Generalized Integrating Factor Method Applied to Coupled n Dimensional Linear Second Order Systems of Ordinary Differential Equations

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Abstract

Coupled systems of second order Ordinary Differential Equations are found in many areas of engineering, such as vibrations and electric circuits. Thus, accurate and efficient solutions are of great interest. This manuscript extends the concept of generalized integrating factor, recently introduced by the authors for one dimensional problems, to coupled n-dimensional systems of second order ODEs. The general formulation for time-varying coefficients is found and particularized to the constant coefficient case. Closed-form integrating factors are obtained for the constant coefficient case under mild assumptions about the damping coefficient matrix. Analytical solutions for different types of continuous and discontinuous excitations are presented for problems with constant coefficients and proportional damping, disregarding the damping level. Examples are provided for each type of excitation studied in this manuscript. Results obtained with the proposed approach are compared to the numerical solution obtained by using the traditional Newmark-beta method. Results show that the proposed procedure is accurate, not suffering with interpolation errors found in traditional numerical methods.

Keywords: coupled system, ordinary differential equation, second order, analytical solution, integrating factor

1. Introduction

Couple systems of second-order Ordinary Differential Equations (ODEs) are found in many applications in applied mathematics, physics and engineering. One example is the time-dependent equation of motion obtained by using the finite-element method, where, after the spatial discretization, results in a set of n coupled ODEs

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \mathbf{f}(t), \tag{1}$$

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where $\mathbf{y}(t)$ is the vector of dependent variables, t is the independent variable, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are coefficient matrices.

There are many different approaches to solve the aforementioned system of coupled ODEs.

In a broad sense, under some circumstances regarding $\mathbf{C}$, it may be possible to convert the original coupled problem into n uncoupled problems [12]. When coefficient matrix $\mathbf{C}$ is written as a linear combination of $\mathbf{M}$ and $\mathbf{K}$ one can use a change of basis given by the eigenvectors of the generalized eigenvalue problem (modal problem), resulting in n independent one-dimensional second order ODEs [17]. Extensions to other general viscous coefficient matrix $\mathbf{C}$ involve the solution of a quadratic eigenvalue problem and are discussed by [15]. This general procedure is known as Modal Superposition and it is common to consider just a subset of the n eigenvectors to exclude (usually larger) modes in the response and to avoid the expensive computation of all the eigenpairs. Nonetheless, each individual equation still needs to be solved either analytically or numerically.

Analytical solutions of one dimensional second order ODEs are generally known for simple harmonic excitations [16]. For other types of periodic functions, it is common to use the Fourier Series to obtain an harmonic approximation [5]. A recent proposal to obtain analytical solutions to a broader cases of excitations is presented by [3] and extended to the multidimensional case in this manuscript.

Other approach is to re-write Eq. 2 by using state variables, converting a coupled second order equation to an extended system of first order coupled ODEs [4]. Analytical solutions can be derived by using exponential maps [14].

The most used approach to solve Eq. 2 is to use numerical methods. In this regard, there are many alternatives in the literature and a complete survey is out of the scope of this manuscript. Again, there are methods to directly solve Eq. 2 [10] and methods to solve the extended first order coupled system obtained by using state variables [4]. Numerical methods have as the main advantage the possibility to consider general forms of excitation.

The objective of the present manuscript is to extend the methodology presented in [3] to analytically solve coupled second order ODE systems. Different forms of excitations are discussed in details.

It is shown that for the constant coefficient case and under mild assumptions over $\mathbf{C}$ the evaluation of the complete response depends on the solution of a quadratic matrix equation with closed form. The analytical solutions do not depend on the level of damping and can be used for any damping ratio.

2. General Formulation

Consider a coupled system of n second order ODEs

$$\mathbf{M}(t)\ddot{\mathbf{y}}(t) + \mathbf{C}(t)\dot{\mathbf{y}}(t) + \mathbf{K}(t)\mathbf{y}(t) = \mathbf{f}(t), \tag{2}$$

where $\mathbf{M}$ and $\mathbf{K}$ are time-dependent, real, positive-definite and symmetric $n \times n$ matrices, $\mathbf{C}$ is a positive semi-definite, time dependent symmetric real $n \times n$ matrix, $\mathbf{f}$ is a general vector also depending on time t and vector $\mathbf{y}$ is the solution. For easy of notation, explicit dependency on time t, (t), will not be carried out in the following equations. Also, as $\mathbf{M}$

is invertible, we multiply Eq. 2 by $\mathbf{M}^{-1}$ to the left, such that

$$\mathbf{I}\ddot{\mathbf{y}} + \bar{\mathbf{C}}\dot{\mathbf{y}} + \bar{\mathbf{K}}\mathbf{y} = \bar{\mathbf{f}}. \tag{3}$$

To solve this equation we start by splitting $\bar{\mathbf{C}}$ as

$$\bar{\mathbf{C}} = \mathbf{F}_{2,1,1} + \mathbf{F}_{2,1,2} \tag{4}$$

where both $\mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,2}$ are also $n \times n$ time dependent matrices, not necessarily real. Thus,

$$\underbrace{\mathbf{I}\ddot{\mathbf{y}} + \mathbf{F}_{2,1,1}\dot{\mathbf{y}}}_{\pi_{2,2}} + \underbrace{\mathbf{F}_{2,1,2}\dot{\mathbf{y}} + \bar{\mathbf{K}}\mathbf{y}}_{\pi_{2,1}} = \bar{\mathbf{f}}. \tag{5}$$

Next step is to multiply Eq. 6 by an invertible $n \times n$ matrix $\boldsymbol{\mu}_2$ (generalized integrating factor) such that

$$\underbrace{\boldsymbol{\mu}_2\mathbf{I}}_{\mathbf{P}_{2,2}}\ddot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,1}}_{\dot{\mathbf{P}}_{2,2}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,2}}_{\mathbf{P}_{2,1}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\bar{\mathbf{K}}}_{\dot{\mathbf{P}}_{2,1}}\mathbf{y} = \boldsymbol{\mu}_2\bar{\mathbf{f}}. \tag{6}$$

Using the definition of $\mathbf{P}_{2,2}$ and its time derivative

$$\dot{\mathbf{P}}_{2,2} = \boldsymbol{\mu}_2\mathbf{F}_{2,1,1} = (\dot{\boldsymbol{\mu}_2\mathbf{I}}) = \dot{\boldsymbol{\mu}}_2 \tag{7}$$

and multiplying by $\boldsymbol{\mu}_2^{-1}$ results in

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2 = \mathbf{F}_{2,1,1} \tag{8}$$

such that

$$\boldsymbol{\mu}_2 = \exp\left(\int \mathbf{F}_{2,1,1}\,\mathrm{d}t\right). \tag{9}$$

The same procedure can be applied to $\mathbf{P}_{2,1}$ and its time derivative

$$\dot{\mathbf{P}}_{2,1} = \boldsymbol{\mu}_2\bar{\mathbf{K}} = \left(\dot{\boldsymbol{\mu}_2\mathbf{F}}_{2,1,2}\right) = \dot{\boldsymbol{\mu}}_2\mathbf{F}_{2,1,2} + \boldsymbol{\mu}_2\dot{\mathbf{F}}_{2,1,2} \tag{10}$$

and multiplying by $\boldsymbol{\mu}_2^{-1}$ to the left

$$\bar{\mathbf{K}} = \boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2\mathbf{F}_{2,1,2} + \dot{\mathbf{F}}_{2,1,2} \tag{11}$$

such that

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2 = \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right)\mathbf{F}_{2,1,2}^{-1} \tag{12}$$

with solution

$$\boldsymbol{\mu}_2 = \exp\left(\int \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right)\mathbf{F}_{2,1,2}^{-1}\,\mathrm{d}t\right). \tag{13}$$

Equating Eqs. 8 and 12

$$\mathbf{F}_{2,1,1} = \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right)\mathbf{F}_{2,1,2}^{-1} \tag{14}$$

such that

$$\mathbf{F}_{2,1,1}\mathbf{F}_{2,1,2} = \left(\bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,2}\right) \tag{15}$$

and using Eq. 4

$$\mathbf{F}_{2,1,1}\left(\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right) = \left(\bar{\mathbf{K}} - \dot{\bar{\mathbf{C}}} + \dot{\mathbf{F}}_{2,1,1}\right) \tag{16}$$

such that

$$\mathbf{F}_{2,1,1}^2 = \dot{\bar{\mathbf{C}}} + \mathbf{F}_{2,1,1}\bar{\mathbf{C}} - \bar{\mathbf{K}} - \dot{\mathbf{F}}_{2,1,1} \tag{17}$$

is a coupled system of Riccati differential equations. Thus, solving Eq. 17 it is possible to find $\boldsymbol{\mu}_2$ using Eq. 9. As both $\mathbf{P}_{2,2}$ and $\mathbf{P}_{2,1}$ depend on known $\boldsymbol{\mu}_2$, it is possible to re-write Eq. 6 as

$$\dot{\left(\mathbf{P}_{2,2}\dot{\mathbf{y}}\right)} + \dot{\left(\mathbf{P}_{2,1}\mathbf{y}\right)} = \boldsymbol{\mu}_2\bar{\mathbf{f}} \tag{18}$$

and integrating w.r.t time

$$\left(\mathbf{P}_{2,2}\dot{\mathbf{y}}\right) + \left(\mathbf{P}_{2,1}\mathbf{y}\right) = \int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2 \tag{19}$$

where $\mathbf{C}_2$ is a constant vector. Multiplying by another invertible time dependent matrix $\boldsymbol{\mu}_1$ (generalized integrating factor)

$$\underbrace{\boldsymbol{\mu}_1\mathbf{P}_{2,2}}_{\mathbf{P}_{1,1}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_1\mathbf{P}_{2,1}}_{\dot{\mathbf{P}}_{1,1}}\mathbf{y} = \boldsymbol{\mu}_1\underbrace{\left(\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2\right)}_{\mathbf{h}} \tag{20}$$

such that

$$\dot{\mathbf{P}}_{1,1} = \boldsymbol{\mu}_1\mathbf{P}_{2,1} = \dot{\left(\boldsymbol{\mu}_1\mathbf{P}_{\mathbf{2,2}}\right)} = \dot{\boldsymbol{\mu}}_1\mathbf{P}_{2,2} + \boldsymbol{\mu}_1\dot{\mathbf{P}}_{2,2} \tag{21}$$

and by multiplying by $\boldsymbol{\mu}_1^{-1}$ to the left

$$\boldsymbol{\mu}_1^{-1}\dot{\boldsymbol{\mu}}_1 = \left(\mathbf{P}_{2,1} - \dot{\mathbf{P}}_{2,2}\right)\mathbf{P}_{2,2}^{-1} \tag{22}$$

with solution

$$\boldsymbol{\mu}_1 = \exp\left(\int \left(\mathbf{P}_{2,1} - \dot{\mathbf{P}}_{2,2}\right)\mathbf{P}_{2,2}^{-1}\,\mathrm{d}t\right). \tag{23}$$

Again, $\mathbf{P}_{1,1}$ is known after evaluating $\boldsymbol{\mu}_1$ and Eq. 20 can be written as

$$\dot{\left(\mathbf{P}_{1,1}\mathbf{y}\right)} = \boldsymbol{\mu}_1\mathbf{h} \tag{24}$$

with solution

$$\mathbf{y} = \mathbf{P}_{1,1}^{-1}\left(\int \boldsymbol{\mu}_1\left\{\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2\right\}\mathrm{d}t + \mathbf{C}_1\right), \tag{25}$$

where $\mathbf{C}_2$ is a constant vector. Complete solution can be split into its particular and homogeneous counterparts as

$$\mathbf{y_p} = \mathbf{P}_{1,1}^{-1}\int \boldsymbol{\mu}_1\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t, \tag{26}$$

and

$$\mathbf{y_h} = \mathbf{P}_{1,1}^{-1}\left(\int \boldsymbol{\mu}_1\mathbf{C}_2\,\mathrm{d}t + \mathbf{C}_1\right). \tag{27}$$

This procedure can be time consuming depending on how matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ vary with time. Other issue is the solution of Eq. 17.

It is worth to mention that the proposed procedure does not depend on a time discretization and, despite being solved by using a computer, is analytical. For example, one can evaluate the response $\mathbf{y}(t)$ at any given time without knowing the solution of previous times. Numerical methods, like the well established Newmark-beta method, work in a total different way, building the solution from time step to time step. Approximation errors associated to the interpolation hypothesis of each particular numerical method are sensitive to "large" time steps, such that approximation errors (as well as numerical errors) are expected when using methods relying on approximations.

In the following we address the special case of constant coefficients.

3. Constant coefficients

We now consider a discrete system of n coupled ODEs with constant coefficients. Recalling Eq. 6

$$\underbrace{\boldsymbol{\mu}_2\mathbf{I}}_{\mathbf{P}_{2,2}}\ddot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,1}}_{\dot{\mathbf{P}}_{2,2}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\mathbf{F}_{2,1,2}}_{\mathbf{P}_{2,1}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_2\bar{\mathbf{K}}}_{\dot{\mathbf{P}}_{2,1}}\mathbf{y} = \boldsymbol{\mu}_2\bar{\mathbf{f}}. \tag{28}$$

For the following steps, the time-derivatives of $\mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,2}$ are considered null, since these partitions are constant just like matrix $\bar{\mathbf{C}}$. Thus,

$$\dot{\mathbf{P}}_{2,2} = \dot{\boldsymbol{\mu}}_2 = \boldsymbol{\mu}_2\mathbf{F}_{2,1,1}. \tag{29}$$

Multiplying to the left by the inverse of the integrating factor yields

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2 = \mathbf{F}_{2,1,1}. \tag{30}$$

The same procedure can be performed to $\mathbf{P}_{2,1}$,

$$\dot{\mathbf{P}}_{2,1} = \boldsymbol{\mu}_2\bar{\mathbf{K}} = \dot{\boldsymbol{\mu}}_2\mathbf{F}_{2,1,2}, \tag{31}$$

which, with further simplifications, yields

$$\boldsymbol{\mu}_2^{-1}\dot{\boldsymbol{\mu}}_2 = \bar{\mathbf{K}}\mathbf{F}_{2,1,2}^{-1}. \tag{32}$$

Equating Eq. 30 to Eq. 34, and multiplying to the right by $\mathbf{F}_{2,1,2} = \bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$ results in

$$\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \bar{\mathbf{K}}, \tag{33}$$

further simplifying this equation yields

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{0}, \tag{34}$$

where $\mathbf{0}$ is the null $n \times n$ matrix. The analytical solution procedure for this matrix polynomial equation is discussed in Subsec. 3.1.

