# QUATERNION DOMAIN LAPLACE TRANSFORM OF THE PULSES VECTOR

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***Abstract*** – Currently, the multi-input-multi-output (MIMO) scheme using complex and hypercomplex signals is widely used in communication theory and technology. This scheme allows to increase the channel capacity and their noise immunity. In the MIMO scheme, the components of the pulse vector are interconnected by a channel matrix. In this regard, the problem arises of obtaining the Laplace transform of such a vector.

In the presented methodology, methods of obtaining the Laplace transform of real signals using a matrix representation of hypercomplex signals were used. Suggested a methodology of obtaining the Laplace transform of a four-dimensional vector of pulses connected by a MIMO scheme. To obtain the Laplace transform, the quaternion in the matrix expansion and the corresponding matrix of quaternion frequencies are used as the kernel of the integral transform. It is shown that to obtain the Laplace quaternion transformation, it is possible to use the well-known transformation formulas, replacing the complex frequency with the quaternion frequency. Examples of the use of the method are presented.

***Keywords***

Quaternion, hypercomplex signals, Laplace transform, MIMO, quaternion frequency, quaternion Laplace transform.

## I. Introduction

Quaternion transformations, including the Laplace transform, have been studied in many works, for example, in [1-3]. However, these studies are often of a general theoretical nature or are used only to obtain the Fractional Quaternion Laplace Transform of two-dimensional images.

The quaternion Fourier series of a vector of 4 pulses interconnected by a multiple-input multiple-output (MIMO) scheme is considered in [4], and the quaternion Fourier transform is considered in [5]. When finding these transformations, only the imaginary part of the quaternion was used as the kernel. As a result, the Fourier series and integral did not converge for some functions. The quaternion Laplace transform, the technique of which is presented below, uses both the imaginary and real parts of the quaternion as a kernel. Since the quaternion can be represented as an analytic and harmonic function, the domain of its definitions is closed. As a result, by choosing the values of the real part, it is possible to determine the kernel of the integral transformation in such a way that the integral will converge for most of the functions considered in radio engineering. In this case, the quaternion Fourier transform of power-limited signals can be obtained from the Laplace transform by equating the real part to zero. As is known, the Laplace transform is widely used in radio engineering for the synthesis and analysis of signals and circuits. There are many formulas for the Laplace transform of various real signals. It is shown that the known formulas for the complex frequency can be used to obtain the quaternion transformation, replacing the complex frequency with the quaternion frequency.

The purpose of the work is to present a technique for obtaining the Laplace transform of a four-dimensional vector of pulses interconnected by a MIMO scheme, and examples of its use for the most common pulses in radio engineering.

## II. Materials and methods for solving the problem, assumptions made

The quaternion refers to hypercomplex numbers and imaginary units $i, j, k$ are used to write the quaternion, which satisfy the following multiplication rules: $ij = -ji = k$, $jk = -kj = i$,

$ki = -ik = j$, $i^2 = j^2 = k^2 = ijk = -1$. The quaternion in algebraic representation has the form [4 - 9]:

$$q = r + ix + jy + kz\,. \tag{1}$$

Numbers *r*, *x*, *y*, *z* take real values, and imaginary units form three orthogonal axes in space. Therefore, taking into account the real part of *r*, the quaternion is located in a 4-dimensional space (4D).

A quaternion with a module equal to one is written in polar representation, as

$$q(\varphi) = e^{\hat{i}\varphi} = \left[\cos\varphi, \hat{i}\sin\varphi\right] = \cos\varphi + \hat{i}\sin\varphi,\ \text{где}\ \hat{i} = (i + j + k)/\sqrt{3},\ \hat{i}^2 = -1\,. \tag{2}$$

As a rule, in problems of radio engineering, functions change their values according to the time parameter, so we write such a quaternion in a parametric representation as:

$$q(t) = r(t) + ix(t) + jy(t) + kz(t)\,. \tag{3}$$

We represent the corresponding function from the parametric quaternion (3) as:

$$f\left[q(t)\right] = p\left[q(t)\right] + iu\left[q(t)\right] + jv\left[q(t)\right] + kw\left[q(t)\right]. \tag{4}$$

As a result of applying functions to a quaternion and after grouping by the resulting imaginary units, we get various quaternion functions. We write (4) as the sum of time functions along orthogonal axes, keeping the same notation:

$$f(t) = p(t) + iu(t) + jv(t) + kw(t)\,. \tag{5}$$

As functions $p(t)$, $u(t)$, $v(t)$, $w(t)$, in (5) one can consider any continuous functions with limited power, for example, pulses of finite length. According to (2), parametric quaternion (3) with a unit modulus $|q| = 1$ and a total circular rotation frequency $\omega$ has the form:

$$q(\omega, t) = e^{\hat{i}\omega t} = \left[\cos\omega t, \hat{i}\sin\omega t\right] = \cos\omega t + \hat{i}\sin\omega t\,. \tag{6}$$

We represent the quaternion (1) in 4D as a 4x4 matrix [4 - 6]:

$$\mathbf{Q} = \begin{bmatrix} r & x & y & z \\ -x & r & -z & y \\ -y & z & r & -x \\ -z & -y & x & r \end{bmatrix}. \tag{7}$$

