Maxwell's Macroscopic Electromagnetic Force

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Abstract

The electromagnetic force, following from Maxwell's minimal macroscopic electromagnetic stress dyad, which reproduces analytically the measurements made by Coulomb[magnet-magnet force], Biot-Savart[current-magnet force], Ampere-Weber[current-current force] and Faraday-Kelvin[polarization current-magnetization current force], is

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presented[1] . This force leads to a space independent internal torque explaining experiments such as the Einstein de Haas effect [the rotation of a material in the presence of a uniform external magnetic field],[2] or the Barnett effect [the creation of the magnetic field by a rotation of a material][3,4]. The cause for the torque should not be, a priori, attributed to the asymmetry in the Cauchy's internal dyad[5] - a conclusion which is unavoidable since the electromagnetic force is obtained from a symmetric dyad which leads to no internal torque.

Introduction

The well known elementary magnetic torque per unit volume $\vec{M} \times \vec{B}$ is shown to follow from Maxwell's macroscopic stress dyad. This torque could explain the initial cause[6] of the rotation of the substance, observed more than a century ago[2]. Hence, in an experimental observation of this mechanism in addition to the intrinsic internal torque (due to the asymmetry of Cauchy's internal stress dyad), the intrinsic external torque (due to the asymmetry of Maxwell's external stress dyad) must be considered.

In section 1 the equation of the motion of the macroscopic angular momentum of a substance in the presence of the electromagnetism is briefly reviewed. In section 2 Maxwell's macroscopic magnetic force and torque are presented. In section 3 Maxwell's macroscopic electromagnetic force and torque are presented, followed with a simple example, discussing the rotation of a soft ferromagnetic sphere in the presence of a uniform static magnetic .

1 The macroscopic angular momentum

In a continuum of density $\rho(\vec{r}, t)$, the elemental angular momentum of a substance $\vec{\ell} = m\vec{r} \times \dot{\vec{r}}$ satisfies the equation of motion:

$$\dot{\vec{\ell}} = m\vec{r} \times \ddot{\vec{r}} \quad = \quad dV\vec{r} \times (\vec{\nabla} \cdot T) \tag{1}$$

where dV(t) is the elemental volume of the substance with the elemental mass $m(= \rho dV)$ at a position $\vec{r}$, velocity $\dot{\vec{r}}$ and acceleration $\ddot{\vec{r}}$ [relative to an arbitrary inertial frame]. And $T = \hat{x}_i \hat{x}_j T_{ij}$ is the net stress dyad acting within the elemental substance. The net force per unit volume on the elemental substance is given by $\vec{\nabla} \cdot T = \partial_i T_{ij} \hat{x}_j$, i.e. the equation of the motion of m is $m\ddot{\vec{r}} = dV\vec{\nabla} \cdot T$ with $\dot{m} = 0$. This force satisfies the third law since by divergence theorem it can be transformed to an arbitrary surface integral enclosing the elemental volume $dV\vec{\nabla} \cdot T = \oint_{\partial A} d\vec{A} \cdot T$ as long as this surface encloses the whole of volume dV but does not enclose any volume of the environment exerting the force. And hence the net force is due to external fields.

If the environment is divided into external and internal domains [the elemental mass m belonging to the internal domain], integrating the equation of the motion over the internal space gives $M\ddot{\vec{r}}_{cm} = \oint d\vec{A} \cdot T$ where $M = \int \rho dV$ is the macroscopic mass of the internal space and $\ddot{\vec{r}}_{cm} = \frac{1}{M} \int m\ddot{\vec{r}}$ is the center of mass acceleration. Similarly integrating the Eq. (1) over the whole volume of the internal space gives:

$$\dot{\vec{L}} = \int dV \vec{r} \times (\vec{\nabla} \cdot T) \tag{2}$$

The macroscopic torque $\dot{\vec{L}}$ is caused by an intrinsic volume torque $\vec{\tau}^{\,i} = -\dot{\vec{S}}$ plus an extrinsic surface torque $\vec{\tau}^{\,e}$, i.e. the right side of Eq. (2) can be transformed to a position independent volume torque $[\vec{\tau}^{\,i} = -\dot{\vec{S}}]$ plus a position dependent extrinsic surface torque $[\vec{\tau}^{\,e}]$, hence $\dot{\vec{L}} = \int dV \vec{r} \times (\vec{\nabla} \cdot T) = -\int dV T_{ij} \hat{x}_i \times \hat{x}_j + \oint \vec{r} \times (d\vec{A} \cdot T) = \vec{\tau}^{\,i} + \vec{\tau}^{\,e} = -\dot{\vec{S}} + \vec{\tau}^{\,e}$ (the intrinsic volume torque $-\dot{\vec{S}}$ is being caused by an external or internal couple, vanishing if the net stress dyad is symmetric $T_{ij} = T_{ji}$) leading to

