{\pi_{n,2}} + \\
&\qquad\qquad \underbrace{f_{n,1,2}(t)\dot{y}(t) + a_0(t)y(t)}_{\pi_{n,1}} = f(t),
\end{aligned} \tag{11}$$

where $\pi_{n,j}$ is the j-th *partition* of Eq. 10, which is the key idea for the proposed approach. Coefficients $f_{n,j,i}$ refer to the order of the differential equation, n, partition j and $i = 1$ or $i = 2$ such that

$$a_{n,j}(t) = f_{n,j,1}(t) + f_{n,j,2}(t), \tag{12}$$

and there are $N_{\pi_n} = 2n - 2$ partitions j.

Multiplying Eq. 11 by a generalized integrating factor $\mu_n(t)$ results in

$$\begin{aligned}
&\underbrace{\mu_n(t)a_n(t)\overset{(n)}{y}(t) + \mu_n(t)f_{n,n-1,1}(t)\overset{(n-1)}{y}(t)}_{\dot{\left(p_{n,n}\overset{(n-1)}{y}(t)\right)}} + \underbrace{\mu_n(t)f_{n,n-1,2}(t)\overset{(n-1)}{y}(t) + \mu_n(t)f_{n,n-2,1}(t)\overset{(n-2)}{y}(t)}_{\dot{\left(p_{n,n-1}\overset{(n-2)}{y}(t)\right)}} + \dots \\
&\qquad + \underbrace{\mu_n(t)f_{n,3,2}(t)\ddot{y}(t) + \mu_n(t)f_{n,2,1}(t)\ddot{y}(t)}_{\dot{\left(p_{n,3}\ddot{y}(t)\right)}} + \underbrace{\mu_n(t)f_{n,2,2}(t)\ddot{y}(t) + \mu_n(t)f_{n,1,1}(t)\dot{y}(t)}_{\dot{\left(p_{n,2}\dot{y}(t)\right)}} + \\
&\qquad\qquad \underbrace{\mu_n(t)f_{n,1,2}(t)\dot{y}(t) + \mu_n(t)a_0(t)y(t)}_{\dot{\left(p_{n,1}y(t)\right)}} = \mu_n(t)f(t),
\end{aligned} \tag{13}$$

such that

$$\dot{\left(p_{n,n}\overset{(n-1)}{y}(t)\right)} + \dot{\left(p_{n,n-1}\overset{(n-2)}{y}(t)\right)} + \dots + \dot{\left(p_{n,1}y(t)\right)} = \mu_n(t)f(t) \tag{14}$$

can be exactly integrated to

$$p_{n,n}\overset{(n-1)}{y}(t) + p_{n,n-1}\overset{(n-2)}{y}(t) + \dots + p_{n,1}y(t) = \int \mu_n(t)f(t)\,\mathrm{d}t + C_n \tag{15}$$

an ordinary differential equation of order $n - 1$, where C_n is an integration constant.

Coefficients $f_{n,j,i}$ can be found by solving

$$\left(\frac{\dot{\mu}_n}{\mu_n}\right)_n = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_{n-1} = \cdots = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_3 = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_2 = \left(\frac{\dot{\mu}_n}{\mu_n}\right)_1 . \tag{16}$$

where

$$\pi_{n,j} \implies \left(\frac{\dot{\mu}_n(t)}{\mu_n(t)}\right)_j = \frac{f_{n,j-1,1}(t) - \dot{f}_{n,j,2}(t)}{f_{n,j,2}(t)}; f_{n,0,1}(t) = a_0(t),\ f_{n,n,2}(t) = a_n(t). \tag{17}$$

It follows that there are $\frac{n(n-1)}{2}$ combinations of these pair-wise equations, along with the $n-1$ equations relating each pair of partitions with their coefficient. Thus, the number of equations available to evaluate coefficients $f_{n,j,i}$, N_{eq}, is

$$N_{eq_n} = \frac{n(n-1)}{2} + n - 1 = \frac{n^2+n-2}{2} \geq N_{\pi_n};\ n \geq 2, \tag{18}$$

being always bigger or equal than the number of partitions, N_{π_n}, such that there are enough equations to solve the problem (actually, not all equations must be used). After finding all coefficients $f_{n,j,i}$, it is possible to find $\mu_n(t)$ by using Eq. 17 for just one partition $\hat{j}$ (any partition can be used)

$$\left(\frac{\dot{\mu}_n(t)}{\mu_n(t)}\right)_{\hat{j}} = \frac{f_{n,\hat{j}-1,1}(t) - \dot{f}_{n,\hat{j},2}(t)}{f_{n,\hat{j},2}(t)}, \tag{19}$$

such that

$$\mu_n(t) = \exp\left(\int \frac{f_{n,\hat{j}-1,1}(t) - \dot{f}_{n,\hat{j},2}(t)}{f_{n,\hat{j},2}(t)}\,\mathrm{d}t\right). \tag{20}$$

The same procedure depicted above can be carried out successively until reaching a first order equation, where the traditional integrating factor $\mu_1(t)$ can be used.

The procedure is applied to a linear ordinary differential equation of order three to exemplify the procedure.

2.1. Example: linear ordinary differential equation of order three

Consider the ODE

$$a_3(t)\dddot{y}(t) + a_2(t)\ddot{y}(t) + a_1(t)\dot{y}(t) + a_0(t)y(t) = f(t). \tag{21}$$

The explicit dependency on t is dropped in the following equations to make the notation more concise. Partitioning coefficients a_2 and a_1 into $a_2 = f_{2,2,1} + f_{2,2,2}$ and $a_1 = f_{2,1,1} + f_{2,1,2}$ results in

$$\underbrace{a_3\dddot{y} + f_{2,2,1}\ddot{y}}_{\pi_{3,3}} + \underbrace{f_{2,2,2}\ddot{y} + f_{2,1,1}\dot{y}}_{\pi_{3,2}} + \underbrace{f_{2,1,2}\dot{y} + a_0 y}_{\pi_{3,1}} = f. \tag{22}$$

Multiplying Eq. 23 by a generalized integrating factor μ_3

$$\underbrace{\mu_3 a_3}_{p_{3,3}}\dddot{y} + \underbrace{\mu_3 f_{2,2,1}}_{\dot{p}_{3,3}}\ddot{y} + \underbrace{\mu_3 f_{2,2,2}}_{p_{3,2}}\ddot{y} + \underbrace{\mu_3 f_{2,1,1}}_{\dot{p}_{3,2}}\dot{y} + \underbrace{\mu_3 f_{2,1,2}}_{p_{3,1}}\dot{y} + \underbrace{\mu_3 a_0}_{\dot{p}_{3,1}} y = \mu_3 f, \tag{23}$$

such that

$$\dot{\left(p_{3,3}\ddot{y}\right)} + \dot{\left(p_{3,2}\dot{y}\right)} + \dot{\left(p_{3,1}y\right)} = \mu_3 f, \tag{24}$$

can be integrated with respect to t, resulting in

$$p_{3,3}\ddot{y} + p_{3,2}\dot{y} + p_{3,1}y = \underbrace{\int \mu_3 f \,\mathrm{d}t + C_3}_{h_2} \tag{25}$$

a second order ordinary differential equation.

The same procedure can be applied to this second order equation. Splitting $p_{3,2} = f_{1,1,1} + f_{1,1,2}$ results in

$$\underbrace{p_{3,2}\ddot{y} + f_{1,1,1}\dot{y}}_{\pi_{2,2}} + \underbrace{f_{1,1,2}\dot{y} + p_{3,1}y}_{\pi_{2,1}} = h_2 \tag{26}$$

and multiplying by a generalized integrating factor μ_2

$$\underbrace{\mu_2 p_{3,2}}_{p_{2,2}}\ddot{y} + \underbrace{\mu_2 f_{1,1,1}}_{\dot{p}_{2,2}}\dot{y} + \underbrace{\mu_2 f_{1,1,2}}_{p_{2,1}}\dot{y} + \underbrace{\mu_2 p_{3,1}}_{\dot{p}_{2,1}} y = \mu_2 h_2, \tag{27}$$

such that

$$\dot{\left(p_{2,2}\dot{y}\right)} + \dot{\left(p_{2,1}y\right)} = \mu_2 h \tag{28}$$

and integrating w.r.t time results in

$$p_{2,2}\dot{y} + p_{2,1}y = \underbrace{\int \mu_2 h \,\mathrm{d}t + C_2,}_{h_1} \tag{29}$$

a first order ODE. Finally, multiplying by an integrating factor μ_1 results in

$$\underbrace{\mu_1 p_{2,2}}_{p_{1,1}}\dot{y} + \underbrace{\mu_1 p_{2,1}}_{\dot{p}_{1,1}} y = \mu_1 h_1 \tag{30}$$

or

$$\dot{\left(p_{1,1}y\right)} = \mu_1 h_1 \tag{31}$$

such that

$$y = \frac{1}{p_{1,1}} \int \mu_1 h_1 \,\mathrm{d}t + C_1, \tag{32}$$

is the solution to Eq. 23.

Solution provided by Eq. 32 depends on μ_3, μ_2 and μ_1. Using Eq. 30, it is possible to write

$$\dot{p}_{1,1} = \mu_1 p_{2,1} = \dot{\left(\mu_1 p_{2,2}\right)} = \dot{\mu}_1 p_{2,2} + \mu_1 \dot{p}_{2,2} \tag{33}$$

such that

$$\frac{\dot{\mu}_1}{\mu_1} = \frac{p_{2,1} - \dot{p}_{2,2}}{p_{2,2}} \tag{34}$$

with solution

$$\mu_1 = \exp\left(\int \frac{p_{2,1} - \dot{p}_{2,2}}{p_{2,2}}\,\mathrm{d}t\right). \tag{35}$$

The same procedure can be applied to Eq. 27. There are two possible combinations:

$$\dot{p}_{2,2} = \mu_2 f_{1,1,1} = \dot{(\mu_2 p_{3,2})} = \dot{\mu}_2 p_{3,2} + \mu_2 \dot{p}_{3,2} \tag{36}$$

and

$$\dot{p}_{2,1} = \mu_2 p_{3,1} = \dot{(\mu_2 f_{1,1,2})} = \dot{\mu}_2 f_{1,1,2} + \mu_2 \dot{f}_{1,1,2}, \tag{37}$$

such that

$$\frac{\dot{\mu}_2}{\mu_2} = \frac{f_{1,1,1} - \dot{p}_{3,2}}{p_{3,2}} = \frac{p_{3,1} - \dot{f}_{1,1,2}}{f_{1,1,2}}. \tag{38}$$

Since $p_{3,2} = f_{1,1,1} + f_{1,1,2}$ it is possible to rewrite previous equation as

$$f_{1,1,2}(p_{3,2} - f_{1,1,2}) - f_{1,1,2}\dot{p}_{3,2} = p_{3,2}p_{3,1} - p_{3,2}\dot{f}_{1,1,2} \tag{39}$$

or

$$f_{1,1,2}^2 = -p_{3,2}p_{3,1} + p_{3,2}\dot{f}_{1,1,2} + (p_{3,2} - \dot{p}_{3,2})f_{1,1,2} \tag{40}$$

a Riccati differential Equation.

Finally, there are three possible combinations for Eq. 23:

$$\dot{p}_{3,3} = \mu_3 f_{2,2,1} = \dot{(\mu_3 a_3)} = \dot{\mu}_3 a_3 + \mu_3 \dot{a}_3, \tag{41}$$

$$\dot{p}_{3,2} = \mu_3 f_{2,1,1} = \dot{(\mu_3 f_{2,2,2})} = \dot{\mu}_3 f_{2,2,2} + \mu_3 \dot{f}_{2,2,2}, \tag{42}$$

and

$$\dot{p}_{3,1} = \mu_3 a_0 = \dot{(\mu_3 f_{2,1,2})} = \dot{\mu}_3 f_{2,1,2} + \mu_3 \dot{f}_{2,1,2}. \tag{43}$$

Thus,

$$\frac{\dot{\mu}_3}{\mu_3} = \frac{f_{2,2,1} - \dot{a}_3}{a_3} = \frac{f_{2,1,1} - \dot{f}_{2,2,2}}{f_{2,2,2}} = \frac{a_0 - \dot{f}_{2,1,2}}{f_{2,1,2}}. \tag{44}$$

Using the splits $a_1 = f_{2,1,1} + f_{2,1,2}$ and $a_2 = f_{2,2,1} + f_{2,2,2}$

$$\frac{(a_2 - f_{2,2,2}) - \dot{a}_3}{a_3} = \frac{(a_1 - f_{2,1,2}) - \dot{f}_{2,2,2}}{f_{2,2,2}} = \frac{a_0 - \dot{f}_{2,1,2}}{f_{2,1,2}}, \tag{45}$$

such that

$$f_{2,2,2}^2 = a_3\dot{f}_{2,2,2} + (a_2 - a_3)f_{2,2,2} - a_3 a_1 + a_3 f_{2,1,2} \tag{46}$$

and

$$f_{2,1,2}^2 = f_{2,2,2}\dot{f}_{2,1,2} + (a_1 - \dot{f}_{2,2,2})f_{2,1,2} - a_0 f_{2,2,2} \tag{47}$$

form a coupled system of Riccati-like differential equations.

Previous equations are much simpler when the coefficients are constant, since most of the rates are null. In this case, Eq. 35 simplifies to

$$\mu_1 = \exp\left(\int \frac{p_{2,1}}{p_{2,2}}\,\mathrm{d}t\right), \tag{48}$$

the Riccati equation, Eq. 40 turns into a second order algebraic equation

$$f_{1,1,2}^2 = -p_{3,2}p_{3,1} + p_{3,2}f_{1,1,2} \tag{49}$$

and the system of Riccati-like differential equations, Eqs. 46 and 47 reduces to

$$f_{2,2,2}^2 = (a_2 - a_3)f_{2,2,2} - a_3a_1 + a_3f_{2,1,2} \tag{50}$$

and

$$f_{2,1,2}^2 = a_1f_{2,1,2} - a_0f_{2,2,2}, \tag{51}$$

a coupled system of quadratic algebraic equations.

Thus, the main shortcoming of the proposed procedure is the evaluation of Equations 16. The larger the order of the differential equation, the harder is to find the coefficients $f_{n,j,i}$ at each step of the procedure to decrease the order. For general coefficients $a_j(t)$, Eqs. 16 result in a system of Riccati-like coupled differential equations and for constant coefficients a system of coupled quadratic equations.

