SOLVING THE CONUNDRUM OF RECOVERING SUFFICIENT PARTICIPATING MASS FOR COMPLEX STRUCTURES

C. Hickey[1], R. Kannan[1], D. Grant[2], S. Hendry[2], K. Hiçyılmaz[3]

[1] Arup, Manchester, United Kingdom

[2] Arup, London, United Kingdom

[3] Arup, Glasgow, United Kingdom

Abstract: *An enduring question for response spectrum analysis of complex structural models is obtaining modes of vibration with a sufficient participating mass to represent the overall seismic response of the structure. Codes of practice for seismic design typically prescribe that a modal dynamic analysis captures a number of modes with a cumulative participating mass of at least 90% of the total model mass. Whilst this might be relatively straightforward for standard, regular structures whose seismic behaviour is governed by a few sway modes, it is far from trivial for complex geometries and models such as multiple towers built on a podium. In these structural models, in which the structure has multiple 'parts', each part can have relatively independent modes of vibration, which must be included in the response spectrum calculation. Effectively, we seek to satisfy the 90% minimum, not just for the entire structure but also for each individual part. In such scenarios, the computational cost (time and memory) of computing these modes can be prohibitively high.*

We introduce a new eigensolver algorithm, implemented in Oasys GSA, that uses a novel filtering technique to compute only the most relevant modes that contribute to the required mass participation. The algorithm also allows for parts of the model to be identified, and mass participation requirements to be checked and enforced on each part. The algorithm is called MASIL, which stands for Mass Accumulating Shift Invert Lanczos, and it returns only those eigenvectors that contribute mass above a certain threshold for each part specified by the engineer.

Using numerical experiments we demonstrate that MASIL can identify the important modes on a part-by-part basis, and that using the conventional method (of checking mass participation only on the structure as a whole can lead to significantly underestimated element forces for design. We conclude that seismic design codes should take into account mass participation requirements for parts of a structure, and further guidance or requirements should be introduced for large irregular structures, especially those made up of many connected parts.

1. Introduction

1.1. Modal analysis and eigenvalue problems

For many forms of structural analysis, one must find the fundamental frequencies of a structural model. The dynamic response of a structure is defined by the equation of motion, the second order differential equation (e.g. Bathe, 2006)

$$M\ddot{u}(t) + D\dot{u}(t) + Ku(t) = f(t). \qquad (1)$$

This equation explains how the displacements of the structure, $u(t)$, arise when subjected to some external force, $f(t)$, given a mass matrix, M, damping matrix, D, and stiffness matrix, K. Each of these matrices has n degrees of freedom; the mass matrix is symmetric positive-semidefinite, and the damping and stiffness matrices are symmetric positive-definite.

If we ignore the force and damping, the natural frequencies of the structure can be found by solving the equation

$$Kx - \lambda Mx = 0. \tag{2}$$

This is known as the Generalised Eigenvalue Problem (Bai et al, 2000), and is well-studied in the field of Numerical Linear Algebra. There will be n eigenpairs (λ_i, x_i) that solve Equation 2 where n is the size of the matrices involved. The vectors x_i are a set of basis vectors that describe the spatial distribution of displacements within the structure; each is associated with a natural frequency given by $\frac{\sqrt{\lambda_i}}{2\pi}$.

By using these fundamental frequencies and the corresponding eigenvectors, also known as normal modes, we can analyse how the structure will respond to external forces. If a full set of n M-orthogonal eigenpairs (λ_i, x_i) can be found, then

$$x_i^T M x_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad x_i^T K x_j = \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases} \tag{3}$$

and the response can be computed exactly. However, n is often very large for real-world structural models, and so this would be prohibitively expensive. Instead, we work with a truncated set of modes; i.e. we find a subset of k eigenpairs that will approximate the response. We use $X = \{x_i\}_{i \in I}$ for our truncated subset I of size k, and reduce the dimensionality of our matrices to get

$$X^T MX\ddot{v}(t) + X^T KXv(t) = X^T f(t) \tag{4}$$

in place of Equation 1, with $v(t) = X^T u(t)$. It reduces our problem from a coupled n-dimensional second order differential which is expensive to solve, to an uncoupled k-dimensional second order differential equation, which is much easier to solve.

There is much study into *modal truncation*, i.e., how to appropriately pick these eigenpairs, such as in Craig Jr et al (2006) or Bathe (2006) and the method ultimately depends on what the modes will be used for.

1.2. Load dependent Ritz vector analysis

To solve Equation 1, we can use modal analysis to decouple the system, as well as allowing us to reduce the dimensionality of the problem. Decoupling the system is what allows us to solve the problem, by choosing a subset of the normal modes and therefore reducing the dimensionality. If our external force takes the form $f(t) = s\,p(t)$ where s is a space vector and $p(t)$ a *scalar* function of time, then we can use an alternative analysis method known as Load Dependent Ritz Vector (LDRV) Analysis, or just Ritz Analysis, from Léger, Wilson and Clough (1986), with a shorter summary in Chopra (2007). Load-Dependent Ritz Analysis is an alternative method to lead us to a reduced-dimensional decoupled system where we first reduce the dimensionality, and then decouple the equation.

