Chapter 2

Analytical Methods in Nonlinear Dynamics

Ed Habtour

This chapter provides an overview of the various nonlinear dynamic behaviors intrinsic to aerospace structures and systems during their operating conditions. Two analytical methods are described, namely the Harmonic Balance and Variation of Parameters, for solving the nonlinear equation of motion. Both methods are relatively simple yet efficient mathematical tools for solving nonlinear ordinary differential equations. The harmonic balance and variation of parameters methods are harmonic probing algorithms, which make them attractive tools in the field of nonlinear system identification (NSI) [1, 2], structural health monitoring (SHM) [3] and nondestructive Inspection (NDI) [4]. Key parameters such as structural stiffness, damping and inertia can be extracted via harmonic probing [5–8].

2.1 Characterizing Nonlinearities

Realistically, aerospace structural systems are not rigorously linear. Consequently, approximation based linear models have serious limitations because of the restrictive conditions of their application, such as low external and internal excitation amplitudes [9]. Linear models apply the principles of superposition and proportionality, which may generate misleading results. For example, to simulate multiaxial vibrations for a structural component in a helicopter, the military standard MIL-STD-810G recommends performing the vibration tests by sequentially applying uniaxial excitation to the component along three orthogonal axes (x, y, z). The test can be accomplished by exciting the structure one axis at a time and rotating it 90^o for each axis. Using the principle of superposition is an additive process. For example, the structure vibration response amplitude in the X direction due to excitation in X, Y and Z directions can be calculated as follows [10]:

$$X_{total} = X_{out}(X_{in}) + X_{out}(Y_{in}) + X_{out}(Z_{in}) \tag{2.1}$$

where, $X_{out}(X_{in})$, $X_{out}(Y_{in})$ and $X_{out}(Z_{in})$ are the output responses as a function input amplitude X_{in}, Y_{in} and Z_{in}, respectively. Another important properties of linear dynamic systems is the principle of proportionality. For example, if the input excitation amplitude is increased by α times, the vibration response is amplified by α. Both principles assume that the structural dynamic response is linear.

In aerospace systems, increasing the amplitude of excitation along multiple degrees of freedom amplifies and often couples multiple nonlinearities within the system, which exacerbate the inaccuracy linear models. To accurately estimate the dynamic response of dynamical systems, it is important to include the nonlinear coefficients in the analysis. Multi axial excitation [11, 12], gyroscopic effect [13], miniaturization [14, 15], complex internal stresses [16, 17], multimodal environmental conditions [10], acoustic thermal coupling [18], microscopic effects such as change in the microstructure [19]. Typically,

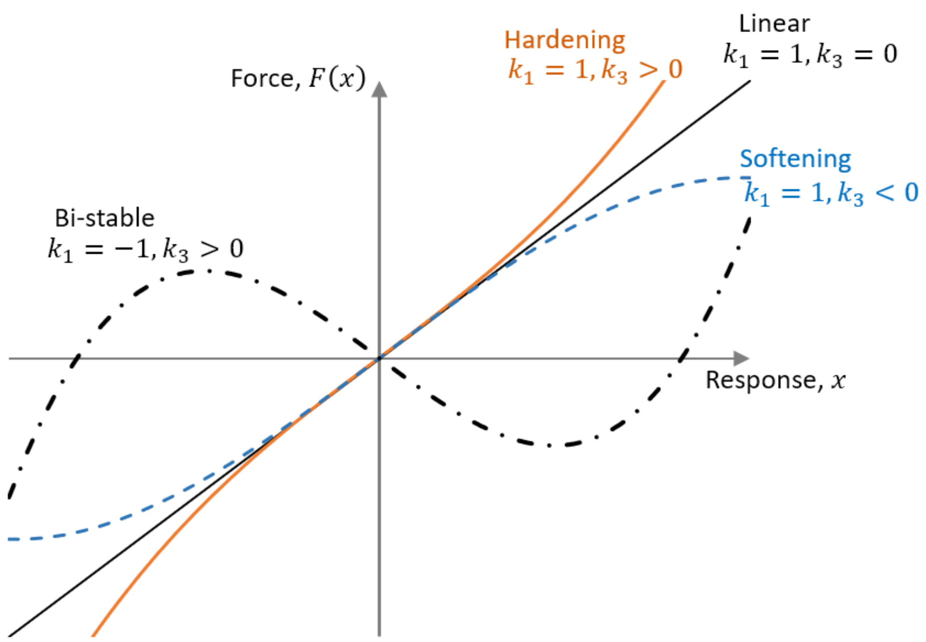

Figure 2.1: Characteristics of symmetrical and non-symmetrical restoring forces

Hooke's law is a good approximation for linear systems exhibiting elastic resistance to the application of external stimuli or loads in the form of restoring forces proportional to a structural displacement, x. For example, thin aerospace structures can endure reversible deformations before their inherent stress-strain characteristics show significant departure from their linear regime [20]. The common assumption is that slender thin structures, such as blades, can encounter kinematic (or geometric) nonlinearities before their materials' nonlinearities are reached [14]. In fact, kinematic nonlinearity generates dynamic hardening (or stiffening from materials prospective) at the first flexural mode, which is a restoring force proportional to a cubic structural displacement, x^3 [13], Figure 2.1. Thus, the total restoring force becomes a function of the linear and nonlinear stiffness coefficients, as follows:

$$F(x) = \frac{dP}{dx} = (k_1 + \alpha k_3 x^2)x \tag{2.2}$$

where, P is the potential energy:

$$P = \frac{1}{2}k_1 x^2 + \frac{1}{4}\alpha k_3 x^4 \tag{2.3}$$

The structural stiffness, and nonlinear geometric coefficients are k_1, and k_3, respectively. The system's equivalent structural stiffness becomes:

$$k_{eq} = k_1 + \alpha k_3 x^2 \tag{2.4}$$

If α is positive, it can augment the linear component of the restoring force, making the

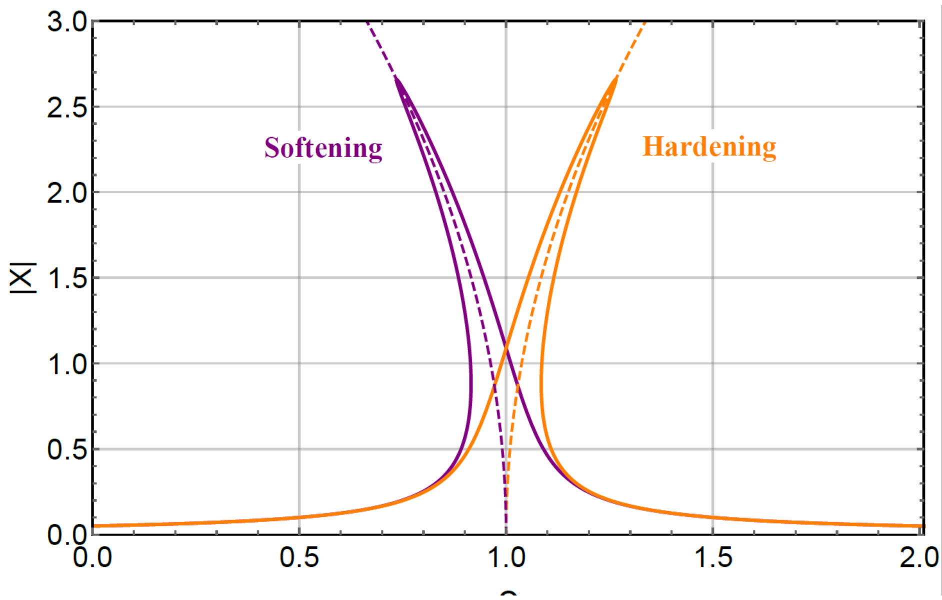

Figure 2.2: Characteristics of symmetrical and non-symmetrical restoring forces

system dynamical stiffer (hardening effect) and, subsequently, increasing the system's resonance [19], as shown in Figure 2.2. Unlike the case of a linear system, the maximum value of the amplitude response occurs at an excitation frequency higher than the structure natural frequency [19], as shown in Figure 2.2. A negative α, on the other hand, opposes the linear restoring force, thus, a structure may appear dynamically softer (softening effect) with reduced natural frequency. Consequently, the linear equation of motion with nonlinear restoring force resembles a damped Duffing oscillator, which contains an additional nonlinear geometric term [19]. The general form of equation of motion is give as:

$$\ddot{x} + c_1\dot{x} + m_n(x\ddot{x} + \dot{x}x^2) + k_1 x + \alpha k_3 x^3 = F\sin(\omega t) \tag{2.5}$$

The over-dots denote derivatives with respect to time, t. The system's distributive mass and displacement are m_s and x, respectively. The forcing amplitude and frequency are

F and ω, respectively. The structure damping coefficient is c_1. The nonlinear inertia is m_n.

The geometric nonlinearity appears in engineering structure where large deformations activates the nonlinear relation between strain and curvature, thus modifying the effective inertia and equivalent stiffness of the system [13]. The inertial nonlinearities, on the other hand, augment the effect of structural mass [5, 21]. Rarely do dynamics engineers take into account fatigue development due to high cycle vibrations as a parameter that could influence the change in the restoring force. Thus, integrating SHM into the aircraft control-system can ameliorate significant engineering challenges such as aeroelastic instability, and fatigue due to vibratory and thermal loads. The basic idea is to correct an aircraft operational drift due to the development of precursors to fatigue by adapting the control laws to operate below high stress regimes based on the aircraft new health state, thus, prolonging the aircraft service life [21].

2.2 Nondimensional Analysis

When analyzing physical systems, it is key to identify the important parameters that govern the physical behaviors of the systems. Thus, it is necessary to identify a complete set of independent quantities in order to simplify and parameterize the physical models that require measured units. Nondimensional procedures are effective in identifying the intrinsic physical attributes of a dynamical system while removing their units. This allows us to scale dynamic problems without compromising the physics. Making the variables dimensionless allows for a meaningful comparative analysis and insightful system identification. Developing dimensionless models can be accomplished by establishing mathematical scaling with an appropriate order of magnitude parameters [22]. The units of physical quantities can be removed from the governing equations by using characteristic units to scale those quantities.