$$\dot{\vec{L}} + \dot{\vec{S}} = \vec{\tau}^{\,e} \tag{3}$$

The intrinsic torque $-\dot{\vec{S}}$ will then be due to *both* Cauchy's internal volume torque $\dot{\vec{s}} = \int dV T_{ij}^{(int)} \hat{x}_i \times \hat{x}_j$ and Maxwell's volume torque $\dot{\vec{\sigma}} = \int dV T_{ij}^{(e.m.)} \hat{x}_i \times \hat{x}_j$, i.e. $T = T^{(int)} + T^{(e.m.)}$

and $\dot{\vec{S}} = \dot{\vec{s}} + \dot{\vec{\sigma}}$. Hence Eq. (3) reads $\dot{\vec{L}} + \dot{\vec{S}} = \dot{\vec{L}} + \dot{\vec{s}} + \dot{\vec{\sigma}} = \dot{\vec{J}} + \dot{\vec{\sigma}} = \vec{\tau}^{\,e} = \vec{\tau}^{e,ext} + \vec{\tau}^{e,int}$ where $\dot{\vec{J}}$ is the net internal torque ($\dot{\vec{J}} = \dot{\vec{L}} + \dot{\vec{s}}$) and where the extrinsic torque $\vec{\tau}^{\,e}$ is the sum of the extrinsic torque due to the external environment $\vec{\tau}^{e,ext}$ and the internal environment $\vec{\tau}^{e,int}$.

If the extrinsic torque $\vec{\tau}^{\,e} = \oint \vec{r} \times (d\vec{A} \cdot T)$ vanishes, the vanishing of the Cauchy's internal volume torque $\dot{\vec{s}} = 0$ does not imply conservation of a net internal angular momentum $\vec{J} = \vec{L} + \vec{s}$ by virtue of the presence of Maxwell's volume torque $\dot{\vec{\sigma}}$: $\dot{\vec{L}} + \dot{\vec{S}} = \dot{\vec{L}} + \dot{\vec{\sigma}} = 0$. However, turning on the external field, keeping in mind that the external force can induce permanent changes in the net internal angular momentum, and then turning off the external field can lead to the conservation of the net internal angular momentum $\vec{J} = \vec{L} + \vec{s}$:

$$0 = \dot{\vec{L}}_0 \quad \overset{\substack{field\ on \\ t=0^+}}{\overbrace{\longrightarrow}} \quad 0 = \dot{\vec{L}}_1 + \dot{\vec{\sigma}}_1 \quad \overset{\substack{field\ on \\ hysteresis}}{\overbrace{\longrightarrow}} \quad 0 = \dot{\vec{L}}_2 + \dot{\vec{s}}_2 + \dot{\vec{\sigma}}_2 \quad \overset{\substack{field\ off \\ hysteresis}}{\overbrace{\longrightarrow}} \quad 0 = \dot{\vec{L}} + \dot{\vec{s}} = \dot{\vec{J}} \qquad (4)$$

Therefore the external force can turn a substance with a symmetric internal Cauchy stress dyad into an asymmetric one.

2 Maxwell's Macroscopic Magnetic Force

Continuity of the fields and their first derivative is assumed and all fields are relative to an arbitrary inertial frame. Experiments performed by Coulomb, Biot-Savart, Ampere-Weber and Faraday-Kelvin suggest the macroscopic force[7] [MKS system of the units are used throughout]

$$\vec{F}^{(m)}_{macro} = \int dV \underbrace{\vec{\nabla} \cdot \vec{B}}_{0} \vec{H} + \int dV (\vec{\nabla} \times \vec{H}) \times (\mu \vec{H}) + \int dV [\vec{B} - \mu \vec{H}] \cdot \vec{\nabla} \vec{H} \tag{5}$$