Matrix (7) is decomposed into basic matrices **E**, **I**, **J**, **K** and written as the sum of matrices:

$$\mathbf{Q} = \mathbf{E}s + \mathbf{I}x + \mathbf{J}y + \mathbf{K}z\,,$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \mathbf{I} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \mathbf{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that when a quaternion is represented as a matrix, the signs of the elements depend on the chosen orthogonal bases. In the matrix representation, we replace the imaginary units with an orthonormal basis. However, the relationship between the real and imaginary elements of a quaternion is preserved in the same way as when it is represented using imaginary units. Such a connection is provided by using the MIMO scheme, in which, when multiplying the vector of elements by an orthogonal channel matrix, we obtain that each element of the original vector participates in the formation of each element of the resulting vector. At the same time, the norm of the resulting vector is preserved.

The corresponding basis matrices of the quaternion, as well as the elements *i*, *j* and *k*, are related by the multiplication rules presented in the table:

$$\begin{array}{c|cccc} \times & \mathbf{E} & \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \hline \mathbf{E} & \mathbf{E} & \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \mathbf{I} & \mathbf{I} & -\mathbf{E} & \mathbf{K} & -\mathbf{J} \\ \mathbf{J} & \mathbf{J} & -\mathbf{K} & -\mathbf{E} & \mathbf{I} \\ \mathbf{K} & \mathbf{K} & \mathbf{J} & -\mathbf{I} & -\mathbf{E} \end{array}. \tag{8}$$

We write the imaginary part of the quaternion using the *imaginary quaternion matrix*:

$$\hat{\mathbf{I}} = \frac{1}{\sqrt{3}}\left(\mathbf{I} + \mathbf{J} + \mathbf{K}\right). \tag{9}$$

The imaginary quaternion matrix (9) has the properties of the imaginary unit of a complex number, since $\hat{\mathbf{I}}\hat{\mathbf{I}} = -\mathbf{E}$, $\hat{\mathbf{I}}^{\mathrm{T}}\hat{\mathbf{I}} = \hat{\mathbf{I}}\hat{\mathbf{I}}^{\mathrm{T}} = -\hat{\mathbf{I}}\hat{\mathbf{I}} = \mathbf{E}$. The inverse operator in matrix notation is expressed by the formula: $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}} / |\,\mathbf{q}\,|^2$. According to this formula, the basis matrices are orthogonal, since $\mathbf{I}^{\mathrm{T}} = \mathbf{I}^{-1} = -\mathbf{I}$, $\mathbf{J}^{\mathrm{T}} = \mathbf{J}^{-1} = -\mathbf{J}$, $\mathbf{K}^{\mathrm{T}} = \mathbf{K}^{-1} = -\mathbf{K}$, and, according to table (8), $\mathbf{I}\mathbf{I}^{\mathrm{T}} = \mathbf{E}$, $\mathbf{J}\mathbf{J}^{\mathrm{T}} = \mathbf{E}$, $\mathbf{K}\mathbf{K}^{\mathrm{T}} = \mathbf{E}$. The unit quaternion (1) with modulus $|q| = 1$ in matrix representation (7) is also orthogonal (orthonormal): $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{E}$.

To obtain the quaternion Laplace transform (QLT), we represent the function of the quaternion (5) as a vector of different pulses:

$$\mathbf{q}(t) = \left[\, p(t) \quad u(t) \quad v(t) \quad w(t) \right]^{\mathrm{T}}.$$

In this case, the pulses $p(t)$, $u(t)$, $v(t)$, $w(t)$ can have, in the general case, a different shape, amplitude and time shift.

## III. Technique of obtaining direct quaternion Laplace transform

It was shown in [5] that the direct single-frequency quaternion Fourier transform in the 4D domain of the quaternion in vector-matrix form has the view:

$$\mathbf{g}(\omega) = \int_{-\infty}^{\infty} \mathbf{\Phi}^{\mathrm{T}}(\omega,t)\mathbf{q}(t)dt\,. \tag{10}$$

where $\mathbf{\Phi}(\omega,t) = e^{\hat{\mathbf{I}}\omega t} = \mathbf{E}\cos\omega t + \hat{\mathbf{I}}\sin\omega t$ is the matrix exponent.

The two-dimensional Laplace transform uses a parameter in the form of a complex frequency $s = \sigma + j\omega$, [10, 11]. In a single-frequency QLT, we will use the *quaternion frequency*:

$$s = \sigma + \tfrac{1}{\sqrt{3}}\omega\left(i + j + k\right) = \sigma + \hat{i}\,\omega. \tag{11}$$

In the matrix representation (7), taking into account (9), the quaternion frequency (11) will look like:

$$\mathbf{S} = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3}\sigma & \omega & \omega & \omega \\ -\omega & \sqrt{3}\sigma & -\omega & \omega \\ -\omega & \omega & \sqrt{3}\sigma & -\omega \\ -\omega & -\omega & \omega & \sqrt{3}\sigma \end{bmatrix} = \mathbf{E}\sigma + \hat{\mathbf{I}}\omega\,. \tag{12}$$

Let's call the matrix **S** the *matrix of quaternion frequencies*, where $\omega = 2\pi/T = 2\pi f$ is the main angular frequency, radian/s; $T = 1/f$ – frequency period, s.

