Fully Distributed Power System State Estimation: A Matrix based Approach

Mohit Sehrawat, Saikat Chakrabarti, *Senior Member, IEEE*, and Ketan Rajawat, *Member, IEEE*

***Abstract*—This paper introduces key propositions and their corresponding proofs pivotal in advancing a matrix-based, fully distributed power system state estimator.**

I. Introduction

In modern power systems, real-time monitoring and control are essential to ensure reliable and efficient operation [1]. State estimation, a pivotal task in power system management, involves the continuous assessment of the system's operating conditions using available measurement data [2]. Traditional centralized state estimation approaches, while effective, face significant challenges in scalability and robustness due to the growing complexity and size of power systems [3]. To address these challenges, fully distributed state estimation methodologies have emerged as a promising alternative.

This paper focuses on the development of mathematical propositions and proofs essential for the advancement of a fully distributed power system state estimator. Unlike centralized methods, distributed state estimation leverages the computational capabilities of multiple nodes, such as distributed estimators and local measurements from phasor measurement units (PMUs), which are dispersed throughout the power grid. This decentralized approach enhances scalability and improves the robustness and fault tolerance of the state estimation process.

The core contributions of this paper are as follows:

1) *Proposition and Proof of Positive Definite Symmetric Matrix:* We establish that the gain matrix, commonly denoted as $\mathbf{A}$ [4], derived from the measurement matrix $\mathbf{H}$, is a positive definite symmetric matrix. This property is fundamental for ensuring the convergence of the state estimation algorithm.
2) *Convergence Conditions for Distributed Estimators:* We derive conditions under which the Jacobi iterative update matrix, $\mathbf{M}_{JOR}$, achieves spectral radius less than one, ensuring the convergence of the distributed state estimation algorithm. Specifically, we show that the optimal relaxation factor, ω, lies within a specific range depending on the minimum eigenvalue of the iteration matrix $\mathbf{M}_J$.
3) *Optimization of the Relaxation Factor:* We propose an optimal value for the relaxation factor, ω_{opt}, that minimizes the spectral radius of $\mathbf{M}_{JOR}$. This optimization significantly enhances the convergence speed of the distributed estimation process.
4) *Case Studies and Practical Implications:* Through illustrative examples, we demonstrate the practical application of our theoretical results. We show how the placement of PMUs affects the reducibility of the gain matrix $\mathbf{A}$, and subsequently, the tuning of the relaxation factor for optimal performance in both reducible and irreducible matrix scenarios.

The propositions and proofs presented in section II lay the groundwork for future research and development in the area of distributed power system state estimation.

The authors are with the Department of Electrical Engineering, Indian Institute of Technology Kanpur, Kanpur, Uttar Pradesh, India 208016. (email: sehrawat@iitk.ac.in; saikatc@iitk.ac.in;ketan@iitk.ac.in)

II. Propositions and Proofs

Proposition A1: Matrix $\mathbf{A}$ is a positive definite symmetric matrix.

Proof. Given $\mathbf{A} = \mathbf{H}^T\mathbf{W}\mathbf{H} \Rightarrow \mathbf{A}^T = (\mathbf{H}^T\mathbf{W}\mathbf{H})^T := (\mathbf{H})^T\mathbf{W}^T(\mathbf{H}^T)^T := \mathbf{H}^T\mathbf{W}\mathbf{H} \Rightarrow \mathbf{A}^T = \mathbf{A} \Rightarrow \mathbf{A}$ is a symmetric matrix.