where $\vec{H}$ is the force per unit magnetic charge(or per unit magnetic flux)$[\frac{Newton}{Volt\ second}]$ and $\vec{B}$ is the magnetic flux per unit area [Tesla] and μ is the permeability of the medium or the magnetomotive force per unit current $[\frac{Newton}{Ampere^2}]$. Hence $\vec{\nabla} \times \vec{H}$ is the true current density [The first term, the Coulomb-Gauss term, although zero is written explicitly to facilitate the transition to the electric sector(see section 3)]. Furthermore, this magnetic force is the divergence of a (minimal) magnetic dyad $T^{(m)}$ given by[7]:

$$\vec{F}^{(m)}_{macro} = \int dV \vec{\nabla} \cdot T^{(m)} = \int dV \vec{\nabla} \cdot [\vec{B}\vec{H} - \frac{\mu}{2} \vec{H} \cdot \vec{H} I] \tag{6}$$

where

$$T^{(m)} = \vec{B}\vec{H} - \frac{\mu}{2} \vec{H} \cdot \vec{H} I \qquad ; \qquad I = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k} \tag{7}$$

Hence this force obeys the third law for arbitrary vector fields $\vec{H}$ and $\vec{B}$. Using Helmholtz' decomposition, the above dyad is written as a combination of a symmetric and asymmetric dyad

$$T^{(m)} = \left[\frac{\vec{B}\vec{H} + \vec{H}\vec{B}}{2} - \frac{1}{2} \mu I H^2 \right] + \frac{\vec{B}\vec{H} - \vec{H}\vec{B}}{2} \tag{8}$$

and then the symmetric part is diagonalized via:

$$\begin{cases} \vec{B} = B(\hat{\xi}_1 \cos\epsilon - \hat{\xi}_2 \sin\epsilon) \ ; \ \vec{H} = H(\hat{\xi}_1 \cos\epsilon + \hat{\xi}_2 \sin\epsilon) \\ I = \hat{\xi}_1\hat{\xi}_1 + \hat{\xi}_2\hat{\xi}_2 + \hat{\xi}_3\hat{\xi}_3 \end{cases} \tag{9}$$

Substituting $\vec{B}$, $\vec{H}$ and I from Eq. (9) in the expression for $T^{(m)}$ given by Eq. (8) leads to:

$$
\begin{aligned}
T^{(m)} &= (BH\cos^2\epsilon - \frac{1}{2}\mu H^2)\hat{\xi}_1\hat{\xi}_1 + BH\sin\epsilon\cos\epsilon(\hat{\xi}_1\hat{\xi}_2 - \hat{\xi}_2\hat{\xi}_1) + (-BH\sin^2\epsilon - \frac{1}{2}\mu H^2)\hat{\xi}_2\hat{\xi}_2 - \frac{1}{2}\mu H^2\hat{\xi}_3\hat{\xi}_3 \\
&= (\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)\begin{pmatrix} BH\cos^2\epsilon - \frac{1}{2}\mu H^2 & 0 & 0 \\ 0 & -BH\sin^2\epsilon - \frac{1}{2}\mu H^2 & 0 \\ 0 & 0 & -\frac{1}{2}\mu H^2 \end{pmatrix}\begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \end{pmatrix} + \\
&\quad +(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)\begin{pmatrix} 0 & BH\sin\epsilon\cos\epsilon & 0 \\ -BH\sin\epsilon\cos\epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \end{pmatrix} \qquad (10)
\end{aligned}
$$

The symmetric part in the above equation gives the principle stresses, while the asymmetric part gives the couple $\frac{1}{2}\vec{B} \times \vec{H}$. Hence[7] "

(1) a pressure equal in all directions $\frac{1}{2}\mu H^2$,

(2) a tension along the line bisecting the angle between the magnetic force $[\vec{H}]$ and the magnetic induction $[\vec{B}]$ $BH\cos^2\epsilon$,

(3) a pressure along the line bisecting the exterior angle between these directions $BH\sin^2\epsilon$, and

(4) a couple tending to turn every element of the substance in the plane of two directions from the direction of the magnetic induction to the direction of the magnetic force $\frac{1}{2}BH\sin 2\epsilon$.".

The couple leads to the intrinsic torque, i.e. the net magnetic external torque, relative to some origin, follows as:

$$
\begin{aligned}
\vec{\tau}^{m.,ext}_{macro} &= \int dV\vec{r} \times (\vec{\nabla} \cdot T^{(m)}) = \int dV\vec{r} \times \left[\vec{\nabla} \cdot \left\{\vec{B}\vec{H} - \frac{1}{2}\mu H^2 I\right\}\right] \\
&= \oint \vec{r} \times \left\{d\vec{S} \cdot \left[\vec{B}\vec{H} - \frac{1}{2}\mu H^2 I\right]\right\} - \int dV\left(\vec{B} \times \vec{H}\right) \qquad (11)
\end{aligned}
$$

the first term [in the second line above] is a surface term which involves the external fields, the second term is Maxwell's intrinsic torque which acts within the body.