Nonetheless, for linear second order equations the procedure can lead to very interesting and practical results, as it will be discussed in the rest of this manuscript.

3. Linear second order ordinary differential equations

Consider a linear second order ordinary differential equation (ODE)

$$m(t)\ddot{y}(t) + c(t)\dot{y}(t) + k(t)y(t) = f(t), \tag{52}$$

with initial conditions $\dot{y}(0) = v_0$ and $y(0) = u_0$, where $y(t)$ and $f(t)$ are functions of t over $\mathbb{R}$. This equation is of great interest in physical problems, like for example the vibration of a mass-spring-damper subjected to a force. Thus, independent variable t is also referred to time in the rest of this manuscript.

For physical reasons, it is assumed that both $m(t)$ and $k(t) \in \mathbb{R}_{>0}$ and that $c(t) \in \mathbb{R}^+$, $\forall t$. The explicit dependency on t will be suppressed in the following equations to simplify the notation.

We start by splitting c as

$$\pi_{2,1} \implies c = f_{2,1,1} + f_{2,1,2}, \tag{53}$$

where $f_{2,1,1}$ and $f_{2,1,2}$ are also function of time, but over $\mathbb{C}$, such that

$$\underbrace{m\ddot{y} + f_{2,1,1}\dot{y}}_{\pi_{2,2}} + \underbrace{f_{2,1,2}\dot{y} + ky}_{\pi_{2,1}} = f. \tag{54}$$

Multiplying the ODE by integrating factor $\mu_2(t)$ results in

$$\underbrace{\mu_2 m}_{p_{2,2}}\ddot{y} + \underbrace{\mu_2 f_{2,1,1}}_{\dot{p}_{2,2}}\dot{y} + \underbrace{\mu_2 f_{2,1,2}}_{p_{2,1}}\dot{y} + \underbrace{\mu_2 k}_{\dot{p}_{2,1}} y = \mu_2 f, \tag{55}$$

or

$$p_{2,2}\ddot{y} + \dot{p}_{2,2}\dot{y} + p_{2,1}\dot{y} + \dot{p}_{2,1}y = \mu_2 f. \tag{56}$$

Analysing the term

$$\dot{p}_{2,2} = \mu_2 f_{2,1,1} = \dot{(\mu_2 m)} = \dot{\mu}_2 m + \mu_2 \dot{m}, \tag{57}$$

it is possible to state that

$$\frac{\dot{\mu}_2}{\mu_2} = \frac{f_{2,1,1} - \dot{m}}{m}. \tag{58}$$

Following the same procedure,

$$\dot{p}_{2,1} = \mu_2 k = \dot{\left(\mu_2 f_{2,1,2}\right)} = \dot{\mu}_2 f_{2,1,2} + \mu_2 \dot{f}_{2,1,2} \tag{59}$$

such that

$$\frac{\dot{\mu}_2}{\mu_2} = \frac{k - f_{2,1,2}}{f_{2,1,2}}. \tag{60}$$

Thus, by relating Eqs. 58 and 60, we obtain the particular form of Eq. 16

$$\frac{f_{2,1,1} - \dot{m}}{m} = \frac{k - f_{2,1,2}}{f_{2,1,2}}. \tag{61}$$

Since $c = f_{2,1,1} + f_{2,1,2}$ it is possible to state that $f_{2,1,1} = c - f_{2,1,2}$. Using Eq. 61

$$\left(c - f_{2,1,2} - \dot{m}\right) f_{2,1,2} = \left(k - \dot{f}_{2,1,2}\right) m \tag{62}$$

such that

$$f_{2,1,2}^2 = \dot{f}_{2,1,2} m - km + (c - \dot{m}) f_{2,1,2}, \tag{63}$$

is a Riccati differential equation [1].

Equation 60 is a first order ODE with known solution

$$\mu_2 = \exp \int \frac{k - \dot{f}_{2,1,2}}{f_{2,1,2}} \, \mathrm{d}t \tag{66}$$

where $f_{2,1,2}$ is obtained from Eq. 63. It is important to stress that only the particular solution of the Riccati equation is needed.

By knowing μ_2 it is possible to re-write Eq. 56 as

$$\dot{\left(p_{2,2} \dot{y}\right)} + \dot{\left(p_{2,1} y\right)} = \mu_2 f \tag{67}$$

such that integrating with respect to t results in

$$(p_{2,2} \dot{y}) + (p_{2,1} y) = \underbrace{\int \mu_2 f \, \mathrm{d}t + C_2}_{h} \tag{68}$$

[1] Conversely, it is also possible to define $f_{2,1,2} = c - f_{2,1,1}$ such that

$$\frac{f_{2,1,1} - \dot{m}}{m} = \frac{k - \dot{c} + \dot{f}_{2,1,1}}{c - f_{2,1,1}} \tag{64}$$

or

$$f_{2,1,1}^2 = -\dot{f}_{2,1,1} m + (\dot{m} + c) f_{2,1,1} - \dot{m} c + (\dot{c} - k) m, \tag{65}$$

also a Riccati differential equation.

a first order ODE. Using another integrating factor μ_1 such that

$$\underbrace{\mu_1 p_{2,2}}_{p_{1,1}} \dot{y} + \underbrace{\mu_1 p_{2,1}}_{\dot{p}_{1,1}} y = \mu_1 h \tag{69}$$

or

$$p_{1,1}\dot{y} + \dot{p}_{1,1} y = \mu_1 h. \tag{70}$$

Following the same procedure

$$\dot{p}_{1,1} = \mu_1 p_{2,1} = \dot{\left(\mu_1 p_{2,2}\right)} = \dot{\mu}_1 p_{2,2} + \mu_1 \dot{p}_{2,2}, \tag{71}$$

such that

$$\frac{\dot{\mu}_1}{\mu_1} = \frac{p_{2,1} - \dot{p}_{2,2}}{p_{2,2}}, \tag{72}$$

with known solution

$$\mu_1 = \exp \int \frac{p_{2,1} - \dot{p}_{2,2}}{p_{2,2}} \,\mathrm{d}t. \tag{73}$$

Equation 70 can be written as

$$\dot{\left(p_{1,1} y\right)} = \mu_1 h \tag{74}$$

integrating with respect to time results in

$$p_{1,1} y = \int \mu_1 h \,\mathrm{d}t + C_1 \tag{75}$$

such that

$$y = \frac{1}{p_{1,1}} \int \mu_1 h \,\mathrm{d}t + \frac{1}{p_{1,1}} C_1. \tag{76}$$

Thus, by using the definition of both $p_{1,1}$ and h

$$y(t) = \frac{1}{\mu_1(t)\mu_2(t)m(t)} \left(\int_0^t \mu_1(t) \left(\int_0^t \mu_2(t) f(t) \,\mathrm{d}t \right) \mathrm{d}t + \int_0^t \mu_1(t) C_2 \,\mathrm{d}t + C_1 \right) \tag{77}$$

is the general solution for the second order ordinary differential equation stated in Eq. 52. This general solution can be split into its particular, $y_p(t)$, and homogeneous, $y_h(t)$ parts

$$y_p(t) = \frac{1}{\mu_1(t)\mu_2(t)m(t)} \int_0^t \mu_1(t) \int_0^t \mu_2(t) f(t) \,\mathrm{d}t \,\mathrm{d}t \tag{78}$$

and

$$y_h(t) = \frac{1}{\mu_1(t)\mu_2(t)m(t)} \left(\int_0^t \mu_1(t) C_2 \,\mathrm{d}t + C_1 \right), \tag{79}$$

such that $y(t) = y_p(t) + y_h(t)$. Constants C_1 and C_2 can be found by considering the solution at known t/values.

Thus, the proposed solution depends on the solution of a Riccati equation and always results in integrating factors $\mu_1(t)$ and $\mu_2(2)$ function of t only, disregarding the form of the coefficients $m(t)$, $c(t)$ and $k(t)$.

Solution of Eq. 63 is fundamental for the success of the proposed formulation. Indeed, the solution of the Riccati equation is not an easy task if we consider general coefficients $m(t)$, $c(t)$ and $k(t)$. Nonetheless, only the particular solution is needed. Various analytical solutions can be found in the literature for specific forms of coefficients [6, 16, 10], as well as numerical methods [9].

Example I - Cauchy-Euler equation
Consider the second order ODE

$$t^2\ddot{y}(t) - 2t\dot{y}(t) + 2y(t) = 6t^2 + 4\ln(t). \tag{80}$$

The coefficients are $m(t) = t^2$, $c(t) = -2t$ and $k(t) = 2$ and the excitation is $f(t) = 6t^2 + 4\ln(t)$.

The Riccati equation, Eq. 63, can be written as

$$f_{2,1,2}^2 = t^2\dot{f}_{2,1,2} - 2t^2 - 4tf_{2,1,2}, \tag{81}$$

with particular solution $f_{2,1,2} = -t$. Using Eq. 66

$$\mu_2 = \exp\int \frac{2+1}{-t}\,\mathrm{d}t = \frac{1}{t^3}. \tag{82}$$

Thus, $p_{2,2} = \mu_2 m = t^{-1}$ and $p_{2,1} = \mu_2 f_{1,2} = -t^{-2}$. Using Eq. 73

$$\mu_1 = \exp\int \frac{-t^{-1}+t^{-2}}{t^{-1}}\,\mathrm{d}t = 1. \tag{83}$$

Complete solution is given by Eq. 77

$$y(t) = \frac{1}{1t^{-3}t^2}\left(\int_0^t 1\left\{\int_0^t t^{-3}\left(6t^2+4\ln(t)\right)\mathrm{d}t\right\}\mathrm{d}t + \int_0^t 1C_2\,\mathrm{d}t + C_1\right) \tag{84}$$

such that

$$y(t) = (6t^2+2)\ln(t) - 6t^2 + 3 + C_2t^2 + C_1t, \tag{85}$$

the analytical solution.

Example II - Bessel equation
Consider EDO

$$t^2\ddot{y}(t) + t\dot{y}(t) + \left(t^2 - \frac{1}{4}\right)y(t) = f(t) = t^{\frac{3}{2}}, \tag{86}$$

with known values $y(t_0) = \dot{y}(t_0) = 0$ at $t_0 = 0.1$. It is known that solution to this equation is singular at $t = 0$.

The corresponding Riccati equation, Eq. 63, is

$$f_{2,1,2}^2 = t^2\dot{f}_{2,1,2} - tf_{2,1,2} + \frac{t^2}{4} - t^4, \tag{87}$$

whose candidate solution is a second order polynomial

$$\tilde{f}_{2,1,2} = z_0 + z_1 t + z_2 t^2. \tag{88}$$

Applying this polynomial into Eq. 87 yields

$$z_0^2+2z_0z_1t+\left(2z_0z_2+z_1^2\right)t^2+2z_1z_2t^3+z_2^2t^4 = z_1t^2+2z_2t^3-z_0t-z_1t^2-z_2t^3+\frac{t^2}{4}-t^4, \tag{89}$$

that simplifies to

$$z_0^2+2z_0z_1t+\left(2z_0z_2+z_1^2\right)t^2+2z_1z_2t^3+z_2^2t^4 = -z_0t+\frac{t^2}{4}+z_2t^3-t^4, \tag{90}$$

whose solution is $z_0 = 0$, $z_1 = \frac{1}{2}$ and $z_2 = i$.

Thus, the integrating factor, Eq. 66, is

$$\mu_2 = \exp\left(\int \frac{t^2-\frac{1}{4}-\frac{1}{2}-2it}{\frac{t}{2}+it^2}\,\mathrm{d}t\right) = \exp\left(\int \frac{t^2-2it-\frac{3}{4}}{\frac{t}{2}+it^2}\,\mathrm{d}t\right). \tag{91}$$

The polynomials in the integrand can be factored and simplified to

$$\mu_2 = \exp\left(\int \frac{\left(t-\frac{i}{2}\right)\left(t-\frac{3i}{2}\right)}{it\left(t-\frac{i}{2}\right)}\,\mathrm{d}t\right) = \exp\left(\int -i\,\mathrm{d}t - \frac{3}{2}\int \frac{1}{t}\,\mathrm{d}t\right), \tag{92}$$

such that

$$\mu_2 = \exp\left(-it - \frac{3}{2}ln\left|t\right|\right) = t^{-\frac{3}{2}}\exp\left(-it\right). \tag{93}$$

The integrating factor to integrate the differential equation from first order to an algebraic equation is evaluated using Eq. 73,

$$\mu_1 = t^{\frac{1}{2}}\exp\left(it\right). \tag{94}$$

Solution is then given by Eq. 77

$$\begin{aligned} y(t) = t^{-\frac{1}{2}}\exp\left(-it\right)\int \exp\left(2it\right)\int t^{-\frac{3}{2}}\exp\left(-t\right)f(t)\,\mathrm{d}t\,\mathrm{d}t + \\ C_1t^{-\frac{1}{2}}\exp\left(it\right)+C_2t^{-\frac{1}{2}}\exp\left(-it\right), \end{aligned} \tag{95}$$

and replacing the expression for $f(t)$

$$y(t) = t^{-\frac{1}{2}} + C_1t^{-\frac{1}{2}}\exp\left(it\right)+C_2t^{-\frac{1}{2}}\exp\left(-it\right). \tag{96}$$

Constants, C_1 and C_2, can be obtained by solving a system of linear equations

$$\begin{bmatrix} t_0^{-\frac{1}{2}} \exp\left(it_0\right) & t_0^{-\frac{1}{2}} \exp\left(-it_0\right) \\ \left(it_0^{-\frac{1}{2}} - \frac{t_0^{-\frac{3}{2}}}{2}\right) \exp\left(it_0\right) & -\left(it_0^{-\frac{1}{2}} + \frac{t_0^{-\frac{3}{2}}}{2}\right) \exp\left(-it_0\right) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} y\left(t_0\right) - t_0^{-\frac{1}{2}} \\ \dot{y}\left(t_0\right) + \frac{t_0^{-\frac{3}{2}}}{2} \end{Bmatrix}. \quad (97)$$

Using values at $t_0 = 0.1$ results in

$$y(t) = t^{-\frac{1}{2}} + (-0.4975 + 0.04992i)\, t^{-\frac{1}{2}} \exp\left(it\right) - (0.4975 + 0.04992i)\, t^{-\frac{1}{2}} \exp\left(-it\right). \quad (98)$$

Figure 1 compares the real part of the solution obtained by the proposed approach, Eq. 95, with the solution obtained by using the Tsitouras 5/4 Runge-Kutta method [18] with adaptative time step. The numerical solution starts at $t = 0.1$ (dotted line) but the analytical solution captures the singularity at $t \to 0$ (blue curve).

