The Einstein's Mass-Energy Equivalence and the Relativistic Mass and Momentum derived from the Newton's Second Law of Motion

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Abstract: In the Einstein's theory of special relativity, the relativistic mechanics is concerned with the motion of bodies whose velocities approach the speed of light. This paper presents the Einstein's mass-energy equivalence and the equations of relativistic mass, momentum and energy from the Newton's second law of motion.

Keywords: relativistic motion, momentum equation, kinetic energy, rest mass

1. Introduction

The equation of Einstein's mass-energy equivalence [1-3] is $E = mc^2$, where E, m, and c denote electromagnetic/light energy, mass of light, and speed of light respectively. The derivation of the equations of relativistic mass and mass-energy equivalence is obtained from the Newton's second law of motion by differentiation and integration [4].

2. Relativistic Mass, Momentum and Energy

In the Einstein's theory of special relativity, the relativistic momentum is concerned with the motion of particle whose velocity approaches the speed of light. Let us derive the relativistic mass, momentum, and energy as follows:

The Newton's second law of motion states that the force (F) acting on a particle is equal to the rate of change of its momentum (p).

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dv}{dt}, \text{where } v \text{ is velocity.}$$

The differential equation for work and kinetic energy is derived as follows:

$$dK = dW = Fds$$

$$dK = Fds = \left(m\frac{dv}{dt} + v\frac{dv}{dt}\right)ds$$

$$dK = Fds = m\frac{ds}{dt}dv + v\frac{ds}{dt}dv, \text{where } \frac{ds}{dt} = v$$

The kinetic energy per second is denoted by dK, which is equivalent to c^2dm. The variable mass of a moving particle at a second is denoted by dm.

$$dK = mvdv + v^2dm$$

$$c^2dm = mvdv + v^2dm$$

$$\frac{dm}{m} = \frac{v}{c^2 - v^2} dv$$

$$\int_{m_0}^{m} \frac{dm}{m} = \int_0^v \frac{v}{c^2 - v^2} dv$$

$$[\ln(m)]_{m_0}^{m} = -\frac{1}{2} [\ln(c^2 - v^2)]_0^v$$

$$\ln m - \ln m_0 = -\frac{1}{2} \ln(c^2 - v^2) + \frac{1}{2} \ln c^2$$

$$\ln \frac{m}{m_0} = \frac{1}{2} \ln \frac{c^2}{c^2 - v^2}$$

$$\frac{m}{m_0} = \sqrt{\frac{c^2}{c^2 - v^2}}$$

$$\text{Relativistic mass } (m) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Where,

- the rest mass of the body is m_0
- the velocity of the body in motion is v
- the speed of the light is c

$$\text{Relativistic momentum } (p) = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{Relativistic Energy } (E) = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The relation between relativistic energy and momentum show below:

$$E^2 = \frac{m_0^2 c^4}{1 - \frac{v^2}{c^2}} \Rightarrow E^2 = \frac{m_0^2 c^2 (v^2 - v^2 + c^2)}{1 - \frac{v^2}{c^2}} \Rightarrow E^2 = \frac{m_0^2 c^2 v^2 - m_0^2 c^2 v^2 + m_0^2 c^4}{1 - \frac{v^2}{c^2}}$$

$$Then, E^2 = \left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \right)^2 c^2 + \frac{m_0^2 c^2 (c^2 - v^2)}{\frac{c^2 - v^2}{c^2}}$$

From the above expression, we obtain the energy-momentum relation

$$E^2 = p^2 c^2 + m_0^2 c^2.$$

If particle is at rest, then $p = 0$. Thus, the rest energy is that $E = m_0 c^2$.

Now, it is understood that the relativistic mass, momentum, and energy are derived from the Newton's second law of motion.

3. Mass-Energy Equivalence

$$\text{Relativistic mass } (m) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$m^2 = \frac{{m_0}^2}{1-\frac{v^2}{c^2}} \Longrightarrow m^2(c^2 - v^2) = {m_0}^2c^2, \text{where rest mass energy} = {m_0}^2c^2$$

$$m^2c^2 - m^2v^2 = {m_0}^2c^2$$

By differentiating the equation, we get

$$2mc^2dm - 2mv^2dm - m^2 2vdv = 0,$$

where the rest mass (m_0) and the speed of light (c) are constant.

The kinetic energy (dK) is equivalent to c^2dm, where c is the speed of light.

$$dK = c^2dm$$

$$\int_0^K dK = \int_{m_0}^m c^2dm$$

$$K = c^2(m - m_0), \text{where } K \text{ is kinetic energy.}$$

Total Energy (E) = Kinetic Energy (K) + Rest Mass-Energy (m_0c^2)

$$E = c^2(m - m_0) + m_0c^2$$

$$E = c^2m - c^2m_0 + m_0c^2$$

$$E = c^2m$$

$$\text{Therefore, } E = mc^2.$$

Hence, the equation of Einstein's mass-energy equivalence is derived from Newton's Second Law of Motion.

4. Conclusion

The mass-energy equivalence along with relativistic mass, momentum, and energy play an important role in the Einstein's theory of special relativity. Also, the relativistic mechanics is concerned with the motion of bodies whose velocities approach the speed of light. In this article, the equation of Einstein's mass-energy equivalence and the relativistic mass and momentum equation have been derived from Newton's Second Law of Motion.

References

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