

State-Dependent Linear Utility Functions for Monetary Returns

By

Somdeb Lahiri

(Former Professor) PD Energy University, Gandhinagar (EU-G), India.

ORCID: <https://orcid.org/0000-0002-5247-3497>

somdeb.lahiri@gmail.com

January 19, 2024.

This version: October 23, 2024.

Abstract

We present a theory of expected utility with state-dependent linear utility function for monetary returns, that includes results on “first order stochastic dominance”, “mean-preserving spread”, “increasing-concave linear utility profiles” and “risk aversion”. As an application of the expected utility theory developed here, we analyze the contract that a monopolist would offer in an insurance market that allowed for partial coverage of loss. We also define a utility function for monetary returns that in a certain sense reconciles state-dependent constant average utility of money with “loss aversion” and the “Friedman-Savage” hypothesis. As an immediate consequence of such a utility function, we obtain a profile of state-dependent linear utility functions for monetary returns, where states of nature correspond to mutually disjoint intervals in which monetary gains and losses may occur.

Keywords: money, utility function, state-dependent, linear, first order stochastic dominance, mean-preserving spread, risk aversion, loss aversion, Friedman-Savage hypothesis

JEL Classifications: D81.

AMS Subject Classification: 90B50

1. Introduction:

A common argument against linear utility function for monetary returns is, that an agent with such a utility function would have no incentive to insure himself against possible “loss”.

However, this argument seems to collapse if the linear utility function for monetary returns is state dependent and the “probability of the gain or loss” is spelt out as the “probability of the state of nature (son) in which there is the gain or loss” with the constant marginal utility of monetary returns in the “worse” state being more than the constant marginal utility of money in the “better” state.

The seminal contribution of Kahneman and Tversky (1979) noted the experimentally verified observation that agents tend to have a marginal utility of loss that is no less- if not higher- than the marginal utility of gain, so that a typical utility function for monetary returns $u: \mathbb{R} \rightarrow \mathbb{R}$ may be of the form $u(x) = u^+ \max\{x, 0\} + u^- \min\{x, 0\}$ with $u^- \geq u^+ > 0$. This phenomenon is known as “loss aversion”. Thus, any utility function of this form can be represented by a pair of real numbers (u^-, u^+) where $u^- \geq u^+ > 0$. Allowance is made for the possibility of $u^- = u^+$. It is generally assumed that under normal circumstances $u^- - u^+$ is a non-decreasing function of

initial wealth, thereby implying that wealthier individuals are more “risk averse” than those individuals who are less wealthy.

The work of Friedman and Savage (1948), spells out clearly that beginning with an initial segment where the utility function for gains and losses is concave, the utility function alternates between convexity and concavity thereafter. This property of utility functions for gains and losses is known as the “Friedman-Savage” hypothesis. In Lahiri (2022), a utility function for gains and losses (expressed in terms of gains and losses) has been suggested that is compatible with “loss aversion” as well as the “Friedman-Savage” hypothesis. However, the utility function in Lahiri (2022) does not display constant average utility in any sub-interval of its domain, and this is a problem for reasons that we now turn to.

The dominant interpretation of probability in expected utility theory, is the one due to Ramsey and de Finetti. Brief discussions along with intuitive motivation of such probabilities are available in Lahiri (2023b, 2023c). The Ramsey-de Finetti subjective probability of an “event” or “state of nature” (say E) that is assessed by an agent is the price (say p) that the agent would be willing to pay for a simple bet that returns one unit of money if state of nature ‘ E ’ occurs and nothing otherwise, so that the expected monetary value of the simple bet to the agent is zero. Thus, if the average utility of money in state of nature E is a constant, say $\mu > 0$, then for one unit of money in state of nature E , the agent will be willing to forego μp units of utility and for ξ units of money in state of nature E the agent will be willing to forego $\mu p \xi$ units of utility, the latter being the utility the agent willingly forgoes for ξ simple bets of the type we have just discussed. ξ simple bets, each of which returns one unit of money if E occurs and nothing otherwise, is identical to a bet that returns ξ unit of money if E occurs and nothing otherwise. Thus, Ramsey-de Finetti subjective probability fits comfortably with “expected utility theory” based on constant average state dependent utility. On the other hand, if the average utility in state of nature E is “non-constant”, then there exists ξ such that the average utility of ξ units of money is not equal to the average utility of $p\xi$ units of money. For a bet that returns ξ units of money in state of nature E and nothing otherwise, the agent will be “willingly foregoing” the utility of $p\xi$ units of money and not ‘ p ’ times the utility of ξ units of money, the latter being the expected utility of the bet to the agent. Hence, on the face of it, there seems to be a mismatch between Ramsey-de Finetti subjective probability and expected utility theory based on such an interpretation, if state-dependent average utility of money is “non-constant”. A comprehensive exposition of the early stages of the analysis of decision making under uncertainty with state dependent preferences is available in Karni (1985). However, the significance of state-dependent linear utility functions for money is that it fits comfortably with the concept of expected utility based on Ramsey-de Finetti probabilities. Thus, with state-dependent linear utility functions for money, Ramsey-de Finetti probabilities and expected utilities are “perfectly economically consistent” with one another.