2.2.1 Linear Systems

The equation of motion for free vibration in a SDoF system takes the following form:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.6}$$

The initial position and velocity of the system are $x(t=0) = L$, and $\dot{x}(t=0) = 0$. The system's characteristic length, L can be the initial displacement of a reference geometric parameter. Time, t, can be transformed into a dimensionless variable by utilizing the inverse of the system's natural frequency, as follows:

$$\tau = \omega_n t, \qquad \frac{d\tau}{dt} = \omega_n \tag{2.7}$$

where, τ is the characteristic time. The nondimensional displacement, η, is related to the characteristic length, L, as follows:

$$\eta = \frac{x}{L} \qquad \frac{d\eta}{dx} = \frac{1}{L} \tag{2.8}$$

Thus, the nondimensional velocity and acceleration are transformed through the following steps:

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = L\omega_n \frac{d\eta}{d\tau} = L\omega_n \eta' \tag{2.9}$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2x}{d\tau}\frac{d\tau^2}{dt^2} = L\omega_n^2 \frac{d^2\eta}{d\tau^2} = L\omega_n^2 \eta'' \tag{2.10}$$

Upon substituting the dimensionless variables in Eqs (5)-(8) into Eq (4) yields the following:

$$mL\omega_n^2\ddot{\eta} + 2\zeta mL\omega_n^2\dot{\eta} + kL\eta = 0 \tag{2.11}$$

Dividing the equation above by the coefficient of the highest ordered term, $mL\omega_n^2$, results in the final dimensionless equation of motion:

$$\eta'' + 2\zeta\eta'' + \eta = 0 \tag{2.12}$$

where,

$$\eta(\tau = 0) = 1, \qquad \dot{\eta}(\tau = 0) = 0 \tag{2.13}$$

The damping ratio, ζ, is related to the damping coefficient c in the following manner:

$$2\zeta = \frac{c}{m\omega_n} \tag{2.14}$$

This system depends only on the parameter, 2ζ, which is the ratio of the damping force to both the inertial and restoring forces. The oscillatory motion in a SDoF system due to forced excitation has the following mathematical form:

$$m\ddot{x} + c\dot{x} + kx = F\sin(\omega t - \phi) \tag{2.15}$$

The time-periodic force function has a forcing amplitude, F, and phase angle ϕ. The frequency of the periodic oscillation of the forcing function is ω. Knowing that:

$$\tau = t\omega_n = t\sqrt{k/m} \qquad \frac{d\tau}{dt} = \omega_n \tag{2.16}$$

The forcing function dimensionless frequency and amplitude become:

$$\Omega = \frac{\omega}{\omega_n} \qquad \hat{F} = \frac{F}{mL\omega_n^2} \tag{2.17}$$

The dimensionless displacement, velocity, and acceleration are:

$$\eta = \frac{x}{L} \qquad \frac{d\eta}{dx} = \frac{1}{L} \tag{2.18}$$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = L\omega_n \frac{d\eta}{d\tau} = L\omega_n \eta' \tag{2.19}$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2x}{d\tau}\frac{d\tau^2}{dt^2} = L\omega_n^2 \frac{d^2\eta}{d\tau^2} = L\omega_n^2 \eta'' \tag{2.20}$$

Substitution into the equation of motion yields:

$$mL\omega_n^2\eta'' + 2\zeta mL\omega_n^2\eta' + kL\eta = \hat{F}mL\omega_n^2\sin(\Omega\tau - \phi) \tag{2.21}$$

Dividing by $mL\omega_n^2$ results in the dimensionless equation of motion.

$$\eta'' + 2\zeta\eta' + \eta = \hat{F}\sin(\Omega\tau - \phi) \tag{2.22}$$

The final dimensionless parameters or coefficients are:

$$2\zeta = \frac{c}{m\omega_n}, \qquad \hat{F} = \frac{F}{mL\omega_n^2} \tag{2.23}$$

Imposing a periodic force on the system, produces only two dimensionless parameters, but all of them are related to the system's resonance frequency, ω_n.

2.3 Nonlinear Systems

2.3.1 Natural Decay

The equation of motion for free vibration single DoF system with cubic nonlinear coefficient is:

$$m\ddot{x} + c\dot{x} + k_1 x + k_2 x^2 + k_3 x^3 = 0 \tag{2.24}$$

Performing the same dimensionless steps in Section 2.3, the cubic equation of motion becomes:

$$mL\omega_n^2\eta'' + 2\zeta mL\omega_n^2\eta' + k_1 L\eta + k_2(L\eta)^2 + k_3(L\eta)^3 = 0 \tag{2.25}$$

Dividing by $mL\omega_n^2$ generates in following dimensionless nonlinear equation of motion:

$$\eta'' + 2\zeta\eta' + \eta + \kappa_2\eta^2 + \kappa_3\eta^3 = 0 \tag{2.26}$$

The system now depends on the following three dimensionless parameters:

$$2\zeta = \frac{c}{m\omega_n}, \qquad \kappa_2 = \frac{k_2 L}{k_1}, \qquad \kappa_3 = \frac{k_3 L^2}{m\omega_n^2} = \frac{k_3 L^2}{k_1} \tag{2.27}$$

2.3.2 Nonlinear Response Due to a Forcing Function

The oscillatory response of a single DoF system with quadratic and cubic stiffness coefficients due to the application of a periodic force is expressed as follows:

$$m\ddot{x} + c\dot{x} + k_1 x + k_2 x^2 + k_3 x^3 = F\sin(\omega t - \phi) \tag{2.28}$$

Following the same process in the previous section, the dimensionless equation of motion becomes:

$$\eta'' + 2\zeta\eta' + \eta + \kappa_2\eta^2 + \kappa_3\eta^3 = \hat{F}\sin(\Omega\tau - \phi) \tag{2.29}$$

The final dimensionless parameters expressed below are related to the system linear restoring force (or its resonance):

$$2\zeta = \frac{c}{m\omega_n} = \frac{c}{k_1}\omega_n, \qquad \kappa_2 = \frac{k_2}{k_1}L, \qquad \kappa_3 = \frac{k_3 L^2}{k_1}, \qquad \hat{F} = \frac{F}{mL\omega_n^2} \tag{2.30}$$

The oscillatory response of a single DoF system with cubic stiffness due to the application of a periodic acceleration is expressed as follows:

$$m\ddot{x} + c\dot{x} + k_1 x + k_2 x^2 + k_3 x^3 = -m\ddot{y} \tag{2.31}$$

Following the same process in the previous section:

$$mL\omega_n^2\ddot{\eta} + 2\zeta mL\omega_n^2\dot{\eta} + k_1 L\eta + k_2(L\eta)^2 + k_3(L\eta)^3 = -mL\omega_n^2 Y'' \tag{2.32}$$

After dividing by $mL\omega_n^2$ dimensionless equation of motion becomes:

$$\eta'' + 2\zeta\eta' + \eta + \kappa_2\eta^2 + \kappa_3\eta^3 = -Y'' \tag{2.33}$$

The final dimensionless parameters expressed below are related to the system linear restoring force (or its resonance):

$$2\zeta = \frac{c}{m\omega_n} = \frac{c}{k_1}\omega_n, \qquad \kappa_2 = \frac{k_2}{k_1}L, \qquad \kappa_3 = \frac{k_3}{k_1}L^2, \qquad Y = \frac{y}{L}, \qquad Y'' = \frac{d^2Y}{d\tau^2} = \frac{\ddot{y}}{L\omega_n^2} \tag{2.34}$$

2.4 Harmonic Balance

The Harmonic Balance Method (HBM) is one of the most widely methods for approximating periodic solutions for oscillatory systems in many applications [21, 23]. The method can be applied to both linear and nonlinear dynamical systems with periodic forcing function. Unlike time-marching numerical techniques, HBC can provide direct steady-state periodic solution of the nonlinear dynamic behavior due to a periodic forcing function. This can be achieved by expressing the solution in the Fourier space via a specified number of terms (harmonics) in the series. When solving oscillatory systems, HBM can be an attractive and efficient alternative to the expensive time-domain methods, if transient is not present. Thus, the response and forcing function have the same minimal period. However, Nyfeh and Mook described the harmonic method as an analytical approach that equivalent to performing stepped-sine testing [24].

2.4.1 Cubic Nonlinear System Under External Force

$$\eta'' + 2\zeta\eta' + \eta + \kappa_3\eta^3 = \hat{F}\sin(\Omega\tau - \phi) \tag{2.35}$$

Using a trial solution $\eta = Y\sin(\Omega\tau)$ yields:

$$-\Omega^2 Y\sin(\Omega\tau) + 2\zeta\Omega Y\cos(\Omega\tau) + Y\sin(\Omega\tau) + \kappa_3 Y^3\sin^3(\Omega\tau) = \hat{F}\sin(\Omega\tau - \phi) \tag{2.36}$$

After applying the following trigonometry identities: $(\sin 3\theta = 3\sin\theta - 4\sin^3\theta)$ & $(\sin(A-B) = \sin A\cos B - \cos A\sin B)$ to Eq. 2.36, the equation of motion becomes:

$$\begin{aligned} -\Omega^2 Y\sin(\Omega\tau) + 2\zeta\Omega Y\cos(\Omega\tau) + Y\sin(\Omega\tau) + \kappa_3 Y^3\left(\tfrac{3}{4}\sin(\Omega\tau) - \tfrac{1}{4}\sin(3\Omega\tau)\right) \\ = \hat{F}\sin(\Omega\tau)\cos\phi - \hat{F}\cos(\Omega\tau)\sin\phi \end{aligned} \tag{2.37}$$

Equating the coefficients of $\sin(\Omega\tau)$ and $\cos(\Omega\tau)$ yields the following equations:

$$-\Omega^2 Y + Y + \tfrac{3}{4}\kappa_3 Y^3 = \hat{F}\cos\phi \tag{2.38}$$

$$2\zeta\Omega Y = -\hat{F}\sin\phi \tag{2.39}$$

Squaring and adding Eqs 2.38 and 2.39 yield:

$$\hat{F}^2 = Y^2\left[\left(-\Omega^2 + 1 + \tfrac{3}{4}\kappa_3 Y^2\right)^2 + 4\zeta^2\Omega^2\right] \tag{2.40}$$

$$\frac{\hat{F}^2}{Y^2} = \Omega^4 + 2\left(2\zeta^2 - 1 - \tfrac{3}{4}\kappa_3 Y^2\right)\Omega^2 + \tfrac{3}{2}\kappa_3 Y^2 + \tfrac{9}{16}\kappa_3^2 Y^4 + 1 \tag{2.41}$$

The frequency response for the nonlinear system can be expressed in a form that resemble the linear FRF, but has a nonlinear geometric term:

$$\frac{\hat{Y}^2}{F^2} = \frac{1}{\left(-\Omega^2 + 1 + \tfrac{3}{4}\kappa_3 Y^2\right)^2 + 4\zeta^2\Omega^2} \tag{2.42}$$