The first generalized integrating factor, $\boldsymbol{\mu}_2$, is found by means of Eq. 30

$$\boldsymbol{\mu}_2 = \exp\left(\int \mathbf{F}_{2,1,1}\,\mathrm{d}t\right) = \exp\left(\mathbf{F}_{2,1,1}t\right). \tag{35}$$

Equation 28 can be written as

$$\dot{\left(\mathbf{P}_{2,2}\dot{\mathbf{y}}\right)} + \dot{\left(\mathbf{P}_{2,1}\mathbf{y}\right)} = \boldsymbol{\mu}_2\bar{\mathbf{f}} \tag{36}$$

which can be integrated w.r.t time, yielding

$$\boldsymbol{\mu}_2\dot{\mathbf{y}} + \boldsymbol{\mu}_2\mathbf{F}_{2,1,2}\mathbf{y} = \int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \mathbf{C}_2, \tag{37}$$

where $\mathbf{C}_2$ is a constant vector. This is a system of first order ordinary differential equations. Thus, by multiplying the equation to the left by $\boldsymbol{\mu}_2^{-1}$ and by another generalized integrating factor $\boldsymbol{\mu}_1$, and using $\mathbf{F}_{2,1,2} = \bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$, yields

$$\underbrace{\boldsymbol{\mu}_1\mathbf{I}}_{\mathbf{P}_{1,1}}\dot{\mathbf{y}} + \underbrace{\boldsymbol{\mu}_1\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]}_{\dot{\mathbf{P}}_{1,1}}\mathbf{y} = \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\mathbf{C}_2. \tag{38}$$

Repeating the same procedure for the second integrating factor,

$$\dot{\mathbf{P}}_{1,1} = \dot{\boldsymbol{\mu}}_1 = \boldsymbol{\mu}_1\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right], \tag{39}$$

or

$$\boldsymbol{\mu}_1^{-1}\dot{\boldsymbol{\mu}}_1 = \bar{\mathbf{C}} - \mathbf{F}_{2,1,1}, \tag{40}$$

such that

$$\boldsymbol{\mu}_1 = \exp\left(\int \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\mathrm{d}t\right) = \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right). \tag{41}$$

Equation 38 can be written as

$$\dot{\left(\mathbf{P}_{1,1}\mathbf{y}\right)} = \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t + \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\mathbf{C}_2, \tag{42}$$

such that by integrating and multiplying to left by $\boldsymbol{\mu}_1^{-1}$ yields

$$\mathbf{y} = \boldsymbol{\mu}_1^{-1}\int \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\int \boldsymbol{\mu}_2\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t + \boldsymbol{\mu}_1^{-1}\int \boldsymbol{\mu}_1\boldsymbol{\mu}_2^{-1}\mathbf{C}_2\,\mathrm{d}t + \boldsymbol{\mu}_1^{-1}\mathbf{C}_1, \tag{43}$$

where $\mathbf{C}_1$ is another constant vector. As the integrating factors are already known for this case, they can be substituted in previous equation, restulting in

$$\mathbf{y} = \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\int \exp\left(\mathbf{F}_{2,1,1}t\right)\bar{\mathbf{f}}\,\mathrm{d}t\,\mathrm{d}t + \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2\,\mathrm{d}t + \exp\left(-\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]t\right)\mathbf{C}_1. \tag{44}$$

According to [7], $\exp\left(\mathbf{A}t\right)$ is an exponential map and, for such map, the property $\exp\left(\mathbf{A}\right)\exp\left(\mathbf{B}\right) = \exp\left(\mathbf{A}+\mathbf{B}\right)$ holds true if $\mathbf{AB} = \mathbf{BA}$ ($\mathbf{A}$ and $\mathbf{B}$ commute). Consequently, if $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute the exponential multiplications in Eq. 44 can be grouped together as

$$\mathbf{y} = \exp\left(\left[\mathbf{F_{2,1,1}} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t +$$

$$\exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \mathbf{C}_2 \, \mathrm{d}t \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1, \quad (45)$$

The time-derivative of the exponential map can be used to evaluate the integral with the $\mathbf{C}_1$ term in Eq. 45,

$$\mathbf{y} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t +$$

$$\exp\left(-\mathbf{F}_{2,1,1} t\right) \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1, \quad (46)$$

where, again, there is a particular solution,

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \, \mathrm{d}t \, \mathrm{d}t, \quad (47)$$

and a homogeneous solution,

$$\mathbf{y_h} = \exp\left(-\mathbf{F}_{2,1,1} t\right) \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1, \quad (48)$$

which can be further simplified if $\mathbf{C}_2$ absorbs the matrices multiplication, *i.e.*,

$$\mathbf{y_h} = \exp\left(-\mathbf{F}_{2,1,1} t\right) \mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \mathbf{C}_1, \quad (49)$$

both valid if

$$\bar{\mathbf{C}} \mathbf{F}_{2,1,1} = \mathbf{F}_{2,1,1} \bar{\mathbf{C}} \quad (50)$$

commute. Complete expressions to evaluate constants $\mathbf{C}_1$ and $\mathbf{C}_2$ are developed in Appendix A and an efficient approach to evaluate Eq. 49 is discussed in Appendix F.

Commutativity of matrices $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}$ is studied in the following.

3.1. Solving the quadratic equation, Eq. 34

Assume that the damping matrix, $\mathbf{C}$, is given by Rayleigh damping model, *i.e.*, proportional damping. Thus

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K}, \quad (51)$$

normalizing by the mass matrix, one gets

$$\bar{\mathbf{C}} = \alpha \mathbf{M}^{-1} \mathbf{M} + \beta \mathbf{M}^{-1} \mathbf{K} = \alpha \mathbf{I} + \beta \bar{\mathbf{K}}, \quad (52)$$

therefore,

$$\bar{\mathbf{C}} \bar{\mathbf{K}} = \alpha \mathbf{I} \bar{\mathbf{K}} + \beta \bar{\mathbf{K}}^2 = \alpha \bar{\mathbf{K}} \mathbf{I} + \beta \bar{\mathbf{K}}^2 = \bar{\mathbf{K}} \bar{\mathbf{C}}, \quad (53)$$

such that $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute.

As stated before, $\mathbf{F}_{2,1,1}$ is given by the following matrix quadratic equation

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{0}. \tag{54}$$

According to [8], a quadratic equation of the form

$$\mathbf{X}^2 + \mathbf{B}\mathbf{X} + \mathbf{D} = \mathbf{0}, \tag{55}$$

has a solution given by

$$\mathbf{X} = -\frac{1}{2}\mathbf{B} + \frac{1}{2}\left[\mathbf{B}^2 - 4\mathbf{D}\right]^{\frac{1}{2}}, \tag{56}$$

if $\mathbf{B}$ and $\mathbf{D}$ commute. When a matrix solves a matrix polynomial equation it is called a *solvent*. In Eq. 54, the unknown matrix variable is at the left of the coefficient, $\mathbf{F}_{2,1,1}\bar{\mathbf{C}}$, hence, it is called a left solvent [9]; while in Eq. 56 the unknown variable is at the right, $\mathbf{B}\mathbf{X}$, thus, a right solvent. The question is whether solvent given by Eq. 56 solves Eq. 54, since Eq. 54 has the first coefficient equal to $\mathbf{I}$ and $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute. To validate it, let the candidate of solvent, $\tilde{\mathbf{F}}_{2,1,1}$, be calculated by Eq. 56 and be substituted into Eq. 54.

$$\tilde{\mathbf{F}}_{2,1,1} = \frac{1}{2}\bar{\mathbf{C}} + \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}, \tag{57}$$

applyied to Eq. 54,

$$\frac{1}{4}\bar{\mathbf{C}}^2 + \frac{1}{4}\bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} + \frac{1}{4}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} + \frac{1}{4}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right] - \tag{58}$$

$$\frac{1}{2}\bar{\mathbf{C}}^2 - \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} + \bar{\mathbf{K}}. \tag{59}$$

The terms without the square roots are cancelled such that

$$\frac{1}{4}\bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} + \frac{1}{4}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} - \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}}. \tag{60}$$

From Corollary 1.34 of [8], a special case of function defined on the spectra of the multiplication of two matrices is the square root function, that gives

$$[\mathbf{A}\mathbf{B}]^{\frac{1}{2}}\mathbf{A} = \mathbf{A}[\mathbf{B}\mathbf{A}]^{\frac{1}{2}}. \tag{61}$$

Let $\mathbf{A} = \bar{\mathbf{C}}$ and $\mathbf{B} = \bar{\mathbf{C}} - 4\bar{\mathbf{C}}^{-1}\bar{\mathbf{K}}$, thus,

$$\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} = \bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{C}}^{-1}\bar{\mathbf{K}}\bar{\mathbf{C}}\right]^{\frac{1}{2}}; \tag{62}$$

as $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ commute,

$$\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}} = \bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{C}}^{-1}\bar{\mathbf{C}}\bar{\mathbf{K}}\right]^{\frac{1}{2}} = \bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}}, \tag{63}$$

hence,

$$\frac{1}{4}\bar{\mathbf{C}}\left[\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}\right]^{\frac{1}{2}}+\frac{1}{4}\left[\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}}-\frac{1}{2}\left[\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}\right]^{\frac{1}{2}}\bar{\mathbf{C}}=\mathbf{0}, \tag{64}$$

therefore Eq. 57 is a solvent to Eq. 54 and

$$\mathbf{F}_{2,1,1}=\frac{1}{2}\bar{\mathbf{C}}+\frac{1}{2}\left[\bar{\mathbf{C}}^2-4\bar{\mathbf{K}}\right]^{\frac{1}{2}}. \tag{65}$$

Another interesting fact is that, as Eq. 57 was derived for a right solvent and satisfies a left solvent as well, such that

$$\mathbf{F}_{2,1,1}^2-\mathbf{F}_{2,1,1}\bar{\mathbf{C}}+\bar{\mathbf{K}}=\mathbf{F}_{2,1,1}^2-\bar{\mathbf{C}}\mathbf{F}_{2,1,1}+\bar{\mathbf{K}}=\mathbf{0}, \tag{66}$$

thus, by inspection, one can realize that $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute too and Eq. 46 is consistent when Rayleigh damping is used.

Another useful relations valid for Rayleigh damping are

$$\bar{\mathbf{C}}\mathbf{F}_{2,1,1}=\left(\alpha\mathbf{I}+\beta\bar{\mathbf{K}}\right)\mathbf{F}_{2,1,1}=\alpha\mathbf{F}_{2,1,1}+\beta\bar{\mathbf{K}}\mathbf{F}_{2,1,1} \tag{67}$$

and

$$\mathbf{F}_{2,1,1}\bar{\mathbf{C}}=\mathbf{F}_{2,1,1}\left(\alpha\mathbf{I}+\beta\bar{\mathbf{K}}\right)=\alpha\mathbf{F}_{2,1,1}+\beta\mathbf{F}_{2,1,1}\bar{\mathbf{K}}, \tag{68}$$

thus, as $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute, by comparing the last two equations if follows that $\bar{\mathbf{K}}$ and $\mathbf{F}_{2,1,1}$ also commute, *i.e.*, $\mathbf{F}_{2,1,1}\bar{\mathbf{K}}=\bar{\mathbf{K}}\mathbf{F}_{2,1,1}$.

Other form of damping leading to commutativity between $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,2}$ is

$$\mathbf{C}=\sum_j \beta_j\mathbf{K}^j, \tag{69}$$

for example.

A numerical method shall be used to find $\mathbf{F}_{2,1,1}$ when other models of constant damping failure to keep commutativity between the damping and the stiffness matrices are used [8].

3.2. Under damped problems

Equations 47 and 49 depend on

$$\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]\right) \tag{70}$$

and

$$\exp\left(-\mathbf{F}_{2,1,1}\right). \tag{71}$$

For under-damped problems, it is possible to show that

$$\exp\left(\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right)^*=\exp\left(-\mathbf{F}_{2,1,1}\right). \tag{72}$$

First, as $\exp\left(\mathbf{A}^*\right)=\exp\left(\mathbf{A}\right)^*$ it is possible to write

$$\mathbf{F}_{2,1,1}^*-\bar{\mathbf{C}}^*=-\mathbf{F}_{2,1,1} \tag{73}$$

and as $\bar{\mathbf{C}}$ is strictly real

$$\mathbf{F}_{2,1,1}^*=-\mathbf{F}_{2,1,1}+\bar{\mathbf{C}} \tag{74}$$

such that

$$\Re\left(\mathbf{F}_{2,1,1}\right) = -\Re\left(\mathbf{F}_{2,1,1}\right) + \bar{\mathbf{C}} \implies \Re\left(\mathbf{F}_{2,1,1}\right) = \frac{1}{2}\bar{\mathbf{C}} \tag{75}$$

and

$$-\Im\left(\mathbf{F}_{2,1,1}\right) = -\Im\left(\mathbf{F}_{2,1,1}\right). \tag{76}$$

Thus, for under-damped problems the imaginary part of $\mathbf{F}_{2,1,1}$ should be only related to the second term in the R.H.S of Eq 65

$$\Im\left(\mathbf{F}_{2,1,1}\right) = \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} \tag{77}$$

and this is true iff the matrix inside the square root is strictly real and has only real negative eigenvalues. The first condition is always fulfilled since both $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ are real matrices. Additionally, the term $\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}$ in Eq. 49 reduces to

$$\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} = -2\Im\left(\mathbf{F}_{2,1,1}\right) i \tag{78}$$

a purely imaginary matrix.

In the following, we investigate the conditions necessary to satisfy that

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} \tag{79}$$

has only real negative eigenvalues.