As the QLT kernel, we use the exponent to the power of $-st$, where $s$ is the quaternion frequency (11):

$$\mathrm{e}^{-st} = \exp\left\{-\left(\sigma + \tfrac{1}{\sqrt{3}}\omega\left(i + j + k\right)\right)t\right\} = \exp\left\{-(\sigma + \hat{i}\,\omega)t\right\}, \tag{13}$$

In the matrix representation, expression (13) takes the form:

$$e^{-\mathbf{S}t} = e^{-\sigma t} e^{-\hat{\mathbf{I}}\omega t} = e^{-\sigma t}\mathbf{\Phi}^{\mathrm{T}}(\omega,t) = e^{-\sigma t}\left(\mathbf{E}\cos\omega t - \hat{\mathbf{I}}\sin\omega t\right) = \tag{14}$$

$$= e^{-\sigma t}\frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3}\cos(\omega t) & -\sin(\omega t) & -\sin(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \sqrt{3}\cos(\omega t) & \sin(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & -\sin(\omega t) & \sqrt{3}\cos(\omega t) & \sin(\omega t) \\ \sin(\omega t) & \sin(\omega t) & -\sin(\omega t) & \sqrt{3}\cos(\omega t) \end{bmatrix}.$$

As you can see, expression (14) differs from the transformation kernel (10) by the presence of a common multiplier $e^{-\sigma t}$.

Thus, in accordance with (10) and (14), we write the QLT in the vector-matrix form in the view of a right representation:

$$\mathbf{q}(\mathbf{S}) = \int_0^{\infty} e^{-\sigma t}\mathbf{\Phi}^{\mathrm{T}}(\omega,t)\mathbf{q}(t)dt = \int_0^{\infty} e^{-\mathbf{S}t}\mathbf{q}(t)dt. \tag{15}$$

QLT (15) for a single-frequency quaternion in the vector-matrix representation associates the 4D vector $\mathbf{q}(t)$ (original) in the quaternion region defined for all possible values of $t$ with the only 4D vector $\mathbf{q}(\mathbf{S})$ (image) in the quaternion frequency region $\mathbf{S}$.

Note that in the notation for QLT (15), we consider only positive time, i.e., unilateral QLT. With negative time, the integral kernel will be $\mathbf{\Phi}(\omega,t) = e^{\mathbf{S}t}$ and the image will simply change to a mirror image, i.e. the rotation of the 4D vector will be in the opposite direction [11].

The determinant of the matrix (12) of the quaternion frequency $\mathbf{S}$ is calculated as $|\mathbf{S}| = \left(\omega^2 + \sigma^2\right)^2$. Accordingly, the inverse matrix is calculated by the formula:

$$\mathbf{S}^{-1} = \frac{1}{\sqrt{|\mathbf{S}|}}\mathbf{S}^{\mathrm{T}} = \frac{1}{\sqrt{|\mathbf{S}|}}\left(\mathbf{E}\sigma + \hat{\mathbf{I}}\omega\right)^{\mathrm{T}} = \frac{1}{\left(\omega^2 + \sigma^2\right)}\left(\mathbf{E}\sigma - \hat{\mathbf{I}}\omega\right). \tag{16}$$

The square of the inverse matrix is calculated as

$$\mathbf{S}^{-2} = \frac{1}{\left(\omega^2 + \sigma^2\right)^2}\left(\mathbf{E}(\sigma^2 - \omega^2) - 2\hat{\mathbf{I}}\omega\sigma\right). \tag{17}$$

The quaternion frequency matrix $\mathbf{S}$ is a damped differential operator for the state-space model [12]:

$$\dot{\mathbf{x}}(t) = \mathbf{S}\mathbf{x}(t). \tag{18}$$

The solution to equation (18) will be the matrix exponent (14).

Since matrix $\mathbf{\Phi}^{\mathrm{T}}(\omega,t) = e^{-\hat{\mathbf{I}}\omega t}$ is a solution of a differential equation, matrix $\mathbf{\Phi}(\omega,t)$ is called the *fundamental matrix*, and matrix $\mathbf{S}$ is called the *state transition matrix* [12, 13]. The fundamental matrix is orthogonal, since $\mathbf{\Phi}(\omega,t)\mathbf{\Phi}^{\mathrm{T}}(\omega,t) = \mathbf{\Phi}^{\mathrm{T}}(\omega,t)\mathbf{\Phi}(\omega,t) = \mathbf{E}$.

