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**UPPER BOUND METHOD FOR YIELD DESIGN OF REINFORCED CONCRETE
SLABS USING CONIC PROGRAMMING AND AN ADAPTIVE REMESHING
STRATEGY**

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Abstract: This study presents a numerical procedure for the analysis of reinforced concrete slabs (RCS) that obey Nielsen's yield criterion (slabs orthogonally reinforced). An upper bound formulation combined with finite elements was established to solve the kinematic theorem as a conic optimization problem with the aim to determine the maximum bearing capacity of RCS. Discrete Kirchhoff finite elements were implemented and adapted to establish a limit state problem for the yield design. By using Nielsen's criterion, a kinematic criterion was established applying the flow rule of plasticity. The kinematic criterion was included in the upper bound formulation with the aim to constraint the curvatures of the slab. The upper bound formulation was organized in the standard form of a second order cone programming (SOCP) problem since the kinematic criterion was formulated in conic form. Numerical examples were proposed to test the accuracy of the method including the adaptive remeshing strategy.

KEYWORDS

Reinforced concrete slabs, upper bound method, collapse load, conic programming, limit state analysis

1. INTRODUCTION

In recent years, different numerical approaches have been proposed for the yield design of reinforced concrete slabs [2, 3-4]. Some of these methodologies have begun to be relevant in the euro code for their applications, especially numerical limit analysis. This method is based on a traditional finite element formulation combined with mathematical programming tools. The capacity of permitting high discontinuities in the derivatives of the elements are an advantage for the estimation of the collapse load of reinforced concrete structures as slabs. In this paper, a kinematic formulation is presented and evaluated with a known example as well as a remeshing strategy is proposed to achieve accuracy in the upper bound solutions.

2. UPPER BOUND THEOREM FOR PLATES INCLUDING A KINEMATIC YIELD SURFACE

Let's consider that Ω is a rigid-perfectly plastic plate subjected to external loads λ , in which the displacement rate field $\dot{\mathbf{u}}$ belongs to an admissible kinematic field $\mathbf{Y}$, such that $\dot{\mathbf{u}}$ are compatibles with the generalized deformations in the following way

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$$\dot{\kappa} = \mathbf{L} \cdot \dot{\mathbf{u}} \text{ on } \Omega, \dot{\mathbf{u}} = 0 \text{ in } \Gamma_u, \quad (1)$$

where $\mathbf{L} = [\partial^2/\partial x^2 \quad \partial^2/\partial y^2 \quad 2\partial^2/\partial x\partial y]^T$ is a differential operator, $\dot{\kappa} = [\dot{\kappa}_x \quad \dot{\kappa}_y \quad 2\dot{\kappa}_{xy}]^T$ are the curvatures (generalized strains) and Γ_u is the boundary of Ω . In a plate, strain energy $W_{\text{int}}(\mathbf{m}, \dot{\mathbf{u}})$ can be maximized from the static field to define a dissipation function as $D_p(\dot{\mathbf{u}}) = \max W_{\text{int}}(\mathbf{m}, \dot{\mathbf{u}})$ (Markov's principle, see [1] where $\mathbf{m}$ are the moments in the plate. Dissipation means that in a rigid-perfectly plastic plate, the energy cannot be stored and it is strictly dissipated in $\dot{\mathbf{u}} \in \mathbf{Y}_p, \mathbf{Y}_p \subset \mathbf{Y}$ (admissible plastic field) when Ω collapses. It is well known that, a yield surface defines the behavior of a rigid-perfectly plastic material, therefore $\mathbf{m} \in \chi(\mathbf{x}), \forall \mathbf{x} \in \Omega$ and $\chi(\mathbf{x})$ is a convex set which represents the yield surface such that

$$\chi(\mathbf{x}) = \{\mathbf{m} \in \mathbf{X} \mid f(\mathbf{m}) \leq 0\}, \quad (2)$$