An expression for Ω is achieved by obtaining the roots of Eq 2.41:

$$\Omega_{1,2}^2 = \left(1 - 2\zeta^2 + \frac{3}{4}\kappa_3 Y^2\right) \pm \sqrt{-4\zeta^2\left(1 - \zeta^2 + \frac{3}{4}\kappa_3 Y^2\right) + \frac{\hat{F}^2}{Y^2}} \tag{2.43}$$

The backbone or the system nonlinear resonance is:

$$\Omega_b^2 = 1 - 2\zeta^2 + \frac{3}{4}\kappa_3 Y^2 \tag{2.44}$$

2.4.2 Cubic Nonlinear Systems Exposed to Based Excitation

$$\eta'' + 2\zeta\eta' + \eta + \kappa_2\eta^2 + \kappa_3\eta^3 = -Y'' \tag{2.45}$$

We set $Y'' = Y_a\sin(\Omega\tau)$, where Y_a is dimensionless acceleration amplitude. Using a trial solution $\eta = \eta_o\sin(\Omega\tau)$:

$$-\Omega^2\eta_o\sin(\Omega\tau) + 2\zeta\Omega\eta_o\cos(\Omega\tau) + \eta_o\sin(\Omega\tau) + \kappa_3\eta_o^3\sin^3(\Omega\tau) = -Y_a\sin(\Omega\tau - \phi) \tag{2.46}$$

After applying some trigonometry: $(\sin 3\theta = 3\sin\theta - 4\sin^3\theta)$ & $(\sin(A-B) = \sin A\cos B - \cos A\sin B)$, the following is obtained:

$$\begin{aligned} -\Omega^2\eta_o\sin(\Omega\tau) + 2\zeta\Omega\eta_o\cos(\Omega\tau) + \eta_o\sin(\Omega\tau) + \kappa_3\eta_o^3\left(\tfrac{3}{4}\sin(\Omega\tau) - \tfrac{1}{4}\sin(3\Omega\tau)\right) \\ = -Y_a\sin(\Omega\tau)\cos\phi + Y_a\cos(\Omega\tau)\sin\phi \end{aligned} \tag{2.47}$$

Equating the coefficients of $\sin(\Omega\tau)$ and $\cos(\Omega\tau)$ yields the following equations:

$$-\Omega^2\eta_o + \eta_o + \tfrac{3}{4}\kappa_3\eta_o^3 = -Y_a\cos\phi \tag{2.48}$$

$$2\zeta\Omega\eta_o = Y_a\sin\phi \tag{2.49}$$

Squaring and adding the equations results in:

$$Y_a^2 = \eta_o^2\left[\left(-\Omega^2 + 1 + \tfrac{3}{4}\kappa_3\eta_o^2\right)^2 + 4\zeta^2\Omega^2\right] \tag{2.50}$$

$$\frac{Y_a^2}{\eta_o^2} = \Omega^4 + 2\left(2\zeta^2 - 1 - \tfrac{3}{4}\kappa_3\eta_o^2\right)\Omega^2 + \tfrac{3}{2}\kappa_3\eta_o^2 + \tfrac{9}{16}\kappa_3^2\eta_o^4 + 1 \tag{2.51}$$

The frequency response for the nonlinear system can be expressed in a form that resemble the FRF:

$$\frac{\eta_o^2}{Y_a^2} = \frac{1}{\left(-\Omega^2 + 1 + \frac{3}{4}\kappa_3\eta_o^2\right)^2 + 4\zeta^2\Omega^2} \tag{2.52}$$

The frequency response for the nonlinear system can be expressed in a form that resemble the FRF:

$$\frac{\eta_o^2}{Y_o^2} = \frac{1}{\left(-\Omega^2 + 1 + \frac{3}{4}\kappa_3\eta_o^2\right)^2 + 4\zeta^2\Omega^2} \tag{2.53}$$

Upon applying the quadratic formula, the roots of the equation are obtained, which yields the following expression for Ω:

$$\Omega_{1,2}^2 = \left(1 - 2\zeta^2 + \frac{3}{4}\kappa_3\eta_o^2\right) \pm \sqrt{-4\zeta^2\left(1 - \zeta^2 + \frac{3}{4}\kappa_3\eta_o^2\right) + \frac{Y_o^2}{\eta_o^2}} \tag{2.54}$$

The backbone is:

$$\Omega_b^2 = 1 - 2\zeta^2 + \frac{3}{4}\kappa_3\eta_o^2 \tag{2.55}$$

The frequency response for the Harmonic Balance and numerical methods for the dy-

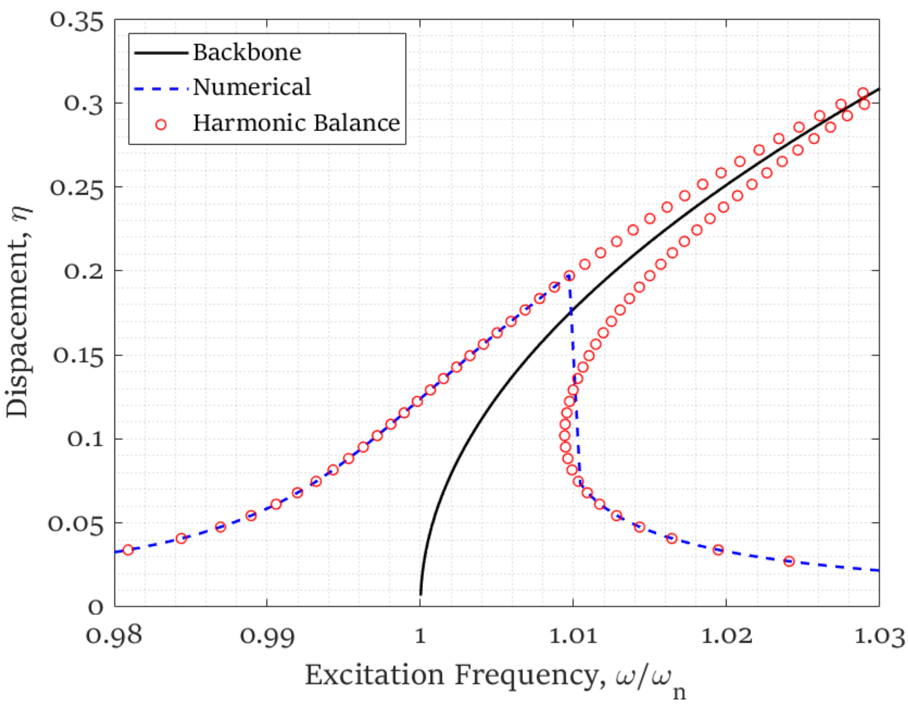

Figure 2.3: Frequency response of a cubic system using the harmonic balance and numerical methods in red and blue curves, respectively. The backbone is the black curve.

namic hardening case are shown in Figure 2.3. The backbone is plotted as well. Notice the sudden drop in the amplitude at the bifurcation point in the numerical solution. The Harmonic balance does not take bifurcation effect into account.

2.4.3 Nonlinear System with Quadratic Restoring Force

$$\ddot{\eta}+2\zeta\dot{\eta}+\eta+\kappa_2\eta^2=\hat{F}\sin(\Omega\tau-\phi) \tag{2.56}$$

The trial solution for the quadratic system is $\eta = Y_o + Y_1\sin(\Omega\tau)$, because $\eta = Y\sin(\Omega\tau)$ would give incomplete results. Therefore,

$$\eta^2 = Y_o^2+2Y_oY_1\sin(\Omega\tau)+Y_1^2\sin^2(\Omega\tau) = Y_o^2+2Y_oY_1\sin(\Omega\tau)+\tfrac{1}{2}Y_1^2-\tfrac{1}{2}Y_1^2\cos(\Omega\tau) \tag{2.57}$$

Substituting the trial solution into the equation of motion, yields:

$$\begin{aligned}-\Omega^2Y_1\sin(\Omega\tau)+2\zeta\Omega Y_1\cos(\Omega\tau)+Y_0+Y_1\sin(\Omega\tau)\\ +\kappa_2Y_0^2+2\kappa_2Y_0Y_1\sin(\Omega\tau)+\tfrac{1}{2}\kappa_2Y_1^2-\tfrac{1}{2}\kappa_2Y_1^2\cos(2\Omega\tau)\\ =\hat{F}\sin(\Omega\tau)\cos\phi-\hat{F}\cos(\Omega\tau)\sin\phi\end{aligned} \tag{2.58}$$

Equating the coefficients of $\sin(\Omega\tau)$ and $\cos(\Omega\tau)$ yields the following equations:

$$\begin{aligned}-\Omega^2Y_1\sin(\Omega\tau)+Y_1\sin(\Omega\tau)+2\kappa_2Y_0Y_1\sin(\Omega\tau)=\hat{F}\sin(\Omega\tau)\cos\phi\\ 2\zeta\Omega Y_1\cos(\Omega\tau)=\hat{F}\cos(\Omega\tau)\sin\phi\end{aligned} \tag{2.59}$$

Squaring and adding the equations results in:

$$\hat{F}^2=Y_1^2\Big[\big(-\Omega^2+1+2\kappa_2Y_0\big)^2+4\zeta^2\Omega^2\Big] \tag{2.60}$$

The interpret this result is not straight forward due to the presence of the two variables Y_0, and Y_1 in the final equation (Fourier Series).

$$\hat{F}^2=Y^2\left[\left(-\Omega^2+1+\tfrac{8\kappa_2Y}{3\pi}\right)^2+4\zeta^2\Omega^2\right] \tag{2.61}$$

$$\frac{\hat{F}^2}{Y^2}=\Omega^4+\big(4\zeta^2-2-\tfrac{16\kappa_2Y}{3\pi}\big)\Omega^2+\frac{64\kappa_2^2Y^2}{9\pi^2}+\tfrac{6}{4}\kappa_3Y^2+1 \tag{2.62}$$

Upon applying the quadratic formula, the roots of the equation are obtained, which yields the following expression for Ω:

$$\Omega_{1,2}^2=\left(1-2\zeta^2+\tfrac{8\kappa_2Y}{3\pi}\right)\pm\sqrt{-4\zeta^2\left(1-\zeta^2+\tfrac{8\kappa_2Y}{3\pi}\right)+\frac{\hat{F}^2}{Y^2}} \tag{2.63}$$

$$Backbone: \Omega_b^2=1-2\zeta^2+\tfrac{8\kappa_2Y}{3\pi} \tag{2.64}$$

The frequency response for the Harmonic Balance for the linear, nonlinear inertia, cubic and quadratic cases with their prospective backbones are shown in Figure 2.4.