Since matrix $\mathbf{\Phi}^{\mathrm{T}}(\omega,t)$ is orthogonal and does not change the modulus of vector $\mathbf{q}(t)$, we write the conditions for the convergence of the integral (15) as

$$\left|\int_0^{\infty} \mathrm{e}^{-\mathbf{E}\sigma t}\,\mathbf{q}(t)dt\right| = \left|\int_0^{\infty} \mathbf{E}\,\mathrm{e}^{-\sigma t}\,\mathbf{q}(t)dt\right| = \left|\int_0^{\infty} \mathrm{e}^{-\sigma t}\,\mathbf{q}(t)dt\right| < \infty. \tag{19}$$

According to (19), the degree of growth of the vector elements $\mathbf{q}(t)$ must be less than the rate of decrease of the common multiplier $\mathrm{e}^{-\sigma t}$ in the QLT (15). Consequently, QLT (15) in the form of writing corresponds to the Laplace transform (LT) for real one-dimensional signals that are components of the vector $\mathbf{q}(t)$. If each element of the vector $\mathbf{q}(t)$ satisfies the convergence condition, then condition (19) will be satisfied.

Consider the main properties of QLT. It is clear that the properties of the QLT in comparison with the LT of real signals have features associated with the non-commutativity of quaternions in the matrix representation.

The linearity property is obvious and shows that a linear combination of originals corresponds to a linear combination of images and vice versa:

$$\sum_{n=1}^{N} a_n \mathbf{q}_n(t) \Leftrightarrow \sum_{n=1}^{N} a_n \mathbf{q}_n(\mathbf{S}), \tag{20}$$

where $a_n$ are real numbers.

Property (20) is proved by interchanging integration in (15) and summation in (20).

When the time scale of the quaternion is increased by a number $a > 0$, the scale of the image is reduced:

$$\mathbf{q}(at) \Leftrightarrow \frac{1}{a}\mathbf{q}\left(\frac{1}{a}\mathbf{S}\right). \tag{21}$$

Property (21) is proved by replacing the variable in (15) with $\tau = at$.

When the elements of the quaternion vector are delayed by the same time $t_0 > 0$,

$\mathbf{q}(t-t_0) = \left[p(t-t_0) \quad u(t-t_0) \quad v(t-t_0) \quad w(t-t_0)\right]^{\mathrm{T}}$, property is true:

$$\mathbf{q}(t-t_0) \Leftrightarrow e^{-\mathbf{S}t_0}\mathbf{q}(\mathbf{S}) = \mathrm{e}^{-\sigma t_0}\,\mathbf{\Phi}^{\mathrm{T}}(\omega,t_0)\mathbf{q}(\mathbf{S}), \tag{22}$$

where the impulses shift matrix of the quaternion vector with a delay on $t_0$ has the form:

$$e^{-\mathbf{S}t_0} = \mathrm{e}^{-\sigma t_0}\,\mathbf{\Phi}^{\mathrm{T}}(\omega,t_0) = \mathrm{e}^{-\sigma t_0}\frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3}\cos\,\omega t_0 & -\sin\,\omega t_0 & -\sin\,\omega t_0 & -\sin\,\omega t_0 \\ \sin\,\omega t_0 & \sqrt{3}\cos\,\omega t_0 & \sin\,\omega t_0 & -\sin\,\omega t_0 \\ \sin\,\omega t_0 & -\sin\,\omega t_0 & \sqrt{3}\cos\,\omega t_0 & \sin\,\omega t_0 \\ \sin\,\omega t_0 & \sin\,\omega t_0 & -\sin\,\omega t_0 & \sqrt{3}\cos\,\omega t_0 \end{bmatrix}.$$

Property (22) is proved by replacing the integration variable in (15) by $\tau = t - t_0$ using the known properties of the fundamental matrix [13].

It is obvious that when advancing the elements of the quaternion vector by the same time $t_0 > 0$, the property is true:

$$\mathbf{q}(t+t_0) \Leftrightarrow e^{\mathbf{S}t_0}\mathbf{q}(\mathbf{S}) = \mathrm{e}^{\sigma t_0}\,\mathbf{\Phi}(\omega,t_0)\mathbf{q}(\mathbf{S}). \tag{23}$$

The quaternion vector consists of 4 pulses that are transmitted over different (orthogonal) channels and, in general, they can have different shapes and delays. We write such a quaternion as: $\mathbf{q}(t-t_m) = \left[p(t-t_0) \quad u(t-t_1) \quad v(t-t_2) \quad w(t-t_3)\right]^{\mathrm{T}}$, where index $m = 0,1,2,3$ corresponds to the element number in the vector representation.

QLT (15) for pulses delayed for different times will have the form:

$$\mathbf{q}(\mathbf{S}) = \int_0^{\infty} e^{-\mathbf{S}t}\mathbf{q}(t-t_m)dt = \int_0^{\infty} e^{-\sigma t}\mathbf{\Phi}^{\mathrm{T}}(\omega,t)\mathbf{q}(t-t_m)dt. \tag{24}$$

Let us represent the vector of pulses of different shapes and amplitudes $\mathbf{q}(t-t_m)$ as

$$\mathbf{q}(t-t_m) = \begin{bmatrix} p(t-t_0) & 0 & 0 & 0 \\ 0 & u(t-t_1) & 0 & 0 \\ 0 & 0 & v(t-t_2) & 0 \\ 0 & 0 & 0 & w(t-t_3) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_0 p(t-t_0) \\ a_1 u(t-t_1) \\ a_2 v(t-t_2) \\ a_3 w(t-t_3) \end{bmatrix},$$

where $\mathbf{a} = \left[a_0 \quad a_1 \quad a_2 \quad a_3\right]^{\mathrm{T}}$ is the vector of pulse amplitudes.