Structural damping

The simpler form of proportional damping is given by $\mathbf{C} = \beta\mathbf{K}$. In this case, the matrix inside the square root is

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} = \beta^2\bar{\mathbf{K}}^2 - 4\bar{\mathbf{K}}. \tag{80}$$

It is worth noticing that the traditional eigenvalue problem is

$$\left(\mathbf{K} - \lambda\mathbf{M}\right)\mathbf{x} = \mathbf{0} \tag{81}$$

where λ is a strictly real positive eigenvalue and $\mathbf{x}$ is the associated eigenvector. Pre-multiplying by $\mathbf{M}^{-1}$

$$\left(\bar{\mathbf{K}} - \lambda\mathbf{I}\right)\mathbf{x} = \mathbf{0}, \tag{82}$$

and multiplying by $\bar{\mathbf{K}}$

$$\bar{\mathbf{K}}^2\mathbf{x} = \lambda\bar{\mathbf{K}}\mathbf{x} = \lambda\lambda\mathbf{x} = \lambda^2\mathbf{x}. \tag{83}$$

Thus, Eq. 80 can also be written as

$$\beta^2\mathbf{X}\mathbf{\Lambda}^2\mathbf{X}^{-1} - 4\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1} \tag{84}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\bar{\mathbf{K}}$ and $\mathbf{X}$ a matrix with its corresponding eigenvectors. Arranging

$$\mathbf{X}\left(\beta^2\mathbf{\Lambda}^2 - 4\mathbf{\Lambda}\right)\mathbf{X}^{-1} \tag{85}$$

such that

$$\beta^2\lambda_i^2 - 4\lambda_i < 0 \;\forall i = 1..n \tag{86}$$

and β must satisfy

$$\beta < \sqrt{\frac{4}{\lambda_i}} \;\forall i = 1..n. \tag{87}$$

Proportional Damping

Considering

$$\bar{\mathbf{C}} = \alpha \mathbf{I} + \beta \bar{\mathbf{K}}. \tag{88}$$

The square of $\bar{\mathbf{C}}$ is

$$\bar{\mathbf{C}}^2 = \alpha^2 \mathbf{I} + 2\beta\alpha \bar{\mathbf{K}} + \beta^2 \bar{\mathbf{K}}^2, \tag{89}$$

such that the term inside the square root is

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} = \alpha^2 \mathbf{I} + 2\beta\alpha \bar{\mathbf{K}} + \beta^2 \bar{\mathbf{K}}^2 - 4\bar{\mathbf{K}}. \tag{90}$$

Using the change of basis

$$\mathbf{X}\left(\alpha^2 \mathbf{I} + 2\beta\alpha \boldsymbol{\Lambda} + \beta^2 \boldsymbol{\Lambda}^2 - 4\boldsymbol{\Lambda}\right)\mathbf{X}^{-1} \tag{91}$$

such that

$$\alpha^2 + 2\beta\alpha\lambda_i + \beta^2\lambda_i^2 - 4\lambda_i < 0 \ \forall i = 1..n. \tag{92}$$

There are many possibilities to find suitable values of α and β in the previous equation. On can notice that the last two terms should be negative or zero if β satisfies Eq. 87. If

$$\beta^2\lambda_i^2 - 4\lambda_i = -\epsilon_i \tag{93}$$

then

$$\bar{\beta} = \sqrt{\frac{-\epsilon_i + 4\lambda_i}{\lambda_i^2}}. \tag{94}$$

Thus,

$$\alpha^2 + \left(2\bar{\beta}\lambda_i\right)\alpha - \epsilon_i < 0 \ \forall i = 1..n, \tag{95}$$

such that

$$\bar{\alpha} < -\left(\bar{\beta}\lambda_i\right) \pm \sqrt{\left(\bar{\beta}\lambda_i\right)^2 + \epsilon_i}. \tag{96}$$

4. Particular solutions obtained by considering specific excitation functions

So far, conditions for obtaining the solution of systems of coupled second order differential equations were discussed. To this end, considering constant coefficients, it was shown that permanent solution is given by Eq. 47. Conditions to solve the quadratic equation associated to the integrating factor $\mathbf{F}_{2,1,1}$ were also discussed in details for proportional damping. Next sections are devoted to discuss further analytical solutions that can be obtained for some particular forms of excitations.

A common benchmark problem is proposed to evaluate the different formulations obtained in the next sections. The problem is a 3 DOFs system described by

$$\mathbf{M} = \begin{bmatrix} 2.0 & 0.0 & 0.0 \\ 0.0 & 2.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \tag{97}$$

$$\mathbf{K} = \begin{bmatrix} 6.0 & -4.0 & 0.0 \\ -4.0 & 6.0 & -2.0 \\ 0.0 & -2.0 & 6.0 \end{bmatrix} \times 10^2 \tag{98}$$

and

$$\mathbf{C} = \beta\mathbf{K}, \tag{99}$$

with

$$\mathbf{y}(0) = \dot{\mathbf{y}}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{100}$$

where the damping parameter β is specified in each example. This small problem was chosen since it makes easy to visualize the response. Nonetheless, it is a coupled system of ODEs and is able to show all the intended characteristics of the proposed approach.

4.1. Periodic excitations

Let the excitation vector be defined as

$$\mathbf{f}(t) = g_1(t)\mathbf{e}_1 + g_2(t)\mathbf{e}_2 + \ldots + g_n(t)\mathbf{e}_n, \tag{101}$$

where n is the dimension of the problem, *i.e.*, the problem has n degrees of freedom, and $g_j(t)$ is a function of time multiplying unitary vector $\mathbf{e}_j$. Normalizing by the mass matrix

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n. \tag{102}$$

Assuming

$$g_j(t) = \sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right), \tag{103}$$

where n_k is the number of terms, $c_{jk} \in \mathbb{R}$ is an amplitude, $\beta_{jk} = i\omega_{jk} \in \mathbb{C}$ a complex angular frequency and $\phi_{jk} \in \mathbb{C}$ a complex phase.

Substituting this excitation vector into the inner convolution of the particular solution in Eq. 47, one gets

$$\int \exp\left(\mathbf{F}_{2,1,1}t\right) \bar{\mathbf{f}}\,\mathrm{d}t = \int \exp\left(\mathbf{F}_{2,1,1}t\right) \sum_{j=1}^{n}\sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk}t + \phi_{jk}\right)\mathbf{v}_j\,\mathrm{d}t, \tag{104}$$

as the vectors $\mathbf{v}_j$ are independent of time, they can be left out of the integral to the right side and as the exponentials $c_{jk}\exp\left(\beta_{jk}t + \phi_{jk}\right)$ are not matrices, they can be commuted with the exponential of matrices,

$$\int \exp\left(\mathbf{F}_{2,1,1}t\right) \bar{\mathbf{f}}\,\mathrm{d}t = \sum_{j=1}^{n}\sum_{k=1}^{n_k} \int c_{jk} \exp\left(\beta_{jk}t + \phi_{jk}\right) \exp\left(\mathbf{F}_{2,1,1}t\right) \mathrm{d}t\mathbf{v}_j. \tag{105}$$

Each one of these integrals can be evaluated by parts

$$\int c_{jk} \exp\left(\beta_{jk}t + \phi_{jk}\right) \exp\left(\mathbf{F}_{\mathbf{2},\mathbf{1},\mathbf{1}}t\right) \mathrm{d}t = \int \left[\frac{c_{jk}}{\beta_{jk}}\dot{\left(\exp\left(\beta_{jk}t + \phi_{jk}\right)\right)}\right] \exp\left(\mathbf{F}_{\mathbf{2},\mathbf{1},\mathbf{1}}t\right) \mathrm{d}t =$$

$$\frac{c_{jk}}{\beta_{jk}} \exp\left(\beta_{jk}t + \phi_{jk}\right) \exp\left(\mathbf{F}_{\mathbf{2},\mathbf{1},\mathbf{1}}t\right)\Bigg|_{t_0}^{t} - \int \frac{c_{jk}}{\beta_{jk}} \exp\left(\beta_{jk}t + \phi_{jk}\right) \exp\left(\mathbf{F}_{\mathbf{2},\mathbf{1},\mathbf{1}}t\right) \mathbf{F}_{2,1,1}\,\mathrm{d}t, \tag{106}$$

where it is possible to neglect the boundary value at t_0 as it is implicit in the integration constant $\mathbf{C}_2$.

Grouping common terms and letting $\mathbf{F}_{2,1,1}$ out of the integral at the right side for it being constant

$$\int c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\mathbf{F}_{2,1,1} t\right) \mathrm{d}t \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right] = \frac{c_{jk}}{\beta_{jk}} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\mathbf{F}_{2,1,1} t\right), \tag{107}$$

such that

$$\int c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\mathbf{F}_{2,1,1}\right) t \,\mathrm{d}t = \frac{c_{jk}}{\beta_{jk}} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\mathbf{F}_{2,1,1} t\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right]^{-1}. \tag{108}$$

Substituting Eq. 108 into Eq. 105,

$$\int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \,\mathrm{d}t = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\mathbf{F}_{2,1,1} t\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j \tag{109}$$

For the second convolution in Eq. 47,

$$\int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \,\mathrm{d}t \,\mathrm{d}t = \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\mathbf{F}_{2,1,1} t\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j \,\mathrm{d}t, \tag{110}$$

as $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}}$ commute,

$$\int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \,\mathrm{d}t \,\mathrm{d}t = \int \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j \,\mathrm{d}t. \tag{111}$$

This equation can also be integrated by parts such that

$$\int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \bar{\mathbf{f}} \,\mathrm{d}t \,\mathrm{d}t = \sum_{j=1}^{n} \sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]^{-1} \left[\mathbf{I} + \frac{1}{\beta_{jk}} \mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j \tag{112}$$

again, as $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute, the particular solution, Eq. 47, is given by

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_j^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{I} + \frac{1}{\omega_j i}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]^{-1} \left[\mathbf{I} + \frac{1}{\beta_{jk}}\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j, \tag{113}$$

as $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$,

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\left[\mathbf{I} + \frac{1}{\beta_{jk}}\mathbf{F}_{2,1,1}\right]\left[\mathbf{I} + \frac{1}{\beta_{jk}}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]\right]^{-1} \mathbf{v}_j, \tag{114}$$

which simplifies to

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{I} + \frac{1}{\beta_{jk}}\bar{\mathbf{C}} - \frac{1}{\beta_{jk}^2}\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\right]^{-1} \mathbf{v}_j, \tag{115}$$

and, using Eq. 33, reduces to

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left(\left[\mathbf{I} + \frac{1}{\beta_{jk}}\bar{\mathbf{C}} - \frac{1}{\beta_{jk}^2}\bar{\mathbf{K}}\right]^{-1} \mathbf{v}_j\right), \tag{116}$$

It is worth noticing that previous expression can be further simplified. First, recall that $\mathbf{v}_j = \mathbf{M}^{-1}\mathbf{e}_j$. Also, as $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$

$$\left[\mathbf{I} + \frac{1}{\beta_{jk}}\bar{\mathbf{C}} + \frac{1}{\beta_{jk}^2}\bar{\mathbf{K}}\right]^{-1} \mathbf{M}^{-1} = \left(\mathbf{M}\left[\mathbf{I} + \frac{1}{\beta_{jk}}\bar{\mathbf{C}} + \frac{1}{\beta_{jk}^2}\bar{\mathbf{K}}\right]\right)^{-1} \tag{117}$$

such that

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=1}^{n_k} \frac{c_{jk}}{\beta_{jk}^2} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{M} + \frac{1}{\beta_{jk}}\mathbf{C} + \frac{1}{\beta_{jk}^2}\mathbf{K}\right]^{-1} \mathbf{e}_j. \tag{118}$$

It is also possible to manipulate the term $1/\beta_{j,k}^2$ by inserting it into the matrix

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\beta_{jk}^2\mathbf{M} + \beta_{jk}\mathbf{C} + \mathbf{K}\right]^{-1} \mathbf{e}_j \tag{119}$$

and as $\beta_{jk} = i\omega_{jk}$

$$\mathbf{y_p} = \sum_{j=1}^{n}\sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk} t + \phi_{jk}\right) \left[\mathbf{K} + i\omega_{jk}\mathbf{C} - \omega_{jk}^2\mathbf{M}\right]^{-1} \mathbf{e}_j \tag{120}$$

or

$$\mathbf{y_p}(t) = \sum_{j=1}^{n}\sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk}t + \phi_{jk}\right) \mathbf{K_D}_{jk}^{-1}\mathbf{e}_j, \tag{121}$$

where

$$\mathbf{K_D}_{jk} = \left[\mathbf{K} + i\omega_{jk}\mathbf{C} - \omega_{jk}^2\mathbf{M}\right] \tag{122}$$

is the Dynamic Stiffness Matrix for jk. It is worth noticing that particular solution given by Eq. 121, a superposition of harmonic responses, can be computed directly using $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ with no need to compute $\mathbf{F}_{2,1,1}$. Solution of Eq. 121 can be written as a linear combination of pre processed vectors $\mathbf{k}_{jk}$

$$\mathbf{K_D}_{jk}\mathbf{k}_{jk} = \mathbf{e}_j, \tag{123}$$

since these operations do not depend on t. Thus, $\mathbf{y_p}(t)$ can be efficiently evaluated as

$$\mathbf{y_p}(t) = \sum_{j=1}^{n}\sum_{k=1}^{n_k} c_{jk} \exp\left(\beta_{jk}t + \phi_{jk}\right) \mathbf{k}_{jk}, \tag{124}$$

for a given time t.

The first derivative of Eq. 121 with respect to time t

$$\dot{\mathbf{y}}_\mathbf{p}(t) = \sum_{j=1}^{n}\sum_{k=1}^{n_k} c_{jk}\beta_{jk} \exp\left(\beta_{jk}t + \phi_{jk}\right) \mathbf{k}_{jk}, \tag{125}$$

is needed to evaluate the constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ (Appendix A).

When there are complex-conjugate pairs in the β coefficients, an interesting implementation optimization can be made for Eq. 123. Let two matrices $\mathbf{K_D}_{j\xi}$ and $\mathbf{K_D}_{j\nu}$ be the dynamic stiffness matrices for the j-th degree of freedom, and let their β coefficients be complex-conjugate, *i.e.*, $\beta_{j\xi} = \beta_{j\nu}^*$; following from Eq. 122, $\mathbf{K_D}_{j\xi}$ and $\mathbf{K_D}_{j\nu}$ are also complex-conjugate. Thus, $\mathbf{k}_{j\xi}$ can be written as

$$\mathbf{k}_{j\xi} = \mathbf{K}_{\mathbf{D}j\xi}^{-1}\mathbf{e}_j = \left(\mathbf{K}_{\mathbf{D}j\nu}^*\right)^{-1}\mathbf{e}_j, \tag{126}$$

which, using the property of inverse of complex-conjugate matrices, is simplified to

$$\mathbf{k}_{j\xi} = \left(\mathbf{K}_{\mathbf{D}j\nu}^{-1}\right)^*\mathbf{e}_j = \left(\mathbf{K}_{\mathbf{D}j\nu}^{-1}\mathbf{e}_j\right)^* = \mathbf{k}_{j\nu}^*. \tag{127}$$

Thereby, when there are pairs of complex-conjugate β coefficients, just half of the dynamic stiffness matrices must be evaluated.