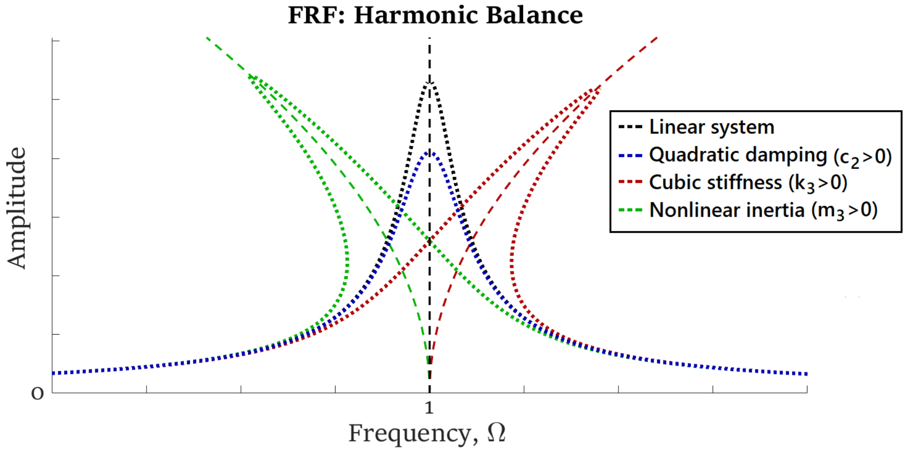

Figure 2.4: Frequency response of linear, cubic, quadratic, nonlinear inertia systems using the harmonic balance and numerical methods with their prospective backbone curves.

2.4.4 Applying Fourier Series

In general, periodic solutions can be estimated using truncated Fourier series, where coefficients are determined by constructing series that satisfy the deferential equations [25]. The solutions accuracy can be improved by increasing the number of terms obtained from the Fourier series into trial solutions. Theoretically, the complete matching can be reached if the solution is represented by a complete Fourier series. For practical reasons, solutions can be approximated using finite sums Fourier series with reasonable accuracy. When dynamic response is represented with a of Fourier series for all time, t, the various variables such as acceleration, velocity and cubic response will have their own Fourier series. Upon assembling the contribution of every terms into the equation of motion, matching of the harmonics (or balancing the harmonics) is performed, which leads to an autoperiodic response to an angular frequency, ω.

Starting with a quadratic damping system:

$$\ddot{\eta} + k_1\eta + c_1\dot{\eta}^2 + c_2\eta\dot{\eta}^2 = 0 \tag{2.65}$$

Assuming: $\eta = \eta_1 \sin(\Omega\tau) = \eta_1 \sin(\phi)$

$$-\phi^2\eta_1 \sin\phi + k_1\eta_1 \sin\phi + c_1\phi^2\eta_1^2 \cos^2\phi + c_2\phi^2\eta_1^3 \sin\phi \cos^2\phi = F\sin(\phi - \theta) \tag{2.66}$$

$$\begin{aligned} -\phi^2\eta_1 \sin\phi + k_1\eta_1 \sin\phi + \tfrac{1}{2}c_1\phi^2\eta_1^2 + \tfrac{1}{2}c_1\phi^2\eta_1^2 \cos 2\phi \\ +\tfrac{1}{4}c_2\phi^2\eta_1^3 \sin\phi + \tfrac{1}{4}c_2\phi^2\eta_1^3 \sin 3\phi = F\sin(\phi) - F\cos(\theta) \end{aligned} \tag{2.67}$$

Balancing the harmonics yields:

$$\frac{\eta_1}{F} = \frac{1}{k_1 - \Omega^2 + \frac{1}{4}c_2\Omega^2\eta_1^2} \tag{2.68}$$

$$\Omega^2 = \left(\frac{F}{\eta_1} - k_1\right)\frac{1}{\frac{1}{4}c_2\eta_1^2 - 1} \tag{2.69}$$

To improve the estimation of the response, an additional term is included in the assume solution:

$$\eta = \eta_o + \eta_1 \sin\phi \tag{2.70}$$

When substituting into the ODE:

$$\begin{aligned}k_1\eta_o + \tfrac{1}{2}c_1\phi^2\eta_1^2 + \tfrac{1}{2}c_1\phi^2\eta_o\eta_1^2 \\ +\left(k_1\eta_1 - \phi^2\eta_1 + \tfrac{1}{4}c_2\phi^2\eta_1^3 - F\right)\sin\phi \\ +\tfrac{1}{2}c_1\phi^2\eta_o\eta_1^2\cos 2\phi \\ +\tfrac{1}{4}c_2\phi^2\eta_1^3\sin 3\phi = -F\cos(\theta)\end{aligned} \tag{2.71}$$

$$\begin{aligned}k_1\eta_o + \tfrac{1}{2}c_1\phi^2\eta_1^2 + \tfrac{1}{2}c_1\phi^2\eta_o\eta_1^2 = 0 \\ k_1\eta_1 - \phi^2\eta_1 + \tfrac{1}{4}c_2\phi^2\eta_1^3 = F\end{aligned} \tag{2.72}$$

Thus,

$$\begin{aligned}\eta_o &= -\frac{k_1 + \frac{1}{2}c_1\phi^2\eta_1^2}{\frac{1}{2}c_1\phi^2\eta_1^2} \\ \Omega^2 &= \left(\frac{F}{\eta_1} - k_1\right)\frac{1}{\frac{1}{4}c_2\eta_1^2 - 1}\end{aligned} \tag{2.73}$$

The response estimate can be improved further by adding a third term to the assumed solution, thus, the output becomes:

$$\eta = \eta_o + \eta_1\sin\phi + \eta_2\sin 2\phi \tag{2.74}$$

The equation of motion describing the vibrations of a slender inextensible cantilever beam carrying a lumped mass with a rotational inertia can be expressed by the nonlinear second order differential equation as follows:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + m_i(x^2\ddot{x} + x\dot{x}^2) + \omega_n^2 x + \alpha k_3 x^3 = -\ddot{y} \tag{2.75}$$

The over-dots denote derivatives with respect time, t. The combined distributive mass and rotational inertia of the structure is m. The beam tip relative displacement, and the steady state excitation amplitude are x, and F, respectively. The coefficient ζ is the system damping ratio, which causes exponential dissipation of the system's natural dynamics over time. The structural stiffness is $k_1 = m\omega_n^2$. The dimensionless nonlinear equation of motion can be expressed as follows:

$$\eta'' + 2\zeta\eta' + m_n(\eta^2\eta'' + \eta\eta'^2) + \eta + k_n\eta^3 = -Y'' \tag{2.76}$$

The dimensionless equation of motion in 2.76 can be normalized by the linear and non-linear inertia of the system, as follows:

$$\eta'' + \frac{m_n}{\left(1+m_n\eta^2\right)}\eta\eta'^2 + \frac{2\zeta}{\left(1+m_n\eta^2\right)}\eta' + \frac{1}{\left(1+m_n\eta^2\right)}\eta + \frac{k_n}{\left(1+m_n\eta^2\right)}\eta^3 = \frac{-1}{\left(1+m_n\eta^2\right)}Y'' \tag{2.77}$$

The dimensionless variables and parameters are:

$$\begin{aligned}\eta = \frac{x}{L}, \qquad \tau = \omega_n t, \qquad \Omega = \frac{\omega}{\omega_n} \\ 2\zeta = \frac{c}{m\omega_n}, \qquad m_n = \frac{m_i}{m}L^2, \qquad k_n = \alpha\frac{k_3}{k_1}L^2,\end{aligned} \tag{2.78}$$

Therefore, a potential approximated periodic solution can be expressed in the form truncated Fourier series using two terms:

$$\eta = A\cos\phi + B\sin\phi \tag{2.79}$$

where, $\phi = \Omega\tau$. Expanding each variable yields:

$$\eta' = -A\phi\sin\phi + B\phi\cos\phi \tag{2.80a}$$

$$\eta'' = -A\phi^2\cos\phi - B\phi^2\sin\phi \tag{2.80b}$$

$$\eta^2 = A^2\cos^2\phi + 2AB\cos\phi\sin\phi + B^2\sin^2\phi \tag{2.80c}$$

$$\eta'^2 = A^2\phi^2\sin^2\phi - 2AB\phi^2\cos\phi\sin\phi + B^2\phi^2\cos^2\phi \tag{2.80d}$$

$$\begin{aligned}
\eta^3 &= A^3\cos^3\phi + 3A^2B\cos^2\phi\sin\phi + 3AB^2\cos\phi\sin^2\phi + B^3\sin^3\phi \\
\eta^3 &= \tfrac{3}{4}A^3\cos\phi + \tfrac{1}{4}A^3\cos(3\phi) + \tfrac{3}{4}A^2B\sin\phi + \tfrac{3}{4}A^2B\sin(3\phi) \\
&\quad + \tfrac{3}{4}AB^2\cos\phi - \tfrac{3}{4}AB^2\cos(3\phi) + \tfrac{3}{4}B^3\sin\phi - \tfrac{1}{4}B^3\sin(3\phi) \\
\eta^3 &= \left(\tfrac{3}{4}A^3 + \tfrac{3}{4}AB^2\right)\cos\phi + \left(\tfrac{3}{4}A^2B + \tfrac{3}{4}B^3\right)\sin\phi \\
&\quad + \left(\tfrac{1}{4}A^3 - \tfrac{3}{4}AB^2\right)\cos(3\phi) + \left(\tfrac{3}{4}A^2B - \tfrac{1}{4}B^3\right)\sin(3\phi)
\end{aligned} \tag{2.80e}$$

$$\begin{aligned}
\eta^2\eta'' &= -A^3\phi^2\cos^3\phi - 3A^2B\phi^2\cos^2\phi\sin\phi - 3AB^2\phi^2\cos\phi\sin^2\phi - B^3\phi^2\sin^3\phi \\
\eta^2\eta'' &= -\left(\tfrac{3}{4}A^3\cos\phi + \tfrac{1}{4}A^3\cos(3\phi)\right)\phi^2 - \left(\tfrac{3}{4}A^2B\sin\phi + \tfrac{3}{4}A^2B\sin(3\phi)\right)\phi^2 \\
&\quad - \left(\tfrac{3}{4}AB^2\cos\phi - \tfrac{3}{4}AB^2\cos(3\phi)\right)\phi^2 - \left(\tfrac{3}{4}B^3\sin\phi - \tfrac{1}{4}B^3\sin(3\phi)\right)\phi^2 \\
\eta^2\eta'' &= \left(-\tfrac{3}{4}A^3\phi^2 - \tfrac{3}{4}AB^2\phi^2\right)\cos\phi + \left(-\tfrac{3}{4}A^2B\phi^2 - \tfrac{3}{4}B^3\phi^2\right)\sin\phi \\
&\quad + \left(-\tfrac{1}{4}A^3\phi^2 + \tfrac{3}{4}AB^2\phi^2\right)\cos(3\phi) + \left(-\tfrac{3}{4}A^2B\phi^2 + \tfrac{1}{4}B^3\phi^2\right)\sin(3\phi)
\end{aligned} \tag{2.80f}$$