To briefly summarise, we aim to compute the Load-Dependent Ritz Vectors, $\psi_1, \psi_2, \ldots, \psi_k$, and use these to reduce Equation 1. The basis Ψ we form is generated from the vector s These vectors are M-orthogonal, however they do not diagonalise K, as in Equation 3. We then use $\Psi^T M\Psi$ and $\Psi^T K\Psi$ in place of M and K as in Equation 4. This reduces our problem down to a coupled k degree of freedom system, which we can then carry out modal analysis on, this time finding all the modes. The results we get from this are no longer normal modes of the original system. They are linear combinations of modes; however they have been used in place of normal modes in analysis.

1.3. Approaches to seismic analysis

For seismic analysis, the external force $f(t)$ takes the form

$$f(t) = g_d(t) M b_d \tag{5}$$

where $g(t)$ is a function of ground accelerations in a particular direction, d, and b_d is the *influence vector*, representing the displacements of the masses from a unit ground displacement in the direction d. The directions are usually picked to be the three global directions x, y and z.

As per Equation 4, our uncoupled external force will be of the form $g_d(t)X^T Mb_d$ and so the higher forces come from the modes where $x_j^T Mb_d$ is large. Modes where this value is high will be modes that are excited by the excitation typically associated with horizontal earthquake ground motion, for example swaying modes.

If we find all n eigenpairs, the sum of $m_{i,d}$ will be the total mass of the structure for each direction d, as we have captured 100% of the mass of the structure. Hence, we define effective mass as

$$m_{i,d} = \frac{(x_i^T Mb_d)^2}{b_d^T Mb_d}, \tag{6}$$

where $b_d^T Mb_d$ is the total mass of the structure. It follows that we have

$$\sum_{i=1}^{n} m_{i,d} = 1. \tag{7}$$

Since for practical models it is prohibitively expensive to compute all n eigenpairs, and we know that the high mass participation modes contribute the greatest to the response, instead we try and find the k lowest frequency modes, (λ_i, x_i) such that

$$\sum_{i=1}^{k} m_{i,d} > t. \tag{8}$$

We look for smallest number of modes, k, that achieves at least a target proportion of the effective mass, t. Codes of practice for seismic design, such as Eurocode 8 (CEN, 2004) and ASCE 7 (ASCE, 2022), often prescribe targets of 90%. Truncating modes to this target limit enables us to model and compute the approximate response of the structure without needing potentially hundreds of thousands of modes. Using the computed modes, we can then do further analysis on the structure.

In this paper, we present a new eigenvalue solver that radically simplifies how truncated modes are found and computed. We then demonstrate how it accelerates Response Spectrum Analysis and simultaneously makes it more accurate. Our solver is implemented in Oasys GSA (2023), a finite element structural analysis suite, and we present examples of structures modelled in this package.

To compare our new method with existing, we will use the following response results:

- Base shear.
- Overturning moments.
- Interstorey drifts – the displacements between the floors of the structure.
- Member forces.

1.4. Response spectrum analysis

Equation (5) is the force imposed on the structure by a *single earthquake ground motion*, and resulting structural response will be specific to that ground motion only. We want to be able to find the maximum responses from a wide range of different potential earthquakes that could affect the structure. To do this, we use Response Spectrum Analysis (RSA). It is a form of linear stochastic analysis that combines site data on the type of ground motion expected in an area with modal responses to estimate the peak maximum response.

RSA is based on the fact that the response of any two systems with equal natural frequency and damping ratios will have the same response to a given input motion. Instead of evaluating the complete response over time, as in the modal superposition method, we can use pre-calculated *response spectra* to give the peak values of these modal responses (see Figure 1). We then need some statistical method for combining together these modal peaks to give an estimate of the peak response of the overall multiple-degree-of-freedom system.

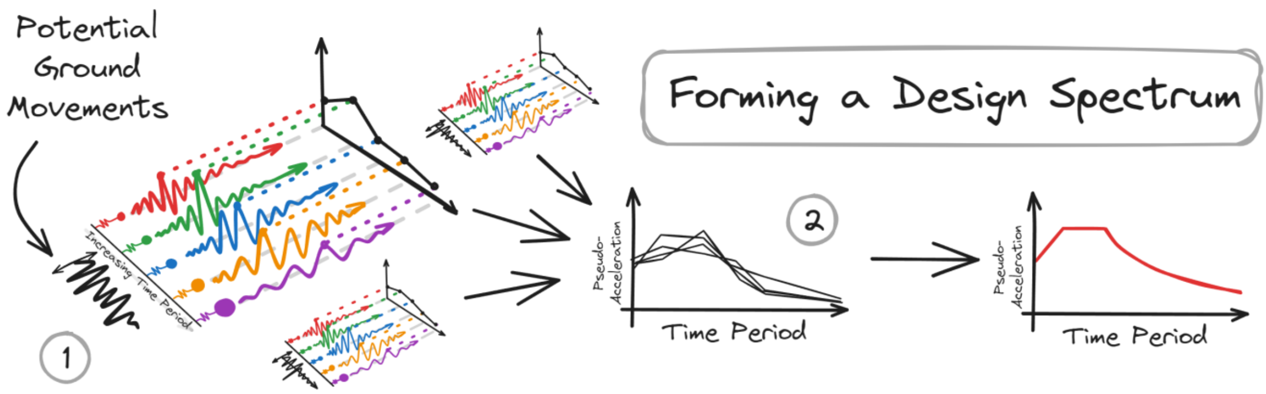


Figure 1. A summary of Response Spectrum Analysis. 1. Compute the time history analysis of several SDOF oscillators with a variety of frequencies and a given damping ratio. 2. Combine these to form the Design Spectrum. Note these steps happen independently of the design of the structure.