$\mathbf{A}$ is a positive definite if $\mathbf{v}^T\mathbf{A}\mathbf{v} > 0$ for any non zero vector $\mathbf{v}$. Let examine $\mathbf{A} : \mathbf{v}^T\mathbf{A}\mathbf{v} := \mathbf{v}^T\mathbf{H}^T\mathbf{W}\mathbf{H}\mathbf{v} := \mathbf{v}^T\mathbf{H}^T\bar{\mathbf{W}}^T\bar{\mathbf{W}}\mathbf{H}\mathbf{v} := \mathbf{y}^T\mathbf{y} > 0 \Rightarrow \mathbf{A}$ is positive definite, where $\mathbf{y} = \bar{\mathbf{W}}\mathbf{H}\mathbf{v}$, $\bar{\mathbf{W}}$ is a diagonal matrix, and its diagonal elements are $\bar{W}_{ii} = \sqrt{W_{ii}}$.

Proposition A2: Let $\mathbf{A}$ be a symmetric positive definite matrix, $\mathbf{B}_J$ be a diagonal matrix, $\mathbf{C}_J = \mathbf{A}-\mathbf{B}_J$, $\mathbf{B}_{JOR} = \frac{\mathbf{B}_J}{\omega}$, $\mathbf{C}_{JOR} = \mathbf{A} - \mathbf{B}_{JOR}$, $\mathbf{M}_J = -\mathbf{B}_J^{-1}\mathbf{C}_J := \mathbf{I} - \mathbf{B}_J^{-1}\mathbf{A}$, and $\mathbf{M}_{JOR} = -\mathbf{B}_{JOR}^{-1}\mathbf{C}_{JOR}$. Then, $\rho(\mathbf{M}_{JOR}) < 1$ if and only if $0 < \omega < \frac{2}{1-\lambda_{M_J min}}$ where $\lambda_{M_J min}$ is the smallest eigenvalue of the $\mathbf{M}_J$.

Proof. By Theorem 1.56 of [5], it is sufficient to prove the matrix $\mathbf{Q} = \mathbf{B}_{JOR} + \mathbf{B}_{JOR}^T - \mathbf{A}$ to be a positive definite for $\rho(\mathbf{M}_{JOR}) < 1$. $\mathbf{B}_{JOR} = \mathbf{B}_{JOR}^T$, since $\mathbf{B}_{JOR}$ is a diagonal matrix. Hence, $\mathbf{Q} = 2\mathbf{B}_{JOR} - \mathbf{A}$. As per *Proposition A3*, $\mathbf{Q}$ is positive definite if and only if $\bar{\mathbf{Q}} = \mathbf{B}_J^{-1/2}\mathbf{Q}\mathbf{B}_J^{-1/2}$ is positive definite. Now we have $\bar{\mathbf{Q}} = 2\mathbf{B}_J^{-1/2}\mathbf{B}_{JOR}\mathbf{B}_J^{-1/2} - \mathbf{B}_J^{-1/2}\mathbf{A}\mathbf{B}_J^{-1/2} := 2\mathbf{B}_J^{-1/2}\frac{\mathbf{B}_J}{\omega}\mathbf{B}_J^{-1/2} - \mathbf{B}_J^{1/2}(\mathbf{I} - \mathbf{M}_J)\mathbf{B}_J^{-1/2} := (\frac{2}{\omega} - 1)\mathbf{I} + \mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2}$. By *Proposition A4*, eigenvalues of $\mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2}$ are the same as of $\mathbf{M}_J$. We can get eigenvalues of $\bar{\mathbf{Q}}$ by $\lambda_{\bar{Q}} = \frac{2}{\omega} - 1 + \lambda_{M_J}, \forall\ \omega > 0$. If the smallest eigenvalue $\lambda_{\bar{Q}min} > 0$, then all the eigenvalues will be $\lambda_{\bar{Q}} > 0$; this will make $\bar{\mathbf{Q}}$ positive definite. Hence, we have the condition $\frac{2}{\omega} - 1 + \lambda_{M_J min} > 0 \Rightarrow 0 < \omega < \frac{2}{1-\lambda_{M_J min}}$.

Proposition A3: Let $\mathbf{B}_J$ is non singular diagonal matrix. Then $\mathbf{Q}$ is positive definite if and only if $\bar{\mathbf{Q}} = \mathbf{B}_J^{-1/2}\mathbf{Q}\mathbf{B}_J^{-1/2}$ is positive definite.