$$\begin{aligned}
\eta\eta'^2 &= AB^2\phi^2\cos^3\phi + (B^3\phi^2 - 2A^2B\phi^2)\cos^2\phi\sin\phi + (A^3\phi^2 - 2AB^2\phi^2)\cos\phi\sin^2\phi + A^2B\phi^2\sin^3\phi \\
\eta\eta'^2 &= \left(\tfrac{3}{4}AB^2\phi^2\cos\phi + \tfrac{1}{4}AB^2\phi^2\cos(3\phi)\right) \\
&\quad + \left(\tfrac{3}{4}(B^3\phi^2 - 2A^2B\phi^2)\sin\phi + \tfrac{3}{4}(B^3\phi^2 - 2A^2B\phi^2)\sin(3\phi)\right) \\
&\quad + \left(\tfrac{3}{4}(A^3\phi^2 - 2AB^2\phi^2)\cos\phi - \tfrac{3}{4}(A^3\phi^2 - 2AB^2\phi^2)\cos(3\phi)\right) \\
&\quad + \left(\tfrac{3}{4}A^2B\phi^2\sin\phi - \tfrac{1}{4}A^2B\phi^2\sin(3\phi)\right) \\
\eta\eta'^2 &= \left(\tfrac{3}{4}A^3\phi^2 - \tfrac{3}{4}AB^2\phi^2\right)\cos\phi + \left(\tfrac{3}{4}A^2B\phi^2 - \tfrac{3}{4}B^3\phi^2\right)\sin\phi \\
&\quad + \left(-\tfrac{3}{4}A^3\phi^2 + \tfrac{7}{4}AB^2\phi^2\right)\cos(3\phi) + \left(\tfrac{3}{4}B^3\phi^2 - \tfrac{7}{2}A^2B\phi^2\right)\sin(3\phi)
\end{aligned} \tag{2.80g}$$

Add the nonlinear inertial variables:

$$\begin{aligned}
\eta^2\eta'' + \eta\eta'^2 &= \left(-\tfrac{3}{2}AB^2\phi^2\right)\cos\phi + \left(-\tfrac{3}{2}A^2B\phi^2\right)\sin\phi \\
&\quad + \left(-A^3\phi^2 + \tfrac{5}{2}AB^2\phi^2\right)\cos(3\phi) + \left(B^3\phi^2 - \tfrac{5}{2}A^2B\phi^2\right)\sin(3\phi)
\end{aligned} \tag{2.81}$$

Next, the periodic solution and its contributions are assembled into the equation of motion. The forcing function is also assumed to be periodic, $Y'' = \Omega^2 Y_o \cos\phi$. Harmonic matching is performed by splitting $\cos\phi$, $\sin\phi$, $\cos(3\phi)$, and $\sin(3\phi)$. Thus,

$$F_c := -A\Omega^2 + 2\zeta B\Omega - \tfrac{3}{2} m_n AB^2\Omega^2 + A + \tfrac{3}{4} k_n\Big(A^3 + AB^2\Big) + \Omega^2 Y_o = 0 \tag{2.82a}$$

$$F_s := -B\Omega^2 - 2\zeta A\Omega - \tfrac{3}{2} m_n A^2 B\Omega^2 + B + \tfrac{3}{4} k_n\Big(B^3 + A^2 B\Big) = 0 \tag{2.82b}$$

$$F_{c3} := m_n\Big(-A^3 + \tfrac{5}{2} AB^2\Big)\Omega^2 + k_n\Big(\tfrac{1}{4} A^3 - \tfrac{3}{4} AB^2\Big) = 0 \tag{2.82c}$$

$$F_{s3} := m_n\Big(B^3 - \tfrac{3}{2} A^2 B\Big)\Omega^2 + k_n\Big(\tfrac{1}{4} A^3 - \tfrac{3}{4} AB^2\Big) = 0 \tag{2.82d}$$

Equations F_c, F_s are the nonlinear governing equations for the periodic force response of the beam for a set of fixed parameters: Y_o, ζ, m_n, and k_n. The base excitation frequency, Ω, can be varied in order to determine the force response amplitudes, A, and B, which are unknowns. To this end, the amplitudes A, and B are transformed to the polar coordinate system by setting $A = r\cos\phi$, and $B = r\sin\phi$. The polar transformation allows for estimating only one variable instead of two. Upon substitution the polar amplitudes into F_c and F_s:

$$\begin{aligned} F_c := &\Big(1-\Omega^2\Big) r\cos\phi + 2\zeta\Omega r\sin\phi - \tfrac{3}{2} m_n\Omega^2 r^3 \cos\phi\sin^2\phi \\ &+ \tfrac{3}{4} k_n r^3\Big(\cos^3\phi + \cos\phi\sin^2\phi\Big) + \Omega^2 Y_o = 0 \end{aligned} \tag{2.83a}$$

$$\begin{aligned} F_s := &\Big(1-\Omega^2\Big) r\sin\phi - 2\zeta\Omega r\cos\phi - \tfrac{3}{2} m_n\Omega^2 r^3 \cos^2\phi\sin\phi \\ &+ \tfrac{3}{4} k_n r^3\Big(\sin^3\phi + \cos^2\phi\sin\phi\Big) = 0 \end{aligned} \tag{2.83b}$$

The above equations can be further simplified by applying trigonometric identities:

$$\Big(1-\Omega^2\Big) r\cos\phi + 2\zeta\Omega r\sin\phi - \tfrac{3}{2} m_n\Omega^2 r^3\Big(\tfrac{3}{4}\cos\phi - \tfrac{1}{4}\cos(3\phi)\Big) + \tfrac{3}{4} k_n r^3\cos\phi = -\Omega^2 Y_o \tag{2.84a}$$

$$\Big(1-\Omega^2\Big) r\sin\phi - 2\zeta\Omega r\cos\phi - \tfrac{3}{2} m_n\Omega^2 r^3\Big(\tfrac{3}{4}\sin\phi + \tfrac{1}{4}\sin(3\phi)\Big) + \tfrac{3}{4} k_n r^3\sin\phi = 0 \tag{2.84b}$$

Multiplying F_c by $\sin\phi$ and F_s by $\cos\phi$, yields the following:

$$\begin{aligned} &\Big(1-\Omega^2\Big) r\cos\phi\sin\phi + 2\zeta\Omega r\sin^2\phi - \tfrac{3}{2} m_n\Omega^2 r^3\Big(\tfrac{3}{4}\cos\phi\sin\phi - \tfrac{1}{4}\cos(3\phi)\sin\phi\Big) \\ &+ \tfrac{3}{4} k_n r^3\cos\phi\sin\phi = -\Omega^2 Y_o\sin\phi \end{aligned} \tag{2.85a}$$

$$\begin{aligned} &\Big(1-\Omega^2\Big) r\cos\phi\sin\phi - 2\zeta\Omega r\cos^2\phi - \tfrac{3}{2} m_n\Omega^2 r^3\Big(\tfrac{3}{4}\cos\phi\sin\phi + \tfrac{1}{4}\cos\phi\sin(3\phi)\Big) \\ &+ \tfrac{3}{4} k_n r^3\cos\phi\sin\phi = 0 \end{aligned} \tag{2.85b}$$

Subtracting equation 2.85b from 2.85a, and balancing the harmonics, provide the following equation:

$$2\zeta\Omega r = -\Omega^2 Y_o\sin\phi \tag{2.86}$$

Repeating the same process, but multiplying F_c by $\cos\phi$ and F_s by $\sin\phi$, yields the following:

$$\begin{aligned}&\left(1-\Omega^2\right)r\cos^2\phi+2\zeta\Omega r\cos\phi\sin\phi-\tfrac{3}{2}m_n\Omega^2r^3\left(\tfrac{3}{4}\cos^2\phi-\tfrac{1}{4}\cos\phi\cos(3\phi)\right)\\&+\tfrac{3}{4}k_nr^3\cos^2\phi+\Omega^2Y_o\cos\phi=0\end{aligned} \tag{2.87a}$$

$$\begin{aligned}&\left(1-\Omega^2\right)r\sin^2\phi-2\zeta\Omega r\cos\phi\sin\phi-\tfrac{3}{2}m_n\Omega^2r^3\left(\tfrac{3}{4}\sin^2\phi+\tfrac{1}{4}\sin\phi\sin(3\phi)\right)\\&+\tfrac{3}{4}k_nr^3\sin^2\phi=0\end{aligned} \tag{2.87b}$$

Adding equations 2.87a and 2.87b, and matching the harmonics, yields:

$$\left[1-\Omega^2-\tfrac{9}{8}m_n\Omega^2r^2+\tfrac{3}{4}k_nr^2\right]r=-\Omega^2Y_o\cos\phi \tag{2.88}$$

Equations 2.86 and 2.88 are squared then added to provide a relationship between the inputs and outputs that resembles an FRF:

$$\left(\frac{Y_o}{X}\right)^2\Omega^4=\left[1+\tfrac{3}{4}k_nX^2-\left(1+\tfrac{9}{8}m_nX^2\right)\Omega^2\right]^2+4\zeta^2\Omega^2 \tag{2.89}$$

Or:

$$\frac{m_eX}{Y_o}=\frac{\Omega^2}{\sqrt{\left(k_e-\Omega^2\right)^2+4\mu^2\Omega^2}} \tag{2.90}$$

where,

$$k_e=\frac{1}{m_e}\left(1+\tfrac{3}{4}k_nX^2\right),\qquad \mu^2=\frac{\zeta^2}{m_e^2},\qquad m_e=1+\tfrac{9}{8}m_nX^2 \tag{2.91}$$

The nonlinear frequency response equation 2.89 are expanded and parameterized in order to find the roots of Ω by simultaneously varying the input force and output response, as follows:

$$\left(1-\frac{Y_o^2}{m_e^2r^2}\right)\Omega^4+\left(4\mu^2-2k_e\right)\Omega^2+k_e^2=0 \tag{2.92}$$