where the yield condition $f(\mathbf{m}) \leq 0$ should be satisfied at every point of the plate Ω . If $f(\mathbf{m}) \leq 0$, the moments $\mathbf{m}$ belong to a statically admissible field $\mathbf{X}$ (equilibrium conditions satisfied), but if $f(\mathbf{m}^*) = 0$, the moments $\mathbf{m}^*$ are plastically admissible on $\mathbf{X}_p \subset \mathbf{X}$. At this state, which is commonly called limit state, the curvatures are determined from the flow rule [1] and these should satisfy the following kinematic condition $f(\dot{\kappa}(\dot{\mathbf{u}})) \leq 0$. From the virtual work principle, we can express that external work $\lambda W_{\text{ext}}(\dot{\mathbf{u}})$ should be equal to the plastic dissipation $D_p(\dot{\mathbf{u}})$ such that an admissible external force is found as $\lambda = D_p(\dot{\mathbf{u}})/W_{\text{ext}}(\dot{\mathbf{u}})$. If the previous expression is minimized, then $\min \lambda = \min D_p(\dot{\mathbf{u}})/W_{\text{ext}}(\dot{\mathbf{u}})$, we can determine the set $C = \{\dot{\mathbf{u}} \in \mathbf{Y} \mid W_{\text{ext}}(\dot{\mathbf{u}}) = 1\}$, $\lambda^+ W_{\text{ext}}(\dot{\mathbf{u}}) > 0$. Therefore, it is obtained the collapse load as $\lambda^+ \geq \min \lambda = \min D_p(\dot{\mathbf{u}})$. Given these conditions, an exact force multiplier λ is the smallest value of all the possible solutions of λ^+ corresponding to a set of all the admissible displacement rates $\dot{\mathbf{u}}$ (kinematic and plastic) in which $\lambda \leq \lambda^+$. Those requirements are defined as a kinematic theorem and may be expressed as an optimization problem as

$$OP \begin{cases} \lambda^+ = \min & D_p(\dot{\mathbf{u}}) \\ \text{subject to:} & \dot{\kappa} = \mathbf{L} \cdot \dot{\mathbf{u}}, f(\dot{\kappa}) \leq 0 \\ & \dot{\mathbf{u}} = 0 \text{ en } \Gamma_u, W_{\text{ext}}(\dot{\mathbf{u}}) = 1 \end{cases} \quad (3)$$

Equation (3) shows the kinematic problem in a standard form of a mathematical programming problem which satisfies the requirements of limit analysis methodology. Detailed descriptions of the formulation are provided by [2] and recently by [3].

3. NIELSEN'S CRITERION FOR REINFORCED CONCRETE SLABS

A yield surface for reinforced concrete slabs with orthogonal reinforcement is described in [4]. The criterion (yield surface) was established on the supposition that the coordinate system of the reinforcement is oriented in the same direction as the principal moments. Based on this supposition, it is presented as a set of inequalities expressed as

$$\begin{aligned} -(m_{px}^- + m_x)(m_{py}^- + m_y) + m_{xy}^2 &\leq 0, m_{px}^- \leq m_x \leq m_{px}^+ \\ -(m_{px}^+ - m_x)(m_{py}^+ - m_y) + m_{xy}^2 &\leq 0, m_{py}^- \leq m_y \leq m_{py}^+ \end{aligned} \quad (4)$$

Where m_{px}^-, m_{py}^- are negative plastic moments and m_{px}^+, m_{py}^+ are yield moments positive in the directions $x-y$.

4. UPPER BOUND FORMULATION FOR REINFORCED CONCRETE SLABS

Kinematic fields should be required to define compatibility conditions and to establish dissipation equations in the upper bound formulation. Let's considers discretized in n triangular finite elements (discrete Kirchhoff elements, described by [5], the strain energy [4] of all triangular slabs is determined by a sum of energy dissipation in each element as

$$D_p(\dot{\mathbf{u}}) = \sum_{e=1}^n A_e \left(-m_{px}^- \dot{\kappa}_x^{-(e)} - m_{py}^- \dot{\kappa}_y^{-(e)} + m_{px}^- \dot{\kappa}_x^{+(e)} + m_{py}^- \dot{\kappa}_y^{+(e)} \right) \quad (5)$$

where A_e is the area of each element, e means an element and the curvatures should satisfy the following inequalities if $\exists: \dot{\kappa}_x^{-(e)} < 0$ and $\dot{\kappa}_y^{-(e)} < 0$ when $\dot{\kappa}_x^{+(e)} = 0$ and $\dot{\kappa}_y^{+(e)} = 0$. Otherwise, if $\exists: \dot{\kappa}_x^{+(e)} > 0$ and $\dot{\kappa}_y^{+(e)} > 0$ when $\dot{\kappa}_x^{-(e)} = 0$ and $\dot{\kappa}_y^{-(e)} = 0$. This due to that in a slab, it is determined that $\dot{\kappa}_x^{(e)} \dot{\kappa}_y^{(e)} \geq 0$ by applying the flow rule. The curvatures are estimated from compatibility equations as constant curvatures to ensure the dissipation within the element, therefore these are determined applying Equation (1) to equation established for DKT element [5], such as