Proof. Given, $\bar{\mathbf{Q}} = \mathbf{B}_J^{-1/2}\mathbf{Q}\mathbf{B}_J^{-1/2}$ is positive definite, for any non zero vector $\mathbf{y}$, we can have $\mathbf{y}^T\bar{\mathbf{Q}}\mathbf{y} :=$

$\mathbf{y}^T\mathbf{B}_J^{-1/2}\mathbf{Q}\mathbf{B}_J^{-1/2}\mathbf{y} > 0 \Rightarrow (\mathbf{B}_J^{T-1/2}\mathbf{y})^T\mathbf{Q}(\mathbf{B}_J^{-1/2}\mathbf{y}) := (\mathbf{B}_J^{-1/2}\mathbf{y})^T\mathbf{Q}(\mathbf{B}_J^{-1/2}\mathbf{y}) := \mathbf{v}^T\mathbf{Q}\mathbf{v} > 0$, Hence $\mathbf{Q}$ is positive definite. where $\mathbf{v} = \mathbf{B}_J^{-1/2}\mathbf{y}$.

Proposition A4: Let $\mathbf{B}_J^{1/2}$ is non singular diagonal matrix. Then eigenvalues of $\mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2}$ are same as of $\mathbf{M}_J$.

Proof. For the eigenvalue λ of $\mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2}$, we can write $\mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2}\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x}$ is corresponding eigenvector. $\mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{M}_J\mathbf{B}_J^{-1/2}\mathbf{x} = \lambda\mathbf{B}_J^{-1/2}\mathbf{x} \Rightarrow \mathbf{M}_J\mathbf{y} = \lambda\mathbf{y}$, hence λ is also the eigenvalue of $\mathbf{M}_J$, where $\mathbf{y} = \mathbf{B}_J^{-1/2}\mathbf{x}$.

Proposition A5: Let $\mathbf{A}$ be a symmetric positive definite matrix and $\mathbf{D}$ is a positive diagonal matrix with diagonal elements of $\mathbf{A}$. Define $\mathbf{E} = \mathbf{A} - \mathbf{D}$ and $\mathbf{B}_J = \mathbf{D} + \mathbf{F}$, where $\mathbf{F}$ is a diagonal matrix with diagonal element $F_{ii} = \eta\sum_j|E_{ij}|$. Define $\mathbf{B}_{JOR} = \frac{\mathbf{B}_J}{\omega}$, $\mathbf{C}_{JOR} = \mathbf{A} - \mathbf{B}_{JOR}$, and $\mathbf{M}_{JOR} = -\mathbf{B}_{JOR}^{-1}\mathbf{C}_{JOR}$. Then, $\rho(\mathbf{M}_{JOR}) < 1, \forall\, \omega \in (0, 2)$ and $\eta \geq 1$.

Proof. We have $\mathbf{M}_J = -\mathbf{B}_J^{-1}\mathbf{C}_J := (\mathbf{D}+\mathbf{F})^{-1}(\mathbf{F}-\mathbf{E})$. The diagonal element of $\mathbf{M}_J$ is $M_{Jii} = \frac{F_{ii}}{D_{ii}+F_{ii}}$ and the non diagonal element is $M_{Jij} = \frac{E_{ij}}{D_{ii}+F_{ii}}$. $M_{Jii} \geq M_{Jij}$ because $F_{ii} \geq \sum_j|E_{ij}|$, implying that $\mathbf{M}_J$ is a diagonally dominant matrix. Hence, by Theorem 1.8 of [6], $\text{Re}(\lambda_{M_J}) > 0$. Imaginary part of the eigenvalues is zero because by *Proposition A4*, eigenvalues of $\mathbf{M}_J$ and $\mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2}$ are same. Furthermore, it can be observed that $\mathbf{B}_J^{1/2}\mathbf{M}_J\mathbf{B}_J^{-1/2} := \mathbf{B}_J^{-1/2}\mathbf{C}_J\mathbf{B}_J^{-1/2}$ is a symmetric matrix so that eigenvalues will be real only. Thus, we have $\lambda_{M_J} > 0 \Rightarrow \lambda_{M_Jmin} > 0$. By *Proposition A2*, the condition over ω for $\rho(\mathbf{M}_{JOR}) < 1$ is $0 < \omega < \frac{2}{1-\lambda_{M_Jmin}}$. Considering $\lambda_{M_Jmin} > 0$ the upper bound of ω is, $\frac{2}{1-\lambda_{M_Jmin}} \geq 2$. Therefore $\rho(\mathbf{M}_{JOR}) < 1, \forall\, \omega \in (0, 2)$.