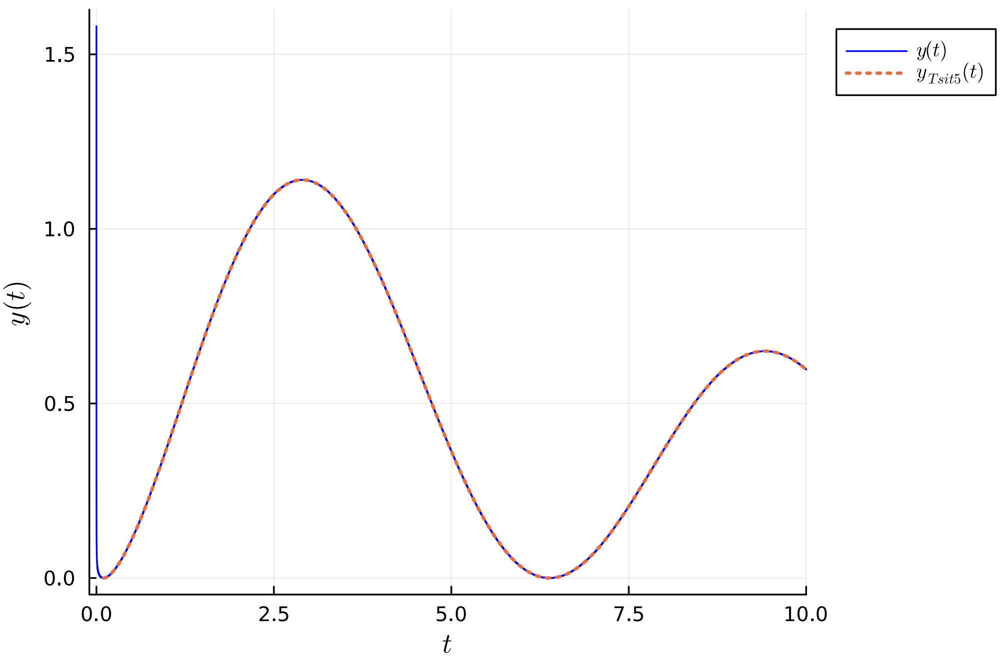

Figure 1: Analytical response $y(t)$ obtained for the Bessel equation (Eq. 86) and the solution obtained by using a numerical method, $y_{Tsit5}(t)$.

3.1. Constant coefficients

Previous equations are much simpler when m, c and k are constant since $\dot{m}$, $\dot{c}$, $\dot{f}_{2,1,2}$ and $\dot{k}$ are zero. As $m > 0$, it is possible to normalize the equation by m such that

$$\ddot{y}(t) + \bar{c}\dot{y}(t) + \bar{k}y(t) = \bar{f}(t), \quad (99)$$

where $\bar{c} = c/m$, $\bar{k} = k/m$ and $\bar{f} = f/m$. It follows that Eq. 63 reduces to an algebraic quadratic equation

$$f_{2,1,2}^2 = -\bar{k} + \bar{c} f_{2,1,2} \tag{100}$$

with **direct solution**

$$f_{2,1,2} = \frac{\bar{c} \pm \sqrt{\bar{c}^2 - 4\bar{k}}}{2}, \tag{101}$$

a complex number when $\bar{c}^2 - 4\bar{k} < 0$ (under damped problems). Any one of the two roots can be used. Equation 66 reduces to

$$\mu_2 = \exp\left(\hat{k}t\right) \tag{102}$$

where

$$\hat{k} = \frac{\bar{k}}{f_{2,1,2}} \tag{103}$$

and Eq. 73 to

$$\mu_1 = \frac{\exp\left(f_{2,1,2}t\right)}{\exp\left(\hat{k}t\right)} = \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right). \tag{104}$$

Thus, the complete solution is

$$y(t) = \frac{1}{\mu_1(t)\mu_2(t)}\left(\int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right)\left(\int_0^t \exp\left(\hat{k}t\right)\bar{f}(t)\,\mathrm{d}t + C_2\right)\mathrm{d}t + C_1\right) \tag{105}$$

where the term

$$\frac{1}{\mu_1(t)\mu_2(t)} = \exp\left(-f_{2,1,2}t\right) \tag{106}$$

such that

$$y(t) = \exp\left(-f_{2,1,2}t\right)\left(\int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right)\left(\int_0^t \exp\left(\hat{k}t\right)\bar{f}(t)\,\mathrm{d}t + C_2\right)\mathrm{d}t + C_1\right), \tag{107}$$

is the complete solution. Additionally, it is possible to split the solution in its particular

$$y_p(t) = \exp\left(-f_{2,1,2}t\right)\left(\int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right)\left(\int_0^t \exp\left(\hat{k}t\right)\bar{f}(t)\,\mathrm{d}t\right)\mathrm{d}t\right), \tag{108}$$

and homogeneous parts

$$y_h(t) = \exp\left(-f_{2,1,2}t\right)\left(\int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right)t\right)C_2\,\mathrm{d}t + C_1\right), \tag{109}$$

which can be further simplified to

$$y_h(t) = \exp\left(-f_{2,1,2}t\right) C_1 + C_2 \exp\left(-\hat{k}t\right). \tag{110}$$

Particular solution for the constant coefficient case, Eq. 108, can be further developed if we assume some particular form for $\bar{f}(t)$. The following subsections are devoted to investigate two situations: Continuous excitation (periodic and polynomial) and discontinuous (unitary impulse and Heaviside).

3.2. Continuous excitation functions

3.2.1. Periodic excitation

Lets consider a further hypothesis: excitation $f(t)$ is periodic or its Fourier series is convergent. Thus, normalized excitation $\bar{f}(t)$ can be represented as a series of exponentials

$$\bar{f}(t) = \sum_{k=1}^{n_k} c_k \exp\left(\beta_k t + \phi_k\right), \tag{111}$$

where n_k is the number of terms, $c_k \in \mathbb{R}$ is an amplitude, $\beta_k = i\omega_k \in \mathbb{C}$ is a complex angular frequency and $\phi_k \in \mathbb{C}$ is a complex phase.

Previously, it was seen that the integrating factor is an exponential by definition, and that the particular solution, *i.e.*, the solution corresponding to the excitation function, naturally appears through the successive integrations. These integrations are convolutions over the integrating factor, thus, if the excitation can be expressed in terms of exponentials, these convolutions can be trivially calculated. Applying Eq. 111 into Eq. 108 yields

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int_0^t \exp\left(\hat{k}t\right) \sum_{k=1}^{n_k} c_k \exp\left(\beta_k t + \phi_k\right) \mathrm{d}t\right) \mathrm{d}t, \tag{112}$$

using the multiplication property between exponentials

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int_0^t \sum_{k=1}^{n_k} c_k \exp\left(\left(\beta_k + \hat{k}\right) t + \phi_k\right) \mathrm{d}t\right) \mathrm{d}t, \tag{113}$$

whose inner integral is trivial and results in

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \sum_{k=1}^{n_k} \frac{c_k}{\beta_k + \hat{k}} \exp\left(\left(\beta_k + \hat{k}\right) t + \phi_k\right) \mathrm{d}t. \tag{114}$$

Again, rearranging the multiplication of exponentials, yields

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int_0^t \sum_{k=1}^{n_k} \frac{c_k}{\beta_k + \hat{k}} \exp\left(\left(\beta_k + f_{2,1,2}\right) t + \phi_k\right) \mathrm{d}t, \tag{115}$$

which is again trivially integrated to

$$y_p(t) = \exp\left(-f_{2,1,2}t\right)\sum_{k=1}^{n_k}\frac{c_k}{\left(\beta_k+\hat{k}\right)\left(\beta_k+f_{2,1,2}\right)}\exp\left(\left(\beta_k+f_{2,1,2}\right)t+\phi_k\right), \tag{116}$$

such that

$$y_p(t) = \sum_{k=1}^{n_k}\frac{c_k\exp\left(\beta_k t+\phi_k\right)}{\left(\beta_k+\hat{k}\right)\left(\beta_k+f_{2,1,2}\right)}. \tag{117}$$

Equation 117 gives a closed-form and analytic particular solution for any periodic excitation or an excitation represented through its Fourier series expansion. It is worth noticing that all constants given by the two integrations at time $t_0 = 0$ were omitted for they can be coupled to the constants C_1 and C_2, which are by definition constants of these very integrations.

Example with periodic excitation

Consider the ODE

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = \bar{f}(t), \tag{118}$$

with initial conditions $y\,(0) = 0.2$ and $\dot{y}\,(0) = 0.0$, where

$$\bar{f}(t) = -cos\,(0.5t) + sin\,(t) + cos\,(1.5t-1.5) - 2sin\,(2t) + 2sin\,(10t)\,. \tag{119}$$

For this example, one gets

$$\bar{k} = 10, \tag{120}$$

$$f_{2,1,2} = 1+3i, \tag{121}$$

$$\hat{k} = 1-3i, \tag{122}$$

By using well known relations

$$sin\left(\omega_\eta t+\phi_\eta\right) = i\frac{e^{-i\left(\omega_\eta t+\phi_\eta\right)} - e^{i\left(\omega_\eta t+\phi_\eta\right)}}{2} \tag{123}$$

and

$$cos\left(\omega_\eta t+\phi_\eta\right) = \frac{e^{-i\left(\omega_\eta t+\phi_\eta\right)} + e^{i\left(\omega_\eta t+\phi_\eta\right)}}{2}, \tag{124}$$

one gets the excitation in a complex form as

$$\begin{aligned}\bar{f} = -\frac{1}{2}e^{0.5it} - \frac{1}{2}e^{-0.5it} - \frac{i}{2}e^{it} + \frac{i}{2}e^{-it} + \frac{1}{2}e^{1.5i(t-1)} + \frac{1}{2}e^{1.5i(1-t)} + \\ ie^{2it} - ie^{-2it} - ie^{10it} + ie^{-10it} = \sum_{j=1}^{10}c_j e^{\beta_j t+\phi_j}.\end{aligned} \tag{125}$$

Looking at Eq. 125 and the series explicit in the end, an algorithmic approach may be derived by writing these coefficients and powers in a tabular manner, as in Tab. 1. Using such matrix representation, all the parameters needed for each operation of the sum can be summarized in each column. Thus, the summation can be carried out column by column, as the row with the index j in Tab. 1 hints.

Table 1: Tabular representation for the excitation.

j	1	2	3	4	5	6	7	8	9	10
c	-0.5	-0.5	$-0.5i$	$0.5i$	0.5	0.5	i	$-i$	$-i$	i
β	$0.5i$	$-0.5i$	i	$-i$	$1.5i$	$-1.5i$	$2i$	$-2i$	$10i$	$-10i$
ϕ	0.0	0.0	0.0	0.0	$-1.5i$	$1.5i$	0.0	0.0	0.0	0.0

Following Tab. 1 to evaluate Eq. 117 yields

$$\begin{aligned} y_p = \frac{-78+8i}{1537}e^{0.5it} - \frac{78+8i}{1537}e^{-0.5it} - \frac{2+9i}{170}e^{it} + \frac{-2+9i}{170}e^{it} + \\ \frac{62-24i}{1105}e^{1.5i(t-1)} + \frac{62+24i}{1105}e^{1.5i(1-t)} + \frac{2+3i}{26}e^{2it} + \\ \frac{2-3i}{26}e^{-2it} + \frac{-2+9i}{850}e^{10it} - \frac{2+9i}{850}e^{-10it}, \end{aligned} \tag{126}$$

and for Eq. 110

$$y_h = C_2 e^{-\frac{\bar{k}}{f_{1,2}}t} + C_1 e^{-f_{1,2}t}, \tag{127}$$

or

$$y_h = C_2 e^{(-1+3i)t} + C_1 e^{-(1+3i)t}. \tag{128}$$

The homogeneous solution, y_h, is straightforward. Using Eq. 126 and Eq. 128 at $t = 0$, the integration constants came out as complex conjugates, $C_2 \approx 0.10564 - 0.11980i$ and $C_1 \approx 0.10564 + 0.11980i$.

Figure 4 shows the real part of the solution for both the homogeneous and the particular solutions (top) as well as the complete solution (bottom). The solution was obtained for a time span of $40s$ with intervals of $0.1s$. It is worth mention that although $y(t)$ is complex, the maximum amplitude of the complex part of the response was 8.3267×10^{-17} in the analysed time interval, which is zero when compared to the real part.

Since it is known that the response to this problem is real it is worth showing that this is indeed just a matter of representation. To this end, Eqs. 126 and 128 can be transformed to a real-valued form by applying Eq. 123 and Eq. 124 in reverse,

$$\begin{aligned} y_p = -\frac{156}{1537}cos\,(0.5t) - \frac{16}{1537}sin\,(0.5t) - \frac{2}{85}cos\,(t) + \frac{9}{85}sin\,(t) + \\ \frac{124}{1105}cos\,\big(1.5\,(t-1)\big) + \frac{48}{1105}sin\,\big(1.5\,(t-1)\big) + \frac{2}{13}cos\,(2t) - \\ \frac{3}{13}sin\,(2t) - \frac{2}{425}cos\,(10t) - \frac{9}{425}sin\,(10t)\,, \end{aligned} \tag{129}$$

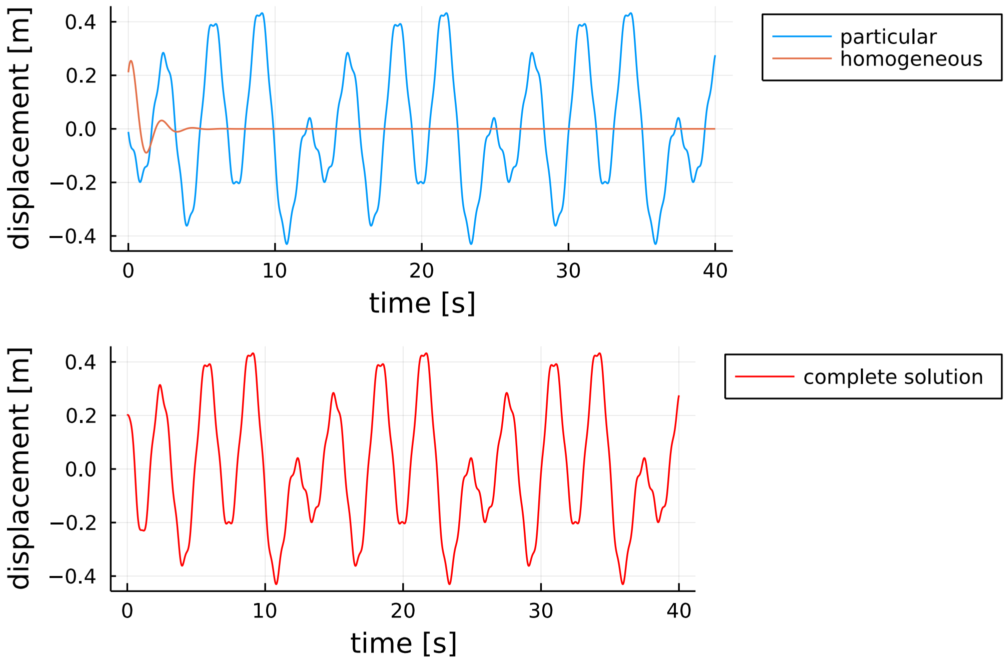

Figure 2: Particular and homogeneous solution (top) and complete solution (bottom).

and

$$y_h = C_2 e^{-t} cos\left(3t\right) + C_1 e^{-t} sin\left(3t\right), \tag{130}$$

with $C_2 \approx 0.2107$ and $C_1 \approx 0.0702$. Complete solution $y(t) = y_p(t) + y_h(t)$ is correct and satisfies Eq. 118 for all t.