Example

Consider the 3 DOFs problem with excitation

$$\mathbf{f} = \begin{Bmatrix} 0 \\ 3sin(4t) \\ 0 \end{Bmatrix}, \tag{128}$$

and $\beta = 1 \times 10^{-2}$. This is an under damped harmonic problem with known permanent solution

$$\mathbf{y_p} = [\mathbf{K} + 4i\mathbf{C} - 16\mathbf{M}]^{-1} \begin{Bmatrix} 0 \\ 3 \\ 0 \end{Bmatrix} \tag{129}$$

with amplitude

$$|\mathbf{y_p}| = \begin{Bmatrix} 9.687 \\ 13.756 \\ 4.711 \end{Bmatrix} \times 10^{-3}, \tag{130}$$

such that

$$\mathbf{y_p}(t) = \sin(4t)|\mathbf{y_p}|. \tag{131}$$

Using the proposed formulation,

$$\mathbf{f} = g_2(t)\mathbf{e}_2 = 3\sin(4t) \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{132}$$

where, by using Euller's identity,

$$g_2(t) = 3\sin(4t) = \frac{3i}{2}\exp(-4it) - \frac{3i}{2}\exp(4it) \tag{133}$$

and $n_k = 2$, $c_{21} = \frac{3i}{2}$, $\omega_{21} = -4$, $\beta_{21} = -4i$, $c_{22} = \frac{-3i}{2}$, $\omega_{22} = 4$, $\beta_{22} = 4i$.

Particular solution can then be found by using Eq. 121

$$\mathbf{y_p} = \sum_{j=2}^{2}\sum_{k=1}^{2} c_{jk}\exp\left(\beta_{jk}t\right)\mathbf{K_D}_{jk}^{-1}\mathbf{e}_j = \frac{3i}{2}\exp(-4it)\,\mathbf{K_D}_{21}^{-1}\mathbf{e}_2 + \frac{-3i}{2}\exp(4it)\,\mathbf{K_D}_{22}^{-1}\mathbf{e}_2 \tag{134}$$

with

$$\mathbf{K_D}_{21} = \mathbf{K} - 4i\mathbf{C} - 16\mathbf{M}, \tag{135}$$

and

$$\mathbf{K_D}_{22} = \mathbf{K} + 4i\mathbf{C} - 16\mathbf{M}. \tag{136}$$

Term $\mathbf{z} = \mathbf{K_D}_{21}^{-1}\mathbf{e}_2$ is conjugate to $\mathbf{K_D}_{22}^{-1}\mathbf{e}_2$. Thus,

$$\mathbf{y_p}(t) = 3\left(\frac{i}{2}\exp(-4it)\mathbf{z} - \frac{i}{2}\exp(4it)\mathbf{z}^*\right) \tag{137}$$

or, by splitting $\mathbf{z}$ and $\mathbf{z}^*$ into their real and imaginary parts

$$\mathbf{y_p}(t) = 3\left(\frac{i}{2}\exp(-4it)\Re(\mathbf{z}) + \frac{i^2}{2}\exp(-4it)\Im(\mathbf{z}) - \frac{i}{2}\exp(4it)\Re(\mathbf{z}) + \frac{i^2}{2}\exp(4it)\Im(\mathbf{z})\right). \tag{138}$$

Collecting common terms

$$\mathbf{y_p}(t) = 3\underbrace{\left(\frac{i}{2}\exp(-4it) - \frac{i}{2}\exp(4it)\right)}_{\sin(4t)}\Re(\mathbf{z}) + 3\underbrace{\left(\frac{i}{2}\exp(-4it) + \frac{i}{2}\exp(4it)\right)}_{0}\Im(\mathbf{z}) \tag{139}$$

such that

$$\mathbf{y_p}(t) = 3\underbrace{\frac{i}{2}\left(\exp\left(-4it\right) - \exp\left(4it\right)\right)}_{\sin(4t)} \begin{Bmatrix} 3.229 \\ 4.585 \\ 1.570 \end{Bmatrix} \times 10^{-3} \tag{140}$$

or

$$\mathbf{y_p}(t) = \sin(4t) \begin{Bmatrix} 9.686 \\ 13.756 \\ 4.711 \end{Bmatrix} \times 10^{-3}, \tag{141}$$

the expected solution.

We can verify some conclusions obtained in previous sections. We start by computing

$$\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}} = \begin{bmatrix} -1187 & 788 & 2 \\ 788 & -1185 & 391 \\ 4 & 782 & -2362 \end{bmatrix} \tag{142}$$

which has real negative eigenvalues -2653.11, -1762.11 and -318.77. Thus, its square root is a complex matrix, as discussed in previous sections

$$\frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} = \begin{bmatrix} 16.026 & -6.29 & -0.410 \\ -6.29 & 15.62 & -2.53 \\ -0.820 & -5.06 & 24.03 \end{bmatrix} i. \tag{143}$$

Matrix $\bar{\mathbf{K}}$ has eigenvalues

$$\mathbf{\Lambda} = \begin{bmatrix} 79.85 & 0.0 & 0.0 \\ 0.0 & 445.49 & 0.0 \\ 0.0 & 0.0 & 674.66 \end{bmatrix} \tag{144}$$

such that inequalities given by Eq. 87 for $\beta = 1 \times 10^{-2}$ are also satisfied

$$1 \times 10^{-2} < \left\{ \sqrt{\frac{4}{79.85}}, \sqrt{\frac{4}{445.49}}, \sqrt{\frac{4}{674.66}} \right\}. \tag{145}$$

Matrix

$$\mathbf{F}_{2,1,1} = \frac{1}{2}\bar{\mathbf{C}} + \frac{1}{2}\left[\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right]^{\frac{1}{2}} = \begin{bmatrix} 1.5 & -1.0 & 0.0 \\ -1.0 & 1.5 & -0.5 \\ 0.0 & -1.0 & 3.0 \end{bmatrix} + \begin{bmatrix} 16.026 & -6.29 & -0.410 \\ -6.29 & 15.62 & -2.53 \\ -0.820 & -5.06 & 24.03 \end{bmatrix} i \tag{146}$$

and we can verify that $\mathbf{F}_{2,1,1}$ and $\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$ are complex-conjugate.

Homogeneous solution, Eq. 49, for this problem is

$$\mathbf{y_h} = \exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2 + \exp\left(\left(\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right)t\right)\mathbf{C}_1 \tag{147}$$

where

$$\mathbf{C}_1 = \begin{Bmatrix} 2.49 \\ 3.38 \\ 1.18 \end{Bmatrix} \times 10^{-4} + i \begin{Bmatrix} 2.36 \\ 2.87 \\ 1.08 \end{Bmatrix} \times 10^{-3} \tag{148}$$

and $\mathbf{C}_2 = \mathbf{C}_1^*$ are obtained by solving Eqs. A.8 and A.9. The maximum order of magnitude of the complex part of the homogeneous solution was 10^{-19}, therefore, the homogeneous solution is real-valued, as expected. Exponentials in Eq. 147 can be computed only once if a constant time step Δt is used, as discussed in Appendix F.

The real part of each DOFs of the complete solution $\mathbf{y}(t) = \mathbf{y_h}(t) + \mathbf{y_p}(t)$ is shown in Fig. 1 (solid lines). The solution $\mathbf{y}$ is compared to $\tilde{\mathbf{y}}$ obtained by using traditional Newmark-beta method with standard parameters and $\Delta t = 0.001$s (dotted lines). It is clear that both solutions match.

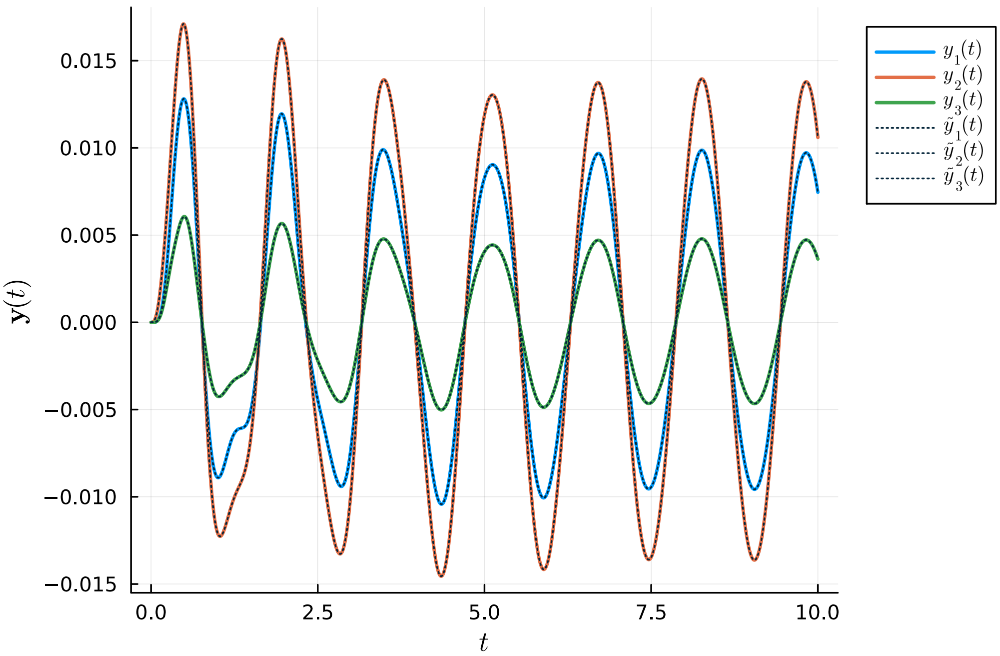

Figure 1: Complete solution for the under damped problem with sinusoidal excitation. Solutions y_1, y_2 and y_3, (real parts) obtained by using the proposed approach, are shown as solid lines. Solutions $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$, obtained by using the Newmark-beta method, are shown as dotted lines.

Example

We study the previous example with $\beta = 10$, an over damped problem. The proposed approach does not make any assumption on the level of damping such that the solution procedure used in the previous example does not change. The main differences are the fact that $\mathbf{F}_{2,1,1}$ is a real matrix

$$\mathbf{F}_{2,1,1} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -2 & 6 \end{bmatrix} \times 10^3, \tag{149}$$

as well as the integration constants

$$\mathbf{C}_1 = \begin{Bmatrix} 1.87 \\ 2.81 \\ 0.94 \end{Bmatrix} \times 10^{-4} \tag{150}$$

and

$$\mathbf{C}_2 = -\begin{Bmatrix} 5.61 \\ 6.23 \\ 2.40 \end{Bmatrix} \times 10^{-9}. \tag{151}$$

Solutions obtained by using the proposed approach (solid lines) and the Newmark-beta method with $\Delta t = 0.01$s (dotted lines) are shown in Fig. 2.

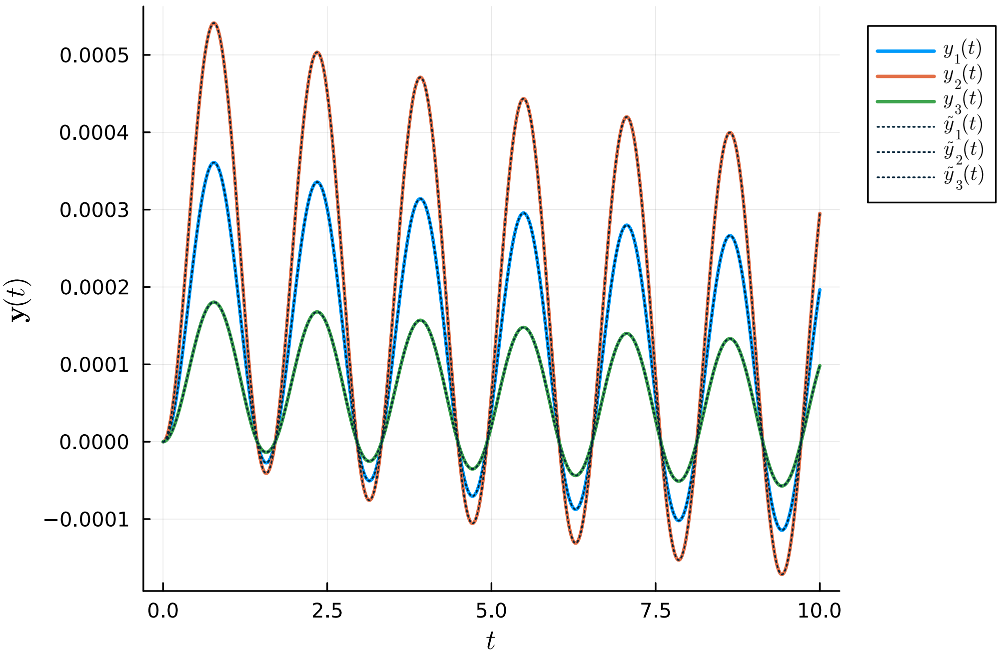

Figure 2: Complete solution for the over damped problem with sinusoidal excitation. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$, obtained by using the Newmark-beta method, are shown as dotted lines.

4.2. Polynomial excitation

Let the normalized excitation vector be defined as

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n, \tag{152}$$

where $g_j(t)$ are polynomial functions

$$g_j(t) = \sum_{k=0}^{n_k} c_{jk} \left(t - t_j\right)^k, \tag{153}$$

$c_{jk} \in \mathbb{R}$ are coefficients, n_k the number of terms and $t_j \in \mathbb{R}$ time shift.