Accordingly, if the magnetic field $\vec{B}$ and the magnetic field intensity $\vec{H}$ are not proportional to one another, they give rise to an elemental internal magnetic torque of the form $dV\vec{B} \times \vec{H}$, even in the absence of the true current density $\vec{\nabla} \times \vec{H} = 0$ [See also the example at the end of the section 3.] .

Maxwell's torque, $dV\vec{B} \times \vec{H}$, can explain the Einstein-de Haas effect[2], i.e. the rotation of the ferromagnetic material in the presence of an external magnetic field. In a similar manner it follows that a forced rotation of a substance could lead to the creation of a magnetic field, which can be observed by noting Maxwell's equation for the conductors in motion, which is given by Eq. (16). The induced $\vec{B}$ and $\vec{H}$ within the substance, when exposed to the external field, could then, when the external field is removed $\dot{\vec{\sigma}} \to 0$, lead to a permanent asymmetry in Cauchy's original symmetric stress dyad. In this light then when the external field is turned off $\dot{\vec{\sigma}} = 0$ the internal angular momentum of the substance $\vec{J}$ will remain constant, according to Eq. (4)

3 Maxwell's Macroscopic Electromagnetic Force

Maxwell's Magnetic force can be transformed into an electric force by virtue of the known Coulomb force $\int dV(\vec{\nabla} \cdot \vec{D})\vec{E}$. Therefore, in equations (5) and (6), the vectors $\vec{H}$ and $\vec{B}$ are replaced with the vectors $\sqrt{\frac{\epsilon}{\mu}}\vec{E}$ and $\sqrt{\frac{\mu}{\epsilon}}\vec{D}$ respectively, resulting in the generalized electric force:

$$
\begin{aligned}
\vec{F}^{(e)}_{macro} &= \int dV(\vec{\nabla}\cdot\vec{D})\vec{E} + \int dV(\vec{\nabla}\times\vec{E})\times(\epsilon\vec{E}) + \int dV[\vec{D}-\epsilon\vec{E}]\cdot\vec{\nabla}\vec{E} \\
&= \int dV\vec{\nabla}\cdot\left[\vec{D}\vec{E} - \frac{1}{2}\epsilon E^2\ I\right]
\end{aligned}
\tag{12}
$$

[page 155 Vol.1 and page 258 Vol.2 (third edition)][7], where ϵ is the permittivity of the medium [Farads/meter], $\vec{E}$ is the electric field [Newton/Coulomb] and $\vec{D}$ is the displacement vector or the true electric flux per unit area [Coulomb/meter^2]. Hence the general macroscopic electromagnetic force motivated by Maxwell's Magnetic force and Coulomb's electric force follows as ($\vec{F}^{e.m.}_{macro} = \vec{F}^{(m)}_{macro} + \vec{F}^{(e)}_{macro}$ [from Eq. (6) and (12)])[7]:

$$
\begin{aligned}
\vec{F}^{e.m.}_{macro} = \int dV\vec{\nabla}\cdot T^{(e.m.)} &= \int dV\vec{\nabla}\cdot\left[\vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}(\epsilon E^2 + \mu H^2)I\right] \\
&= \oint d\vec{S}\cdot\left[\vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}(\epsilon E^2 + \mu H^2)I\right]
\end{aligned}
\tag{13}
$$

where the permittivity ϵ and the permeability μ are that of the medium.

The net electromagnetic external torque, relative to some origin, follow as:

$$
\begin{aligned}
&\vec{\tau}^{e.m.,ext}_{macro} = \int dV\vec{r}\times\left\{\vec{\nabla}\cdot\left[\vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}(\epsilon E^2 + \mu H^2)I\right]\right\} = \\
&= \oint \vec{r}\times\left\{d\vec{S}\cdot\left[\vec{D}\vec{E} + \vec{B}\vec{H} - \frac{1}{2}(\epsilon E^2 + \mu H^2)I\right]\right\} - \int dV\left(\vec{D}\times\vec{E} + \vec{B}\times\vec{H}\right)
\end{aligned}
\tag{14}
$$

the second term [in the second line above] is Maxwell's torque which acts within the body. This torque vanishes if the dyad is symmetric, i.e. when $\vec{B}$ and $\vec{E}$ are along $\vec{H}$ and $\vec{D}$ respectively. Maxwell's torque can be written in the familiar form [$\vec{H} = \frac{\vec{B}}{\mu} - \vec{M}$, $\vec{D} = \epsilon\vec{E} + \vec{P}$]:

$$
\vec{\tau}^{Maxwell,ext}_{macro} = \int dV\left(\vec{D}\times\vec{E} + \vec{B}\times\vec{H}\right) = \int dV\left(\vec{P}\times\vec{E} + \vec{M}\times\vec{B}\right)
\tag{15}
$$