3.2.2. Polynomial excitation

Assuming a polynomial excitation in the form

$$\bar{f}(t) = \sum_{k=0}^{n_p} c_k \left(t - t_s\right)^k, \tag{131}$$

where $c_k \in \mathbb{R}$ are coefficients, n_p is the number of terms and $t_s \in \mathbb{R}$ a time shift. Applying Eq. 131 into Eq. 108 yields

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int_0^t \exp\left(\hat{k}t\right) \sum_{k=0}^{n_p} c_k \left(t - t_s\right)^k \mathrm{d}t\right) \mathrm{d}t, \tag{132}$$

as integration is a linear operator, the summation can be transferred to the whole integration and the coefficient of each power of t can also be put out of the integral,

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int_0^t \exp\left(\hat{k}t\right) (t - t_s)^k \,\mathrm{d}t\right) \mathrm{d}t. \tag{133}$$

The convolution of a polynomial over an exponential is recursively evaluated using integration by parts. For it, let the power of t be a positive integer α. The first integration by parts is

$$\int_0^t \exp(\beta t) (t - t_s)^\alpha \,\mathrm{d}t = \int_0^t \left(\frac{1}{\beta} \exp(\beta t)\right)^{\cdot} (t - t_s)^\alpha \,\mathrm{d}t = \\ \frac{1}{\beta} \exp(\beta t) (t - t_s)^\alpha \Big|_0^t - \frac{1}{\beta} \alpha \int \exp(\beta t) (t - t_s)^{\alpha - 1} \,\mathrm{d}t, \tag{134}$$

such that another convolution with a smaller power appeares in the RHS of Eq. 134. This procedure can be used recursively until the null power, where the integral is over the exponential only,

$$\int_0^t \exp(\beta t) (t - t_s)^\alpha \,\mathrm{d}t = \frac{1}{\beta} (t - t_s)^\alpha \exp(\beta t) \Big|_0^t \\ - \left(\frac{1}{\beta}\right)^2 \alpha (t - t_s)^{\alpha - 1} \exp(\beta t) \Big|_0^t + \left(\frac{1}{\beta}\right)^3 \alpha(\alpha - 1) (t - t_s)^{\alpha - 2} \exp(\beta t) \Big|_0^t \\ - \left(\frac{1}{\beta}\right)^4 \alpha(\alpha - 1)(\alpha - 2) (t - t_s)^{\alpha - 3} \exp(\beta t) \Big|_0^t \ldots \\ + (-1)^\alpha \left(\frac{1}{\beta}\right)^\alpha \alpha! \int \exp(\beta t) (t - t_s)^{\alpha - \alpha} \,\mathrm{d}t. \tag{135}$$

Rarranging the terms,

$$\int_0^t \exp(\beta t) (t - t_s)^\alpha \,\mathrm{d}t = \\ \sum_{l=1}^{\alpha} (-1)^{l+1} \left(\frac{1}{\beta}\right)^l \frac{\alpha!}{(\alpha - l + 1)!} (t - t_s)^{\alpha - l + 1} \exp(\beta t) \Big|_0^t \\ + (-1)^\alpha \left(\frac{1}{\beta}\right)^\alpha \alpha! \; \exp(\beta t) \left(\frac{1}{\beta}\right), \tag{136}$$

which can be further simplified to

$$\int_0^t \exp(\beta t)(t-t_s)^\alpha \,\mathrm{d}t = \\ \sum_{l=1}^{\alpha+1} (-1)^{l+1} \left(\frac{1}{\beta}\right)^l \frac{\alpha!}{(\alpha-l+1)!} (t-t_s)^{\alpha-l+1} \exp(\beta t)\Bigg|_0^t . \tag{137}$$

Applying this result to Eq. 133 and neglecting the evaluation of this integral at $t=0$, since it can be summed with constant C_1, yields

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \int_0^t \exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right) \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \\ \frac{k!}{(k-l+1)!} (t-t_s)^{k-l+1} \exp\left(\hat{k}t\right) \mathrm{d}t. \tag{138}$$

Rearranging terms, using the linearity of the integration operator and defining $r = k-l+1$

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{r!} \int_0^t \exp\left(f_{2,1,2}t\right)(t-t_s)^r \,\mathrm{d}t. \tag{139}$$

Using the result from Eq. 137,

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \sum_{k=0}^{n_p} c_k \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{(r)!} \\ \sum_{p=1}^{r+1} (-1)^{p+1} f_{2,1,2}^{-p} \frac{r!}{(r+1-p)!} (t-t_s)^{r+1-p} \exp\left(f_{2,1,2}t\right), \tag{140}$$

that can be further simplified to

$$y_p(t) = \sum_{k=0}^{n_p} c_k \sum_{l=1}^{k+1} (-1)^{l+1} \left(\hat{k}\right)^{-l} \frac{k!}{r!} \sum_{p=1}^{r+1} (-1)^{p+1} f_{2,1,2}^{-p} \frac{r!}{(r+1-p)!} (t-t_s)^{r+1-p}, \tag{141}$$

which is itself another polynomial.

Example with polynomial excitation

A common example of excitation in electric circuitry analysis is the ramp function, a polynomial of first degree. Consider the ode subjected to

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = \bar{f}(t) = t, \tag{142}$$

with $\dot{y}(0) = y(0) = 0$, $t \in [0,4]$s.

Steps used to solve Eq. 141, $y_p(t)$, are depicted in an algorithm form in Alg. 1, resulting in $y_p(t) = 0.1t - 0.02$. The homogeneous solution is given by Eq. 110, that yields a system of equations for its integration constants, C_1 and C_2,

$$\begin{bmatrix} \exp(0) & \exp(0) \\ -f_{2,1,2}\exp(0) & -\hat{k}\exp(0) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} y(0) - y_p(0) \\ \dot{y}(0) - \dot{y}_p(0) \end{Bmatrix}, \tag{143}$$

which simplifies to

$$\begin{bmatrix} 1 & 1 \\ -1-3i & -1+3i \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0.02 \\ -0.1 \end{Bmatrix}, \tag{144}$$

whose solution is $C_1 = 0.0100 - 0.0133i$ and $C_2 = 0.0100 + 0.0133i$. Thus, the complete solution is

$$\begin{aligned} y(t) = 0.1t - 0.02 + (0.0100 - 0.0133i)\exp\big(-(1+3i)\,t\big) + \\ (0.0100 + 0.0133i)\exp\big(-(1-3i)\,t\big) \end{aligned} \tag{145}$$

Algorithm 1: Evaluation of Eq. 141

$k = 0$
  $c_0 = 0$
$k = 1$
  $c_1 = 1$
  $l = 1 \implies (-1)^{l+1}\left(\hat{k}\right)^{-l}\frac{k!}{r!} = 0.1 + 0.3i$
    $p = 1 \implies (-1)^{p+1} f_{2,1,2}^{-p}\frac{r!}{(r+1-p)!}(t-t_s)^{r+1-p} = (0.1 - 0.3i)\,t$
    $p = 2 \implies (-1)^{p+1} f_{2,1,2}^{-p}\frac{r!}{(r+1-p)!}(t-t_s)^{r+1-p} = 0.08 + 0.06i$
  $l = 2 \implies (-1)^{l+1}\left(\hat{k}\right)^{-l}\frac{k!}{r!} = 0.08 - 0.06i$
    $p = 1 \implies (-1)^{p+1} f_{2,1,2}^{-p}\frac{r!}{(r+1-p)!}(t-t_s)^{r+1-p} = 0.1 - 0.3i$

$y_p = \big(1\,(0.1+0.3i)\,(0.1-0.3i)\,t\big) + \big(1\,(0.1+0.3i)\,(0.08+0.06i)\big) +$
$\big(1\,(0.08-0.06i)\,(0.1-0.3i)\big)$
$y_p = 0.1t - 0.02$

The complete solution, Eq. 145, is then compared to the numerical solution obtained by using the Tsitouras 5/4 Runge-Kutta method [18], Fig. 3.2.2. Both solutions converge to the same result.

3.3. Discontinuous excitation functions

3.3.1. Particular solution due to unitary impulses - Dirac's deltas

Consider that the normalized excitation $\bar{f}(t)$ is given by n_δ Dirac's deltas at times t_k

$$\bar{f}(t) = \sum_{k=1}^{n_\delta} c_k \delta(t - t_k), \tag{146}$$

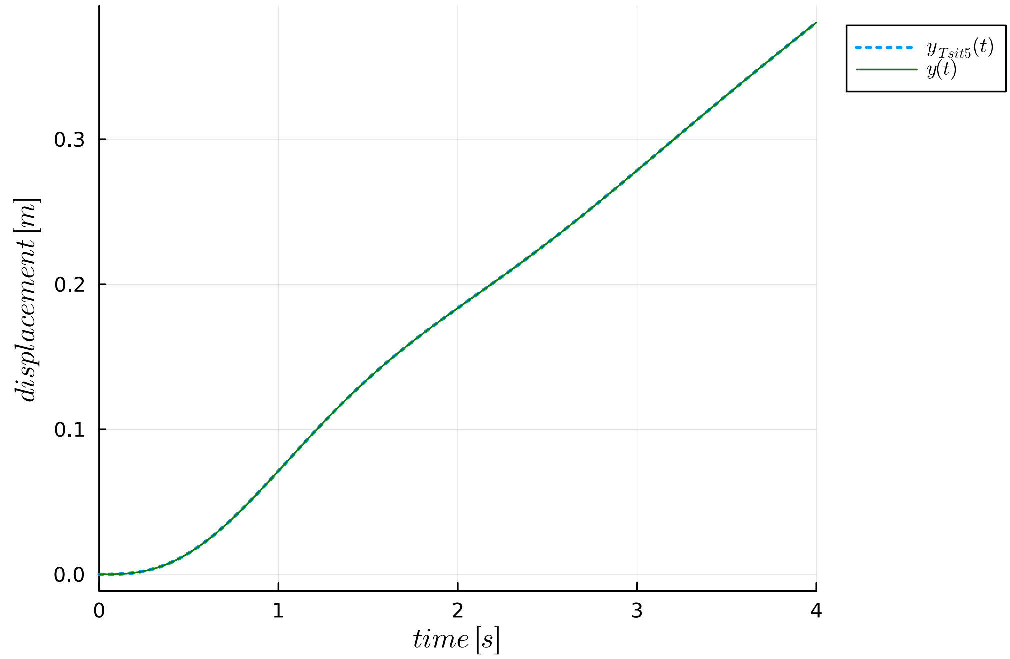

Figure 3: Comparison between ramp solutions using Eq. 145 (green) and using the numerical method [18] (blue).

with coefficients $c_k \in \mathbb{R}$. Particular solution given by Eq. 108 can be written as

$$y_p(t) = \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int_0^t \exp\left(\hat{k}t\right) \sum_{k=1}^{n_\delta} c_k \delta(t - t_k)\, \mathrm{d}t\right) \mathrm{d}t, \tag{147}$$

or, using the linearity of the integral

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\int_0^t \exp\left(\hat{k}t\right) \delta(t - t_k)\, \mathrm{d}t\right) \mathrm{d}t. \tag{148}$$

The inner integral can be found by using the Filtering property of the Dirac's delta (Appendix, Eq. 214), such that

$$\int_0^t \exp\left(\hat{k}t\right) \delta(t - t_k)\, \mathrm{d}t = \exp\left(\hat{k}t_k\right) \mathcal{H}(t - t_k) \tag{149}$$

where $\mathcal{H}(t - t_k)$ is the Heaviside function at t_k. Thus,

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(-f_{2,1,2}t\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \exp\left(\hat{k}t_k\right) \mathcal{H}(t - t_k)\, \mathrm{d}t, \tag{150}$$

or

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(-f_{2,1,2}t\right) \exp\left(\hat{k}t_k\right) \int_0^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \mathcal{H}(t - t_k)\,\mathrm{d}t. \tag{151}$$

This convolution over a function multiplied by the Heaviside function can be evaluated splitting the integration domain according to the step function, thus, resulting in a change of integration limits,

$$\int_0^t \exp\left(\beta t\right) f_k(t)\mathcal{H}(t-t_k)\,\mathrm{d}t = \underbrace{\int_0^{t_k} \exp\left(\beta t\right) f_k(t)\mathcal{H}(t-t_k)\,\mathrm{d}t}_{=0} + \int_{t_k}^t \exp\left(\beta t\right) f_k(t)\mathcal{H}(t-t_k)\,\mathrm{d}t, \tag{152}$$

where β is a generic exponent, such that

$$\int_0^t \exp\left(\beta t\right) f_k(t)\mathcal{H}(t-t_k)\,\mathrm{d}t = \left(\int_{t_k}^t \exp\left(\beta t\right) f_k(t)\,\mathrm{d}t\right) \mathcal{H}(t-t_k). \tag{153}$$

Thus, Eq. 151 can be written as

$$y_p(t) = \sum_{k=1}^{n_\delta} c_k \exp\left(-f_{2,1,2}t\right) \exp\left(\hat{k}t_k\right) \int_{t_k}^t \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \mathrm{d}t\ \mathcal{H}(t - t_k), \tag{154}$$

or

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{f_{2,1,2} - \hat{k}} \left(\exp\left(\hat{k}(t_k - t)\right) - \exp\left(f_{2,1,2}(t_k - t)\right)\right) \mathcal{H}(t - t_k). \tag{155}$$