Over the years, several methods have been proposed to combine the responses of these single-degree-of-freedom systems to simulate a multiple-degree-of-freedom system. We have the peak responses for each decoupled system, and so one method to combine them is to simply sum their absolute values. This is the absolute sum method, and often leads to a significant overestimation of peak response, as the peak modal responses for each system are unlikely to occur simultaneously. More statistically-based methods, such as the square-root-of-sum-of-squares (SRSS) rule of Rosenblueth (1951) (which works well for well-separated natural frequencies) and the complete-quadratic-combination (CQC) method from Wilson et al (1981) (an improvement on SRSS for models with closely-spaced frequencies), are generally used in practice.

As with modal superposition method (Section 1.3), we need to use not just the values recovered from the design spectrum, but also the modal masses, the mode shapes, and the damping, hence why we aim to get the higher mass participation modes, as these have the larger modal masses and contribute more to the forces, moments, and deformations. An overview of RSA is given in Figure 2. Further information can be found in standard dynamics textbooks, such as Chopra (2007).

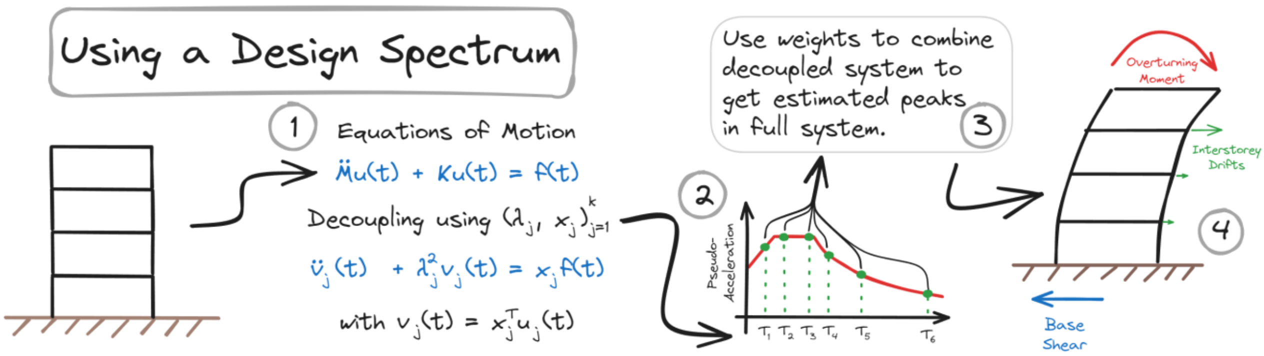


Figure 2. A summary of Response Spectrum Analysis. 1. Solve the equations of motion for your structure, either with Ritz or Modal analysis, to decouple the linear system. 2. Compute the peak responses of each decoupled SDOF system from the previous stage. 3. Combine to find the peak responses of the original MDOF using some combination method, such as CQC or SRSS. 4. Use these peak responses to determine the required values: base shear, overturning moment, and inter-storey drift.

1.5. Motivation for a better algorithm

An RSA for a complex structure can be computationally expensive. There are two main bottlenecks that slow down RSA. Firstly, the computation of free body vibration modes (i.e. decoupling the original system into k SDOF systems via modal analysis), and secondly, running RSA. The larger the value of k, the more expensive the first decoupling stage. The cost of running RSA with k SDOF systems will be of an order that is quadratic with k, and hence again, we want as small a value of k as possible. Figure 3 shows how the number of modes required to reach a cumulative 90% participating mass can be unreasonably high for some structures.