$$\dot{\kappa}_m^{(e)} = \frac{1}{3} \sum_{j=1}^3 \mathbf{B}^{(e)}(\mathbf{x}_j) \dot{\mathbf{u}}_e, \text{ where } \dot{\kappa}_m^{(e)} = \begin{bmatrix} \dot{\kappa}_x^{(e)} & \dot{\kappa}_y^{(e)} & 2\dot{\kappa}_{xy}^{(e)} \end{bmatrix}^T \text{ and } \mathbf{x}_k \in (\xi_k, \eta_k), \forall k=1, 2, \dots, p, \quad (6)$$

vectors are evaluated on p nodes of the finite element mesh. To satisfy the signs of the curvatures, it is necessary to establish a set of equations. The following is proposed

$$\mathbf{d} \dot{\kappa}_{(e)}^+ + \mathbf{d} \dot{\kappa}_{(e)}^- = \dot{\kappa}_{(m)}^{(e)}, \text{ such that } \mathbf{d} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \dot{\kappa}_{(e)}^+ = \begin{bmatrix} \dot{\kappa}_{x(e)}^+ & \dot{\kappa}_{y(e)}^+ \end{bmatrix}^T \text{ and } \dot{\kappa}_{(e)}^- = \begin{bmatrix} \dot{\kappa}_{x(e)}^- & \dot{\kappa}_{y(e)}^- \end{bmatrix}^T. \quad (7)$$

To satisfy the kinematic yield criterion in all the slab domain, it is defined that

$$\mathbf{d} \dot{\kappa}_{(e)}^+ - \mathbf{d} \dot{\kappa}_{(e)}^- = \frac{\dot{\kappa}_{(m)}^{(e)}}{\sqrt{2}}, \text{ where } \dot{\kappa}_{(e)}^+ = \begin{bmatrix} \dot{\kappa}_{x(e)}^+ & \dot{\kappa}_{y(e)}^+ \end{bmatrix}^T \text{ and } \dot{\kappa}_{(e)}^- = \begin{bmatrix} \dot{\kappa}_{x(e)}^- & \dot{\kappa}_{y(e)}^- \end{bmatrix}^T. \quad (8)$$

Equations (7) and (8) are established because these permit that the curvatures are converted into a standard conic form. Therefore, the set $D^{(e)}$ may be defined for an admissible plastic field as a kinematic yield criterion, as follows

$$D_e = \left\{ \dot{\kappa}^{(e)} \in \mathbf{Y}_p \mid 2\dot{\kappa}_{x(p)}^{(e)} \dot{\kappa}_{y(p)}^{(e)} = \left(\dot{\kappa}_{xy}^{(e)} \right)^2, \dot{\kappa}_{x(p)}^{(e)}, \dot{\kappa}_{y(p)}^{(e)} \geq 0 \right\}. \quad (9)$$

To formulate the kinematic theorem, Equations (7) and (8) should be rewritten in a standard form of an optimization problem, thus Equation (9) is rewritten as $W_p(\dot{\mathbf{u}}) = \mathbf{c}^T \dot{\mathbf{u}}$, where $\dot{\mathbf{u}} = \begin{bmatrix} \mathbf{u} & \dot{\kappa}_m & \dot{\kappa}^- & \dot{\kappa}^+ & \dot{\kappa} \end{bmatrix}^T$ and $\mathbf{u} \in \mathbb{R}^{9n}$, $\dot{\kappa}_m \in \mathbb{R}^{3n}$, $\dot{\kappa}^- \in \mathbb{R}^{2n}$, $\dot{\kappa}^+ \in \mathbb{R}^{2n}$ and $\dot{\kappa} \in \mathbb{R}^{2n}$. Equations (6), (7) and (8) are organized in matrix form such that $\mathbf{H} \dot{\mathbf{u}} = 0$, where $\mathbf{H} \in \mathbb{R}^{7n \times 18n}$. In a similar way, external work is determined by application of external forces on nodes $\mathbf{x}_k$ in such a way that

$$W_{ext}(\dot{\mathbf{u}}) = \mathbf{f} \cdot \dot{\mathbf{u}} = \sum_{e=1}^n \sum_{j=1}^3 \frac{A_e}{3} (\dot{u}_{je} + \dot{\theta}_{jxe} + \dot{\theta}_{jye}) = 1, \quad (10)$$

where $\mathbf{f} = [f_1 \ f_2 \ \dots \ f_n \ 0]$, $f_i = [1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0]$, $\forall i=1, \dots, 9n$, $\dot{\theta}_{jxe}$ and $\dot{\theta}_{jye}$ are the rotations about x and y at the node j , $\forall j=1, 2, 3$. Boundary conditions should be specified for kinematic boundaries since the dual spaces are not considered in this study. Continuity relations are kept in the curvatures but these are discontinuous in the rotations. The kinematic theorem in limit analysis is implemented as a conic optimization problem and it is expressed in its standard form as

$$OP \begin{cases} \lambda^+ = \min & D_p(\dot{\mathbf{u}}) \\ \text{subject to:} & \mathbf{H} \dot{\mathbf{u}} = 0, \mathbf{G} \dot{\mathbf{u}} = 0 \text{ en } \Gamma \\ & \mathbf{f} \dot{\mathbf{u}} = 1, \dot{\mathbf{u}} \in D \end{cases} \quad (11)$$