In the next section of the paper, we provide a motivation for our discussion in the subsequent sections, by considering a “toy model” of insurance against a risky loss. We apply expected state dependent linear utility analysis in this model and show that insurance is possible under state-dependent “risk neutrality”. In the third section, we present the formal framework for “expected utility theory with state-dependent linear utility functions for monetary returns”. Using concepts introduced in this section, in subsequent sections we introduce “first order

stochastic dominance”, “mean-preserving spread”, “increasing-concave linear utility profiles” and “risk aversion”. As an application of the expected utility theory developed here, we analyze the contract that a monopolist would offer in an insurance market that allowed for partial coverage of loss. In the third section we also provide a way of incorporating “ambiguity” in the definition of a portfolio of risky assets, along the lines suggested in Gilboa and Schmeidler (1989).

Our final endeavor here concerns amplification of ideas implicit in “increasing-concave linear utility profiles”. We define a utility function for monetary gains and losses that in a certain sense reconciles state-dependent constant average utility of money with “loss aversion” and the “Friedman-Savage” hypothesis. Such utility functions, that we refer to as “Almost Linear” utility functions are characterized by “jump” discontinuities between successive intervals, and the risk perception of the agent relative to a pora that involves equally probable small gains and losses at such points, depends on whether the utility function at such a point is discontinuous from the right or from the left. If it is “continuous from the right” at such a point, the agent is risk averse. If it is “continuous from the left” at such a point, the agent is risk loving/seeking. As an immediate consequence of such a utility function, we obtain a profile of state-dependent linear utility functions for monetary gains and losses, where states of nature correspond to mutually exclusive intervals in which monetary gains and losses may occur and the average utility in each interval is constant. In such a framework, we show that relative to poras involving two equally likely gains that are equidistant from a common endpoint of two successive intervals, the agent is risk loving/seeking, whereas, relative to poras involving two equally likely losses equidistant from a common endpoint of two successive intervals, the agent is risk averse. The result seems quite plausible, since random outcomes that yield nothing but gains are welcome, whereas random outcomes that involves nothing but losses are unwelcome.

In what follows we often refer to “monetary gains and losses” as “monetary returns”. All proofs of major results are relegated to an appendix of this paper. We hope that with this paper, we are able to provide an incremental impetus for further development for decision analysis with linear utility functions for money.

2. Motivation- Insuring Against Risky Loss:

Consider a situation with 2 states of nature 1,2, where an agent with initial wealth $w > 0$ may face a loss of $L \in (0, w)$ units of money in the second son. Let $p \in (0,1)$ be the probability of loss. Suppose that his utility function for monetary returns in son i , is a function of the above form with (u_i^-, u_i^+) being the slopes for losses and gains respectively in son ‘ i ’.

There are two ways in which insurance can be introduced in this setting. First is a variation of the traditional textbook setting where we assume $u_2^- > u_1^-$. Even an individual who is not affected by the loss, would react to the news of the loss- by leaning closer towards caution and hence a higher marginal utility of money- than in the absence of such news, however small the difference in the marginal utilities may. If one hears about frequent bicycle thefts in the neighbourhood that one lives in, then the same person is likely to be concerned more about the safety of his/her bicycle than he/she would be in the absence of such news, regardless of whether the person has been a victim of such theft or not. For an agent with a stake in the loss, the difference gets more pronounced.

In the absence of an insurance policy the expected utility of the agent is $-pu_2^-L$.

An insurance policy that provides complete coverage is available for a premium π which if “actuarially fair” would satisfy $\pi = pL$.

The expected utility from buying this policy is $-[(1-p)u_1^- + pu_2^-]\pi = -p[(1-p)u_1^- + pu_2^-]L$.