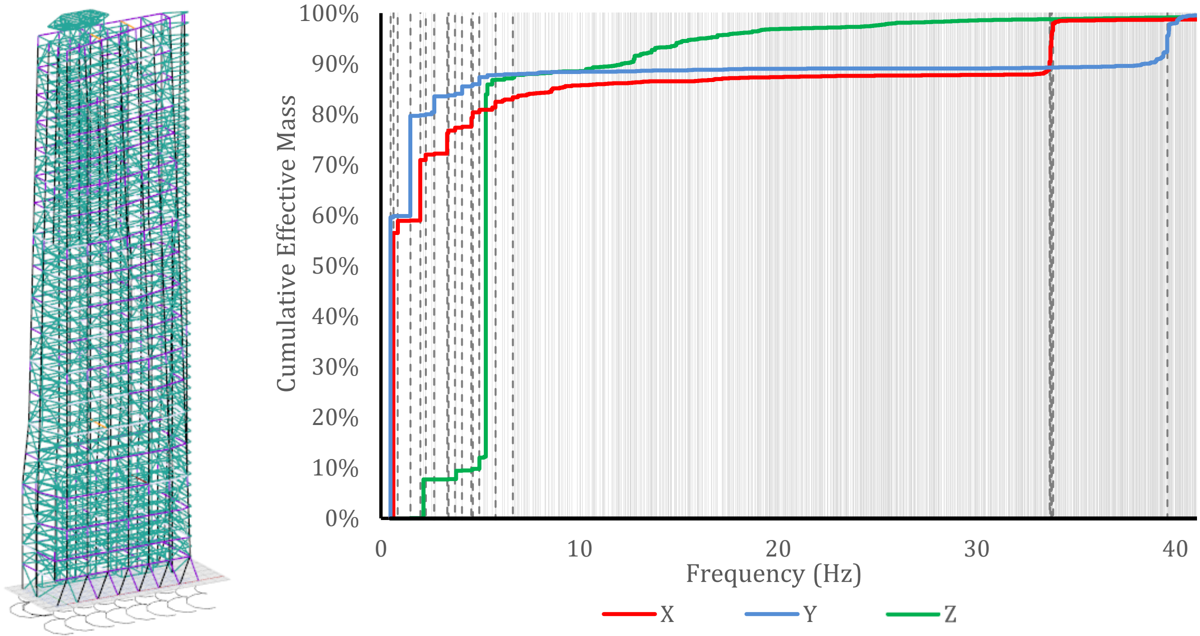


Figure 3. The Cumulative Effective Mass for this tower. The thin light grey lines on the graph mark the first 788 modes of the problem, the dark grey dashed lines are the 22 modes required to achieve 90% in the X and Y directions.

Not all normal modes are equally important. Some contribute significantly less mass than others. In a complex FE model, often there are vibration modes that only activate a small, localized part of the structure. It therefore follows that by selecting just the lowest frequency modes with high mass, we can significantly accelerate RSA whilst simultaneously reducing the memory footprint of computing the modes. Since typically such analyses are carried out on the engineer's desktop computer rather than large supercomputers, a reduction in analysis time and memory footprint leads to faster design iteration cycles. Purging unnecessary modes also has the advantage that the engineer has a smaller dataset to deal with, and therefore reduces the risk of errors and oversight that arise from being inundated with data.

This is the key insight of the MASIL algorithm in in Zemaityte et al. (2019). MASIL is a novel algorithm to find the most important eigenmodes without computing the irrelevant ones. By virtue of being normal modes, we retain the accuracy and interpretability of modal analysis.

It is worth noting that Ritz analysis is designed to solve the above problem of finding effective mass quickly; however, it does not find true modes. It decouples a *reduced* form of Equation 1, and thus some information is lost, with no certainty of how this has happened. This is often to the engineer's advantage, as it can reduce the system down by ignoring unimportant modes. However, Ritz vectors are linear combinations of normal modes, and the combination can be arbitrary with no *a priori* error control. The use of Ritz Vectors for computing dynamic responses gets discussed in Chen, Harn-ching and Robert Taylor (1989). As they discuss, there is no basis to support using Ritz Vectors, especially if we move beyond simple structures that only require a few modes to reach the target mass.

In the remainder of this paper, we briefly describe how the MASIL algorithm works and present how it can be deployed to capture the vibration behaviour of all parts of a complex structure, thereby avoiding a common pitfall with RSA. We compare results of RSA using modes computed using MASIL and (normal) modal analysis and demonstrate that the truncation error is negligible.

2. The MASIL algorithm

When solving the sparse symmetric generalized eigenvalue problem (Equation 2), we will be solving for a relatively small number of eigenpairs in comparison to the size of the matrices. To do this, we typically use projection-based methods such as Subspace Iteration or Lanczos (Bai et al., 2000). These solvers typically reduce the problem down to a much smaller dense matrix problem, which are typically cheap to solve. The solution to the original problem is then recovered from the solution to the dense problem.

MASIL, Zemaityte et al (2019), uses a special property of both the Lanczos algorithm to estimate the effective mass located around unconverged eigenvalue estimates. This allows us to find a sequence of shifts that are near modes of high effective mass, thereby skipping entirely the modes that contribute little mass, rather than computing them first and then purging the unnecessary ones. Subsequently we can move away from Equation 8 and instead solve

$$\sum_{i \in I} m_{i,d} > t \tag{9}$$

where I is no longer $[1, k]$ but is some minimal subset of $[1, n]$. A conventional eigensolver can solve the problem of equation 9 only by trial and error, whereby the solver is run and the cumulative mass of converged eigenvalues computed repeatedly until the desired target is reached. As such, MASIL solves a problem that would otherwise consume significantly more computational time and memory. From Zemaityte et al (2019), the recommendation following error analysis is to pick the modes that contribute most to the response, and these can be found by considering $\frac{m_{i,d}}{f_i}$ where f_i is the frequency of the mode of interest.

Once we have decoupled Equation 1 by computing vibration modes, we can now use these decoupled modes to run RSA. As mentioned in Section 1.6, we only need low frequency and high effective mass modes to capture the response of the structure.

3. The inaccuracy of effective mass-led modal truncation

Most design codes require 90% of the effective mass in the X and Y direction, with occasional further guarantees that we have all modes with greater than 5% effective mass. This approach might produce accurate responses for simple structures. However, for structures such as Figure 4, consisting of multiple sub-structures (a situation that can arise when there are multiple towers on a podium) it is possible we miss critical modes that can be *hiding* in the missing 10% effective mass.