Additionally, for under damped problems, $f_{2,1,2}$ is complex and $\hat{k} = \frac{\bar{k}}{f_{2,1,2}} = f^*_{2,1,2}$ (where $*$ stands for complex-conjugate). Thus

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{2i\Im(f_{2,1,2})} \left(e^{f^*_{2,1,2}(t_k-t)} - e^{f_{2,1,2}(t_k-t)}\right) \mathcal{H}(t - t_k), \tag{156}$$

or

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{2i\Im(f_{2,1,2})} e^{\Re\left(f_{2,1,2}\right)(t_k-t)} \left(e^{-i\Im\left(f_{2,1,2}\right)(t_k-t)} - e^{i\Im\left(f_{2,1,2}\right)(t_k-t)}\right) \mathcal{H}(t - t_k) \tag{157}$$

and user Euler's identity for sin

$$y_p(t) = \sum_{k=1}^{n_\delta} \frac{c_k}{2\Im(f_{2,1,2})} e^{\Re\left(f_{2,1,2}\right)(t_k-t)} \sin\left(\Im(f_{2,1,2})(t - t_k)\right) \mathcal{H}(t - t_k). \tag{158}$$

Example with unitary impulse
Consider the problem

$$2\ddot{y}(t)+\dot{y}(t)+2y(t)=\delta(t-5), \tag{159}$$

with $y(0)=\dot{y}(0)=0$. The analytical solution is [5]

$$y_p(t)=\begin{cases} 0 & t<5 \\ \frac{2}{\sqrt{15}}e^{\frac{5-t}{4}}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right) & t\geq 5 \end{cases}. \tag{160}$$

From the data, $\bar{c}=1/2$, $\bar{k}=1$, $n_\delta=1$ and $t_1=5$. Constant $f_{2,1,2}$ can be found by solving Eq. 101 such that $f_{2,1,2}=0.25+0.96824i$. Equation 155 reduces to

$$y_p(t)=\frac{1}{2(0.96824)i}\left(e^{(0.25-0.9682i)(5-t)}-e^{(0.25+0.96824i)(5-t)}\right)\mathcal{H}(t-5) \tag{161}$$

and, although complex, has negligible complex values, matching the analytical solution. Alternatively, using Eq. 158

$$y_p(t)=\frac{1}{2(0.96824)}e^{0.25(5-t)}\sin\left(0.96824(t-5)\right)\mathcal{H}(t-5) \tag{162}$$

which is also identical to the analytical solution provided by [5].

3.3.2. Particular solution due to step functions - Heaviside

Let the excitation be defined as a sum of functions $f_k(t)$ multiplied by a step functions,

$$\bar{f}(t)=\sum_{k=1}^{n_k}f_k(t)\mathcal{H}(t-t_k). \tag{163}$$

Inserting Eq. 163 into Eq. 108 yields

$$y_p(t)=\exp\left(-f_{2,1,2}t\right)\int_0^t\exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\int_0^t\exp\left(\hat{k}t\right)\sum_{k=1}^{n_k}f_k(t)\mathcal{H}(t-t_k)\,\mathrm{d}t\,\mathrm{d}t, \tag{164}$$

as integration is a linear operator,

$$y_p(t)=\sum_{k=1}^{n_k}\exp\left(-f_{2,1,2}t\right)\int_0^t\exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\int_0^t\exp\left(\hat{k}t\right)f_k(t)\mathcal{H}(t-t_k)\,\mathrm{d}t\,\mathrm{d}t. \tag{165}$$

Using Eq. 153 in Eq. 165 for the inner integral results in

$$y_p(t)=\sum_{k=1}^{n_k}\exp\left(-f_{2,1,2}t\right)\int_0^t\exp\left(\left(f_{2,1,2}-\hat{k}\right)t\right)\int_{t_k}^t\exp\left(\hat{k}t\right)f_k(t)\,\mathrm{d}t\,\mathcal{H}(t-t_k)\,\mathrm{d}t, \tag{166}$$

and for the outer integral

$$y_p(t) = \sum_{k=1}^{n_k} \exp\left(-f_{2,1,2}t\right) \int_{t_k}^{t} \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \int_{t_k}^{t} \exp\left(\hat{k}t\right) f_k(t)\,\mathrm{d}t\,\mathrm{d}t\,\mathcal{H}(t - t_k). \tag{167}$$

Example with Heaviside function
Consider the problem from [5]

$$\ddot{y}(t) + 4y(t) = \bar{f}(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{t-5}{5}, & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$
$$\dot{y}(0) = y(0) = 0, \tag{168}$$

such that the excitation function can be written as

$$\bar{f}(t) = \frac{t-5}{5}\mathcal{H}(t-5) - \frac{t-10}{5}\mathcal{H}(t-10). \tag{169}$$

Using Eq. 101

$$f_{2,1,2} = \frac{0 + \sqrt{0-16}}{2} = 2i \tag{170}$$

and Eq. 167

$$\begin{aligned} y_p(t) = e^{-2it} \Bigg(\int_t^t e^{4it} \int_5^t e^{-2it} \left(\frac{t-5}{5}\right) \mathrm{d}t\,\mathrm{d}t\mathcal{H}(t-5) + \\ \int_t^t e^{4it} \int_5^t e^{-2it} \left(\frac{10-t}{5}\right) \mathrm{d}t\,\mathrm{d}t\mathcal{H}(t-10) \Bigg) \end{aligned} \tag{171}$$

such that

$$\begin{aligned} y_p(t) = &\frac{\left(ie^{4it} + \left(4e^{10i}t - 20e^{10i}\right)e^{2it} - ie^{20i}\right)e^{-2it-10i}}{80}\mathcal{H}(t-5) + \\ &-\frac{\left(ie^{4it} + \left(4e^{20i}t - 40e^{20i}\right)e^{2it} - ie^{40i}\right)e^{-2it-20i}}{80}\mathcal{H}(t-10). \end{aligned} \tag{172}$$

Figure 4 shows the real part of the solution given by Eq. 172 (the maximum complex part has a magnitude of 1×10^{-18} when Eq. 172 was evaluated). This solution matches the result given by [5].

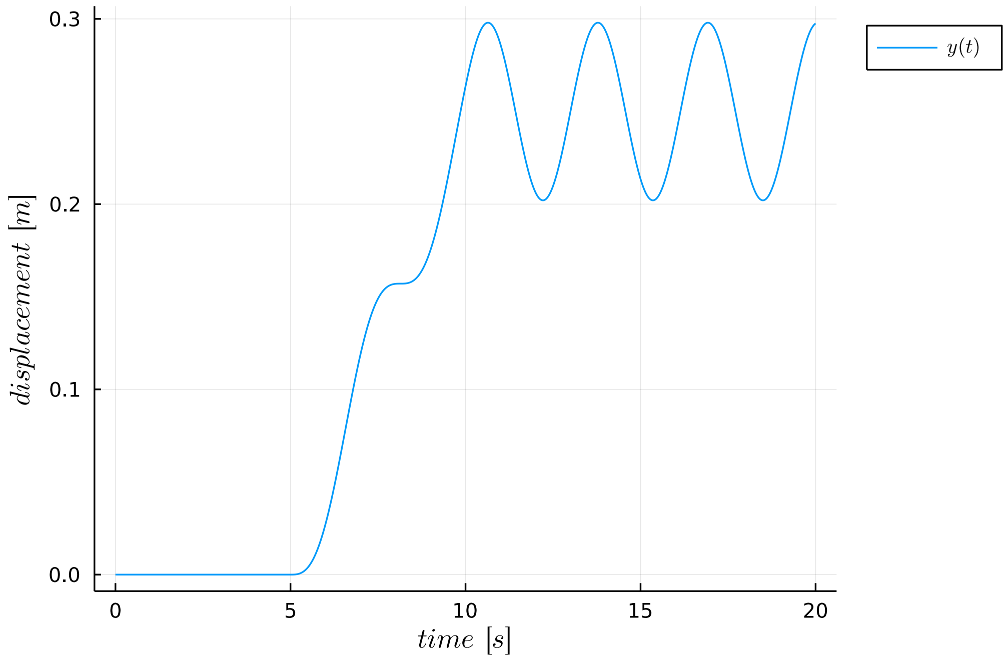

Figure 4: Real part of the solution, Eq. 172, for problem defined in Eq. 168.

4. Further extensions - discrete or hard to convolute excitations

The general solution, Eq. 108, can be used for general as well as some particular continuous forms of excitation $f(t)$. Nonetheless, sometimes we are interested in using excitations only known at discrete time steps t_k or using some $f(t)$ hard to evaluate by convolution, although the integral of $f(t)$ itself can be easy to evaluate (at least numerically). These cases can be addressed by using Heaviside series.

4.1. Heaviside series

Let a function $f(t)$ be approximated as a series of polynomials $c_{k,m}t^m$ up to power n_m multiplied by Heaviside step functions at n_k discrete time steps t_k,

$$f(t) \approx \tilde{f}(t) = \sum_{k=0}^{n_k} \sum_{m=0}^{n_m} c_{k,m} t^m \mathcal{H}\left(t - t_k\right). \tag{173}$$

As the function is essentially approximated by a polynomial of order n_m, it is fair to say that Eq. 173 provides an approximation of order n_m for $f(t)$, much like using Taylor series. Nonetheless, the coefficients of the polynomials are updated according to t, such that there is no need to center the approximation around a point like in Taylor series. Thus, the calculation of coefficients $c_{k,m}$ is carried out differently.

The main idea for evaluating the coefficients $c_{k,m}$ is to preserve the integral of the original function, $f(t)$ and its derivatives at each point t_k. Examples with zero, fist and second order approximations are presented in the following and provide a glance at the niche of application of this series representation. The application and advantage of representing the excitation function as a series of Heaviside steps is shown in Fig. 5, where the use of the integrating factor (and its convolution) is straightforward even beyond the subspace of excitation functions whose convolution is simply calculated.

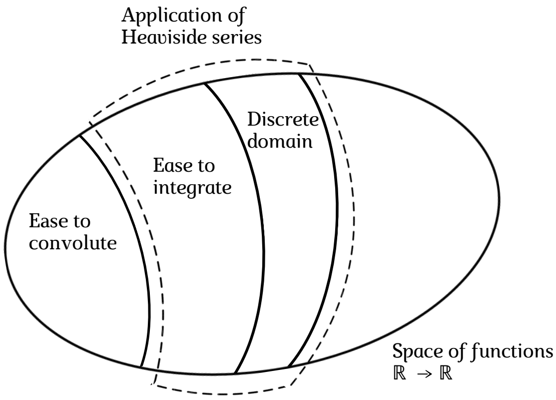

Figure 5: Diagram of the extension of the function space covered by the use of Heaviside series representation as excitation function, despite the simple convolution of the original excitation with the exponential function.

The Heaviside series extends the viability of the integrating factor technique to excitation functions that are easy to integrate (inner integration in Eq. 108) but are hard to convolute. These series also allow the consideration of excitation functions (or distributions) defined on discrete domains, as summarized in Fig. 5.

4.1.1. Zero order approximation

For zero order approximation, $n_m = 0$, Eq. 173 reduces to

$$\tilde{f}(t) = \sum_{k=0}^{n_k} c_k \mathcal{H}\left(t - t_k\right). \tag{174}$$

Using the preservation of the integral of $f(t)$ at each interval $t \in [t_l, t_{l+1}]$ and the mean value theorem for integrals

$$\tilde{f}(t) = \sum_{k=0}^{l} c_k = \frac{1}{\Delta t_l} \int_{t_l}^{t_{l+1}} f(t)\, \mathrm{d}t, \quad t_l \leq t \leq t_{l+1}, \tag{175}$$

where $\Delta t_l = t_{l+1} - t_l$. Thus, the l-th coefficient, c_l, is evaluated by

$$c_l = \frac{1}{\Delta t_l} \int_{t_l}^{t_{l+1}} f(t)\, \mathrm{d}t - \sum_{k=0}^{l-1} c_k. \tag{176}$$

4.1.2. First order approximation

For first order approximation, $n_m = 1$, Eq. 173 reduces to

$$\tilde{f} = \sum_{k=0}^{n_k} \left(c_{k,0} + c_{k,1} t\right) \mathcal{H}\left(t - t_k\right). \tag{177}$$

The approximation at interval $t \in [t_l, t_{l+1}]$ is given by

$$\tilde{f}(t) = \sum_{k=0}^{l} c_{k,0} + \sum_{k=0}^{l} c_{k,1} t = a_{0,l} + a_{1,l} t, \quad t_l \leq t \leq t_{l+1}. \tag{178}$$

The slope of this polynomial, $a_{1,l}$, can be tailored to be equal to

$$a_{1,l} = \frac{f\left(t_{l+1}\right) - f\left(t_l\right)}{\Delta t_l}, \tag{179}$$

which, by the mean value theorem for derivatives, guarantees that the derivative of $f(t)$ coincides with the derivative of $\tilde{f}$ at one point within the given interval, at least. Coefficient $a_{0,l}$ is obtained by

$$\int_{t_l}^{t_{l+1}} \tilde{f}(t)\,\mathrm{d}t = a_{0,l} \Delta t_l + \frac{a_{1,l}}{2}\left(t_{l+1}^2 - t_l^2\right) = \int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t, \tag{180}$$

such that,

$$a_{0,l} = \frac{1}{\Delta t_l} \int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{a_{1,l}}{2\Delta t_l}\left(t_{l+1}^2 - t_l^2\right). \tag{181}$$

Thus, using Eq. 178, the coefficients are evaluated as

$$c_{l,0} = \frac{1}{\Delta t_l} \int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{a_{1,l}}{2\Delta t_l}\left(t_{l+1}^2 - t_l^2\right) - \sum_{k=0}^{l-1} c_{k,0} \tag{182}$$

and

$$c_{l,1} = \frac{f\left(t_{l+1}\right) - f\left(t_l\right)}{\Delta t_l} - \sum_{k=0}^{l-1} c_{k,1}. \tag{183}$$

4.1.3. Second order approximation

For second order approximation, $n_m = 2$, Eq. 173 reduces to

$$\tilde{f}(t) = \sum_{k=0}^{n_k} \left(c_{k,0} + c_{k,1} t + c_{k,2} t^2\right) \mathcal{H}\left(t - t_k\right). \tag{184}$$