The solution of the problem proposed in Equation (11) is obtained using a conic optimization algorithm, for this purpose we used software MOSEK [6]. The remeshing strategy is taken from the study done by [3].

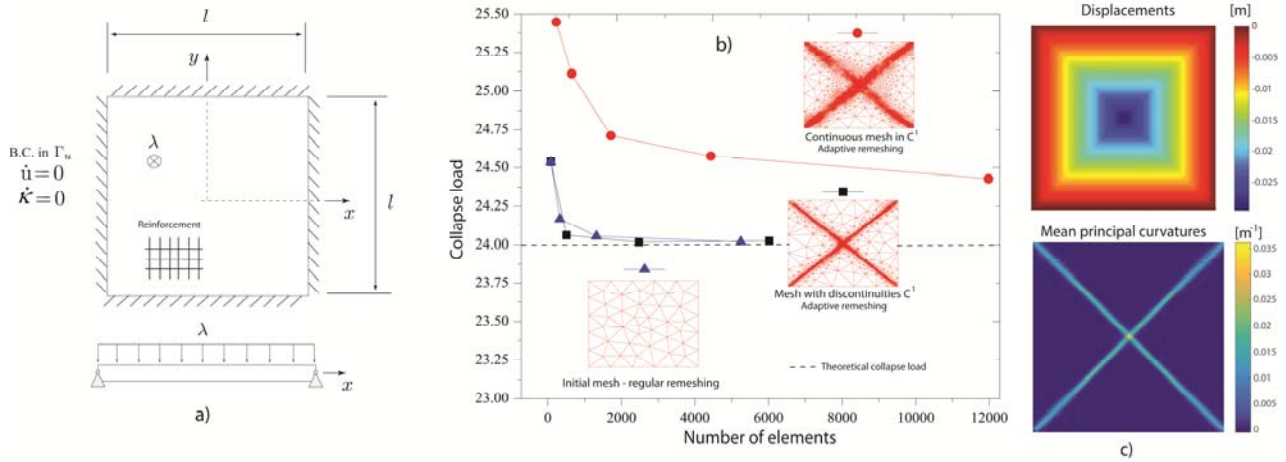


Figure 1. a) Simply supported slab. b) Collapse load convergence. c) Displacements rate field and collapse mechanism from curvatures.

2. RESULTS

A simply supported square slab (see Figure 1a) is considered as an example in Figure 1a. Plastic moments for the yield criterion were chosen as $m_{px}^- = m_{py}^- = -100N \cdot m$, $m_{px}^+ = m_{py}^+ = 100N \cdot m$ with a length $l = 10m$. The collapse load of this example, it is well known from the yield line theory which is $\lambda = 24m_p / l^2$.

A convergence analysis was carried out to evaluate the performance of the upper bound solution proposed for this study. Three different cases of remeshing were considered on the same initial mesh; a regular remeshing, remeshing strategy with discontinuity C^1 (free rotations between elements) and remeshing strategy with continuity C^1 (traditional finite element mesh). In Figure 1b, it is observed that the adaptive remeshing converged faster than the regular mesh, the achieved values were 24.01 (adaptive) and 24.02 (regular). These meshes permitted discontinuities in the rotations, it means that discontinuities in the derivatives of the displacement field are accepted. Additionally, a continuous mesh was proposed to observe the disadvantages of imposing full continuity conditions in the elements; then 24.42 (adaptive) was the best solution for the collapse load.

Figure 1c shows the displacement field and mean principal curvatures obtained with the results. The principal curvatures are associated to yield lines that conform the collapse mechanism of the slab and this mechanism is the correct. It is also observed that the adaptive remeshing took the shape of the yield lines showing that the algorithm propose by [3] works.

CONCLUSIONS

An upper bound method has been presented and described to determine the load carrying capacity of a reinforced concrete slab. The method is based on the DKT (discrete Kirchhoff theory) element which permits to impose partially free rotations in its formulation and this is evidenced as an advantage in the obtained solutions.

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