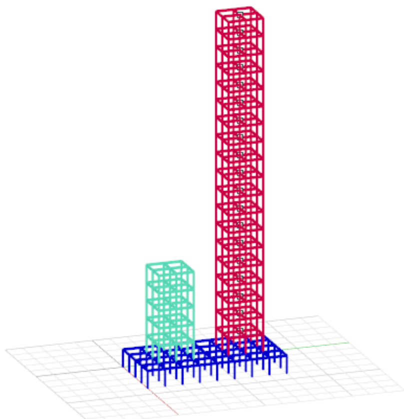


Figure 4. Structure with two towers on a plinth.

To demonstrate, we model such a structure in Oasys GSA. Beams and columns are modelled as 1D beam elements with rectangular steel sections, with slabs being modelled as rigid diaphragms. The model has 625 elements, leading to 1110 degrees of freedom. Using a lumped mass model yields a semi-definite mass matrix, which subsequently leads to 498 modes as the remaining are associated with infinite eigenvalues. Further we run RSA for several variations of input modes and execute an RSA using Eurocode 8. To establish a baseline, we also compute all 498 modes of the model and run RSA using these modes. Table 1 compares storey drifts from each analysis for the two towers. We observe that we get significant differences depending on the number of modes used in RSA. For the large tower, we get approximately the expected drift with all inputs. This is to be expected as the critical modes are the lower frequency modes found within 0–3Hz.

Input	Number of Modes	Maximum Frequency (Hz)	Storey Drift (mm)								
			Small Tower			Large Tower					
			Floor 2	Floor 4	Floor 6	Floor 2	Floor 4	Floor 6	Floor 10	Floor 15	Floor 20
Sequential 90%	8	3.5	**0.00**	**0.01**	**0.01**	11.9	33.4	55.1	96.8	144	181
Ritz 90%	4*	2.7	**8.24**	**17.2**	**21.3**	12.3	33.0	54.0	96.0	145	180
Sequential 99%	71	44.7	0.79	1.89	2.46	11.9	33.5	55.1	96.8	144	181
Full System	498	1134	0.79	1.89	2.46	11.9	33.5	55.1	96.8	144	181

*Table 1. Storey Drift in the X direction for the towers in Figure 3. *Note that Ritz analysis does not producing normal modes but approximate mode-shapes.*

Evidently, the typical approach of running Modal Analysis sequentially to produce the first k modes until we achieve the desired effective mass does **not** work. If we only have 90%, we do not capture the seismic response for the smaller tower. As expected, when we increase our requirement to 99% we do agree with the full solution, although this requires 71 modes, which is a significantly high proportion of the total number of degrees of freedom. The inaccuracies of using a Ritz basis are reflected at 90%, where the influence of the low frequency motions of the large tower is incorrectly imposed onto the small tower, leading us to get massively exaggerated forces on the small tower.

3.1. Finding effective mass of substructures

To analyse this inability of a purely effective mass-led truncation of eigenmodes, we introduce the notion of effective mass of a substructure. In equation 5, we defined b_d to be the influence vector of a structure in a particular direction corresponding to a unit displacement in that direction. Vector b_d is then used to compute the effective mass of a mode shape v, as the inner product $b_d^T M v$ tells us how much of the structures' mass shifts in the direction of our influence vector. It follows that if we consider the influence vector of a *substructure* in a particular direction, this will tell us how much the mass of substructure moves in under a particular mode. Therefore, we introduce an influence vector $b_d^{(i)}$, for substructure i along direction d. Using this definition of influence vectors for substructures, we can capture effective mass of individual substructures.

Firstly, we identify substructures in Figure 4, with the small tower being the green, and the large tower being the red. We can then compute the effective masses for each mode, as in Table 2. In , we plot effective masses and cumulative effective mass for each substructure using the same colour codes in Figure 4. There are several insights that emerge from the graph in . Firstly, we can clearly see why MASIL itself is useful, as there are several modes between 10Hz and 25hZ that do not contribute to the effective mass. As such skipping these does not affect the outcome of the analysis. Secondly, the dominance of the big tower in the overall response is patently obvious. As we see, the cumulative masses for the large tower and entire structure follow a very similar trajectory, but the cumulative mass for the small tower has a relatively minimal effect.

Table 2. Individual modes and their effective masses for the two towers model. The blank cells have negligible effective mass.

Mode Index	Frequency (Hz)	Effective Mass					
		Total Structure		Small Tower		Big Tower	
		X	Y	X	Y	X	Y
1	0.44		77.7%				80.6%
2	0.58	75.2%				78.1%	
3	1.35		11.5%				11.8%
4	1.88	13.6%				14.1%	
12	6.14		1.50%		33.8%		0.15%
13	6.37		0.30%		51.1%		0.21%
22	9.13			88.4%			
40	18.60		0.30%		7.60%		
51	27.46	0.40%		7.00%			

The effective mass for each substructure across various modes are reported in Table 3. We see that the critical modes for the large tower have a lower frequency, and they also contribute the most to the total structure effective mass.