Proposition A6: The spectral radius $\rho(\mathbf{M}_{JOR})$ is minimum at optimal value of ω, which is given by $\omega_{opt} = \frac{2}{2-(\lambda_{M_Jmin}+\lambda_{M_Jmax})}$, where λ_{M_Jmin} and λ_{M_Jmax} are the smallest and largest eigenvalues of $\mathbf{M}_J$.

Proof. We have $\mathbf{M}_{JOR} = -\mathbf{B}_{JOR}^{-1}\mathbf{C}_{JOR} := -\mathbf{B}_{JOR}^{-1}(\mathbf{A} - \mathbf{B}_{JOR}) := -\mathbf{B}_{JOR}^{-1}(\mathbf{B}_J + \mathbf{C}_J - \mathbf{B}_{JOR}) := -(\frac{\mathbf{B}_J}{\omega})^{-1}(\mathbf{B}_J + \mathbf{C}_J - \frac{\mathbf{B}_J}{\omega}) := (1-\omega)\mathbf{I} - \omega\mathbf{B}_J^{-1}\mathbf{C}_J := (1-\omega)\mathbf{I} + \omega\mathbf{M}_J$. Hence, we can establish a relationship between the eigenvalues of $\mathbf{M}_{JOR}$ and $\mathbf{M}_J$, given by $\lambda_{M_{JOR}} = 1 - \omega(1 - \lambda_{M_J})$. Variation in different eigenvalues $|\lambda_{M_{JOR}i}|$ with ω are shown in Fig.1. The trajectory of the spectral radius, which is the largest magnitude of eigenvalues of a matrix, is shown by red dotted lines. It can be observed that the ω_{opt} lies on the intersection of lines $-1 + \omega(1 - \lambda_{M_Jmin})$ and $1 - \omega(1 - \lambda_{M_Jmax})$, which gives $\omega_{opt} = \frac{2}{2-(\lambda_{M_Jmin}+\lambda_{M_Jmax})}$.

Example Ex1: Optimal value of ω for distributed estimators: For a system with gain matrix $\mathbf{A}$ we can find the ω_{opt} with the help of *Proposition A6.* we can further improve the performance by finding the optimal value of ω for each estimator based on reducibility or irreducibility of $\mathbf{A}$. The irreducibility of matrix $\mathbf{A}$ is contingent upon the placement strategy of the PMUs. By Theorem 1.6 of [6], a square matrix $\mathbf{A}$ is irreducible if and only if its associated directed graph $G(\mathbf{A})$ is strongly connected, where each non-zero element of matrix $\mathbf{A}$ corresponds to a link in $G(\mathbf{A})$. This property is particularly useful for identifying the group of distributed estimators that contribute to making a portion of the power system observable and obtaining the optimal weighting factor, denoted as ω_{opt}, for that PMU set.