At interval $t \in [t_l, t_{l+1}]$ the approximation function is given by

$$\tilde{f}(t) = \sum_{k=0}^{l} c_{k,0} + \sum_{k=0}^{l} c_{k,1} t + \sum_{k=0}^{l} c_{k,2} t^2 = a_{0,l} + a_{1,l} t + a_{2,l} t^2, \quad t_l \leq t \leq t_{l+1}. \tag{185}$$

For this order of approximation, the first derivative of the Heaviside series representation is set to be equal to the derivative of $f(t)$ at the vicinity of each time point t_k, what gives the advantage that the first derivative is smooth, although not defined at points t_k. Thus, the derivative of the representation, $\tilde{f}$, is continuous in the domain $\Omega: (-\infty, \infty) - \{t_l\}, l \in 0, 1, \ldots, n_k$. It is straightforward to observe that, if f is differentiable at t_l, t_l is an accumulation point and the limit of the derivative of $\tilde{f}$ converges from both the left and the right sides. These conditions yield a linear system of equations

$$2a_{2,l}t_{l+1} + a_{1,l} = \dot{f}\left(t_{l+1}\right), \tag{186}$$

and

$$2a_{2,l}t_l + a_{1,l} = \dot{f}\left(t_l\right) \tag{187}$$

whose solutions are the coefficients $a_{1,l}$ and $a_{2,l}$,

$$a_{2,l} = \frac{\dot{f}\left(t_{l+1}\right) - \dot{f}\left(t_l\right)}{2\Delta t_l}, \tag{188}$$

and

$$a_{1,l} = \dot{f}\left(t_{l+1}\right) - 2a_{2,l}t_{l+1}. \tag{189}$$

Using the mean value theorem for integrals, one gets the coefficient a_3 to make the integral of the representation equal to the integral of the represented function at each point t_k. Hence, the integral of the representation is

$$\int_{t_l}^{t_{l+1}} a_2t^2 + a_1t + a_0\,\mathrm{d}t = \frac{a_2}{3}\left(t_{l+1}^3 - t_l^3\right) + \frac{a_1}{2}\left(t_{l+1}^2 - t_l^2\right) + a_0\left(t_{l+1} - t_l\right), \tag{190}$$

which, comparing to the integral of the represented function, reduces to

$$a_0 = \frac{1}{\Delta t_l}\int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{1}{\Delta t_l}\frac{a_2}{3}\left(t_{l+1}^3 - t_l^3\right) - \frac{1}{\Delta t_l}\frac{a_1}{2}\left(t_{l+1}^2 - t_l^2\right). \tag{191}$$

Turning these equations in terms of $c_{k,j}$ yields

$$c_{l,2} = \frac{\dot{f}\left(t_{l+1}\right) - \dot{f}\left(t_l\right)}{2\Delta t_l} - \sum_{k=0}^{l-1} c_{k,2}, \tag{192}$$

$$c_{l,1} = \dot{f}\left(t_{l+1}\right) - 2a_{2,l}t_{l+1} - \sum_{k=0}^{l-1} c_{k,1} \tag{193}$$

and

$$c_{l,0} = \frac{1}{\Delta t_l}\int_{t_l}^{t_{l+1}} f(t)\,\mathrm{d}t - \frac{1}{\Delta t_l}\frac{a_2}{3}\left(t_{l+1}^3 - t_l^3\right) - \frac{1}{\Delta t_l}\frac{a_1}{2}\left(t_{l+1}^2 - t_l^2\right) - \sum_{k=0}^{l-1} c_{k,0}. \tag{194}$$

4.2. Heaviside series as excitation function

Let the approximation function, $\tilde{f}$, of Subsec. 4.1.1, 4.1.2 and 4.1.3, be used as excitation functions for for ordinary second order differential equations with constant coefficients. Let the order of the approximation be equal to n_{lk},

$$\tilde{f} = \sum_{k=0}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} t^l \mathcal{H}\left(t - t_k\right). \tag{195}$$

Thus, applying Eq. 195 in Eq. 167 and using the linearity of the integration operator yields

$$y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} \exp\left(-f_{2,1,2} t\right) \int_{t_k}^{t} \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \int_{t_k}^{t} \exp\left(\hat{k} t\right) c_{k,l} t^l \,\mathrm{d}t \,\mathrm{d}t \,\mathcal{H}(t - t_k). \tag{196}$$

The inner convolution is evaluated by using Eq. 137

$$\begin{aligned} y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} \exp\left(-f_{2,1,2} t\right) \int_{t_k}^{t} \exp\left(\left(f_{2,1,2} - \hat{k}\right) t\right) \left(\sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t^s \exp\left(\hat{k} t\right) \right. \\ \left. - \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \exp\left(\hat{k} t_k\right) \right) \mathrm{d}t \,\mathcal{H}(t - t_k) \end{aligned} \tag{197}$$

where $s = l - p + 1$. The integral of the outer convolution is split among its integrands,

$$\begin{aligned} y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} \exp\left(-f_{2,1,2} t\right) \left(\int_{t_k}^{t} \exp\left(f_{2,1,2} t\right) \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t^s \,\mathrm{d}t \right. \\ \left. - \int_{t_k}^{t} \exp\left(f_{2,1,2} t - \hat{k}\left(t_k - t\right)\right) \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \,\mathrm{d}t \right) \mathcal{H}(t - t_k). \end{aligned} \tag{198}$$

Taking the constant terms out of the integrals

$$\begin{aligned} y_p(t) = \sum_{k=1}^{n_k} \sum_{l=0}^{n_{lk}} c_{k,l} \exp\left(-f_{2,1,2} t\right) \left(\sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \int_{t_k}^{t} \exp\left(f_{2,1,2} t\right) t^s \,\mathrm{d}t \right. \\ \left. - \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \int_{t_k}^{t} \exp\left(f_{2,1,2} t - \hat{k}\left(t_k - t\right)\right) \mathrm{d}t \right) \mathcal{H}(t - t_k), \end{aligned} \tag{199}$$

which has, then, the first convolution evaluated again using Eq. 137, and the second convolution evaluated by simple integration,

$$y_p(t) = \sum_{k=1}^{n_k} \mathcal{H}(t-t_k) \sum_{l=0}^{n_{lk}} c_{k,l} \exp\left(-f_{2,1,2}t\right) \left\{ \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \left\{ \sum_{q=1}^{s+1} (-1)^{q+1} \left(\frac{1}{f_{2,1,2}}\right)^q \right.\right.$$

$$\left.\frac{(s)!}{(s+1-q)!} t^{s+1-q} \exp\left(f_{2,1,2}t\right) - \sum_{q=1}^{s+1} (-1)^{q+1} \left(\frac{1}{f_{2,1,2}}\right)^q \frac{s!}{(s+1-q)!} t_k^{s+1-q} \exp\left(f_{2,1,2}t_k\right)\right\} -$$

$$\left.\sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \left\{ \exp\left(f_{2,1,2}t + \hat{k}\left(t_k - t\right)\right) - \exp\left(f_{2,1,2}t_k + \hat{k}\underbrace{\left(t_k - t_k\right)}_{0}\right)\right\} \frac{1}{f_{2,1,2} - \hat{k}} \right\} \quad (200)$$

Rearranging the terms

$$y_p(t) = \sum_{k=1}^{n_k} \mathcal{H}(t-t_k) \sum_{l=0}^{n_{lk}} c_{k,l} \left\{ \left\{ \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \sum_{q=1}^{s+1} (-1)^{q+1} \left(\frac{1}{f_{2,1,2}}\right)^q \right.\right.$$

$$\left.\frac{s!}{(s+1-q)!} t^{s+1-q} \right\} + \exp\left(f_{2,1,2}\left(t_k - t\right)\right) \left\{ \frac{1}{f_{2,1,2} - \hat{k}} \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \right.$$

$$\left. - \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \sum_{q=1}^{s+1} (-1)^{q+1} \left(\frac{1}{f_{2,1,2}}\right)^q \frac{s!}{(s+1-q)!} t_k^{s+1-q} \right\}$$

$$\left. - \exp\left(\hat{k}\left(t_k - t\right)\right) \left(\frac{1}{f_{2,1,2} - \hat{k}}\right) \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \right\}, \quad (201)$$

which, with further simplification, can be written as

$$y_p(t) = \sum_{k=1}^{n_k} \mathcal{H}(t-t_k) \sum_{l=0}^{n_{lk}} c_{k,l} \left\{ \left\{ \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \sum_{q=1}^{s+1} (-1)^{q+1} \left(\frac{1}{f_{2,1,2}}\right)^q \right.\right.$$

$$\left.\frac{s!}{(s+1-q)!} t^{s+1-q} \right\} + \exp\left(f_{2,1,2}\left(t_k - t\right)\right) \left\{ \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} \left\{ \frac{t_k^s}{f_{2,1,2} - \hat{k}} \right.\right.$$

$$\left.\left. - \sum_{q=1}^{s+1} (-1)^{q+1} \left(\frac{1}{f_{2,1,2}}\right)^q \frac{s!}{(s+1-q)!} t_k^{s+1-q} \right\}\right\}$$

$$\left. - \exp\left(\hat{k}\left(t_k - t\right)\right) \left(\frac{1}{f_{2,1,2} - \hat{k}}\right) \sum_{p=1}^{l+1} (-1)^{p+1} \left(\hat{k}\right)^{-p} \frac{l!}{s!} t_k^s \right\}. \quad (202)$$

4.2.1. Initial conditions and the Heaviside series

Let a second order ordinary differential equation have a function $\bar{f}(t)$ multiplied by a Heaviside step as excitation function,

$$\ddot{y}(t) + \bar{c}\dot{y}(t) + \bar{k}y(t) = \bar{f}(t)\mathcal{H}\left(t - t_0\right), \tag{203}$$

which is the same as solving two differential equations,

$$\begin{cases} \ddot{y}_1(t) + \bar{c}\dot{y}_1(t) + \bar{k}y_1(t) = 0, & \text{if } t \leq t_0 \\ \ddot{y}_2(t) + \bar{c}\dot{y}_2(t) + \bar{k}y_2(t) = \bar{f}(t). & \text{otherwise} \end{cases}, \tag{204}$$

At t_0, the initial conditions of y_2 must be equal to the values of y_1 and its derivative at this point, *i.e.*, $y_1\left(t_0\right) = y_2\left(t_0\right)$ and $\dot{y}_1\left(t_0\right) = \dot{y}_2\left(t_0\right)$. As there is no excitation before t_0 and a purely particular solution is sought after, the solution between $t = 0$ and $t = t_0$ is $y_1(t) = 0$ and, consequently $\dot{y}_1(t) = 0$. Thus, it follows that $y_2(t_0) = 0$ and $\dot{y}_2(t_0) = 0$. This phenomenon is illustrated in Fig. 6.

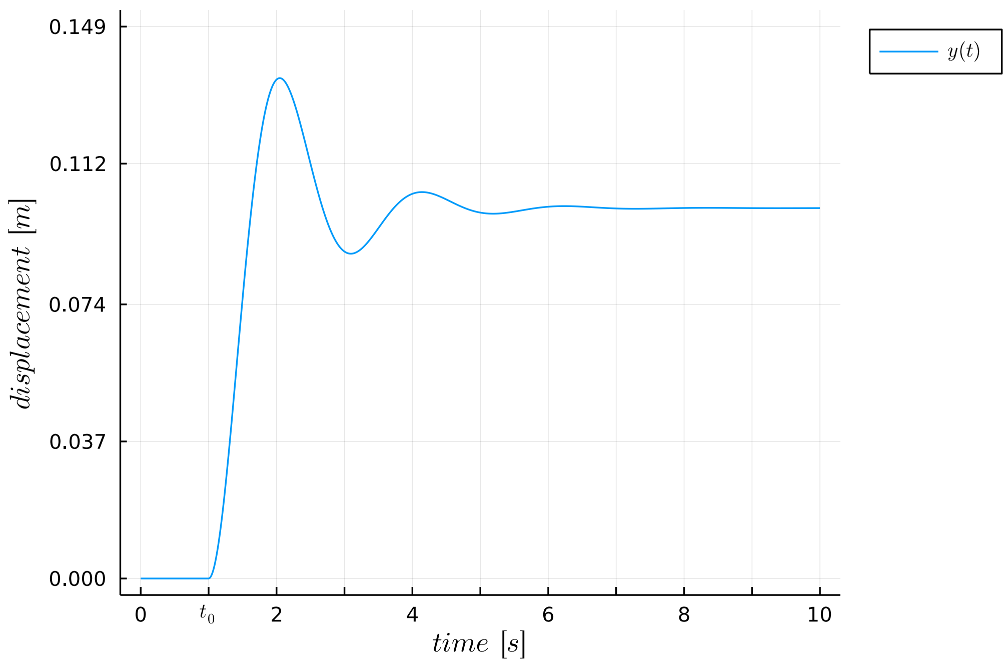

Figure 6: Solution of a linear second order differential equation with constant coefficients due to unitary Heaviside step at t_0 as excitation function (under damped case).

It is straightforward that this holds true even when $t_0 \to 0$. Therefore, all solutions given by Eq. 202, *i.e.* using Heaviside series as excitation function, have $y_p(t) = 0$ and $\dot{y}_p(t) = 0$ as fixed initial conditions. Thus, the imposition of non-homogeneous initial conditions, $y\left(t_{in}\right)$ and $\dot{y}\left(t_{in}\right)$, at $t_{in} \leq t_0$ gets even simpler, through the following system of linear equations,

$$\begin{bmatrix} \exp\left(-f_{2,1,2}t_{in}\right) & \exp\left(-\hat{k}t_{in}\right) \\ -f_{2,1,2}\exp\left(-f_{2,1,2}t_{in}\right) & -\hat{k}\exp\left(-\hat{k}t_{in}\right) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} y\left(t_{in}\right) \\ \dot{y}\left(t_{in}\right) \end{Bmatrix}, \tag{205}$$

which, particularized for $t_{in} = 0$, simplifies to

$$\begin{bmatrix} 1 & 1 \\ -f_{2,1,2} & -\hat{k} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} y\left(0\right) \\ \dot{y}\left(0\right) \end{Bmatrix}, \tag{206}$$

whose solution is

$$C_2 = \frac{\dot{y}(0) + f_{2,1,2}y(0)}{f_{2,1,2} - \hat{k}}, \tag{207}$$

$$C_1 = y(0) - C_2. \tag{208}$$

Thus, the constants C_1 and C_2 can be evaluated without any knowledge about the derivative of the particular response.