By choosing modes in such a way to capture 90% effective mass for both the large and small tower, our response results are a very close match for RSA with the full system. Furthermore, we can do this with just 9 modes, substantially fewer than all other successful methods.

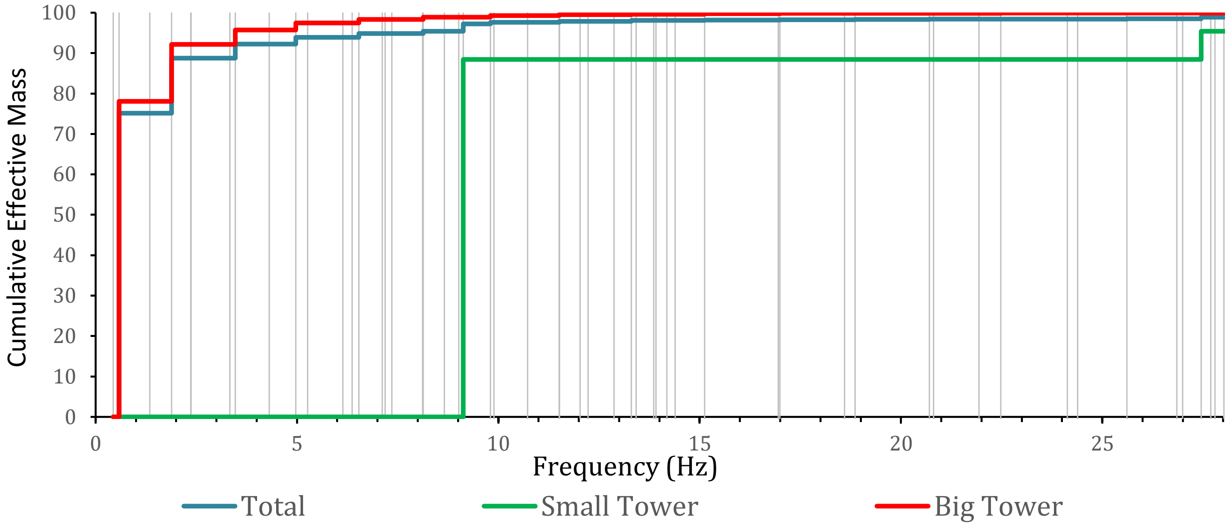


Figure 5. Cumulative effective mass for the substructures of Figure 4. The grey lines represent individual modes.

*Table 3. Storey Drift in the X direction for the towers in Figure 4. * Note that Ritz analysis does not producing normal modes but approximate mode-shapes.*

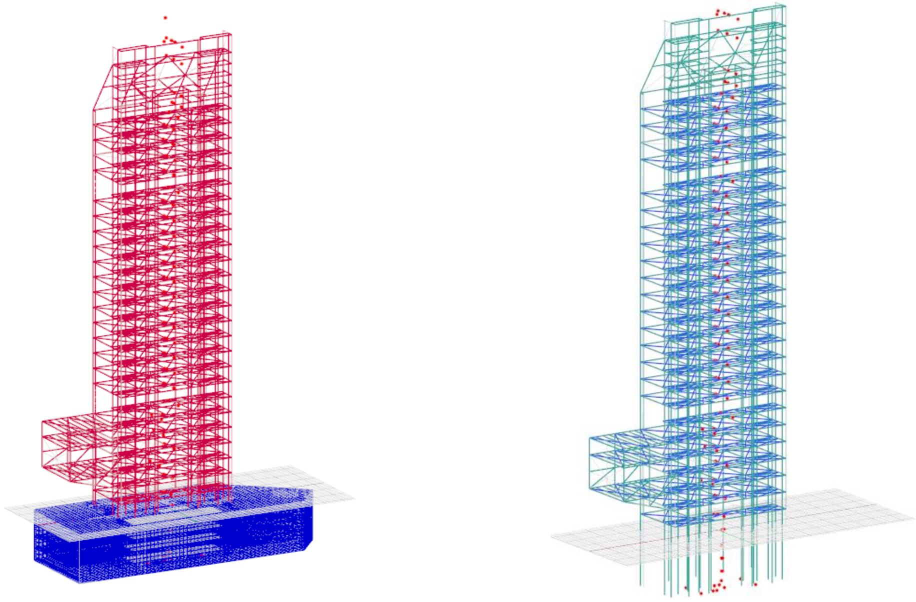
Input	Number of Modes	Maximum Frequency (Hz)	Storey Drift (mm)								
			Small Tower			Large Tower					
			Floor 2	Floor 4	Floor 6	Floor 2	Floor 4	Floor 6	Floor 10	Floor 15	Floor 20
Sequential 90%	8	3.5	**0.00**	**0.01**	**0.01**	11.9	33.4	55.1	96.8	144	181
Ritz 90%	4*	2.7	**8.24**	**17.2**	**21.3**	12.3	33.0	54.0	96.0	145	180
Sequential 99%	71	44.7	0.79	1.89	2.46	11.9	33.5	55.1	96.8	144	181
Ritz 99%	19*	9.3	0.80	1.93	2.51	11.9	33.5	55.1	96.8	144	181
MASIL by Parts 90%	**9**	**27.5**	**0.79**	**1.89**	**2.46**	**11.9**	**33.4**	**55.1**	**96.8**	**144**	**181**
Full System	498	1134	0.79	1.89	2.46	11.9	33.5	55.1	96.8	144	181

4. MASIL by Parts

The previous section demonstrates how the effective mass of substructures is a useful metric for analysis of a complex or irregular structure. Building on this intuition, we can use the influence vectors for each substructure of interest as an initial vector for MASIL. If we run MASIL with all the influence vectors we are interested in, we can find the most relevant modes with relatively low computational cost. This is the *MASIL by Parts* algorithm, and we present two examples that showcase its utility.