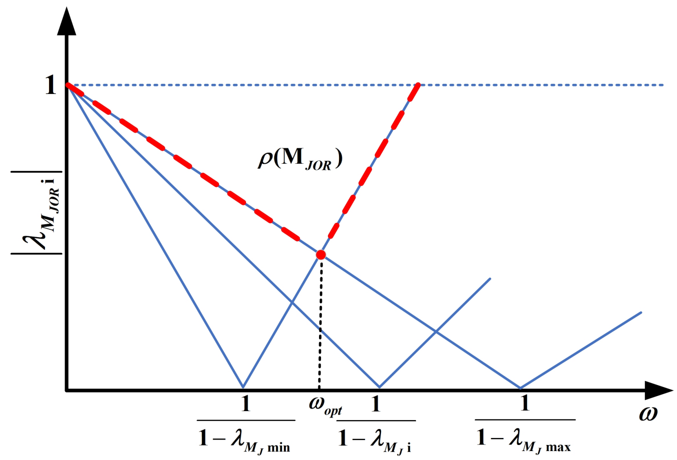

Fig. 1: Trajectory of the spectral radius $\rho(\mathbf{M}_{JOR})$

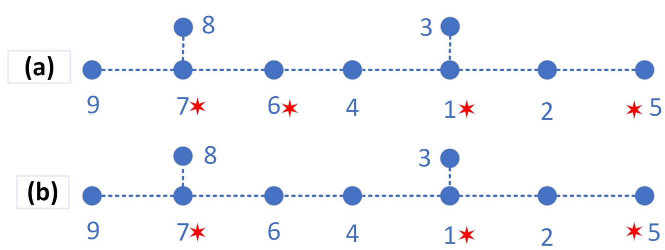

Fig. 2: PMU placement Example; Star represent PMU bus

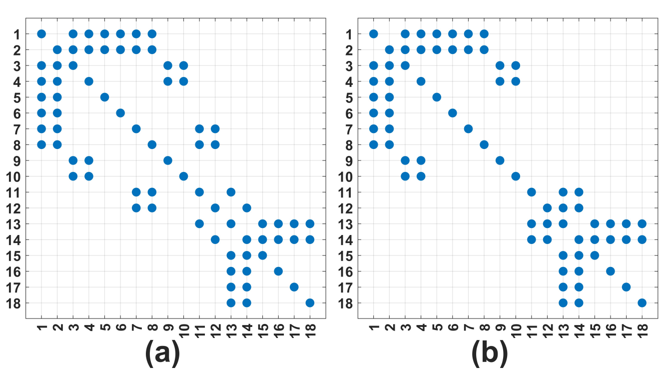

Fig. 3: (a) matrix $\mathbf{A}$ for 2(a) (b) matrix $\mathbf{A}$ for 2(b)

Case 1. $\mathbf{A}$ is irreducible: For the PMU placement as shown in Fig.2(a), PMUs are placed to ensure system observability. The corresponding matrix $\mathbf{A}$ is shown in Fig. 3(a), and its irreducibility can be verified by examining its associated directed graph $G(\mathbf{A})$, as illustrated in Fig.14.

Case 2. $\mathbf{A}$ is reducible: In Fig.2(b), PMUs are optimally placed to ensure system observability. $\mathbf{A}$ is reducible to irreducible sub-matrices. With the help of basic matrix operations, a reducible matrix can be written in terms of its irreducible sub-matrices, as shown in Fig. 3(b). We can directly identify $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_{22} \end{bmatrix}$.

We can use this property of the matrix $\mathbf{A}$ to tune the relaxation factor ω for each sub-matrix exclusively as per *Proposition A6*, thereby obtaining the optimal convergence. $\mathbf{A}_{11}$ pertains to buses 1, 2, 3, 4, and 5, with estimators positioned at buses 1 and 5. Consequently, both of these estimators will share the same ω_{opt}, which can be computed using $\mathbf{M}_{JOR}$ derived from $\mathbf{A}_{11}$. Similarly, ω_{opt} for the estimator situated at bus 7 can be derived using $\mathbf{M}_{JOR}$ obtained from $\mathbf{A}_{22}$.