4.2.2. Examples

The following second order differential equation with constant coefficients is used to show the use of Heaviside series as excitation function

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = f(t). \tag{209}$$

The periodic excitation was first addressed to compare the analytical solution (Subsubsec. 3.2.1) with a solution using Heaviside series representation as excitation function. Then, an example using a random (Gaussian-distribution) was evaluated. The objective of the first example, beside comparing the representation with a purely analytical solution, is to show a common case of excitation in Engineering and Applied Mathematics, but without representing the periodic signal as Fourier series. The objective of the second example is to show the usage of Heaviside series representation when there is no explicit formula for the excitation function.

Example I: Let the excitation function $f(t) = sin(4t)$ be represented by Heaviside series with four fixed Δt: $0.2s$, $0.4s$, $0.6s$ and $0.8s$. Figure 7 shows the original excitation function with blue cross symbols and four different zero order approximations, one for each Δt. Figures 8 and 9 do the same but for first and second order approximations, respectively.

One may observe that the zero order approximation represents the function rather poorly and, thus, needs even smaller steps of time. Quadratic approximation takes the best shape, with smoother corners, but gets out of phase and reduces its amplitude prematurely, much like what happens with zero order approximation. The odd behavior observed in the second order approximation is due to Eq. 188, since for large Δt the approximation enforces two very different slopes in the same interval. The amplitude of the linear approximation proved to be less vulnerable to the increase of Δt in this example.

Figure 10 shows the solution obtained with Eq. 202 due to excitation of zero order approximation of the sine function. Again, the same four different Δt were tested, and the solution using Eq. 202 was compared to the solution due to harmonic (Eq. 117) excitation with initial conditions of $y(0) = \dot{y}(0) = 0$. Figure 11 and Figure 12 show the same procedure with first and second order approximations, respectively.

It is straightforward to see that the zero order approximation performed worst, losing amplitude even for small Δt. The second order approximation yielded slightly better results, but out of phase for $\Delta t = 0.8$, as expected. Solutions obtained by using the first order for Δt $0.2s$ and $0.4s$ matches the result given by Eq. 117. It is also perceptible that the loss of amplitude is not linearly dependent to the increase of Δt, as the rate of decay of amplitude grows with the increase in Δt.

It is worth noticing that the analytical solutions obtained herein are not dependent on Δt. The dependence only appears when using the Heaviside series representation for $\bar{f}(t)$. In this case, one can compare the dependence of the solution with respect to Δt with traditional numerical integration methods, like the Newmark-Beta, for example. In such case, even when using an unconditionally stable integration methods, there are some constraints on Δt to avoid aliasing. In the example studied in this section, the smaller period associated to the loading is $\pi/2$ seconds, such that $\Delta t < 0.15s$ (preferable less) are to be used to obtain a good representation of the response [11].

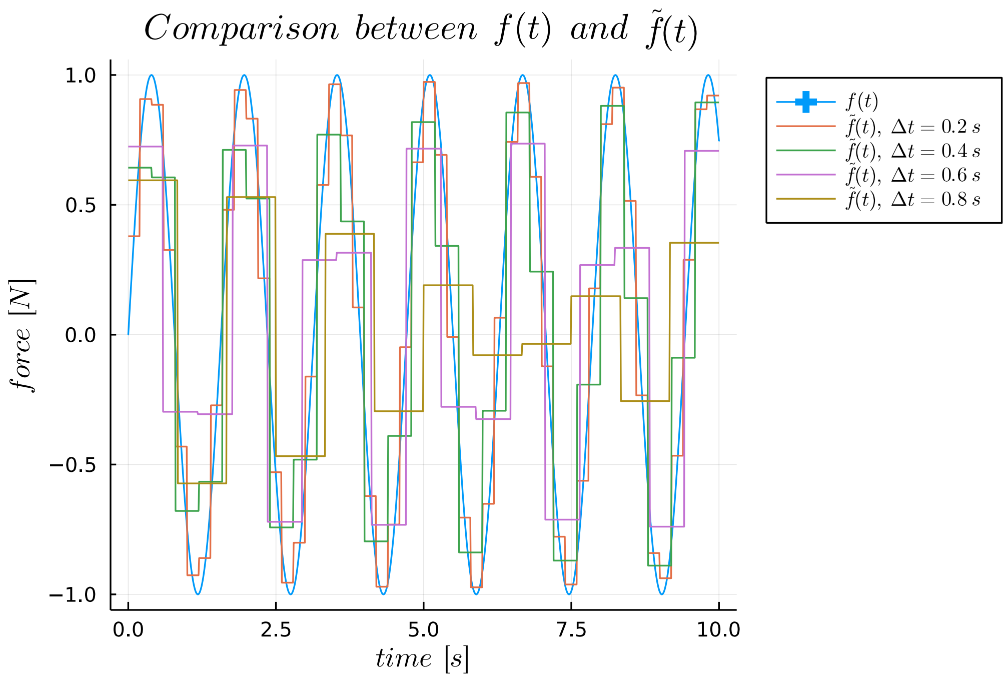

Figure 7: Comparison between the original excitation function (blue) and the representations using Heaviside series, $\tilde{f}(t)$, using zero order approximation with different discretizations.

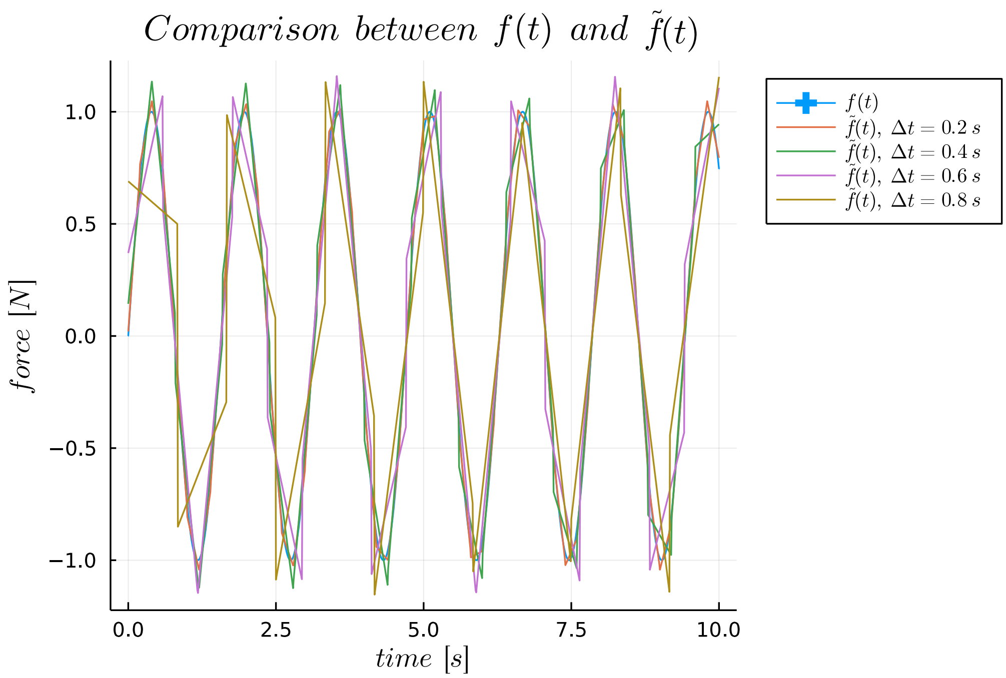

Figure 8: Comparison between the original excitation function (blue) and the representations using Heaviside series, $\tilde{f}(t)$, using first order approximation.

Example II

A set of randomly generated points, obtained by using a Gaussian distribution, is used as the excitation function in this example. Figure 13 shows the random points as blue crosses and the zero order approximation. Figure 14 also shows the random points and the linear approximation.

The second order approximation cannot be used in this example, since there is no information about the derivative of the original excitation function. Constants a for the approximations are evaluated by using the mean value theorem for integrals. Thereby, the integral for a discrete set of points is taken using trapezoid rule between each adjacent pair of points. Consequently, the step Δt is equal to the time between two subsequent points.

Figure 13 shows the comparison of the solution obtained by using first order approximation by Heaviside series, $\tilde{y}_{H1}(t)$, by using zero first order approximation by Heaviside series, $\tilde{y}_{H0}(t)$ and by using the Tsitouras 5/4 Runge-Kutta method [18] with linear interpolation for $f(t)$. A total of 91 time steps were needed for the Tsitouras method to match the solution obtained by using the first order Heaviside series (50 points). Solution obtained with the zero order Heaviside series (red) show some discrepancies with respect to the amplitude at most of the peaks.

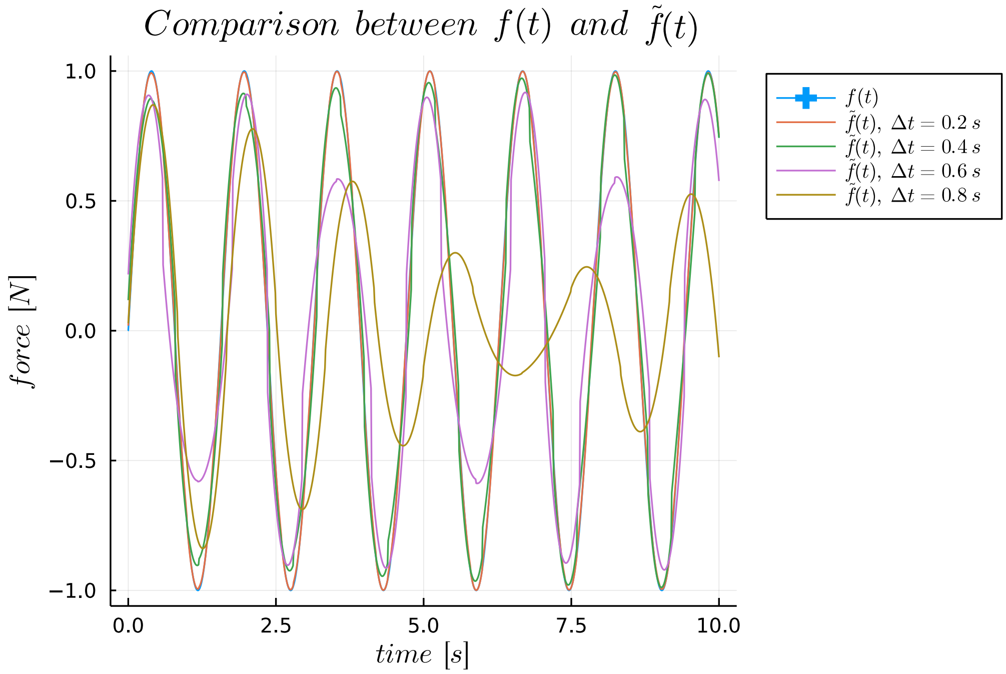

Figure 9: Comparison between the original excitation function (blue) and the representations using Heaviside series, $\tilde{f}(t)$, using second order approximation.

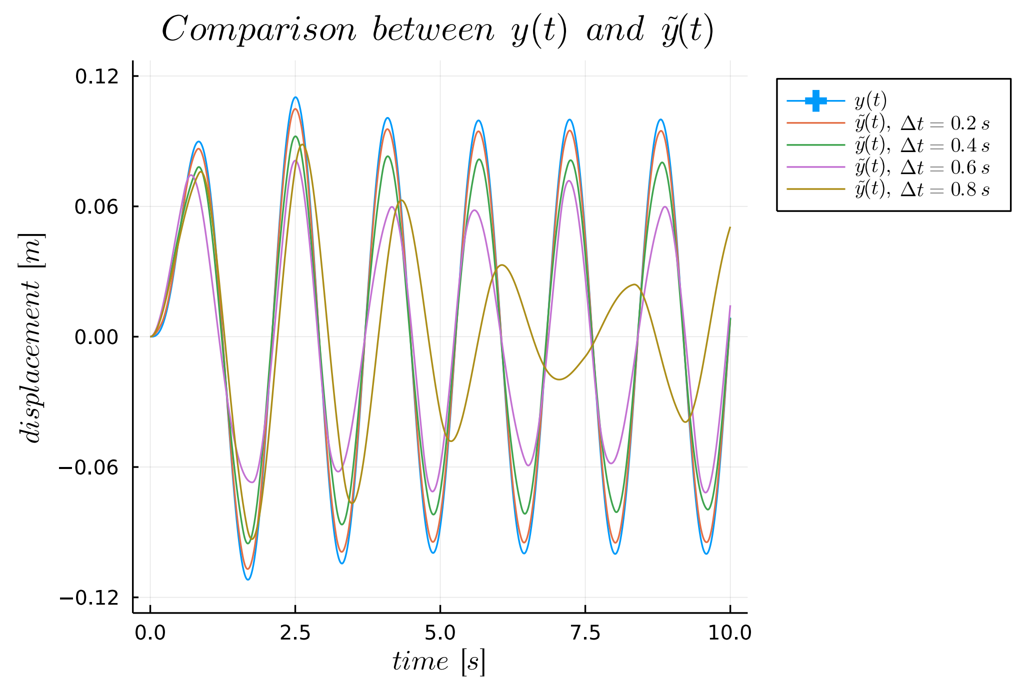

Figure 10: Comparison between the solutions of the differential equation using the original excitation function (blue), and the solutions with Heaviside series as excitation function, $\tilde{y}$, using zero order approximation.

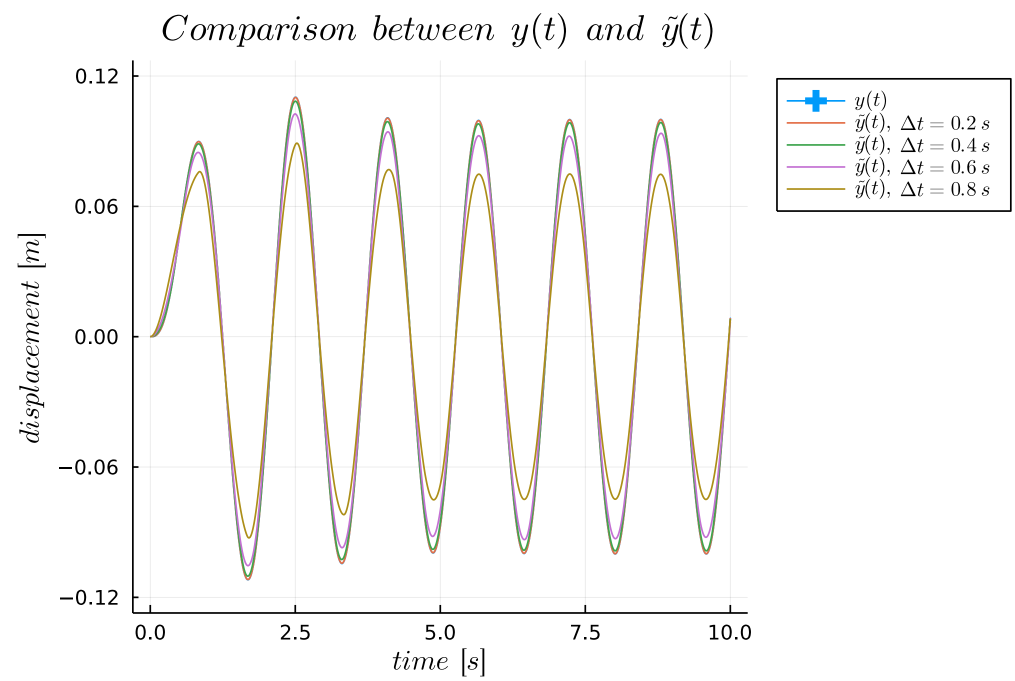

Figure 11: Comparison between the solutions of the differential equation using the original excitation function (blue), and the solutions with Heaviside series as excitation function, $\tilde{y}$, using first order approximation.