While the above equation is fourth order, it is treated as a quadratic equation with two roots, $\Omega_{1,2}^2$, which can be obtained using the quadratic formula. Setting:

$$\alpha=1-\frac{Y_o^2}{m_e^2r^2},\qquad \beta=4\mu^2-2k_e \tag{2.93}$$

provides the following solution for the system frequency in terms of its natural resonance:

$$\Omega_{1,2}^2=\frac{1}{2\alpha}\left(-\beta\pm\sqrt{\beta^2+4\alpha k_e^2}\right) \tag{2.94}$$

$$\Omega_{1,2}^2=\frac{1}{\alpha}\left(\left(k_e-2\mu^2\right)\pm\sqrt{4\mu^2\left(\mu^2-4\mu^2k_e\right)+\frac{Y_o^2}{m_e^2r^2}k_e^2}\right), \tag{2.95}$$

and the backbone response is:

$$\Omega_b=\sqrt{\frac{k_e-2\mu^2}{1-\frac{Y_o^2}{m_e^2r^2}}} \tag{2.96}$$

To evaluate in terms of constant amplitude input acceleration:

$$\left(\frac{G}{r}\right)^2 = \left[1 + \tfrac{3}{4}k_n r^2 - \left(1 + \tfrac{9}{8}m_n r^2\right)\Omega^2\right]^2 + 4\zeta^2\Omega^2 \tag{2.97}$$

Or:

$$\frac{m_e r}{G} = \frac{1}{\sqrt{\left(k_e - \Omega^2\right)^2 + 4\mu^2\Omega^2}} \tag{2.98}$$

The nonlinear frequency response equation 2.89 are expanded and parameterized in order to find the roots of Ω by simultaneously varying the input force and output response, as follows:

$$\Omega^4 + \left(4\mu^2 - 2k_e\right)\Omega^2 + k_e^2 - \frac{G^2}{m_e^2 r^2} = 0 \tag{2.99}$$

While the above equation is fourth order, it is treated as a quadratic equation with two roots, $\Omega_{1,2}^2$, which can be obtained using the quadratic formula. Setting:

$$\alpha = 1, \qquad \beta = 4\mu^2 - 2k_e, \qquad c = k_e^2 - \frac{G^2}{m_e^2 r^2} \tag{2.100}$$

provides the following solution for the system frequency in terms of its natural resonance:

$$\Omega_{1,2}^2 = \frac{1}{2\alpha}\left(-\beta \pm \sqrt{\beta^2 + 4\alpha c}\right) \tag{2.101}$$

$$\Omega_{1,2}^2 = \left(k_e - 2\mu^2\right) \pm \sqrt{4\mu^2\left(\mu^2 - 4\mu^2 k_e\right) + \frac{G^2}{m_e^2 X^2}}, \tag{2.102}$$

and the backbone response is:

$$\Omega_b = \sqrt{k_e - 2\mu^2} \tag{2.103}$$

2.5 Variation of Parameters Techniques

The method of averaging is an analytical approximation technique for solving nonlinear systems, which is based on variation of parameters method for solving linear ODEs. The method of averaging can be utilized to obtain a first-order uniform expansion of the solution of the nonlinear system. However, it is necessary to transform a second-order system into two first-order equations, which can be realized by applying the method of variation of parameters. The analytical approximation can be achieved by introducing trail harmonic functions, and minimizing the approximation error.

There are many variation of parameters techniques such as method of averaging, van der Pol's technique, Krylov-Bogoliubov, the generalized method of averaging, the Krylov-Bogoliubov-Mitropolsky technique Additional details can be found in study of this method can be found in [24].

2.5.1 Method of Averaging

Systems with Nonlinear Stiffness

A nonlinear dynamic system with linear damping, and cubic stiffness can be expressed as follows:

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x + k_3 x^3 = 0 \tag{2.104}$$

If $k_3 = 0$, the system is linear with the following solution:

$$x(t) = X\cos(\omega_n t + \phi) = X\cos(\gamma) \tag{2.105}$$

where, the amplitude X, and phase ϕ are constants. The first temporal derivative becomes:

$$\dot{x} = -X\omega_n \sin\gamma \tag{2.106}$$

When the cubic stiffness is not zero, the solution provided in Eq 2.105 can be used, however, X, and ϕ are no longer not constants. To apply the method of averaging to equation of motion, $X(t)$, and $\phi(t)$ will be time-varying parameters in the assume solution as follows:

$$x(t) = X(t)\cos(\omega_n t + \phi(t)) = X(t)\cos(\gamma(t)) \tag{2.107}$$

The amplitude $X(t)$, and phase $\phi(t)$ are slowly varying functions of the characteristic time, t. The solution contains three unknowns, namely, $x(t)$, $X(t)$, and ϕ. For notational convenience set:

$$\gamma = \omega_n t + \phi(t) \tag{2.108}$$

The first temporal derivative is:

$$\dot{x} = -X\omega_n \sin\gamma + \dot{X}\cos\gamma - \dot{\phi}X\sin\gamma \tag{2.109}$$

When comparing the first Equating derivatives of x in Eq. 2.109 to Eq. 2.106, the following can be extracted:

$$\dot{X}\cos\gamma - \dot{\phi}X\sin\gamma = 0 \tag{2.110}$$

Subsequently, the first derivative is reduced to:

$$\dot{x} = -X\omega_n \sin\gamma \tag{2.111}$$

The velocity derivative with respect to time becomes:

$$\ddot{x} = -X\omega_n^2 \cos\gamma - \dot{X}\omega_n \sin\gamma - \dot{\phi}X\omega_n \cos\gamma \tag{2.112}$$

Substituting x, $\dot{x}$, and $\ddot{x}$ into 2.104 yields:

$$-X\omega_n^2 \cos\gamma - \dot{X}\omega_n \sin\gamma - \dot{\phi}X\omega_n^2 \cos\gamma - 2\zeta X\omega_n \sin\gamma + \omega_n^2 X\cos\gamma + k_3 X^3 \cos^3\gamma = 0 \tag{2.113}$$

Rearrange:

$$\dot{\phi}X\omega_n \cos\gamma + (2\zeta X + \dot{X})\omega_n sin\gamma = k_3 X^3 cos^3\gamma \tag{2.114}$$

Equations 2.110, and 2.114 are two first-order ODEs for the amplitude and phase. Multiplying 2.110 by $\cos\gamma$, and 2.110 by $\sin\gamma$ yields the following:

$$\dot{X}\cos^2\gamma - \dot{\phi}X\cos\gamma\sin\gamma = 0, \tag{2.115a}$$

$$-\dot{\phi}X\omega_n\cos\gamma\sin\gamma-(2\beta_1X+\dot{X})\omega_n\sin^2\gamma+k_3X^3\sin\gamma\cos^3(\gamma)=0 \tag{2.115b}$$

Adding equations 2.115:

$$\dot{X}=-2\zeta X\sin^2\gamma+\frac{k_3}{\omega_n}X^3\sin\gamma\cos^3(\gamma), \tag{2.116a}$$

$$\dot{\phi}=-2\zeta\cos\gamma\sin\gamma+\frac{k_3}{\omega_n}X^2\cos^4(\gamma) \tag{2.116b}$$

While the nonlinearities in the transformed first-order ODEs appear to be higher than the second-order ODE system, the transformation can be advantageous if X, and ϕ vary slower than x. From a numerical computation prospective, unlike the second-order systems, the transformed equations allows for using larger step size in the numerical integration. Thus, when dealing with complex systems, it is reasonable gain insights into the influence of important parameters by utilizing the variation of parameters method to obtain the governing equations that describe the system's parameters.

Averaging by integrating over one period $[0,2\pi]$ for the following functions:

$$\begin{gathered}
\int_0^{2\pi}Xd\gamma=2\pi X,\quad \int_0^{2\pi}X\cos^2\gamma d\gamma=\pi X,\\
\int_0^{2\pi}X\cos^2\gamma\sin^2\gamma d\gamma=\frac{1}{4}\pi X,\quad \int_0^{2\pi}X^2\cos^4\gamma d\gamma=\frac{3}{4}\pi X^2,\\
\int_0^{2\pi}X^3\dot{\phi}\cos^4\gamma d\gamma=\frac{3}{4}\pi X^3\dot{\varphi},\qquad \int_0^{2\pi}\sin^3\gamma\cos\gamma d\gamma=0
\end{gathered} \tag{2.117}$$

Applying the van der Pol method by substituting the averages into equations 2.116 yields:

$$\dot{X}=-2\zeta X, \tag{2.118a}$$

$$\dot{\phi}=\frac{3}{8}\frac{k_3}{\omega_n}\pi X^2 \tag{2.118b}$$

Systems with Nonlinear Damping

$$\ddot{x}+2\zeta\omega_n(1+\zeta_2x^2)\dot{x}+\omega_n^2x+k_3x^3=0 \tag{2.119}$$

Damping causes oscillations to decay due to the dissipation of nonconservative energy in the system. For mildly nonlinear dynamical system with light damping, the oscillations decay is relatively slow to appreciably reduce their amplitudes. The linear damping can be expressed in terms of the inverse time scale that governs the decay in a free vibration system:

$$\beta_1=\zeta\omega_n,\quad \beta_2=\zeta_2 \tag{2.120}$$

After scaling the equation of motion becomes:

$$\ddot{x}+2\beta_1(1+\beta_2x^2)\dot{x}+x+k_3x^3=0 \tag{2.121}$$

To apply the method of averaging to equation of motion an assume solution that retains the same displacement, x, is required:

$$x = X(t)\cos(\omega_n t + \phi) = X(t)\cos(\gamma) \quad (2.122)$$

Equating the first derivatives of x, produces:

$$\dot{X}\omega_n \cos\gamma - \dot{\phi} X\omega_n \sin\gamma = 0, and \qquad \dot{X}\omega_n \cos\gamma \sin\gamma - \dot{\phi} X\omega_n \sin^2\gamma = 0 \quad (2.123)$$

The first derivative is reduced to:

$$\dot{x} = -X\omega_n \sin\gamma \quad (2.124)$$

The second derivative becomes:

$$\ddot{x} = -X\omega_n^2 \cos\gamma - \dot{X}\omega_n \sin\gamma - \dot{\phi} X\omega_n \cos\gamma \quad (2.125)$$

Substituting x, $\dot{x}$, and $\ddot{x}$ into 2.121 yields:

$$-X\omega_n^2 \cos\gamma - \dot{X}\omega_n \sin\gamma - \dot{\phi} X\omega_n^2 \cos\gamma - 2\beta_1(1+\beta_2 X^2\cos^2\gamma) X\omega_n^2 \sin\gamma + \omega_n X\cos\gamma + k_3 X^3\cos^3\gamma = 0 \quad (2.126)$$