Substituting in Eq. 47, yields

$$\mathbf{y_p} = e^{\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1}t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \left(t - t_j\right)^k \mathbf{v}_j \,\mathrm{d}t\,\mathrm{d}t, \tag{154}$$

using the linearity of the integral operator and as $\mathbf{v}_j$ does not depend on time results in

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1}t\right) \left(t - t_j\right)^k \mathrm{d}t\,\mathrm{d}t\,\mathbf{v}_j. \tag{155}$$

The inner convolution can be evaluated by parts (here we use e^x instead of $\exp(x)$ to shortening the equations),

$$\begin{aligned}
\int e^{\mathbf{A}t} \left(t - t_j\right)^\alpha \mathrm{d}t &= \int \left(e^{\mathbf{A}t}\dot{\mathbf{A}}^{-\mathbf{1}}\right) \left(t - t_j\right)^\alpha \mathrm{d}t \\
&= \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \int e^{\mathbf{A}t} \left(t - t_j\right)^{\alpha-1} \mathrm{d}t\alpha\mathbf{A}^{-1} \\
&= \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \alpha\left(t - t_j\right)^{\alpha-1} e^{\mathbf{A}t}\mathbf{A}^{-2} + \int e^{\mathbf{A}t} \left(t - t_j\right)^{\alpha-2} \mathrm{d}t\alpha\left(\alpha - 1\right)\mathbf{A}^{-2} \\
&= \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \alpha\left(t - t_j\right)^{\alpha-1} e^{\mathbf{A}t}\mathbf{A}^{-2} + \alpha\left(\alpha - 1\right)\left(t - t_j\right)^{\alpha-2} e^{\mathbf{A}t}\mathbf{A}^{-3} + \cdots + \\
&\quad \left(-1\right)^{m-1}\alpha\left(\alpha - 1\right)\ldots\left(\alpha - m + 2\right)\left(t - t_j\right)^{\alpha-m+1} e^{\mathbf{A}t}\mathbf{A}^{-m} + \\
&\quad \left(-1\right)^m \int e^{\mathbf{A}t} \left(t - t_j\right)^{\alpha-m} \mathrm{d}t\alpha\left(\alpha - 1\right)\ldots\left(\alpha - m + 1\right)\mathbf{A}^{-m} \\
&\overset{m=\alpha}{=} \left(t - t_j\right)^\alpha e^{\mathbf{A}t}\mathbf{A}^{-1} - \alpha\left(t - t_j\right)^{\alpha-1} e^{\mathbf{A}t}\mathbf{A}^{-2} + \alpha\left(\alpha - 1\right)\left(t - t_j\right)^{\alpha-2} e^{\mathbf{A}t}\mathbf{A}^{-3} + \cdots + \\
&\quad \left(-1\right)^{\alpha-1}\alpha!\left(t - t_j\right) e^{\mathbf{A}t}\mathbf{A}^{-\alpha} + \left(-1\right)^\alpha \alpha! e^{\mathbf{A}t}\mathbf{A}^{-\alpha-1}
\end{aligned} \tag{156}$$

which, by arranging all the terms in a single sum, yields

$$\int \exp\left(\mathbf{A}t\right)\left(t - t_j\right)^\alpha \mathrm{d}t = \exp\left(\mathbf{A}t\right) \sum_{l=1}^{\alpha+1} \left(-1\right)^{l+1} \frac{\alpha!}{\left(\alpha - l + 1\right)!} \left(t - t_j\right)^{\alpha-l+1} \mathbf{A}^{-l}. \tag{157}$$

Using this integral formula with Eq. 155 and the considering that $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute,

$$\begin{aligned}
\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \\
\sum_{l=1}^{k+1} \left(-1\right)^{l+1} \frac{k!}{\left(k - l + 1\right)!} \left(t - t_j\right)^{k-l+1} \mathbf{F}_{2,1,1}^{-l} \,\mathrm{d}t\,\mathbf{v}_j.
\end{aligned} \tag{158}$$

As the integral is a linear operator

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\int \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right) \left(t - t_j\right)^{k-l+1} \mathbf{F}_{2,1,1}^{-l} \,\mathrm{d}t \,\mathbf{v}_j, \tag{159}$$

and using again Eq. 157,

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!} \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t\right)$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-p} \mathbf{F}_{2,1,1}^{-l} \mathbf{v}_j. \tag{160}$$

Since $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute for Rayleigh damping, the matrix exponentials can be cancelled. From Eq. 33, it follows that $\bar{\mathbf{C}} - \mathbf{F}_{2,1,1} = \mathbf{F}_{2,1,1}^{-1}\bar{\mathbf{K}}$. Substituting this relation and applying the inverse property $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$ results in

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{F}_{2,1,1}^{l} \mathbf{F}_{2,1,1}^{-p} \bar{\mathbf{K}}^{p}\right]^{-1} \mathbf{v}_j, \tag{161}$$

and as $\mathbf{v}_j = \mathbf{M}^{-1}\mathbf{e}_j$ and using again the inverse property,

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\mathbf{F}_{2,1,1}^{l-p} \bar{\mathbf{K}}^{p}\right]^{-1} \mathbf{e}_j. \tag{162}$$

As $\bar{\mathbf{K}}$ and $\mathbf{F}_{2,1,1}$ commute for Rayleigh damping, Eq. 67, and p and l are integers, property in Eq. C.1 holds,

$$\mathbf{y_p}(t) = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$
$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^{p} \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j. \tag{163}$$

Derivative of previous equation with respect t is needed to evaluate constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$ (Appendix A)

$$\dot{\mathbf{y}}_{\mathbf{p}}(t) = \sum_{j=1}^{n}\sum_{k=0}^{n_k} c_{jk} \sum_{l=1}^{k+1} (-1)^{l+1} \frac{k!}{(k-l+1)!}$$

$$\sum_{p=1}^{k-l+2} (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} (k-l-p+2) \left(t-t_j\right)^{k-l-p+1} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j, \quad (164)$$

valid when $k-l-p+2$ is not zero.

Example

Consider the 3 DOF problem with $\beta = 1\times 10^{-6}$ and excitation

$$\mathbf{f} = \begin{Bmatrix} 0 \\ 10t - t^2 \\ 0 \end{Bmatrix}, \quad (165)$$

such that $g_2(t) = c_{20} + c_{21}t + c_{22}t^2$ with $c_{20} = 0$, $c_{21} = 10$, $c_{22} = -1$ and $t_2 = 0$.

Evaluation of particular solution $\mathbf{y}_{\mathbf{p}}(t)$, Eq. 163, is shown by means of Alg. 1 where $j = 2$ and $n_k = 2$. The terms obtained in the Algorithm result in

$$\mathbf{y}_{\mathbf{p}}(t) = (2t-10)\left(\left(\mathbf{M}\bar{\mathbf{K}}^2\mathbf{F}_{2,1,1}^{-1}\right)^{-1} + \left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1}\right)\mathbf{e}_2 + \left(10t - t^2\right)\left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1}\mathbf{e}_2 -$$
$$2\left(\mathbf{M}\bar{\mathbf{K}}^2\right)^{-1}\mathbf{e}_2 - 2\left(\mathbf{M}\bar{\mathbf{K}}^3\mathbf{F}_{2,1,1}^{-2}\right)^{-1}\mathbf{e}_2 - 2\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}^{2}\right)^{-1}\mathbf{e}_2 \quad (166)$$

The derivative of the particular response w.r.t time t is given by

$$\dot{\mathbf{y}}_{\mathbf{p}} = 2\left(\left(\mathbf{M}\bar{\mathbf{K}}^2\mathbf{F}_{2,1,1}^{-1}\right)^{-1} + \left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1}\right)\mathbf{e}_2 + (10-2t)\left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1}\mathbf{e}_2 \quad (167)$$

such that by using Eqs. A.8 and A.9, one gets the integration constants,

$$\mathbf{C}_1 = \begin{Bmatrix} 8.8792 \\ 0.1457 \\ 4.6479 \end{Bmatrix} \times 10^{-5} + i \begin{Bmatrix} 1.5639 \\ 1.9436 \\ 0.7237 \end{Bmatrix} \times 10^{-3}, \quad (168)$$

with $\mathbf{C}_2 = \mathbf{C}_1^*$.

Thus, by using the real part of Eq. 49, one gets the complete solution $\mathbf{y}(t) = \mathbf{y}_{\mathbf{h}}(t) + \mathbf{y}_{\mathbf{p}}(t)$ for the three DOFs, as shown in Fig. 3 (solid lines). The solution $\mathbf{y}$ is compared to the solution $\tilde{\mathbf{y}}$ obtained by using traditional Newmark-beta method with standard parameters and $\Delta t = 0.001$s (dotted lines). It is clear that both solutions match.

4.3. Dirac's delta distribution

Let the normalized excitation vector be defined as

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n, \quad (169)$$

Algorithm 1: Evaluation of Eq. 163 for $j = 2$ and $n_k = 2$.

$k = 0$

$c_{1,0} = 0$

$k = 1$

$c_{1,1} = 10$

$l = 1 \implies (-1)^{l+1} \frac{k!}{(k-l+1)!} = 1$

$p = 1 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$t \left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1} \mathbf{e}_2$

$p = 2 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$-\left(\mathbf{M}\bar{\mathbf{K}}^2 \mathbf{F}_{2,1,1}^{-1}\right)^{-1} \mathbf{e}_2$

$l = 2 \implies (-1)^{l+1} \frac{k!}{(k-l+1)!} = -1$

$p = 1 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1} \mathbf{e}_2$

$k = 2$

$c_{1,2} = -1$

$l = 1 \implies (-1)^{l+1} \frac{k!}{(k-l+1)!} = 1$

$p = 1 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$t^2 \left(\mathbf{M}\bar{\mathbf{K}}\right)^{-1} \mathbf{e}_2$

$p = 2 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$-2t \left(\mathbf{M}\bar{\mathbf{K}}^2 \mathbf{F}_{2,1,1}^{-1}\right)^{-1} \mathbf{e}_2$

$p = 3 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$2 \left(\mathbf{M}\bar{\mathbf{K}}^3 \mathbf{F}_{2,1,1}^{-2}\right)^{-1} \mathbf{e}_2$

$l = 2 \implies (-1)^{l+1} \frac{k!}{(k-l+1)!} = -2$

$p = 1 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$t \left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}\right)^{-1} \mathbf{e}_2$

$p = 2 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$-\left(\mathbf{M}\bar{\mathbf{K}}^2\right)^{-1} \mathbf{e}_2$

$l = 3 \implies (-1)^{l+1} \frac{k!}{(k-l+1)!} = 2$

$p = 1 \implies (-1)^{p+1} \frac{(k-l+1)!}{(k-l-p+2)!} \left(t - t_j\right)^{k-l-p+2} \left[\mathbf{M}\bar{\mathbf{K}}^p \mathbf{F}_{2,1,1}^{l-p}\right]^{-1} \mathbf{e}_j =$
$\left(\mathbf{M}\bar{\mathbf{K}}\mathbf{F}_{2,1,1}^2\right)^{-1} \mathbf{e}_2$

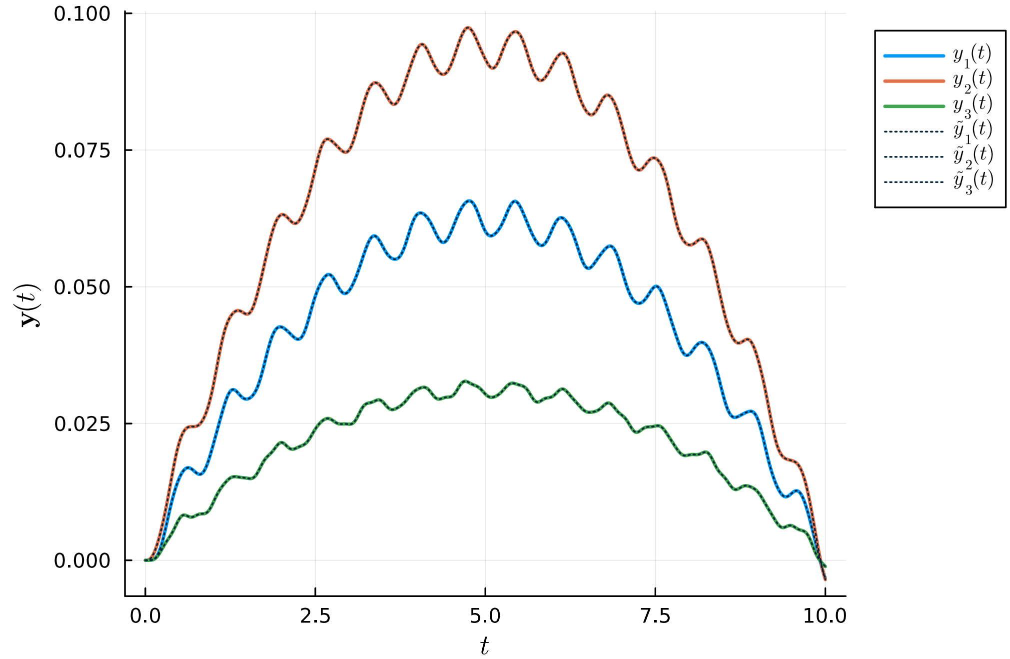

Figure 3: Complete solution for the three DOFs example subjected to a polynomial excitation. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$, obtained by using the Newmark-beta method are shown as dotted lines.

where $g_j(t)$ are Dirac's delta distributions

$$g_j(t) = \sum_{k=0}^{n_k} c_{jk}\delta\left(t - t_{jk}\right), \tag{170}$$

$c_{jk} \in \mathbb{R}$ are coefficients, n_k the number of terms and $t_{jk} \in \mathbb{R}$ time shifts.