Since $u_2^- > u_1^- > 0$ and $p \in (0,1)$, $(1-p)u_1^- + pu_2^- < u_2^-$ and so $-p[(1-p)u_1^- + pu_2^-]L > -pu_2^-L$.

Actually, it would be more realistic to consider three sons: 1-where there is no loss, 2- where there is a loss and the agent “has not” bought the insurance policy and 3- where there is a loss and the agent “has” bought the insurance policy, with $u_2^- > u_3^- > u_1^- > 0$, since having bought the insurance policy, the agent is somewhat more relaxed than what it would be had it not purchased the insurance policy, but since recovering the insurance payment involves some transaction cost (e.g. paper work, etc.) the agent’s disutility from expenditure incurred on buying the premium could be expected to be higher than what it would be had there been no loss.

A second way in which insurance can be introduced in this context, which may be more realistic is to assume that the seller of the insurance policy has recourse to an investment opportunity, which for some $r > 0$, returns $1 + r$ units of money for every unit of money invested in the current period. In this case, we can weaken the restriction on the slopes of the utility functions and assume $u_2^- \geq u_1^-$, i.e., allow for $u_2^- = u_1^-$.

In this case an insurance policy that provides complete coverage for a premium π , yields an expected return of $(1+r)\pi - pL$ to the seller of the insurance policy which is non-negative if $\pi \geq \frac{pL}{1+r}$. Since $r > 0$, $\frac{pL}{1+r} < pL$, so that the seller of the policy can make a profit by selling it for a premium $\pi \in (\frac{pL}{1+r}, pL)$.

In this case, the expected utility from buying this policy for a premium of π is $-[(1-p)u_1^- + pu_2^-]\pi$ and $-[(1-p)u_1^- + pu_2^-]\pi > -pu_2^-L$, since $0 < (1-p)u_1^- + pu_2^- \leq u_2^-$ and $\pi < pL$.

Now let us consider an agent whose initial monetary wealth is $w > 0$ and an investible amount $I \in (0,w)$ can either be diversified equally between two-risky investment opportunities or invested entirely in one investment opportunity, with each investment opportunity having a probability $p \in (0,1)$ of failing.

This is a situation where there are three states of nature denoted by 1,2,3 with (u_i^-, u_i^+) being the slopes for losses and gains respectively in son ‘i’ > 0 Son 1 is the situation where neither investment opportunity fails, son 2 is the situation where 50% of the invested amount is lost and son 3 is the situation where the entire invested amount is lost.

Suppose $0 < u_1^- < u_2^- < u_3^-$.

Even if the agent was not an investor, the news of an investment opportunity crashing would very likely have the effect of increasing its disutility of expenditure and such disutility would further increase if it were to hear the news of two investment opportunities crashing simultaneously. In the case of an investor, the effect of such news could only be expected to be more pronounced.

If the agent invests the entire amount I in exactly one investment opportunity, then his expected utility is $-pu_3^-I$.

If the agent spreads his investment opportunity equally between the 2 investment opportunities, then his expected utility is $-2p(1-p)u_2^- \frac{I}{2} - p^2u_3^-I = -p[(1-p)u_2^- + pu_3^-]I$.

Since $u_3^- > (1-p)u_2^- + pu_3^-$, we have $-p[(1-p)u_2^- + pu_3^-]I > -pu_3^-I$, and hence there is always an incentive for “spreading risks”.

3. The Framework of Analysis:

Let us now set up the general framework of analysis with linear utility functions for monetary returns. For a more general framework of analysis, one may refer to Bonanno (2019).

For some positive integer $L \geq 2$, let $\{1, 2, \dots, L\}$ denote the finite set of states of nature.

A (column) vector $x \in \mathbb{R}^L$ where for each $j \in \{1, \dots, L\}$, the j^{th} coordinate of x denotes the monetary return in son j , is said to be a **return vector**.

A (column) vector $p \in \mathbb{R}_{++}^L$ satisfying $\sum_{j=1}^L p_j = 1$, such that for $j \in \{1, \dots, L\}$, p_j is the probability of occurrence of son j , is a **probability vector**.

Given $x, y \in \mathbb{R}^L$, let $y^T x$ denote $\sum_{j=1}^L y_j x_j$.

A **portfolio of risky assets** (briefly referred to as a **pora**) is a pair (x, p) where x is a return vector and p is a probability vector.