For the first example, we revisit the model from the previous section. We define parts for the large and small tower, and run MASIL by parts on them, and subsequently RSA. Table 3 extends Table 1 with our MASIL by Parts run. As we can see, RSA results computed using MASIL by Parts are of the same accuracy as the 'Full System' results, but only need 9 modes.

The two towers model shows us that MASIL by Parts is a highly effective method for such structures with clearly defined connected 'parts'. Next we wish to investigate how MASIL by Parts can be used for a different type of structure, where we have a massive, rigid substructure that drags the cumulative effective mass down without impacting the overall response. Figure 6a shows a building model of a moderately large mixed-use skyscraper. The model has steel beams and columns supporting RC slabs on 25 floors. The basement consists of a 4-storey car park, modelled using beam and shell elements. The model has more than 16000 elements, over 10,000 nodes and 60,000 degrees of freedom. We can see from Table 4 that the number of sequential modes to achieve 90% effective mass is nearly 5000, which is not only impractical to compute, but also to run RSA on.



(a) (b)

Figure 6. (a) Large tower with a heavy basement. (b) Substructure consisting of the tower and the main foundation columns.

Figure 7 shows the cumulative effective mass over the spectral decomposition of the structure for several substructures. Clearly, the structure above ground level accumulates 90% effective mass in X and Y with relatively few modes, as it is the flexible part of the global structure and its mass is excited by lower frequency modes. The basement is only excited by higher frequency modes, and this is reflected in the slow accumulation of the effective mass. Subsequently, the effective mass for the whole structure also accumulates slowly.

To ignore the largely irrelevant modes of the basement, we define a substructure as in Figure 6b. This is the entire tower with the main foundation columns. This allows us to account for displacements affecting the entire height of the tower, from the base of the basement through the actual tower.

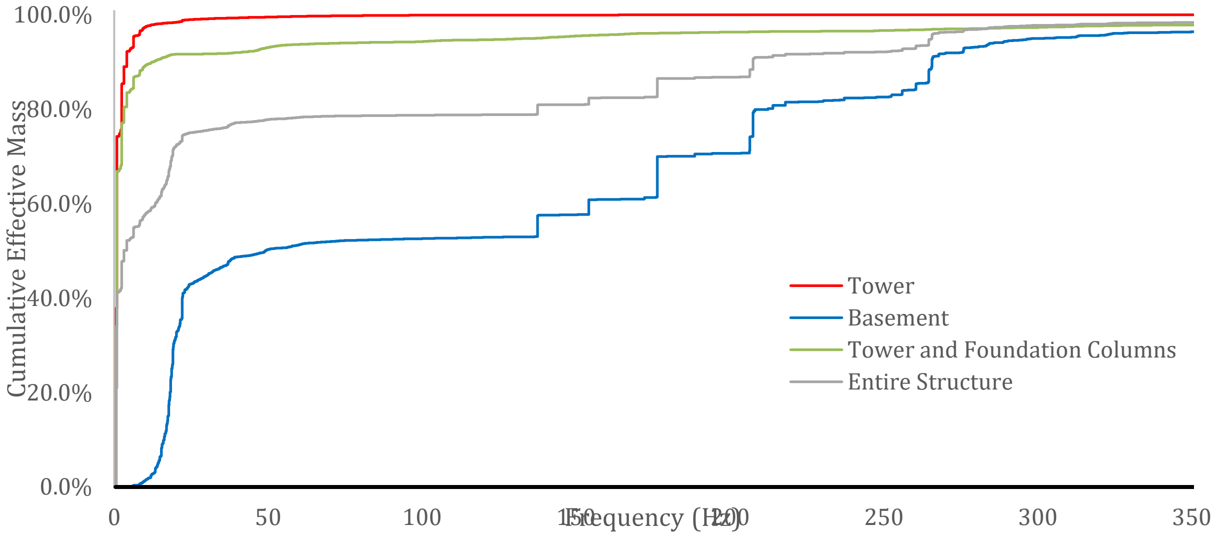


Figure 7. Cumulative effective mass across the spectrum for various substructures of Figure 6.

Table 4 shows that by using this as our only 'part' in MASIL by Parts requires only 98 modes to achieve the necessary cumulative effective mass. This compares to the 433 modes required for MASIL, and almost 5000 required using the sequential modal analysis technique. This amounts to a major memory and time improvement for storing and using the required modes.

Table 4. Differing total effective masses for various subsets of modes for the structure in Figure 5.