Substituting in Eq. 47, yields

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk}\delta\left(t - t_{jk}\right) \mathbf{v}_j \,\mathrm{d}t\,\mathrm{d}t, \tag{171}$$

using the linearity of the integral operator and as $\mathbf{v}_j$ does not depend on time results in

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \delta\left(t - t_{jk}\right) \mathrm{d}t\,\mathrm{d}t\,\mathbf{v}_j. \tag{172}$$

Equation B.5, Appendix B, is used to evaluate the inner convolution

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \exp\left(\mathbf{F}_{2,1,1} t_k\right) \mathcal{H}\left(t - t_{jk}\right) \mathrm{d}t \, \mathbf{v}_j, \tag{173}$$

and, as $\bar{C} - 2\mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,1}$ commute, the two exponential maps can be simplified to

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t + \mathbf{F}_{2,1,1} t_k\right) \mathcal{H}\left(t - t_{jk}\right) \mathrm{d}t \, \mathbf{v}_j, \tag{174}$$

hence, by using the relation derived in [3]

$$\int_0^t \exp\left(\mathbf{A}t\right) f_k(t) \mathcal{H}(t - t_k) \, \mathrm{d}t = \left(\int_{t_k}^t \exp\left(\mathbf{A}t\right) f_k(t) \, \mathrm{d}t \right) \mathcal{H}(t - t_k), \tag{175}$$

where $\mathbf{A}$ is a matrix. The outer convolution is evaluated to

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \mathcal{H}\left(t - t_{jk}\right) \int_{t_k}^t \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t + \mathbf{F}_{2,1,1} t_k\right) \mathrm{d}t \, \mathbf{v}_j, \tag{176}$$

which is

$$\begin{aligned} \mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \mathcal{H}\left(t - t_{jk}\right) \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \Big[& \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t + \mathbf{F}_{2,1,1} t_k\right) \\ & - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t_k\right) \Big] \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{v}_j. \end{aligned} \tag{177}$$

The commutativity of the first power is checked,

$$\begin{aligned} \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t \left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t + \mathbf{F}_{2,1,1} t_k\right) = \\ t^2 \left(\mathbf{F}_{2,1,1}\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}^2 - \bar{\mathbf{C}}^2 + 2\bar{\mathbf{C}}\mathbf{F}_{2,1,1}\right) + t t_k \left(\mathbf{F}_{2,1,1}^2 - \bar{\mathbf{C}}\mathbf{F}_{2,1,1}\right) \end{aligned} \tag{178}$$

$$\begin{aligned} \left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] + \mathbf{F}_{2,1,1} t_k\right) \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t = \\ t^2 \left(\bar{\mathbf{C}}\mathbf{F}_{2,1,1} - 2\mathbf{F}_{2,1,1}^2 - \bar{\mathbf{C}}^2 + 2\mathbf{F}_{2,1,1}\bar{\mathbf{C}}\right) + t t_k \left(\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}}\right), \end{aligned} \tag{179}$$

as $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute, Equation 178 and Equation 179 are equal. Commutativity for the other power is straightforward, since a matrix commutes with itself regardless of different constants multiplying it. Hence, also substituting $\mathbf{v}_j = \mathbf{M}^{-1}\mathbf{e}_j$,

$$\mathbf{y_p} = \sum_{j=1}^{n} \left(\sum_{k=0}^{n_k} c_{jk} \mathcal{H}\left(t - t_{jk}\right) \Big[\exp\left(-\mathbf{F}_{2,1,1} t + \mathbf{F}_{2,1,1} t_k\right) \right.$$

$$\left. - \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t + \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] t_k \right) \Big] \right) \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{M}^{-1} \mathbf{e}_j, \tag{180}$$

rearranging the terms and applying the property $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$ results in

$$\mathbf{y_p} = \sum_{j=1}^{n} \left(\sum_{k=0}^{n_k} c_{jk} \mathcal{H}\left(t - t_{jk}\right) \Big[\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t\right)\right) \right.$$

$$\left. - \exp\left(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] \left(t_k - t\right) \right) \Big] \right) \left[\mathbf{C} - 2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{e}_j. \tag{181}$$

In Eq. 73, it was shown that, for under-damped problems, $\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$ is the complex-conjugate of $\mathbf{F}_{2,1,1}$, thus, according to [7], $\exp(\mathbf{A}^*) = \exp(\mathbf{A})^*$, and it follows that

$$\mathbf{y_p} = \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \mathcal{H}\left(t - t_{jk}\right) \Big[\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t\right)\right) - \exp\left(\mathbf{F}_{2,1,1}\left(t_k - t\right)\right)^* \Big] \left[\mathbf{C} - 2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{e}_j, \tag{182}$$

which further simplifies to

$$\mathbf{y_p} = 2i \sum_{j=1}^{n} \sum_{k=0}^{n_k} c_{jk} \mathcal{H}\left(t - t_{jk}\right) \Im\Big(\exp\left(\mathbf{F}_{2,1,1}\left(t_k - t\right)\right) \Big) \left[\mathbf{C} - 2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{e}_j. \tag{183}$$

Example

Consider the 3 DOF problem with $\beta = 1 \times 10^{-2}$ and excitation

$$\mathbf{f} = \begin{Bmatrix} 0 \\ \delta(t-1) - \delta(t-5) \\ 0 \end{Bmatrix}, \tag{184}$$

such that $g_2(t) = c_{20}\delta(t - t_0) + c_{21}\delta(t - t_1)$ with $c_{20} = 1$, $c_{21} = -1$, $t_0 = 1$ and $t_0 = 5$.
Using the conjugacy of $\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}$ and $\mathbf{F}_{2,1,1}$, one can use Eq. 183 directly,

$$\mathbf{y_p} = 2i \Big(\mathcal{H}(t-1) \Im\Big(\exp\left(\mathbf{F}_{2,1,1}(1-t)\right)\Big) - \mathcal{H}(t-5) \Im\Big(\exp\left(\mathbf{F}_{2,1,1}(5-t)\right)\Big) \Big) \left[\bar{\mathbf{C}} - 2\mathbf{M}\mathbf{F}_{2,1,1}\right]^{-1} \mathbf{e}_2,$$

and as shown, for homogeneous initial conditions and impulse excitation, $\mathbf{y} = \mathbf{y_p}$.

The response was compared to the solution obtained by using the traditional Newmark-beta method with $\Delta t = 0.001$. Each impulse was approximated by [6]

$$\delta(t - t_0) \approx \frac{1}{2\epsilon}\left(1 + \cos\left(\frac{\pi(t - t_0)}{\epsilon}\right)\right) \quad t_0 - \epsilon \leq t \leq t_0 + \epsilon. \tag{185}$$

with $\epsilon = \Delta t$ to impose the δs to the numerical method (the proposed methodology does not need such approximation). Solutions are shown in Fig. 4 where the solid lines correspond to the real part of the proposed approach and the dotted lines to the numerical solution (Newmark-beta method with $\Delta t = 0.001$s).

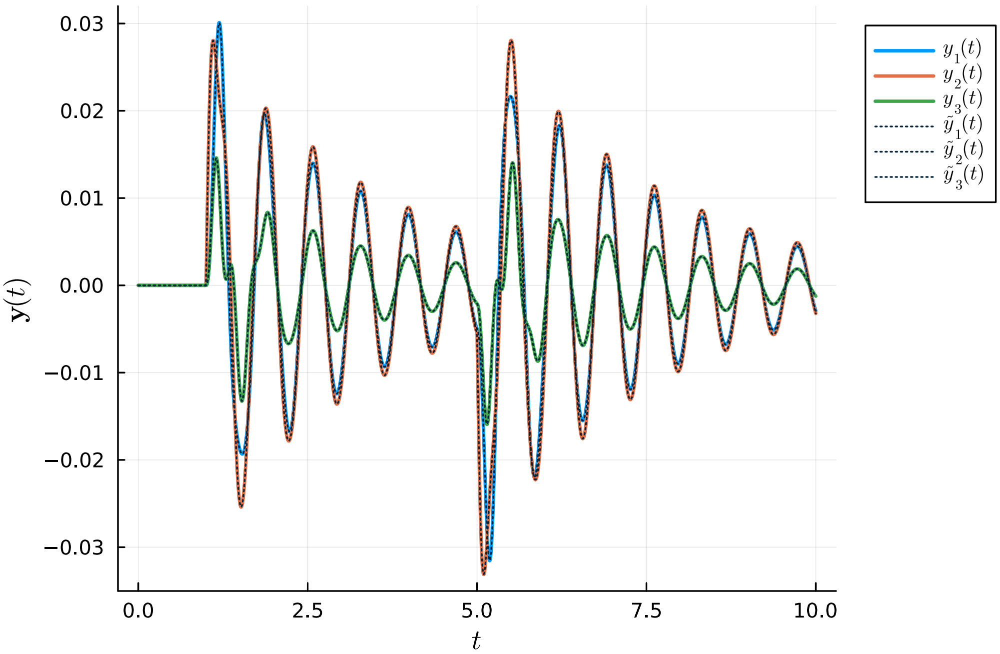

Figure 4: Complete solution for the three DOFs example subjected to two opposed Dirac's deltas at $t = 1$ and $t = 5$, respectively. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$, obtained by using the Newmark-beta method, are shown as dotted lines.

4.4. Heaviside step function

Let the normalized excitation vector be defined as

$$\bar{\mathbf{f}} = g_1(t)\mathbf{M}^{-1}\mathbf{e}_1 + \ldots + g_n(t)\mathbf{M}^{-1}\mathbf{e}_n = g_1(t)\mathbf{v}_1 + \ldots + g_n(t)\mathbf{v}_n, \tag{186}$$

where $g_j(t)$ are functions multiplied by other time-dependent functions,

$$g_j(t) = \sum_{k=0}^{n_k} f_{jk}(t)\mathcal{H}\left(t - t_{jk}\right), \tag{187}$$

$c_{jk} \in \mathbb{R}$ are coefficients, n_k the number of terms and $t_{jk} \in \mathbb{R}$ time shifts.

Substituting in Eq. 47, yields

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} f_{jk}(t) \mathcal{H}\left(t - t_{jk}\right) \mathbf{v}_j \,\mathrm{d}t \,\mathrm{d}t, \tag{188}$$

using the linearity of the integral operator and as $\mathbf{v}_j$ does not depend in time results in

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t) \mathcal{H}\left(t - t_{jk}\right) \mathrm{d}t \,\mathrm{d}t \,\mathbf{v}_j. \tag{189}$$

Equation 175 is used to evaluate the inner convolution,

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \int \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t) \,\mathrm{d}t \mathcal{H}\left(t - t_{jk}\right) \mathrm{d}t \,\mathbf{v}_j, \tag{190}$$

and, again, Equation 175 is used to evaluate the outer integral,

$$\mathbf{y_p} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathcal{H}\left(t - t_{jk}\right) \int_{t_{jk}}^{t} \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) f_{jk}(t) \,\mathrm{d}t \,\mathrm{d}t \,\mathbf{v}_j. \tag{191}$$

This solution depends on the function that multiplies the Heaviside function; however, the cases derived in previous subsections cover an extent set of widely used excitation functions, *e.g.*, harmonic, polynomial and constant (a constant function is a polynomial of order 0). Thus, the integrals in Eq. 191 can be found by using previous results as reference. In the following, we present the result for a second order polynomial.

4.4.1. Particularizing Heaviside excitation for second order polynomial

In Eq. 187, $f_{j,k}$ can be particularized to be a second order polynomial, *i.e.*,

$$g_j(t) = \sum_{k=0}^{n_k} \left(c_{jk0} + c_{jk1} t + c_{jk2} t^2\right) \mathcal{H}\left(t - t_{jk}\right) = \sum_{k=0}^{n_k} f_{jk}. \tag{192}$$

Applying this excitation in Eq. 191 yields

$$\begin{aligned}
\mathbf{y_p}^{(2)} = \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] t\right) \sum_{j=1}^{n} \sum_{k=0}^{n_k} \mathcal{H}\left(t - t_{jk}\right) \int_{t_{jk}}^{t} \exp\left(\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] t\right) \\
\left(\int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) c_{jk0} \,\mathrm{d}t + \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) c_{jk1} t \,\mathrm{d}t + \right. \\
\left. \int_{t_{jk}}^{t} \exp\left(\mathbf{F}_{2,1,1} t\right) c_{jk2} t^2 \,\mathrm{d}t \right) \mathrm{d}t \,\mathbf{v}_j.
\end{aligned} \tag{193}$$

The inner convolutions are analytically integrated using Eq. 157, and, rearranging the terms, one gets

$$\mathbf{y_p}^{(2)} = \exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t\right)\sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathcal{H}\left(t-t_{jk}\right)\int_{t_{jk}}^{t} c_{jk2}t^2\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\mathbf{F}_{2,1,1}^{-1}+ \\ t\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]t\right)\left(2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\right. \\ \left.-c_{jk1}\mathbf{F}_{2,1,1}^{-2}+c_{jk0}\mathbf{F}_{2,1,1}^{-1}\right)+\exp\left(\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]t\right)\left(-c_{jk2}t_{jk}^2\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}+\right. \\ 2c_{jk2}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}-c_{jk1}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1} \\ -2c_{jk2}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-3}+c_{jk1}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2} \\ \left.-c_{jk0}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}\right)\mathrm{d}t\,\mathbf{v}_j \quad (194)$$

which can be again convoluted by using Eq. 157. Multiplying by $\exp\left(\left[\mathbf{F}_{2,1,1}-\bar{\mathbf{C}}\right]t\right)$, yields

$$\mathbf{y_p}^{(2)} = \sum_{j=1}^{n}\sum_{k=0}^{n_k}\mathcal{H}\left(t-t_{jk}\right)\left(c_{jk2}t^2\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-1}+t\left(\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\left(\right.\right.\right. \\ \left.c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)-2c_{jk2}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}\left.\mathbf{F}_{2,1,1}^{-1}\right)+2c_{jk2}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-3}\mathbf{F}_{2,1,1}^{-1} \\ -\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\left(2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\right. \\ \left.-c_{jk1}\mathbf{F}_{2,1,1}^{-2}+c_{jk0}\mathbf{F}_{2,1,1}^{-1}\right)+\exp\left(-\mathbf{F}_{2,1,1}t\right)\left(\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right)^{-1}\left(\right. \\ -c_{jk2}t_{jk}^2\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}+2c_{jk2}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2} \\ -c_{jk1}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-3} \\ \left.+c_{jk1}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}-c_{jk0}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}\right)+\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{jk}-t\right)\right)\left(\right. \\ -c_{jk2}t_{jk}^2\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\mathbf{F}_{2,1,1}^{-1}+2c_{jk2}t_{jk}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}\mathbf{F}_{2,1,1}^{-1} \\ -2c_{jk2}\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-3}\mathbf{F}_{2,1,1}^{-1}-t_{jk}\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-2}\left(\right. \\ \left.c_{jk1}\mathbf{F}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)-\left(\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right)^{-1}\left.\left(2c_{jk2}\mathbf{F}^{-3}-c_{jk1}\mathbf{F}^{-2}+c_{jk0}\mathbf{F}^{-1}\right)\right)- \\ \exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{jk}-t\right)-\mathbf{F}_{2,1,1}t_{jk}\right)\left(\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right)^{-1}\left(\right. \\ -c_{jk2}t_{jk}^2\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}+2c_{jk2}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}- \\ c_{jk1}t_{jk}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-3}+ \\ \left.\left.c_{jk1}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-2}-c_{jk0}\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)\mathbf{F}_{2,1,1}^{-1}\right)\right)\mathbf{v}_j \quad (195)$$