In this case, the images of pulses $\mathbf{q}(t-t_m)$ will be written in the form of vectors: $\mathbf{q}_m(\mathbf{S})=\begin{bmatrix} q_m(\mathbf{S}) & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$, where $m=0,1,2,3$ is the number of the pulse. Then, using the sum property (20), the vector of all images (24) can be written as

$$\mathbf{q}(\mathbf{S})=a_0\mathbf{E}\mathbf{q}_0(\mathbf{S})+a_1\mathbf{I}^{\mathrm{T}}\mathbf{q}_1(\mathbf{S})+a_2\mathbf{J}^{\mathrm{T}}\mathbf{q}_2(\mathbf{S})+a_3\mathbf{K}^{\mathrm{T}}\mathbf{q}_3(\mathbf{S}). \tag{25}$$

When selecting the corresponding columns in (25), we get that the matrix of different delays will look like:

$$e^{-\mathbf{S}t_m}=\frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3}\,\mathrm{e}^{-\sigma t_0}\cos\,\omega t_0 & -\mathrm{e}^{-\sigma t_1}\sin\,\omega t_1 & -\mathrm{e}^{-\sigma t_2}\sin\,\omega t_2 & -\mathrm{e}^{-\sigma t_3}\sin\,\omega t_3 \\ \mathrm{e}^{-\sigma t_0}\sin\,\omega t_0 & \sqrt{3}\,\mathrm{e}^{-\sigma t_1}\cos\,\omega t_1 & \mathrm{e}^{-\sigma t_2}\sin\,\omega t_2 & -\mathrm{e}^{-\sigma t_3}\sin\,\omega t_3 \\ \mathrm{e}^{-\sigma t_0}\sin\,\omega t_0 & -\mathrm{e}^{-\sigma t_1}\sin\,\omega t_1 & \sqrt{3}\,\mathrm{e}^{-\sigma t_2}\cos\,\omega t_2 & \mathrm{e}^{-\sigma t_3}\sin\,\omega t_3 \\ \mathrm{e}^{-\sigma t_0}\sin\,\omega t_0 & \mathrm{e}^{-\sigma t_1}\sin\,\omega t_1 & -\mathrm{e}^{-\sigma t_2}\sin\,\omega t_2 & \sqrt{3}\,\mathrm{e}^{-\sigma t_3}\cos\,\omega t_3 \end{bmatrix}.$$

Thus, using the introduced notation, we write the property of different delays of impulses as follows:

$$\mathbf{q}(t-t_m)\Leftrightarrow e^{-\mathbf{S}t_m}\mathbf{q}(\mathbf{S}). \tag{26}$$

Similarly, we write the different advance property as

$$\mathbf{q}(t+t_m)\Leftrightarrow e^{\mathbf{S}t_m}\mathbf{q}(\mathbf{S}). \tag{27}$$

It can be shown that the differentiation property of the original has the form:

$$\mathrm{L}\left\{\mathbf{q}'(t)\right\}=\mathbf{S}\mathbf{q}(\mathbf{S})-\mathbf{q}(0). \tag{28}$$

To prove property (28), we used the integration-by-parts formula in the vector-matrix representation: $\int\mathbf{U}\,\mathrm{d}\,\mathbf{v}=\mathbf{U}\mathbf{v}-\int\mathbf{U}'\mathbf{v}\,\mathrm{d}\,t$.

The differentiation property can be generalized to the case of the second derivative:

$$\mathrm{L}\left\{\mathbf{q}''(t)\right\}=\mathbf{S}\,\mathrm{L}\left\{\mathbf{q}'(t)\right\}-\mathbf{q}'(0)\}=\mathbf{S}\left[\mathbf{S}\mathbf{q}(\mathbf{S})-\mathbf{q}(0)\right]-\mathbf{q}'(0)=\mathbf{S}^2\mathbf{q}(\mathbf{S})-\mathbf{S}\mathbf{q}(0)-\mathbf{q}'(0).$$

According to (18), **S** is a differentiation operator.

In general, if $\mathbf{q}^{(n)}(t)$ exists and satisfies the conditions for the existence of an image, then from $\mathbf{q}(t)\Leftrightarrow\mathbf{q}(\mathbf{S})$ it follows that

$$\mathbf{q}^{(n)}(t)\Leftrightarrow\mathbf{S}^n\mathbf{q}(\mathbf{S})-\left[\mathbf{S}^{n-1}\mathbf{q}(0)+\mathbf{S}^{n-2}\mathbf{q}'(0)+\cdots+\mathbf{q}^{(n-1)}(0)\right],$$

where $\mathbf{q}(0)$ is the initial state at $t=0$.