Rearrange:

$$-\dot{\phi} X\omega_n \cos\gamma + k_3 X^3 cos^3\gamma - (2\beta_1 X + \dot{X})\omega_n sin\gamma - 2\beta_1\beta_2 X^3\omega_n \sin\gamma\cos^2\gamma = 0 \quad (2.127)$$

Multiply 2.127 by $\cos\gamma$:

$$\begin{aligned} &-\dot{\phi} X\omega_n \cos^2\gamma + k_3 X^3\cos^4\gamma \\ &-(2\beta_1 X + \dot{X})\omega_n \cos\gamma\sin\gamma - 2\beta_1\beta_2 X^3\omega_n \sin\gamma\cos^3\gamma = 0 \end{aligned} \quad (2.128)$$

Multiply 2.127 by $\sin\gamma$:

$$\begin{aligned} &-\dot{\phi} X\omega_n \cos\gamma\sin\gamma + k_3 X^3\sin\gamma\cos^3(\gamma) \\ &-(2\beta_1 X + \dot{X})\omega_n \sin^2\gamma - 2\beta_1\beta_2 X^3\omega_n \sin^2\gamma\cos^2(\gamma) = 0 \end{aligned} \quad (2.129)$$

Adding equation 2.123 to 2.128:

$$\begin{aligned} &-\dot{\phi} X\omega_n + k_3 X^3\cos^4\gamma - \\ &2\beta_1 X\omega_n \cos\gamma\sin\gamma - 2\beta_1\beta_2 X^3\omega_n \sin\gamma\cos^3\gamma = 0 \end{aligned} \quad (2.130)$$

Adding equation 2.123 to 2.129:

$$\begin{aligned} &-\dot{X}\omega_n + k_3 X^3\sin\gamma\cos^3\gamma \\ &-2\beta_1 X\omega_n \sin^2\gamma - 2\beta_1\beta_2 X^3\omega_n \sin^2\gamma cos^2\gamma = 0 \end{aligned} \quad (2.131)$$

Averaging by integrating over one period:

$$\begin{aligned} &\int_0^{2\pi} X d\gamma = 2\pi\chi, \quad \int_0^{2\pi} X\cos^2\gamma d\gamma = \pi\chi, \\ &\int_0^{2\pi} X\cos^2\gamma sin^2\gamma d\gamma = \frac{1}{4}\pi\chi, \quad \int_0^{2\pi} X^3\cos^4\gamma d\gamma = \frac{3}{4}\pi\chi^3, \\ &\int_0^{2\pi} X^3\dot{\phi}\cos^4\gamma d\gamma = \frac{3}{4}\pi\chi^3\dot{\varphi}, \qquad \int_0^{2\pi} \sin^3\gamma\cos\gamma d\gamma = 0 \end{aligned} \quad (2.132)$$

Substituting the averages into equations 2.130 and 2.131 yields:

$$-2\dot{\varphi}\chi\omega_n + \frac{3}{4}k_3\chi^3 = 0 \tag{2.133a}$$

$$2\dot{\chi} + 2\beta_1\chi + \frac{1}{2}\beta_1\beta_2\chi^3 = 0 \tag{2.133b}$$

Finally the rate of change in the amplitude and phase are:

$$\dot{\phi} = \frac{3}{8}\frac{k_3}{\omega_n}X^2 \tag{2.134a}$$

$$\dot{X} = -2\zeta\omega_n\Big(1 + \frac{1}{4}\beta_2 X^2\Big)X \tag{2.134b}$$

Integrating over time:

$$\phi(t) = \frac{3}{8}\frac{k_3}{\omega_n}X^2 t + C_1 \tag{2.135a}$$

$$X - \frac{1}{2}\beta_2 X^2(t) = C_2 e^{-2\zeta\omega_n\ t} \tag{2.135b}$$

The initial velocity is zero, which implies that $C_1 = 0$, and $C_2 = (1 - \frac{1}{2}\beta_2 X_o)X_o$.

$$\phi = \frac{3}{8}\frac{k_3}{\omega_n}X^2 t \tag{2.136a}$$

$$X - \frac{\beta_2}{2}X^2 = X_o e^{-2\zeta\omega_n t} - \frac{1}{2}\beta_2 X_o^2 e^{-2\zeta\omega_n t} \tag{2.136b}$$

When the damping is linear, the amplitude rate of change is:

$$X(t) = X_o e^{-\beta_1 t} \tag{2.137}$$

$$\begin{aligned} x(t) &= X\cos(\omega_n t + \phi), \\ x(t) &= X_o e^{-2\zeta\omega_n t}\cos\Big[\Big(\omega_n + \frac{3}{8}\frac{k_3}{\omega_n}X^2\Big)t\Big] \end{aligned} \tag{2.138}$$

The system frequency is:

$$\omega = \omega_n + \frac{3}{8\omega_n}k_3 X_o^2, \quad or \quad \Omega \ = 1 + \frac{3}{8}k_3\Big(\frac{X_o}{\omega_n}\Big)^2 \tag{2.139}$$

The time history amplitude decay envelopes using the method of averaging are shown in Figure 2.5 for dynamic system with viscous, quadratic and combined viscous-quadratic.

2.5.2 Forced System

Consider a dynamic system with nonlinear stiffness and damping undergoing a harmonic driving force with oscillations of frequency ω, and strength magnitude F. The governing equation can be expressed as follows:

$$m\ddot{x} + c_1(1 + \beta_2 x^2)\dot{x} + \omega_n^2 x + K_3 x^3 = F_o\cos(\omega t) \tag{2.140}$$

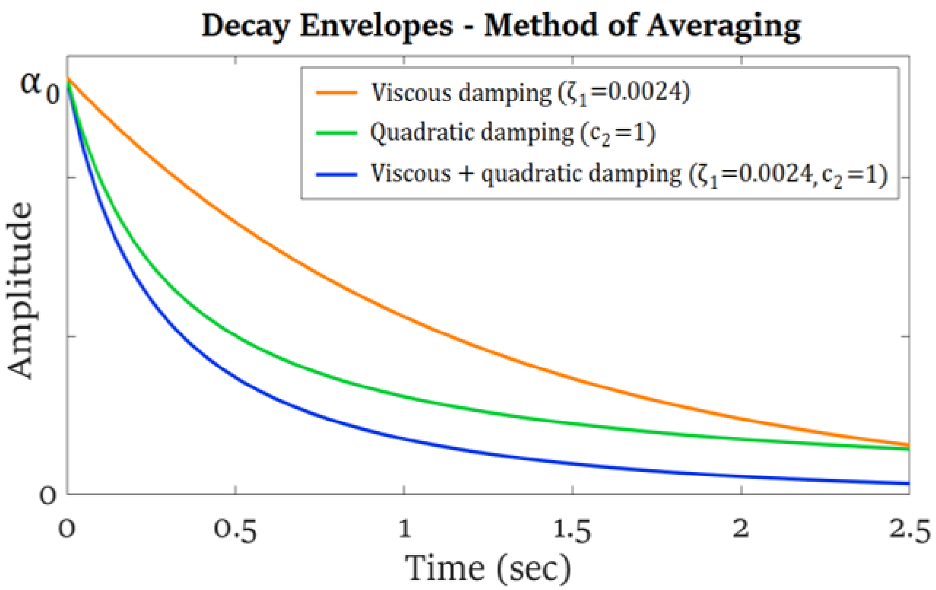

Figure 2.5: The decay envelopes using the method for dynamic systems with viscous, quadratic and combined viscous-quadratic

Normalizing the equation of motion by the system's mass, and introducing a dimensionless time quantity, $\tau = \omega_n t$, the ODE becomes:

$$x'' + 2\beta_1(1+\beta_2 x^2)x' + x + k_3 x^3 = F\cos(\Omega\tau) \tag{2.141}$$

where,

$$2\beta_1 = \frac{c}{m\omega_n}, \quad k_3 = \frac{K_3}{m\omega_n^2}, \quad F = \frac{F_o}{m\omega_n^2}, \quad \Omega = \frac{\omega}{\omega_n} \tag{2.142}$$

The assumed solution is:

$$x = X(\tau)\cos(\tau + \phi(\tau)) = X(\tau)\cos(\gamma) \tag{2.143}$$

The first and second derivatives of x are:

$$x' = -X\sin\gamma \tag{2.144a}$$
$$x'' = -X\cos\gamma - X'\sin\gamma - \phi' X\cos\gamma \tag{2.144b}$$

Substituting x, x', and x'' into 2.141 yields:

$$-X'\sin\gamma - \phi' X\cos\gamma - 2\beta_1(1+\beta_2 X^2\cos^2\gamma)X\sin\gamma + k_3 X^3\cos^3\gamma = F\cos(\Omega\tau) \tag{2.145}$$

Multiply 2.145 by $\cos\gamma$, and $\sin\gamma$ separately, yields the following:

$$\begin{aligned} -\phi' X\cos^2\gamma + k_3 X^3\cos^4\gamma - (2\beta_1 X + X')\cos\gamma\sin\gamma \\ 2\beta_1\beta_2 X^3\sin\gamma\cos^3\gamma = F\cos(\Omega\tau)\cos\gamma \end{aligned} \tag{2.146a}$$

$$\begin{aligned} -\phi' X\cos\gamma\sin\gamma + k_3 X^3\sin\gamma\cos^3(\gamma) - (2\beta_1 X + X')\sin^2\gamma \\ -2\beta_1\beta_2 X^3\sin^2\gamma\cos^2(\gamma) = F\cos(\Omega\tau)\sin\gamma \end{aligned} \tag{2.146b}$$

Adding equation 2.128 to 2.146a and 2.146b separately, yields the following:

$$-\phi' X + k_3 X^3 \cos^4\gamma - 2\beta_1 X \cos\gamma \sin\gamma - 2\beta_1\beta_2 X^3 \sin\gamma \cos^3\gamma = F\cos(\Omega\tau)\cos\gamma \tag{2.147a}$$
$$-X' + k_3 X^3 \sin\gamma \cos^3(\gamma) - 2\beta_1 X \sin^2\gamma - 2\beta_1\beta_2 X^3 \sin^2\gamma \cos^2(\gamma) = F\cos(\Omega\tau)\sin\gamma \tag{2.147b}$$