Input	Number of Modes	Maximum Frequency	Total Effective Mass							
			Total Structure		Tower		Basement		Tower and Foundations	
			X	Y	X	Y	X	Y	X	Y
MASIL by Parts 90% Tower and Foundation Only	98	265Hz	63.1	63.7	97.3	97.0	12.4	11.4	90.1	90.1
MASIL 90%	433	350Hz	90.9	91.0	98.9	99.0	78.6	78.3	93.2	93.4
Sequential 90%	4835	234Hz	91.9	91.4	100	100	81.9	80.8	96.6	96.7
First 350Hz	7815	350Hz	98.5	97.8	100	100	96.6	95.1	98.0	97.8

In order to demonstrate we have still meaningfully captured the response, some analysis results are presented in Table 5 and Table 6. Table 5 shows that the storey-drifts for RSA based on Eurocode 8 design spectrum. We can see that despite running with a tiny fraction of the modes, our maximum drifts are accurate to within tenths or hundredths of millimetres.

Table 6 shows the errors in the axial forces and overturning moments for individual elements. For the Element Axial Force Difference column, we consider the 200 elements with the largest axial forces for the Tower and the Basement independently, when RSA is run with the 7815 modes with frequency under 350 Hz in order to establish as close a baseline to running RSA with all modes of the system. We consider the percentage difference for these 200 elements with the results they have from our alternative inputs. We see that both MASIL and MASIL by Parts produce a similar error for axial forces, within a couple tenths of a percent. As expected, sequential matches almost exactly, as it contains all modes with frequency below 234 Hz, and so modes beyond this frequency will not affect the RSA results. We also compare the 200 elements with the highest overturning moments. We see here again the results are sufficiently similar, even with just our 98 modes. The results for the basement can be seen to have a slightly higher relative error with MASIL by Parts, however this is largely due to the overturning moments for the basement being substantially smaller, due to the rigidity of the structure.

The results for both Sequential 90% and the First 350Hz require thousands of modes, which would take many hours to run and generate gigabytes of data. Our analysis was run on an 8-core Intel Xeon machine with 3.7 MHz clock speed and 256 GB of RAM. For our model, RSA took approximately 30 hours to run on 7815 modes and required over 150 GB of memory. The 90% sequential analysis took approximately 13 hours. The analysis for MASIL by Parts with 98 modes was completed in 15 minutes.

Table 5. Storey drift for the RSA Y response, for the tower in Figure 5.

Input	Storey Drift (mm)					
	Floor 5	**Floor 10**	**Floor 15**	**Floor 20**	**Floor 25**	**Floor 30**
MASIL by Parts 90%	45.743	111.26	183.20	255.17	323.10	369.42
MASIL 90%	45.829	111.30	183.21	255.17	323.09	369.41
Sequential 90%	45.830	111.30	183.21	255.17	323.09	369.41
First 350Hz	45.830	111.30	183.21	255.17	323.09	369.41

Table 6. Comparing the RSA analysis from the first 350 Hz to our subsets of modes. The errors shown are for the 200 largest axial forces (for the left) or overturning moments (for the right) of individual elements from the First 350 Hz RSA.

Input	Element Axial Force Difference				Element Overturning Moment Difference			
	Tower		**Basement**		**Tower**		**Basement**	
	Average	**Maximum**	**Average**	**Maximum**	**Average**	**Maximum**	**Average**	**Maximum**
MASIL 90% Combined	0.06%	0.39%	0.07%	0.48%	0.10%	0.62%	0.52%	2.70%
MASIL 90% Structure	0.06%	0.34%	0.05%	0.22%	0.08%	0.44%	0.19%	0.69%
Sequential 90% Total	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
First 350Hz	-	-	-	-	-	-	-	-

5. Conclusions

MASIL by Parts is a new approach to compute a truncated eigensystem for Response Spectrum Analysis that is computationally cheaper and can be substantially more accurate. Within the realm of numerical algorithms, solutions usually only have one of the following traits: they are either more accurate or they are computationally more efficient, but rarely are they both.

We have demonstrated that the code-prescribed rule of using a basis with 90% effective mass can produce large errors, and entirely miss the response of certain parts of the structure. Furthermore, the Load-dependent Ritz Vector analysis, which produce Ritz Vectors that are not modes, can induce artificial movements. The most concerning limitation of using LDRV analysis is that the user gets no feedback about when they are using Ritz vectors incorrectly.

The MASIL algorithm (Zemaitye et al., 2019) provides a way to speed up Modal Analysis by skipping modes with low effective mass, however this can still struggle for irregular structures with clear and distinct parts. MASIL by Parts overcomes this limitation by allowing users to partition structures for analysis, and examine the local responses of these parts, without losing the global structural context. This new approach can allow us to predict responses of complex and irregular structures with RSA without losing accuracy arising from the inevitable modal truncation.

Future work in this area would be to study exactly what makes a ‘part’. We showed pathological examples where MASIL by Parts can produce results not immediately possible with previous methods. This raises the question of whether there are other structural geometries that are susceptible to large errors in RSA responses. Such models are likely to be ones where large, stiff parts dominate effective mass and conceal critical modes of the more flexible and lighter parts of the structure.

The problems with using Ritz vectors as a basis and the 90% effective mass rule show that there are pitfalls in seismic codes that are not immediately obvious, and more research into the consequences of these pitfalls is needed.

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