We write the integration property of the original as

$$\int_{-\infty}^{t}\mathbf{q}(\tau)\,\mathrm{d}\,\tau\Leftrightarrow\mathbf{S}^{-1}\left[\mathbf{q}(\mathbf{S})+\mathbf{q}(0)\right]. \tag{29}$$

To prove property (29), we represent the integral as the sum of two integrals:

$$\int_{-\infty}^{t}\mathbf{q}(\tau)\,\mathrm{d}\,\tau=\int_{-\infty}^{0}\mathbf{q}(\tau)\,\mathrm{d}\,\tau+\int_{0}^{t}\mathbf{q}(\tau)\,\mathrm{d}\,\tau.$$

Since $\mathbf{q}(0)$ is a constant value, the first integral is calculated as

$$\mathrm{L}\{\mathbf{q}(0)\}=\int_{0}^{\infty}\mathrm{e}^{-\mathbf{S}t}\,\mathbf{q}(0)\,\mathrm{d}\,t=\left(\int_{0}^{\infty}\mathrm{e}^{-\mathbf{S}t}\,\mathrm{d}\,t\right)\mathbf{q}(0)=-\mathbf{S}^{-1}\,\mathrm{e}^{-\mathbf{S}t}\Big|_{0}^{\infty}\,\mathbf{q}(0)=\mathbf{S}^{-1}\mathbf{q}(0). \tag{30}$$

When integrating from 0, we get that

$$\mathrm{L}\{\mathbf{q}(0)\}=\int_{0}^{0}\mathrm{e}^{-\mathbf{S}t}\,\mathbf{q}(0)\,\mathrm{d}\,t=\left(\int_{0}^{0}\mathrm{e}^{-\mathbf{S}t}\,\mathrm{d}\,t\right)\mathbf{q}(0)=-\mathbf{S}^{-1}\,\mathrm{e}^{-\mathbf{S}t}\Big|_{0}^{0}\,\mathbf{q}(0)=\mathbf{0}.$$

We use the differentiation property (28) and calculate the second integral in terms of the antiderivative. As a result, we get:

$$\int_0^t \mathbf{q}(\tau)d\tau \Leftrightarrow \mathbf{S}^{-1}\mathbf{q}(\mathbf{S}). \tag{31}$$

Combining results (30) and (31), we obtain (29). When integrating from 0, expression (29) will be equal to expression (31).

The integration property (31) can be generalized for a positive integer *n*:

$$\mathrm{L}\left\{\int_0^{t_0} d\tau_0 \int_0^{t_1} d\tau_1 \cdots \int_0^{t_{n-2}} \mathbf{q}(\tau_{n-1})d\tau_{n-1}\right\} = \mathbf{S}^{-n}\mathbf{q}(\mathbf{S}).$$

Consider a MIMO device whose transmission function of elements $h_{n,m}(t)$ is represented as a square matrix $\mathbf{H}(t)$. We represent the vector acting on the input of the device as a vector of impulses $\mathbf{q}(t)$ defined in a certain time interval [0, t]. In general, matrix and vector will be complicated functions of time. Their convolution on this segment is a new vector function $\mathbf{f}(t)$ defined by the equality:

$$\mathbf{f}(t) = \int_0^t \mathbf{H}(\tau)\mathbf{q}(t-\tau)\mathrm{d}\tau = \mathbf{H}(t) * \mathbf{q}(t). \tag{32}$$

Expression (32) is defined as a convolution in the vector-matrix representation. Note that for matrix convolution the law of commutativity does not hold, since $\mathbf{Hq} \neq \mathbf{Qh}$.

Let us show that

$$\int_0^\infty \mathbf{H}(\tau)\mathbf{q}(t-\tau)\mathrm{d}\tau \Leftrightarrow \mathbf{H}(\mathbf{S})\mathbf{q}(\mathbf{S}) \text{ or } \mathbf{H}(t) * \mathbf{q}(t) \Leftrightarrow \mathbf{H}(\mathbf{S})\mathbf{q}(\mathbf{S}).$$

Let's take the QLT from expression (32):

$$\mathrm{L}\left\{\int_0^\infty \mathbf{H}(\tau)\mathbf{q}(t-\tau)\mathrm{d}\tau\right\} = \int_0^\infty \left[\int_0^\infty \mathrm{e}^{-\mathbf{S}t}\,\mathbf{H}(\tau)\mathbf{q}(t-\tau)\mathrm{d}\tau\right]\mathrm{d}t. \tag{33}$$

Let's make a change of variables. Let's put $\lambda = t - \tau$, $t = \tau + \lambda$, and $\mathrm{d}t = \mathrm{d}\lambda$. Substituting these values into (33), we obtain

$$\int_0^\infty \mathrm{e}^{-\mathbf{S}\lambda}\left[\int_0^\infty \mathrm{e}^{-\mathbf{S}\tau}\,\mathbf{H}(\tau)d\tau\right]\mathbf{q}(\lambda)\mathrm{d}\lambda = \int_0^\infty \mathrm{e}^{-\mathbf{S}\lambda}\,\mathbf{H}(\mathbf{S})\mathbf{q}(\lambda)\mathrm{d}\lambda = \mathbf{H}(\mathbf{S})\mathbf{q}(\mathbf{S}).$$

Thus, from the expressions obtained, it can be concluded that, in terms of the form of writing, the properties of the QLT are similar to the properties of the LT of real signals when the complex frequency is replaced by a quaternion frequency. In this case, it is necessary to take into account the property of non-commutativity of matrices.