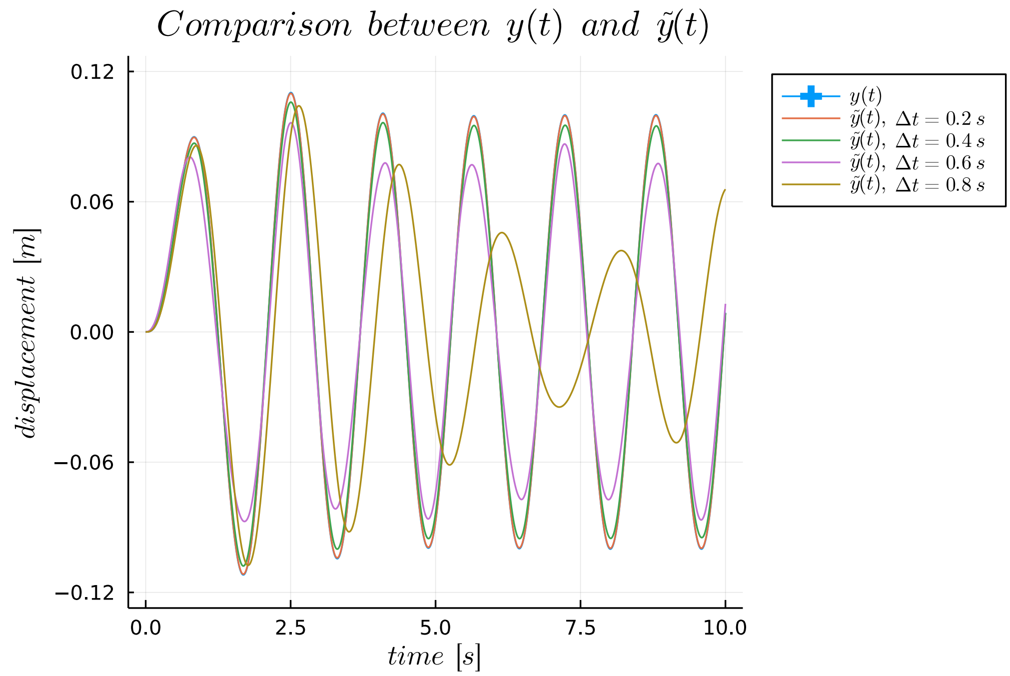

Figure 12: Comparison between the solutions of the differential equation using the original excitation function (blue), and the solutions with Heaviside series as excitation function, $\tilde{y}$, using second order approximation.

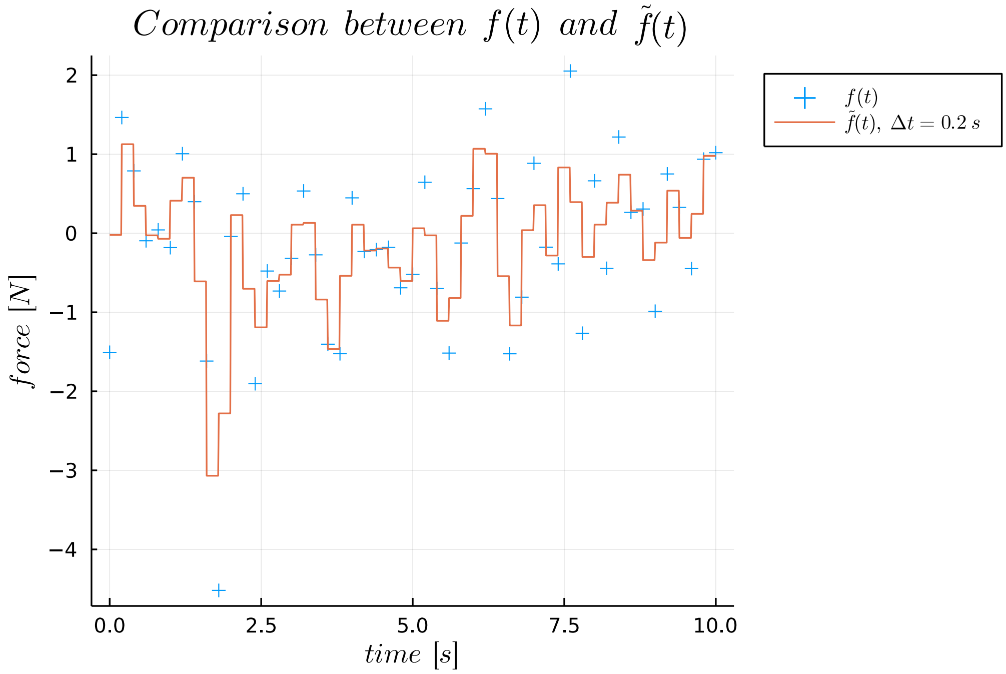

Figure 13: Comparison between the original excitation function (blue) and the representations using Heaviside series, $\tilde{f}(t)$, using zero order approximation with $\Delta t = 0.2s$.

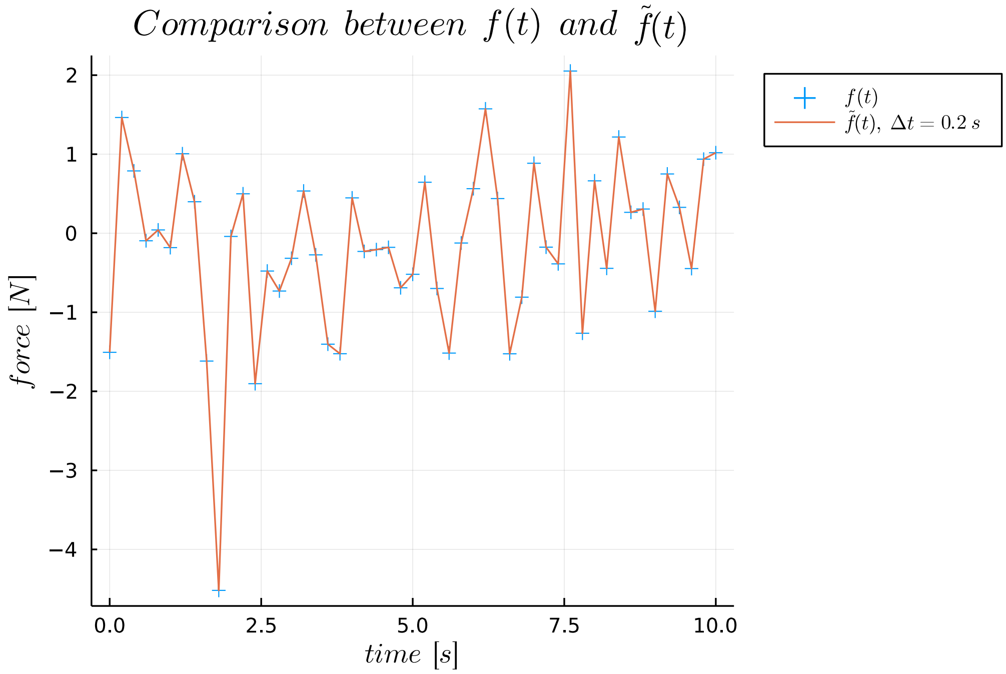

Figure 14: Comparison between the original excitation function (blue) and the representations using Heaviside series, $\tilde{f}(t)$, using first order approximation with $\Delta t = 0.2s$.

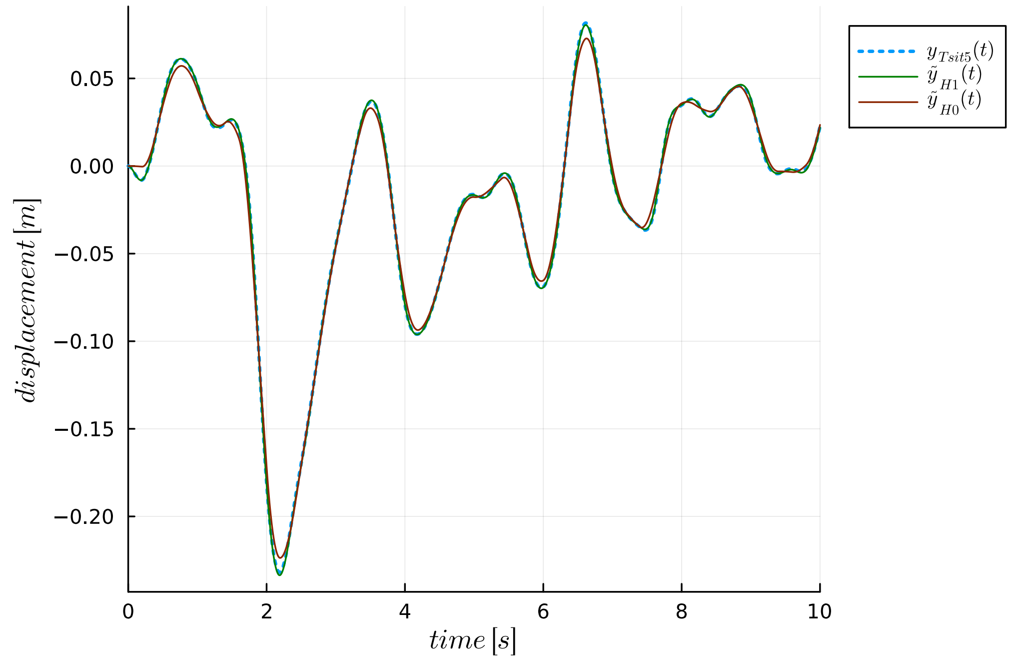

Figure 15: Solution of the differential equation using first order approximation Heaviside series excitation, $\tilde{y}_{H1}(t)$ (green line), using zero order approximation Heaviside series excitation, $\tilde{y}_{H0}(t)$ (red line), and using a numerical solver, $y_{Tsit5}(t)$ (blue dotted line).

5. Conclusion

This work proposes a new technique to solve one dimensional linear ordinary differential equations of order $n \geq 2$ by a generalization of the Leibniz integrating factor for first order linear differential equations.

The methodology was applied to second-order ordinary differential equation, since this type of ODE is of great importance in Applied Mathematics, Physics and Engineering. Special care was devoted to the constant coefficient case for different types of excitations. Various analytical excitation functions yielded analytical solutions in closed form, like periodic and polynomial functions. Closed form solutions were also obtained for discontinuous excitation, as for Dirac's delta impulse and Heaviside step function. The Heaviside case was particularized for steps multiplied by polynomials, enlarging the range of applications with analytical solutions. In contrast to previous and well established methods, like undetermined coefficients and Laplace transform, no knowledge of a candidate solution nor the calculation of a inverse transformation was needed. The solution was instead analytically derived by means of double convolutions.

The Heaviside series representation was derived by using the closed form solution obtained for Heaviside excitation multiplied by polynomials. Examples showed the effectiveness of representing a function as a series of Heaviside steps, thus, augmenting the set of excitation functions solved within the generalized integrating factor method. A discussion about the discretization of the series was performed and shed light about the advantages of the proposed methodology compared to numerical alternatives, due to size of discretization, representation capability and the use of analytical solutions.

For differential equations with general coefficients, the integrating factor depends on a particular solution of a Riccati differential equation. It was shown that the coefficients themselves might help finding a particular solution rather easily. Thus, the Riccati differential equation poses no strictly direct barrier to the wide application of the method, capable of giving accurate results and requiring no assumptions of solution candidates.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request

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Appendix - Convolution over Dirac's delta distribution

The convolution of a function over a delta of Dirac is usually defined as [13]

$$\int_{-\infty}^{\infty} f(t)\delta\left(t-t_k\right)dt = f\left(t_k\right). \tag{210}$$

The integration limits can be split as

$$\int_{-\infty}^{\infty} f(t)\delta\left(t-t_k\right)dt = \int_{-\infty}^{0} f(t)\delta\left(t-t_k\right)dt + \int_{0}^{t} f(t)\delta\left(t-t_k\right)dt + \int_{t}^{\infty} f(t)\delta\left(t-t_k\right)dt, \tag{211}$$

for t_k strictly positive, the integral from $-\infty$ to 0 is 0 by definition, thus, the integral from 0 to t can be rewritten as

$$\int_{0}^{t} f(t)\delta\left(t-t_k\right)dt = \int_{-\infty}^{\infty} f(t)\delta\left(t-t_k\right)dt - \int_{t}^{\infty} f(t)\delta\left(t-t_k\right)dt. \tag{212}$$

The *filter* or *sifting* property of the delta of Dirac is due to the shape of this distribution, *i.e.*, it is null everywhere except in its discontinuity, thus, the function that multiplies the Dirac's delta is constant at this point, for the discontinuity of the delta distribution is infinitely close to the t_k point. Therefore, the value of the function can be taken out of the integral and the definition of the Dirac's delta is used to show that

$$\int_{-\infty}^{\infty} f(t)\delta\left(t-t_k\right)dt = \int_{t_k-\tau}^{t_k+\tau} f(t)\delta\left(t-t_k\right)dt = f(t_k)\int_{t_k-\tau}^{t_k+\tau} \delta\left(t-t_k\right)dt = f(t_k), \tag{213}$$

hence, Equation 212 can be rewritten to

$$\int_{0}^{t} f(t)\delta\left(t-t_k\right)dt = f(t_k) - f(t_k)\begin{cases} 0, & t \geq t_k \\ 1, & t < t_k \end{cases} = f(t_k)\mathcal{H}(t-t_k). \tag{214}$$

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