The matrices, shown in Fig 3, have directed graph $G(\mathbf{A})$, as shown in Fig.4. In the case of a reducible matrix, the directed graph divides.

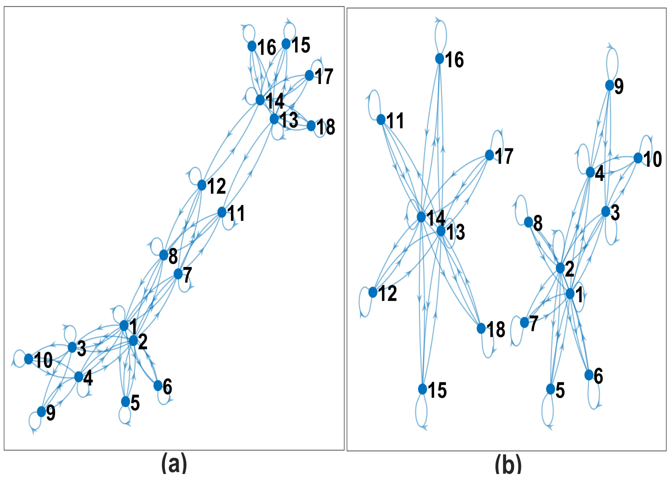

Fig. 4: (a) $G(\mathbf{A})$ of irreducible matrix $\mathbf{A}$ (b) $G(\mathbf{A})$ of reducible matrix $\mathbf{A}$

Proposition A7. If the iteration matrix $\mathbf{M}$ is convergent, then $\boldsymbol{\delta}^{(m)} \to \boldsymbol{\delta}$ as $m \to \infty$, and $\boldsymbol{\delta} = \mathbf{A}^{-1}\boldsymbol{\beta}$, represents the unique solution.

Proof. Let an error vector $\mathbf{e}^{(m)}$ be defined by $\mathbf{e}^{(m)} = \boldsymbol{\delta}^{(m)} - \boldsymbol{\delta}$. Similarly, $\mathbf{e}^{(m+1)} = \boldsymbol{\delta}^{(m+1)} - \boldsymbol{\delta}$, with the help of $\boldsymbol{\delta}^{(m+1)} = \mathbf{M}\boldsymbol{\delta}^{(m)} + \mathbf{B}^{-1}\boldsymbol{\beta}$, can be expressed as $\mathbf{M}\boldsymbol{\delta}^{(m)} + \mathbf{B}^{-1}\boldsymbol{\beta} - \boldsymbol{\delta} := \mathbf{M}\boldsymbol{\delta}^{(m)} + \mathbf{B}^{-1}\mathbf{A}\boldsymbol{\delta} - \boldsymbol{\delta} := \mathbf{M}\boldsymbol{\delta}_m + \mathbf{B}^{-1}(\mathbf{B} + \mathbf{C})\boldsymbol{\delta} - \boldsymbol{\delta} := \mathbf{M}\boldsymbol{\delta}^{(m)} + (\mathbf{I} + \mathbf{B}^{-1}\mathbf{C})\delta - \boldsymbol{\delta} := \mathbf{M}(\boldsymbol{\delta}^{(m)} - \boldsymbol{\delta}) := \mathbf{M}\mathbf{e}^{(m)}$. We establish a relation between two consecutive error vectors, i.e, $\mathbf{e}^{(m+1)} = \mathbf{M}\mathbf{e}^{(m)}$, and this can be extended up to the initial error vector as $\mathbf{e}^{(m+1)} = \mathbf{M}\mathbf{e}^{(m)} = \ldots = \mathbf{M}^{m+1}\mathbf{e}^{(0)}$. Since $\mathbf{M}$ is a convergent matrix ($\mathbf{M}^{\infty} \to \mathbf{O}$ Null matrix), the error vector $\mathbf{e}^{(m+1)} \to \mathbf{0}$ as $m \to \infty \Rightarrow \boldsymbol{\delta}^{(m)} \to \boldsymbol{\delta}$.

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