## IV. Examples of using the technique for calculating the direct quaternion Laplace transform

We write the vector of real exponents with different attenuation and different amplitudes as

$$\mathbf{ex}(\alpha_m, a_m, t) = \begin{bmatrix} a_0 e^{-\alpha_0 t} \\ a_1 e^{-\alpha_1 t} \\ a_2 e^{-\alpha_2 t} \\ a_3 e^{-\alpha_3 t} \end{bmatrix} = \begin{bmatrix} e^{-\alpha_0 t} & 0 & 0 & 0 \\ 0 & e^{-\alpha_1 t} & 0 & 0 \\ 0 & 0 & e^{-\alpha_2 t} & 0 \\ 0 & 0 & 0 & e^{-\alpha_3 t} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = e^{-\mathbf{A}t}\mathbf{a},$$

where

$$\mathbf{A}=\begin{bmatrix}\alpha_0 & 0 & 0 & 0\\ 0 & \alpha_1 & 0 & 0\\ 0 & 0 & \alpha_2 & 0\\ 0 & 0 & 0 & \alpha_3\end{bmatrix}$$ - damping matrix, $\mathbf{a}=\left[a_0\ \ a_1\ \ a_2\ \ a_3\right]^{\mathrm{T}}$ - amplitude vector.

Using formula (15), we directly obtain the QLT of the matrix of real exponentials with the damping matrix **A** and the amplitude vectors **a** in the form: $\mathbf{ex}(\alpha_m,a_m,t)\Leftrightarrow\mathbf{Ex}(\mathbf{S})=\left(\mathbf{S}+\mathbf{A}\right)^{-1}\mathbf{a}$.

We write the vector of lines with different slopes as

$\mathbf{l}(t)=\mathbf{L}(t)\mathbf{a}=t\mathbf{E}\mathbf{a}=\left[a_0t\quad a_1t\quad a_2t\quad a_3t\right]^{\mathrm{T}}$.

We get the QLT of the vector of lines with different slopes: $\mathbf{l}(t)\Leftrightarrow\mathbf{S}^{-2}\mathbf{a}$.

The QLT matrix of cosines has the form: $\mathbf{Cs}(\mathbf{S})=\mathbf{S}\left(\mathbf{S}^2+\mathbf{E}\omega^2\right)^{-1}$.

The QLT matrix of the sine vector will be: $\mathbf{Sn}(\mathbf{S})=\omega\left(\mathbf{S}^2+\mathbf{E}\omega^2\right)^{-1}$.

The QLT matrix of the vector of steps is calculated as $\mathbf{U}(\mathbf{S})=\mathbf{S}^{-1}$.

The QLT matrix of the delta pulses vector: $\delta(t)\mathbf{E}\Leftrightarrow\mathbf{E}$.

The QLT matrix of rectangular pulses symmetrical to zero time has the form:

$$\mathbf{R}(\mathbf{S})=\mathbf{S}^{-1}\left(e^{\frac{T}{2}\mathbf{S}}-e^{-\frac{T}{2}\mathbf{S}}\right).$$

Let us substitute formula (12) for the matrix of quaternion frequencies into the resulting expression in brackets:

$$e^{\frac{T}{2}(\mathbf{E}\sigma+\hat{\mathbf{I}}\omega)}-e^{-\frac{T}{2}(\mathbf{E}\sigma+\hat{\mathbf{I}}\omega)}=e^{\frac{T}{2}\mathbf{E}\sigma}e^{\frac{T}{2}\hat{\mathbf{I}}\omega}-e^{-\frac{T}{2}\mathbf{E}\sigma}e^{-\frac{T}{2}\hat{\mathbf{I}}\omega}.$$

To obtain the spectrum matrix, we equate $\sigma$ to zero, as a result, we obtain that

$$e^{\frac{T}{2}\hat{\mathbf{I}}\omega}-e^{-\frac{T}{2}\hat{\mathbf{I}}\omega}=2\hat{\mathbf{I}}\sin\left(\frac{\omega T}{2}\right).$$

We multiply the resulting expression by the inverse matrix of quaternion frequencies (16):

$$\mathbf{R}(\omega)=\mathbf{S}^{-1}\left(e^{\frac{T}{2}\mathbf{S}}-e^{-\frac{T}{2}\mathbf{S}}\right)=\frac{1}{\omega^2}\hat{\mathbf{I}}^{\mathrm{T}}\omega 2\hat{\mathbf{I}}\sin\left(\frac{\omega T}{2}\right)=\frac{2}{\omega}\sin\left(\frac{\omega T}{2}\right)\mathbf{E}=T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)\mathbf{E}.$$

The resulting expression corresponds to the quaternion Fourier transform [5]. When multiplying the spectral matrix by the amplitude vector **a**, we obtain the vector of the spectra of rectangular pulses of different amplitudes.

To obtain the QLT matrix of rectangular pulses with the same or different shift, we use the properties of the same or different delays (22), (26). Similar properties (23) and (27) are used to obtain the advance of impulses.