Averaging by integrating over one period, 2.147a and 2.147b become the first order system of equations for expressing the motion near the primary resonance. Thus, the phase and amplitude are:

$$\phi' X = \frac{3}{8}k_3 X^3 - \frac{1}{4}F\cos\Big((1-\Omega)\tau + \phi\Big) - \frac{1}{4}F\cos\Big((1+\Omega)\tau + \phi\Big) \tag{2.148a}$$
$$X' = -\beta_1\Big(1 + \frac{1}{4}\beta_2 X^2\Big)X - \frac{1}{4}F\sin\Big((1-\Omega)\tau + \phi\Big) - \frac{1}{4}F\sin\Big((1+\Omega)\tau + \phi\Big) \tag{2.148b}$$

Maintaining the slowly time varying terms, rates of change in the amplitude and phase are:

$$\phi' X = \frac{3}{8}k_3 X^3 - \frac{1}{4}F\cos\Big((1-\Omega)\tau + \phi\Big) \tag{2.149a}$$
$$X' = -2\zeta\Big(1 + \frac{1}{4}\beta_2 X^2\Big)X - \frac{1}{4}F\sin\Big((1-\Omega)\tau + \phi\Big) \tag{2.149b}$$

Integrating over time:

$$\phi(t) = \frac{3}{8}\frac{k_3}{\omega_n}X^2 t + C_1 \tag{2.150a}$$
$$X - \frac{1}{2}\beta_2 X^2(t) = C_2 e^{-2\zeta\omega_n\ t} \tag{2.150b}$$

The first order system of equations for the amplitude and phase are:

$$-2\dot{\varphi} X\omega_n + \frac{3}{4}k_3 X^3 = F\cos(\omega t)\cos\gamma \tag{2.151a}$$
$$2\dot{X}\omega_n + 2\beta_1\omega_n X + \frac{1}{2}\beta_1\beta_2\omega_n X^3 = F\cos(\omega t)\sin\gamma \tag{2.151b}$$

Finally the rate of change in the amplitude and phase are:

$$\dot{X} = \frac{F}{2\omega_n}\cos(\omega t)\sin\gamma - \frac{1}{2}\beta_1\Big(1 + \frac{1}{4}\beta_2 X^2\Big)X \tag{2.152a}$$
$$\dot{\phi} = \frac{3}{8}\frac{k_3}{\omega}X^2 - \frac{F}{2X}\cos(\omega t)\cos\gamma \tag{2.152b}$$

The equation of motion in the frequency domain is:

$$\frac{F^2}{X^2} = \frac{9}{16}k_3^2 X^4 + \frac{1}{16}\beta_2^2\omega^2 X^4 + \omega_n\zeta\beta_2\omega^2 X^2 + 4\omega_n^2\zeta^2\omega^2 - 3\omega_n k_3 X^2 + 4\omega_n^2 \tag{2.153}$$

Rearrange:

$$\Big(\frac{F}{2X}\Big)^2 = \Big(\frac{3}{8}k_3 X^2\Big)^2 + \Big(\frac{1}{8}\beta_2 X^2\Big)^2\omega^2 + \frac{1}{4}\omega_n\zeta\beta_2\omega^2 X^2 - 2\frac{3}{8}k_3\omega_n X^2 + \omega_n^2 + \omega_n^2\zeta^2\omega^2 \tag{2.154}$$

Normalize by the system's natural resonance

$$\left(\frac{F}{2\omega_n X}\right)^2 = \left(\frac{3}{8}\frac{k_3}{\omega_n}X^2\right)^2 + \left(\frac{1}{8}\frac{\beta_2}{\omega_n}X^2\right)^2\omega^2 + \frac{1}{4}\frac{\zeta\beta_2}{\omega_n}\omega^2 X^2 - 2\frac{3}{8}\frac{k_3}{\omega_n}X^2 + 1 + \frac{\zeta^2}{\omega_n^2}\omega^2 \tag{2.155}$$

The backbone, ω_b can be calculated by setting kinetic, and non-conservative energy parameters equal to zero. We solve for ω_n as follows:

$$\left(\frac{3}{8}\frac{k_3}{\omega_n}X^2\right)^2 - 2\frac{3}{8}\frac{k_3}{\omega_n}X^2 + 1 = 0 \tag{2.156}$$

After calculating the roots:

$$\omega_b = 1 + \omega_n = 1 + \frac{3}{8}k_3 X^2 \tag{2.157}$$

$$\omega = 1 + \frac{3}{8}k_3 X^2 \pm \sqrt{\left(\frac{F}{2X}\right)^2 - \left(\frac{1}{8}\beta_2\right)^2 X^4 - \frac{1}{8}\zeta\beta_2 X^2 - \frac{1}{4}\zeta^2} \tag{2.158}$$

Bibliography

[1] J. P. Noël and G. Kerschen, "Nonlinear system identification in structural dynamics: 10 more years of progress," *Mechanical Systems and Signal Processing*, vol. 83, pp. 2–35, 2017.

[2] A. Montoya, E. Habtour, and F. Moreu, "Quantifying information without entropy: Identifying intermittent disturbances in dynamical systems," *Entropy*, vol. 22, no. 11, p. 1199, 2020.

[3] R. A. Haynes, E. Habtour, T. C. Henry, D. P. Cole, and B. Wisner, "Damage Precursor Indicator for Aluminum 7075-T6 based on Nonlinear Dynamics," 2019.

[4] E. M. Habtour, D. P. Cole, C. M. Kube, T. C. Henry, R. A. Haynes, F. Gardea, T. Sano, and T. Tinga, "Structural state awareness through integration of global dynamic and local material behavior," *Journal of Intelligent Material Systems and Structures*, 2019.

[5] E. H. Ernst, Matthew and A. Dasgupta., "Examining Steinberg's Octave rule applicability for electronic systems exposed to multiaxial vibration.," *IEEE Transactions on Components, Packaging and Manufacturing Technology 6.4*, 2016.

[6] J. M. Londoño, S. A. Neild, and J. E. Cooper, "Identification of backbone curves of nonlinear systems from resonance decay responses," *Journal of Sound and Vibration*, vol. 348, pp. 224–238, 2015.

[7] J. M. Londoño, J. E. Cooper, and S. A. Neild, "Identification of systems containing nonlinear stiffnesses using backbone curves," *Mechanical Systems and Signal Processing*, vol. 84, 2017.

[8] E. Habtour, D. Di Maio, T. Masmeijer, L. Cordova Gonzalez, and T. Tinga, "Highly sensitive nonlinear identification to track early fatigue signs in flexible structures," *Journal of Nondestructive Evaluation, Diagnostics and Prognostics of Engineering Systems*, vol. 5, no. 2, 2022.

[9] E. Habtour, M. Paulus, and A. Dasgupta, "Modeling approach for predicting the rate of frequency change of notched beam exposed to gaussian random excitation," *Shock and Vibration*, vol. 2014, 2014.

[10] E. Habtour, W. S. Connon, M. F. Pohland, S. C. Stanton, M. Paulus, and A. Dasgupta, "Review of response and damage of linear and nonlinear systems under multiaxial vibration," *Shock and Vibration*, vol. 2014, 2014.

[11] J. M. Hooper, J. Marco, G. H. Chouchelamane, J. S. Chevalier, and D. Williams, "Multi-axis vibration durability testing of lithium ion 18650 nca cylindrical cells," *Journal of energy storage*, vol. 15, pp. 103–123, 2018.

[12] E. Habtour, R. Sridharan, A. Dasgupta, M. Robeson, and S. Vantadori, "Phase influence of combined rotational and transverse vibrations on the structural response,"

Mechanical Systems and Signal Processing, vol. 100, pp. 371–383, 2018.

[13] E. Habtour, D. P. Cole, S. C. Stanton, R. Sridharan, and A. Dasgupta, "Damage precursor detection for structures subjected to rotational base vibration," *International Journal of Non-Linear Mechanics*, vol. 82, pp. 49–58, 2016.

[14] L. G. Villanueva, R. B. Karabalin, M. H. Matheny, D. Chi, J. E. Sader, and M. L. Roukes, "Nonlinearity in nanomechanical cantilevers," *Physical Review B - Condensed Matter and Materials Physics*, vol. 87, no. 2, pp. 1–8, 2013.

[15] D. Tan and A. Erturk, "On the Origin of the Nonclassical Softening Nonlinearity in MEMS/NEMS Cantilevers.," *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference.*, 2017.

[16] S. Vantadori, R. Haynes, G. Fortese, E. Habtour, C. Ronchei, D. Scorza, and A. Zanichelli, "Methodology for assessing embryonic cracks development in structures under high-cycle multiaxial random vibrations," *Fatigue and Fracture of Engineering Materials and Structures*, vol. 41, no. 1, pp. 20–28, 2018.

[17] D. A. Ehrhardt, T. L. Hill, S. A. Neild, and J. E. Cooper, "Veering and nonlinear interactions of a clamped beam in bending and torsion," *Journal of Sound and Vibration*, vol. 416, pp. 1–16, 2018.

[18] A. E. Aliev, N. K. Mayo, R. H. Baughman, B. T. Mills, and E. Habtour, "Subwoofer and nanotube butterfly acoustic flame extinction," *Journal of physics D: applied physics*, vol. 50, no. 29, p. 29LT01, 2017.

[19] D. P. Cole, E. M. Habtour, T. Sano, S. J. Fudger, S. M. Grendahl, and A. Dasgupta, "Local Mechanical Behavior of Steel Exposed to Nonlinear Harmonic Oscillation," *Experimental Mechanics*, vol. 57, no. 7, pp. 1027–1035, 2017.

[20] L. N. Virgin, " Introduction to Experimental Nonlinear Dynamics: A Case Study in Mechanical Vibration.," 2000.

[21] E. Habtour, T. Dragman, T. Masmeije, D. Di Maio, R. Haynes, A. Homborg, and T. Tinga, "Connecting nonlinearities: Damage precursors detection and control methodology," in *28th International Conference on Noise and Vibration Engineering, ISMA 2018 and 7th International Conference on Uncertainty in Structural Dynamics, USD 2018*, pp. 3723–3730, Katholieke Universiteit Leuven, 2018.

[22] A. H. Nayfeh, *Introduction to perturbation techniques*. John Wiley & Sons, 2011.

[23] J. Liu, D. T. Martin, T. Nishida, L. N. Cattafesta, M. Sheplak, and B. P. Mann, "Harmonic Balance Nonlinear Identification of a Capacitive Dual-Backplate MEMS Microphone," 2008.

[24] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*. John Wiley & Sons, Inc., 1995.

[25] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers*. Oxford University Press, 4 ed., 2007.