This is a remarkable finding[by Maxwell]: a classical *macroscopic* torque leading to a macroscopic spin. It states that the field *outside* the substance [which itself gets modified] can *induce* an electric displacement as well as a magnetic induction within the substance, which can then lead to a rotation of the substance.

An elementary example (involving two parameters β and γ) is given in this conclusion:

Maxwell's macroscopic equations for a substance moving with variable velocity $\vec{v}$ is given by [all vector fields are relative to an inertial frame]:

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= \quad -\tfrac{\partial \vec{B}}{\partial t} + [-(\underbrace{\vec{\nabla} \cdot \vec{B}}_{0})\vec{v} + \vec{\nabla} \times (\vec{v} \times \vec{B})] \\ \vec{\nabla} \times \vec{H} &= \vec{J}_f + \tfrac{\partial \vec{D}}{\partial t} + [+(\vec{\nabla} \cdot \vec{D})\vec{v} - \vec{\nabla} \times (\vec{v} \times \vec{D})] \end{aligned} \tag{16}$$

Consider the case where $\vec{E} = \vec{D} = \vec{J}_f = 0$. The above equations simplify to:

$$0 = -\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{v} \times \vec{B}) \quad ; \quad \vec{\nabla} \times \vec{H} = 0$$

Let the above equations hold within and without a solid soft ferromagnetic sphere of radius R, initially in the presence of a space and time independent magnetic field $B_0 (= \mu H_0)$ along the z axis. The following fields satisfy the above equations:

$$\begin{aligned} &\vec{v} = \omega r \sin\theta \hat{\phi} \;\; ; \;\; \vec{B}_{in} = 3B_0 \hat{z} + \tfrac{B_0}{\beta} r \sin\theta \hat{\phi} \;\; ; \;\; \vec{H}_{in} = \tfrac{H_0}{(-\gamma)} r \hat{r} \;\; ; \;\; 0 \le r \le R \\ &\vec{v} = 0 \;\; ; \;\; \vec{B}_{out} = \mu \vec{H}_{out} = B_0 \left[\hat{z} + \tfrac{R^3}{r^3}(2\cos\theta \hat{r} + \sin\theta \hat{\theta})\right] \;\; ; \;\; R \le r \le \infty \end{aligned} \tag{17}$$

where the angular velocity $\omega(t)$ is independent of space but may depend on time, explicitly. The parameters β and γ have units of length. β is a constant independent of time and space and $\gamma(t)$ is independent of space but may depend on time explicitly. The Cartesian and spherical base vectors are $(\hat{x}, \hat{y}, \hat{z})$, $(\hat{r}, \hat{\theta}, \hat{\phi})$ respectively. Here $\vec{r} = r\hat{r}$ is relative to the geometric center of the sphere and $\vec{v} = \dot{\vec{r}}$ is the velocity of a point within (without) the sphere. The above solution satisfies the following boundary conditions for the magnetization $\vec{M}|_{r=R}$:

$$\begin{aligned} -\hat{r} \cdot \vec{M}_{in} &= \frac{RH_0}{(-\gamma)} - \frac{3B_0 \cos\theta}{\mu} \\ -\hat{r} \times \vec{M}_{in} &= \frac{3B_0 \sin\theta}{\mu} \left(\hat{\phi} + \frac{R}{3\beta} \hat{\theta} \right) \end{aligned}$$

The force follows from equation (13) which vanishes:

$$\begin{aligned} \vec{F}^{(m)}_{em} &= \oint d\vec{S} \cdot [\vec{B}_{out}\vec{H}_{out} - \frac{1}{2}\mu I \vec{H}_{out} \cdot \vec{H}_{out}] \\ &= \int_0^{\pi} \int_0^{2\pi} R^2 \sin\theta d\theta d\phi [\hat{r} \cdot \vec{B}_{out}\vec{H}_{out} - \frac{1}{2}\mu \hat{r} \vec{H}_{out} \cdot \vec{H}_{out}]|_{r=R} = 0 \end{aligned}$$