The QLT matrix of the vector of sawtooth pulses symmetrical with duration *T* has the form:

$$\mathbf{W}(\mathbf{S})=\mathbf{S}^{-2}\left(e^{\frac{T}{2}\mathbf{S}}-e^{-\frac{T}{2}\mathbf{S}}\right)-\mathbf{S}^{-1}e^{-\frac{T}{2}\mathbf{S}}.$$

The QLT of the sawtooth pulse vector can be obtained from the QLT of the square wave vector by integration, i.e. multiplying the image by $\mathbf{S}^{-1}$ and then subtracting the QLT of the step vector with a delay of $T/2$. We obtain the matrices of the spectrum of sawtooth pulses by substituting the values for the inverse matrices (16) and (17):

$$\mathbf{W}(\mathbf{S})=-2\hat{\mathbf{I}}\frac{1}{\omega^2}\sin\left(\frac{\omega T}{2}\right)+\frac{1}{\omega}\hat{\mathbf{I}}e^{\frac{-T}{2}\hat{\mathbf{I}}\omega}=\mathbf{E}\frac{T}{2}\mathrm{sinc}\left(\frac{\omega T}{2}\right)+\hat{\mathbf{I}}\frac{1}{\omega}\left(\cos\left(\frac{\omega T}{2}\right)-T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)\right).$$

The resulting expression corresponds to the quaternion Fourier transform [5] for pulses with duration $T$=1.

The QLT matrix of triangular pulses symmetrical $t=0$ with duration $T$ has the form:

$$\mathbf{T}(\mathbf{S}) \Leftrightarrow \mathbf{S}^{-2}\left(e^{\frac{T}{2}\mathbf{S}} + e^{-\frac{T}{2}\mathbf{S}} - 2\mathbf{E}\right).$$

With $\sigma = 0$ and normalizing the amplitude to 1 by dividing by $T$/2, we get the spectrum matrix:

$$\mathbf{T}(\omega) = 4\mathbf{E}\frac{2}{\omega^2 T}\sin^2\left(\frac{\omega T}{4}\right) = \mathbf{E}\frac{T}{2}\operatorname{sinc}^2\left(\frac{\omega T}{4}\right).$$

The QLT matrix of meanders symmetrical to $t=0$ with duration $T$, has the form:

$$\mathbf{M}(\mathbf{S}) \Leftrightarrow \mathbf{S}^{-1}\left(e^{\frac{T}{2}\mathbf{S}} + e^{-\frac{T}{2}\mathbf{S}} - 2\mathbf{E}\right).$$

The QLT matrix for the meander vector can be calculated from the QLT matrix for the triangular pulses vector by taking the derivative, i.e. by multiplying the image by **S**. For $\sigma = 0$ we get the spectrum matrix:

$$\mathbf{M}(\omega) = \frac{4}{\omega}\hat{\mathbf{I}}\sin^2\left(\frac{\omega T}{4}\right) = \frac{\omega T^2}{4}\hat{\mathbf{I}}\operatorname{sinc}^2\left(\frac{\omega T}{4}\right).$$

The QLT of pulses with different amplitudes is obtained by multiplying the QLT in the matrix representation by the amplitude vector **a**. Different pulses shifts are obtained by multiplying by matrices of different shifts (26) or (27).

Consider a method for obtaining a QLT vector consisting of various elements with different shifts, for example, a rectangular pulse with an amplitude of $a_0$, a sawtooth pulse with an amplitude of $a_1$, a triangular pulse with an amplitude of $a_2$, and a meander with an amplitude of $a_3$. First, you need to take the first column of the QLT matrix of rectangular pulses $\mathbf{R}(\mathbf{S})^{\langle 0\rangle}$, the second - sawtooth pulses $\mathbf{W}(\mathbf{S})^{\langle 1\rangle}$ , the third - triangular pulses $\mathbf{T}(\mathbf{S})^{\langle 2\rangle}$ , the fourth - meanders $\mathbf{M}(\mathbf{S})^{\langle 3\rangle}$. Next, you need to multiply the resulting columns by the corresponding elements of the amplitude vector:

$$\mathbf{q}(\mathbf{S}) = a_0\mathbf{R}(\mathbf{S})^{\langle 0\rangle} + a_1\mathbf{W}(\mathbf{S})^{\langle 1\rangle} + a_2\mathbf{T}(\mathbf{S})^{\langle 2\rangle} + a_3\mathbf{M}(\mathbf{S})^{\langle 3\rangle}.$$

To obtain the QLT of the pulses vector with different shifts, it is necessary to multiply the QLT matrices of the pulses by the shift matrices (26), (27). As a result, we obtain the QLT matrix in the form (25), after multiplying it by the amplitude vector, we obtain the QLT vector consisting of different pulses.

## V. Conclusion

Thus, a technique for obtaining the quaternion Laplace transform of a four-dimensional vector of pulses interconnected by a many-input-many output scheme is presented, and examples of its use for the most common pulses in radio engineering. It is shown that the quaternion Laplace transform can be obtained using the well-known one-dimensional Laplace transform of real signals by replacing the complex frequency with the quaternion frequency. This technique will simplify the analysis of signal vectors, the elements of which are interconnected by the MIMO scheme, that is, hypercomplex signals, which are currently increasingly used in communication systems to increase the capacity and noise immunity of channels.

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