By using Eqs. G.1 and D.1, the term $\exp\left(\mathbf{F}_{2,1,1}t_{jk}\right)$ can be commuted with $\left(\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right)^{-1}$. Thus, the expression can be further simplified to

$$
\begin{aligned}
\mathbf{y_p}^{(2)} = \sum_{j=1}^{n}\sum_{k=0}^{n_k} \mathcal{H}\left(t-t_{jk}\right)\Bigg\{ & c_{jk2}t^2\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1} + t\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\right. \\
& \left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right) - 2c_{jk2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\left.\mathbf{F}_{2,1,1}^{-1}\right) + 2c_{jk2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-3}\mathbf{F}_{2,1,1}^{-1} \\
& -\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\left(c_{jk1}\mathbf{F}_{2,1,1}^{-1}-2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\right)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\left(2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\right. \\
& \left.-c_{jk1}\mathbf{F}_{2,1,1}^{-2}+c_{jk0}\mathbf{F}_{2,1,1}^{-1}\right)+\exp\left(\mathbf{F}_{2,1,1}\left(t_{jk}-t\right)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1} \\
& \left(-\left(c_{jk2}t_{jk}^2+c_{jk1}t_{jk}+c_{jk0}\right)\mathbf{F}_{2,1,1}^{-1}+\left(2c_{jk2}t_{jk}+c_{jk1}\right)\mathbf{F}_{2,1,1}^{-2}-2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\right) \\
& +\exp\left(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]\left(t_{jk}-t\right)\right)\Bigg(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\Bigg(\mathbf{F}_{2,1,1}^{-1}\left(-c_{jk2}t_{jk}^2-c_{jk1}t_{jk}-c_{jk0}\right) \\
& +\mathbf{F}_{2,1,1}^{-2}\left(2c_{jk2}t_{jk}+c_{jk1}\right)-2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\Bigg)+\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-2}\Bigg(\mathbf{F}_{2,1,1}^{-1}\left(2c_{jk2}t_{jk}+c_{jk1}\right) \\
& -2c_{jk2}\mathbf{F}_{2,1,1}^{-2}\Bigg)-2c_{jk2}\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-3}\mathbf{F}_{2,1,1}^{-1}-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\Bigg(-\mathbf{F}_{2,1,1}^{-1}\Big(c_{jk2}t_{jk}^2 \\
& +c_{jk1}t_{jk}+c_{jk0}\Big)+\mathbf{F}_{2,1,1}^{-2}\left(2c_{jk2}t_{jk}+c_{jk1}\right)-2c_{jk2}\mathbf{F}_{2,1,1}^{-3}\Bigg)\Bigg)\Bigg\}\mathbf{M}^{-1}\mathbf{e}_j
\end{aligned} \tag{196}
$$

Two examples are discussed to show the flexibility provided by the proposed approach, since complicate loading scenarios cab be built by a linear combination of polynomials and Heavisides.

Example

Consider the 3 DOF problem with $\beta = 1\times 10^{-2}$ and excitation

$$
f_2(t) = \begin{cases} -30+40t-10t^2 & 1\le t\le 3 \\ 0 & t\in[0,1]\cup[3,\inf] \end{cases} \tag{197}
$$

or

$$
\mathbf{f} = \begin{Bmatrix} 0 \\ (-30+40t-10t^2)\mathcal{H}\left(t-1\right)+(30-40t+10t^2)\mathcal{H}\left(t-3\right) \\ 0 \end{Bmatrix}, \tag{198}
$$

such that $c_{200}=-30$, $c_{201}=40$, $c_{202}=-10$, $t_{20}=1$, $c_{210}=30$, $c_{211}=-40$, $c_{212}=10$ and $t_{21}=3$.

Using the real part of the solution given by Eq. 196 and also solving the problem by using the standard Newmark-beta numerical method lead to the complete responses shown in Fig. 5. Solution obtained by using Eq. 196 is shown as solid lines and solution obtained by using the Newmark method with $\Delta t = 0.001$ s is shown as dotted lines.

Example

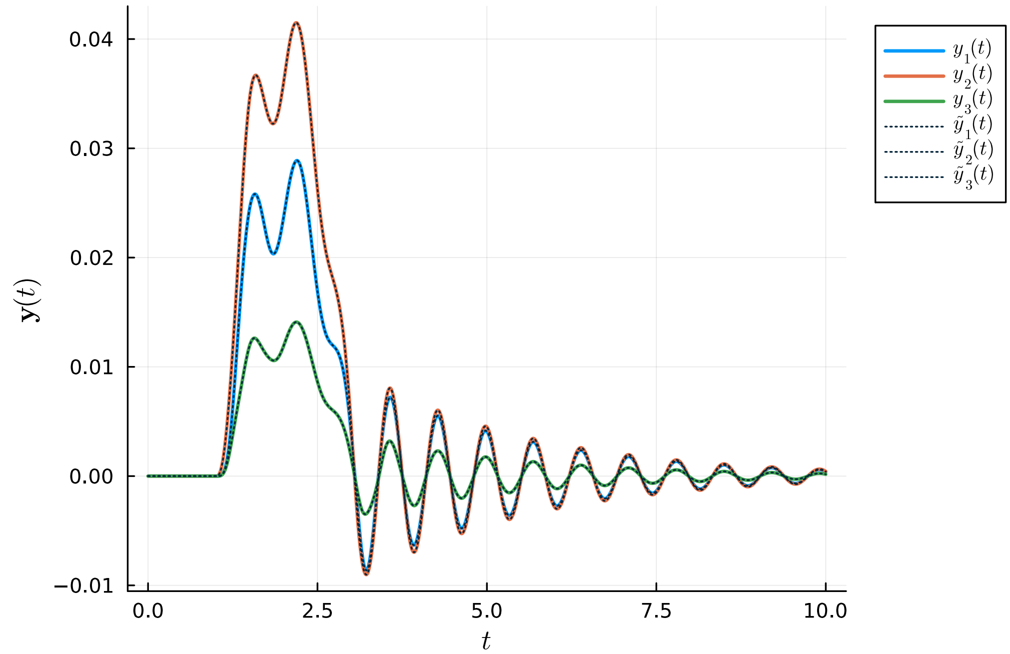

Figure 5: Complete solution for the three DOFs example subjected to a quadratic load between $t = 1$ and $t = 3$s. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$, obtained by using the Newmark-beta method, are shown as dotted lines.

Consider the 3 DOF problem with $\beta = 1 \times 10^{-2}$. Assuming an unitary step at DOF 2 between $t = 1$ and $t = 5$

$$\mathbf{f} = \begin{Bmatrix} 0 \\ \mathcal{H}(t-1) - \mathcal{H}(t-5) \\ 0 \end{Bmatrix}, \tag{199}$$

such that $g_2(t) = c_{200}\mathcal{H}(t - t_{20}) + c_{210}\mathcal{H}(t - t_{21})$ with $c_{200} = 1$, $c_{210} = -1$, $t_{20} = 1$ and $t_{21} = 5$. Solution provided by Eq. 196 can be particularized to order zero

$$\mathbf{y_p}^{(0)} = \sum_{j=1}^{n}\sum_{k=0}^{n_k} \mathcal{H}\left(t - t_{jk}\right) \left\{ \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} c_{jk0}\mathbf{F}_{2,1,1}^{-1} \right.$$
$$-c_{jk0}\exp\Big(\mathbf{F}_{2,1,1}\left(t_{jk} - t\right)\Big)\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1} + \exp\Big(\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\left(t_{jk} - t\right)\Big)$$
$$\left.\Big(-c_{jk0}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1} + c_{jk0}\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\Big)\right\}\mathbf{M}^{-1}\mathbf{e}_j. \tag{200}$$

Using the data for this example

$$
\begin{aligned}
\mathbf{y_p}^{(0)} = \mathcal{H}(t-1)\Big\{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1} \\
-\exp\left(\mathbf{F}_{2,1,1}(1-t)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+\exp\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](1-t)\Big) \\
\Big(-\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\Big)\Big\}\mathbf{M}^{-1}\mathbf{e}_2+ \\
\mathcal{H}(t-5)\Big\{\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}(-1)\mathbf{F}_{2,1,1}^{-1} \\
+\exp\left(\mathbf{F}_{2,1,1}(5-t)\right)\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}+\exp\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right](5-t)\Big) \\
\Big(\left[\bar{\mathbf{C}}-\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}-\left[\bar{\mathbf{C}}-2\mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1}\Big)\Big\}\mathbf{M}^{-1}\mathbf{e}_2. \qquad (201)
\end{aligned}
$$

Using the real part of the solution given by previous equation and also solving the problem using the standard Newmark-beta numerical method lead to the complete responses shown in Fig. 6. Solution obtained by using Eq. 201 is shown as solid lines and solution obtained by using the Newmark method with $\Delta t = 0.001$s is shown as dotted lines.

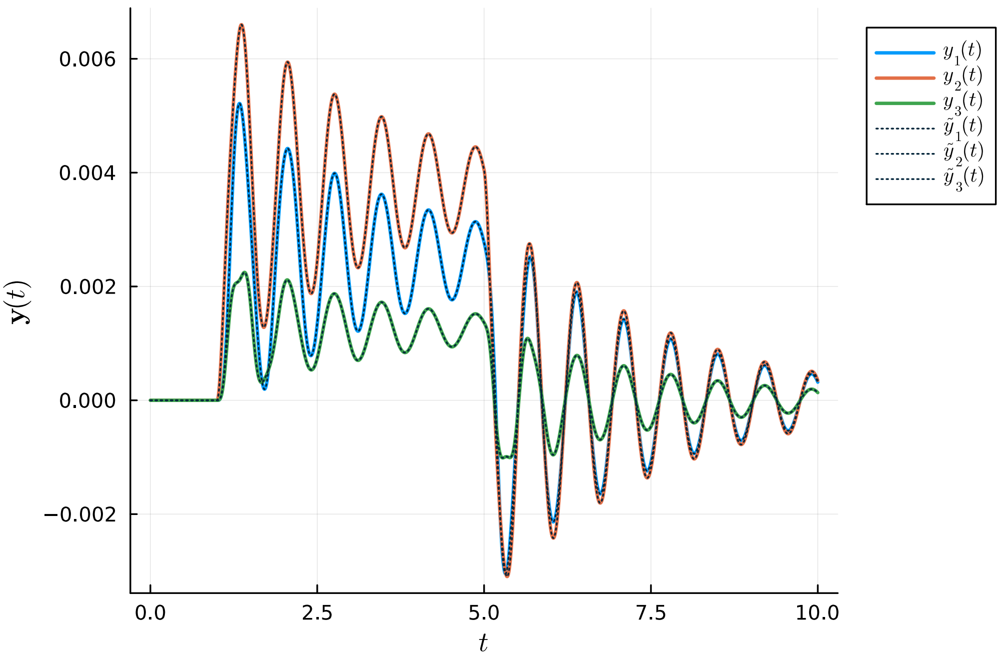

Figure 6: Complete solution for the three DOFs example subjected to an unitary step between $t = 1$ and $t = 5$s. Solutions y_1, y_2 and y_3, (real part) obtained by using the proposed approach, are shown as solid lines. Solutions $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_3$, obtained by using the Newmark-beta method are shown as dotted lines.

4.4.2. Initial conditions of Heaviside and Dirac's delta excitation

As all previous excitation functions were broken in the canonical basis of the $\mathbb{R}^n$, each integration was evaluated separately for each functional component and, then, multiplied by the respective basis vector. Nonetheless, for both the Heaviside and the Dirac's delta excitations, the component of the solution that multiply each basis vector has null response and derivative at the initial point. Consequently, the initial conditions for these excitations are zero such that $\mathbf{y_p}(t) = \mathbf{0}\ \forall t$ [3]. This result was observed in previous examples. The consideration of non homogeneous initial conditions is discussed in Appendix A.

4.4.3. Matrix complexity for polynomial particularized Heaviside

Heaviside excitation particularized to polynomial f_{jk} yields expensive matrices evaluations, like power and inverse operations. This is expected, since similar complexity is observed when addressing polynomial excitations. As it can be observed in Eq. 196, this complexity increases with the order of f_{jk}.

However, some calculation can be avoided by using the quadratic matrix equation, Eq. 34. For example, by factoring Eq. 34,

$$\mathbf{F}_{2,1,1}^2 - \mathbf{F}_{2,1,1}\bar{\mathbf{C}} + \bar{\mathbf{K}} = \mathbf{F}_{2,1,1}\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] + \bar{\mathbf{K}} = \mathbf{0}, \tag{202}$$

which is then rearranged to

$$\mathbf{F}_{2,1,1}\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right] = \bar{\mathbf{K}}. \tag{203}$$

Taking the inverse of both sides, results in

$$\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1}\mathbf{F}_{2,1,1}^{-1} = \bar{\mathbf{K}}^{-1} = \mathbf{K}^{-1}\mathbf{M}, \tag{204}$$

thus, the inverse of $\mathbf{F}_{2,1,1}$ can be calculated by

$$\mathbf{F}_{2,1,1}^{-1} = \left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]\mathbf{K}^{-1}\mathbf{M}; \tag{205}$$

conversely, the inverse of $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$ can be evaluated by

$$\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]^{-1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{F}_{2,1,1}. \tag{206}$$

Hence, the two most common inverses in Eq. 196 are evaluated using just one inverse, the inverse of $\mathbf{K}$. These relations can also be used to decrease the computational cost for polynomial excitation. The inverse of $\mathbf{K}$ is particularly interesting in the Finite Element Analysis (FMA) context, since the stiffness matrix is sparse, enabling the use of fast, efficient and tailored algorithms for this kind of matrices [13, 2]. Therefore, the inverse of more complicate matrices, like $\mathbf{F}_{2,1,1}$ and $\left[\bar{\mathbf{C}} - \mathbf{F}_{2,1,1}\right]$, can be computed by using an inverse that is cheaper to evaluate.