The torque relative to the geometric center of the sphere follows from equation (14):

$$\begin{aligned} \vec{\tau}^{ext} &= \oint \vec{r} \times \left\{ d\vec{S} \cdot \left[\vec{B}\vec{H} - \frac{1}{2}\mu H^2 I \right] \right\} - \int dV \left(\vec{B} \times \vec{H} \right) \\ &= \underbrace{\oint \vec{R} \times \left\{ d\vec{S} \cdot \left[\vec{B}_{out}\vec{H}_{out} - \frac{1}{2}\mu H^2_{out} I \right] \right\}}_{\vec{\tau}^{e,ext}=0} - \underbrace{\int dV \left(\vec{B}_{in} \times \vec{H}_{in} \right)}_{\dot{\vec{\sigma}} = -\frac{8B_0H_0}{15\beta(-\gamma)}\pi R^5 \hat{z}} \\ &= (B_0 H_0 \frac{4\pi R^3}{3})(\frac{2}{5}R^2 \frac{1}{\beta(-\gamma)})\hat{z} \end{aligned} \quad (18)$$

where the factor $\frac{2}{5}R^2$ is the moment of inertia of a uniform solid sphere per unit mass. From Euler's equation of motion $[\dot{\vec{L}} + \dot{\vec{S}} = \dot{\vec{L}} + \underbrace{\dot{\vec{s}}}_{0} + \dot{\vec{\sigma}} = \vec{\tau}^{e} = \underbrace{\vec{\tau}^{e,ext}}_{0} + \underbrace{\vec{\tau}^{e,int}}_{0} = 0$ where Cauchy's stress dyad is assumed to be symmetric $\dot{\vec{s}}$=0 and $\vec{\tau}^{e,int} = 0]$:

$$\frac{\partial I_{cm}\vec{\omega}}{\partial t} + \underbrace{\vec{\omega} \times \vec{L}}_{0} + \underbrace{\dot{\vec{s}}}_{0} + \dot{\vec{\sigma}} = 0 \quad (19)$$

Hence, equations (18) and (19) give

$$\boxed{\frac{\partial}{\partial t} I_{cm}\omega(t) = (B_0 H_0 \frac{4\pi R^3}{3})(\frac{2}{5}R^2 \frac{1}{\beta(-\gamma(t))})} \quad (20)$$

Introducing the new time dependent magnetic field $b(t)$

$$-g\frac{q_e}{2m_e}\frac{\partial}{\partial t}I_{cm}b(t) = (B_0H_0\frac{4\pi R^3}{3})(\frac{2}{5}R^2\frac{1}{\beta(-\gamma(t))}) \tag{21}$$

and comparing Eq. (21) with Eq. (20) gives

$$\omega = -g\frac{q_e}{2m_e}b \tag{22}$$

which is a well known macroscopic result connecting the angular velocity ω of a "fluid" to the magnetic field b involving Thomson's parameter $\frac{q_e}{m_e}$.

References

[1] All the relevant references to experiments with the exception of the Einstein-de Haas experiment and Barnett Effect, can be found in reference [7].

[2] Einstein, A.; de Haas, W. J. (1915). "Experimental proof of the existence of Ampre's molecular currents" . Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings. 18: 696711. Bibcode:1915KNAB...18..696E.

[3] S. J. Barnett, The magnetization of iron, nickel, and cobalt by rotation and the nature of the magnetic molecule, Phys. Rev. 10, 7 (1917).

[4] Mohsen Arabgol and Tycho Sleator "Observation of the Nuclear Barnett Effect" Physical Review Letters 122, 177202 (2019)

[5] O.W. Richardson "A Mechanical Effect Accompanying Magnetization" Phys. Rev. (Series I) 26, 248 Published 1 March 1908

[6] As discussed in the "Angular Momentum" section, the initial torque on the substance is due to the external field. The subsequent deformation of the substance will lead to a mechanical complication, i.e. to the transformation of Cauchy's internal stress tensor [assuming the initial stress tensor is known] where the details are beyond the scope of the present discussion.

[7] James Clerk Maxwell, "A treatise on Electricity and Magnetism" Unabridged Third Edition Vol. 1 and 2 Dover Publication INC. New york 1954 [Page 277 as well as 285]