The matrix $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]$ is trickier to simplify, but it can be done using Eq. 65,

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] = -\left(\bar{\mathbf{C}}^2 - 4\bar{\mathbf{K}}\right)^{\frac{1}{2}}; \tag{207}$$

if $\bar{\mathbf{C}}$ is given by proportional damping,

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] = -\left(\beta^2\bar{\mathbf{K}}^2 + (2\alpha\beta - 4)\,\bar{\mathbf{K}} + \alpha^2\mathbf{I}\right)^{\frac{1}{2}}. \tag{208}$$

Equation 208 can be approximated if we assume both α and β as very small values. In such case

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right] \approx -2i\bar{\mathbf{K}}^{\frac{1}{2}}, \tag{209}$$

and the inverse of $\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]$ can be approximated to

$$\left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1} \approx \frac{i}{2}\left(\mathbf{M}^{-1}\mathbf{K}\right)^{-\frac{1}{2}} = \frac{i}{2}\left(\mathbf{K}^{-1}\mathbf{M}\right)^{\frac{1}{2}}. \tag{210}$$

Again, a matrix inverse can be calculated using the inverse of the stiffness matrix. The approximation presented in Eq. 209, however, can also be used in Eq. 65 to reduce computation costs. We did not use such approximations in the examples discussed in this manuscript.

5. Conclusion

This work extended the use of the Generalized Integrating Factor [3] to coupled systems of second order ODEs. Expressions for the time-dependent coefficient case were derived in general form. These forms were later particularized to the constant coefficient case where it was shown that under mild assumptions about coefficient matrix $\mathbf{C}$ the integrating factor can be found in closed form. Analytical particular solutions were derived for different forms of continuous and discontinuous excitations. Complicate expressions for loading can be generated by using a linear combination of polynomials multiplied by Heavisides and also by combining the analytical solutions derived in this manuscript.

Examples showed that the proposed approach is accurate and can be made efficient, not suffering with common issues found in traditional numerical approaches, like stability associated to time discretization and interpolation errors. Initial conditions can be imposed at any given time t_0, not only in the extremes of the interval. Actually, no time span is needed to evaluate the response when using the proposed approach.

An optimized way to calculate the matrix exponential in a discrete set of points was proposed to greatly reduce computational costs, which benefits the evaluation of the homogeneous solution and the solutions due to Dirac's delta and Heaviside series. Thereby, important real-world problems can be analytically solved by using the proposed approach.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The complete computer implementation of the theory discussed in this manuscript can be found at https://github.com/CodeLenz/Giffndof.jl with the scripts used to generate the images shown in this manuscript. The interested reader can freely change the parameters to study the proposed formulation.

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Appendix A. Evaluation of integration constants for coupled systems of ODEs with constant coefficients

The homogeneous solution with constant coefficients and when $\bar{\mathbf{C}}$ and $\mathbf{F}_{2,1,1}$ commute is given by Eq. 49. The complete solution

$$\mathbf{y} = \mathbf{y_h} + \mathbf{y_p} = \exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right)\mathbf{C}_1 + \mathbf{y_p}, \tag{A.1}$$

and its derivative w.r.t. time is given by

$$\dot{\mathbf{y}} = -\mathbf{F}_{2,1,1}\exp\left(-\mathbf{F}_{2,1,1}t\right)\mathbf{C}_2 + \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t\right)\mathbf{C}_1 + \dot{\mathbf{y}}_\mathbf{p}. \tag{A.2}$$

Considering initial conditions at a time t_0 yields

$$\mathbf{y}\left(t_0\right) = \exp\left(-\mathbf{F}_{2,1,1}t_0\right)\mathbf{C}_2 + \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t_0\right)\mathbf{C}_1 + \mathbf{y_p}\left(t_0\right), \tag{A.3}$$

and

$$\dot{\mathbf{y}}\left(t_0\right) = -\mathbf{F}_{2,1,1}\exp\left(-\mathbf{F}_{2,1,1}t_0\right)\mathbf{C}_2 + \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t_0\right)\mathbf{C}_1 + \dot{\mathbf{y}}_\mathbf{p}\left(t_0\right) \tag{A.4}$$

which can be summarized in a linear system

$$\begin{bmatrix} \exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t_0\right) & \exp\left(-\mathbf{F}_{2,1,1}t_0\right) \\ \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\exp\left(\left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]t_0\right) & -\mathbf{F}_{2,1,1}\exp\left(-\mathbf{F}_{2,1,1}t_0\right) \end{bmatrix} \begin{Bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 - \mathbf{y_p}\left(t_0\right) \\ \mathbf{v}_0 - \dot{\mathbf{y}}_\mathbf{p}\left(t_0\right) \end{Bmatrix}. \tag{A.5}$$

Hence, $\mathbf{C}_1$ and $\mathbf{C}_2$ can be found by solving the above linear system with standard techniques, much alike the evaluation of C_1 and C_2 for a single degree of freedom problem.

The most common choice for t_0 is 0 such that Eq. A.5 reduces to

$$\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right] & -\mathbf{F}_{2,1,1} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 - \mathbf{y_p}\left(0\right) \\ \mathbf{v}_0 - \dot{\mathbf{y}}_\mathbf{p}\left(0\right) \end{Bmatrix}. \tag{A.6}$$

Pivoting, it further simplifies to

$$\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_0 - \mathbf{y_p}\left(0\right) \\ \mathbf{v}_0 - \dot{\mathbf{y}}_\mathbf{p}\left(0\right) - \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\left(\mathbf{u}_0 - \mathbf{y_p}\left(0\right)\right) \end{Bmatrix}. \tag{A.7}$$

thus, it follows that

$$\mathbf{C}_2 = \left[\bar{\mathbf{C}} - 2\mathbf{F}_{2,1,1}\right]^{-1}\left(\mathbf{v}_0 - \dot{\mathbf{y}}_\mathbf{p}\left(0\right) - \left[\mathbf{F}_{2,1,1} - \bar{\mathbf{C}}\right]\left(\mathbf{u}_0 - \mathbf{y_p}\left(0\right)\right)\right), \tag{A.8}$$

and

$$\mathbf{C}_1 = \mathbf{u}_0 - \mathbf{y_p}\left(0\right) - \mathbf{C}_2. \tag{A.9}$$

A very important particularization for previous equations is when evaluating the constants $\mathbf{C}_1$ and $\mathbf{C}_2$ for excitations described by Dirac's deltas, Heavisides and Heaviside series. As discussed in [3], $\mathbf{y}_p(t)$ and $\dot{\mathbf{y}}(t)_p$ are zero for $t \leq t_H$, where t_H is the first time with non null excitation. Therefore, if $t_0 \leq t_H$, both $\mathbf{y}_p(t_0)$ and $\dot{\mathbf{y}}_p(t_0)$ are $\mathbf{0}$ in Eqs. A.5, A.8 and A.9 and there is no need to evaluate the derivative of the particular response with respect to time.

Appendix B. Convolution over Dirac's delta distribution

The convolution of a function over a Dirac's delta is usually defined as [11]

$$\int_{-\infty}^{\infty} f(t)\delta\left(t - t_k\right) dt = f\left(t_k\right). \tag{B.1}$$

The integration limits can be split as

$$\int_{-\infty}^{\infty} f(t)\delta\left(t - t_k\right) dt = \int_{-\infty}^{0} f(t)\delta\left(t - t_k\right) dt + \int_{0}^{t} f(t)\delta\left(t - t_k\right) dt + \int_{t}^{\infty} f(t)\delta\left(t - t_k\right) dt, \tag{B.2}$$

for t_k strictly positive, the integral from $-\infty$ to 0 is 0 by definition. Thus, the integral from 0 to t can be rewritten as

$$\int_{0}^{t} f(t)\delta\left(t - t_k\right) dt = \int_{-\infty}^{\infty} f(t)\delta\left(t - t_k\right) dt - \int_{t}^{\infty} f(t)\delta\left(t - t_k\right) dt. \tag{B.3}$$

The *filter* or *sifting* property of the delta of Dirac is due to the shape of this distribution, *i.e.*, it is null almost everywhere except at its discontinuity. As a consequence, the function multiplying the Dirac's delta is constant at this point, for the discontinuity of the delta distribution is infinitely close to t_k. Therefore, the value of the function can be taken out of the integral and the definition of the Dirac's delta is used to show that

$$\int_{-\infty}^{\infty} f(t)\delta\left(t-t_k\right)dt = \int_{t_k-\tau}^{t_k+\tau} f(t)\delta\left(t-t_k\right)dt = f(t_k)\int_{t_k-\tau}^{t_k+\tau} \delta\left(t-t_k\right)dt = f(t_k), \tag{B.4}$$

hence, Equation B.3 can be rewritten as

$$\int_0^t f(t)\delta\left(t-t_k\right)dt = f(t_k) - f(t_k)\begin{cases} 0, & t \geq t_k \\ 1, & t < t_k \end{cases} = f(t_k)\mathcal{H}(t-t_k). \tag{B.5}$$

Appendix C. Commutativity of powers of two matrices

Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices $n \times n$ that commute. For positive integer powers a and b, the following property holds

$$\mathbf{A}^a\mathbf{B}^b = \mathbf{B}^b\mathbf{A}^a. \tag{C.1}$$

Proof.

$$\begin{aligned}
\mathbf{A}^a\mathbf{B}^b &= \underbrace{\mathbf{AAA}\ldots\mathbf{AAA}}_{a\ \ times}\underbrace{\mathbf{BBB}\ldots\mathbf{BBB}}_{b\ \ times} \\
&= \mathbf{AAA}\ldots\mathbf{AABABB}\ldots\mathbf{BBB} \\
&= \mathbf{AAA}\ldots\mathbf{ABABAB}\ldots\mathbf{BBB} \\
&= \mathbf{AAA}\ldots\mathbf{BABABA}\ldots\mathbf{BBB} \\
&= \mathbf{AAA}\ldots\mathbf{ABABAB}\ldots\mathbf{BBB} \\
&\ \ \vdots \\
\underbrace{\mathbf{BBB}\ldots\mathbf{BBB}}_{b\ \ times}\underbrace{\mathbf{AAA}\ldots\mathbf{AAA}}_{a\ \ times} &= \mathbf{B}^b\mathbf{A}^a.
\end{aligned}$$

□

Appendix D. Commutativity of inverse of matrix

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ square matrices that commute and $\mathbf{B}$ be invertible. Then

$$\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A}. \tag{D.1}$$

Proof.

$$\begin{aligned}
\mathbf{AB}^{-1} &= \mathbf{D}, \\
\mathbf{A} &= \mathbf{DB}. \\
\mathbf{BA} = \mathbf{BDB} &= \mathbf{AB}, \\
\mathbf{BD} &= \mathbf{A}, \\
\mathbf{D} &= \mathbf{B}^{-1}\mathbf{A}, \\
\implies \mathbf{AB}^{-1} &= \mathbf{B}^{-1}\mathbf{A}.
\end{aligned}$$

□

Appendix E. Exponential map

An exponential map is defined as [7],

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2 t^2 + \ ... \ + \frac{1}{n!}\mathbf{A}^n t^n = \sum_{j=0}^{\infty} \frac{1}{j!}\mathbf{A}^j t^j, \tag{E.1}$$

with time-derivative given by

$$\frac{d}{dt}\big[\exp(\mathbf{A}t)\big] = \mathbf{A} + \mathbf{A}^2 t + \ ... \ + \frac{1}{(n-1)!}\mathbf{A}^n t^{n-1} = \sum_{j=0}^{\infty} \frac{1}{j!}\mathbf{A}^{j+1} t^j. \tag{E.2}$$

By direct comparison of Eq. E.1 to Eq. E.2, one immediately gets

$$\frac{d}{dt}\big[\exp(\mathbf{A}t)\big] = \mathbf{A}\exp(\mathbf{A}t) = \exp(\mathbf{A}t)\,\mathbf{A}. \tag{E.3}$$

Appendix F. Efficient evaluation of exp ($\mathbf{A}t$)

One critical component of the proposed formulation is the efficient evaluation of $\exp(\mathbf{A}t)$ for various values of t. According to [1], the following relation holds,

$$\exp(\mathbf{A})^{\alpha} = \exp(\alpha\mathbf{A}) \quad \forall\, \alpha \in \mathbb{Z}, \tag{F.1}$$

for general matrix $\mathbf{A}$

Assuming that the time span $t \in [t_i, t_f]$ is discretized in n_t intervals Δt, it is possible to write $t = k\Delta t$ such that

$$\mathcal{E}(t) = \exp(\mathbf{A}t) = \exp(k\Delta t\mathbf{A}) = \exp(\Delta t\mathbf{A})^k, \tag{F.2}$$

where $k \in \mathbb{Z}$. Hence, the exponential map, $\mathcal{E}(t)$, in a discrete set of time can be evaluated through the following recursion

$$\begin{aligned}
t_1 :&\quad \mathcal{E}(t_1) = \exp(\Delta t\mathbf{A}) \\
t_2 :&\quad \mathcal{E}(t_2) = \mathcal{E}(t_1)\exp(\Delta t\mathbf{A}) \\
t_3 :&\quad \mathcal{E}(t_3) = \mathcal{E}(t_2)\exp(\Delta t\mathbf{A}) \\
&\quad \vdots \\
t_k :&\quad \mathcal{E}(t_k) = \mathcal{E}(t_{k-1})\exp(\Delta t\mathbf{A}),
\end{aligned} \tag{F.3}$$

where matrix $\exp(\Delta t\mathbf{A})$ has to evaluated just once.

Appendix G. Commutativity of exponential map and matrix

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ square matrices. If $\mathbf{A}$ and $\mathbf{B}$ commute, then

$$\exp(\mathbf{A})\,\mathbf{B} = \mathbf{B}\exp(\mathbf{A})\,. \tag{G.1}$$

Proof. By definition

$$\begin{aligned}\exp(\mathbf{A})\,\mathbf{B} &= \left[\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 + \ \dots\ + \frac{1}{n!}\mathbf{A}^n t^n\right]\mathbf{B} \\ &= \mathbf{IB} + \mathbf{AB}t + \frac{1}{2}\mathbf{A}^2\mathbf{B}t^2 + \ \dots\ + \frac{1}{n!}\mathbf{A}^n\mathbf{B}t^n\end{aligned} \tag{G.2}$$

such that

$$\exp(\mathbf{A})\,\mathbf{B} = \mathbf{BI} + \mathbf{BA}t + \frac{1}{2}\mathbf{BA}^2t^2 + \ \dots\ + \frac{1}{n!}\mathbf{BA}^n t^n = \mathbf{B}\exp(\mathbf{A})\,.$$

□

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