Given a pora (x, p) with X denoting the random monetary return for (x, p) and $\alpha \in \mathbb{R}$, let $\{X = \alpha\}$ denote the event that the realized son yields a monetary return of α , $\{X \leq \alpha\}$ denote the event that the realized son yields a monetary return less than or equal to α , $\{X \geq \alpha\}$ denote the event that the realized son yields a monetary return greater than or equal to α , $\{X < \alpha\}$ denote the event that the realized son yields a monetary return less than α , $\{X > \alpha\}$ denote the event that the realized son yields a monetary return greater than α .

Thus, for all $\alpha \in \mathbb{R}$, Probability of $\{X \leq \alpha\} = 1 - \text{Probability of } \{X > \alpha\}$

The **expected value** of a pora (x, p) denoted $E(x, p)$ is $p^T x = \sum_{j=1}^L p_j x_j$.

Note: One possible generalization of the concept of portfolio of risky assets that can be inferred from the solution proposed by Gilboa and Schmeidler (see Gilboa and Schmeidler (1989)) as a response to the Ellsberg Paradox may be the following. Let a **generalized portfolio of risky assets** be a pair (ξ, p) where $p \in \mathbb{R}_{++}^L$ is a probability vector and for each $j \in \{1, \dots, L\}$, ξ_j is a non-empty finite set of returns, exactly one from which is realized if SON j occurs. In order to incorporate “ambiguity aversion” one may associate with (ξ, p) , the **min portfolio of risky assets (min-PORA)** (x, p) which is defined as follows: for each $j \in \{1, \dots, L\}$, $x_j = \min\{\alpha \mid \alpha \in \xi_j\}$.

A **linear utility profile** is a vector $u \in \mathbb{R}_{++}^L$ such that the trader’s (Bernoulli) utility for monetary returns (gains or losses) in son $j \in \{1, \dots, L\}$ is $u_j \alpha$ for all real numbers α , with α denoting the monetary return in son j .

Given a linear utility profile u and a pora (x, p) the **expected utility** of (x, p) for u , denoted by $Eu(x, p)$ is $\sum_{j=1}^L p_j u_j x_j$.

Clearly $Eu(x, p) = p_1(u_1x_1 - u_2x_2) + (p_1 + p_2)(u_2x_2 - u_3x_3) + (p_1 + p_2 + p_3)(u_3x_3 - u_4x_4) + \dots + (p_1 + \dots + p_{L-1})(u_{L-1}x_{L-1} - u_Lx_L) + (p_1 + p_2 + \dots + p_L)x_L = \sum_{j=1}^{L-1} (\sum_{k=1}^j p_k) (u_j x_j - u_{j+1}x_{j+1}) + (\sum_{k=1}^L p_k)u_Lx_L$.

Given a linear utility profile u and a pora (x, p) the **certainty equivalent** of (x, p) for u , denoted by $CE(u, x, p)$ is the scalar that satisfies $u(CE(u, x, p)) = Eu(x, p)$.

Suppose that (x, p) is a pora satisfying $x_j < x_{j+1}$ for all $j \in \{1, \dots, L-1\}$. Then, for all $k \in \{1, \dots, L-1\}$ and $\alpha, \beta \in (x_k, x_{k+1})$, Probability of $\{X > \alpha\} =$ Probability of $\{X > x_k\} =$ Probability of $\{X > \beta\}$ and Probability of $\{X \leq \alpha\} =$ Probability of $\{X \leq x_k\} =$ Probability of $\{X \leq \beta\}$.

4. First Order Stochastic Dominance:

Given two poras (x, p) and (y, q) with X denoting the random monetary return for (x, p) and Y denoting the random monetary return for (y, q) , we say that (x, p) **stochastically dominates** (y, q) **in the first order**, denoted by $(x, p) >_{\text{FSD}} (y, q)$ if for all $\alpha \in \mathbb{R}$, Probability of $\{X > \alpha\} \geq$ Probability of $\{Y > \alpha\}$ and for some $\alpha \in \mathbb{R}$, Probability of $\{X > \alpha\} >$ Probability of $\{Y > \alpha\}$.

The intuitive interpretation of $(x, p) >_{\text{FSD}} (y, q)$ is that given any monetary return α , the probability that the monetary return from (x, p) is greater than α is greater than or equal to the probability that the monetary return from (y, q) is at greater α , and for some monetary return the first probability is strictly greater than the second probability i.e., (x, p) is consistently “more likely” to yield better rewards better than (y, q) .

We know that for a linear utility profile and a pora (x, p) , $Eu(x, p) = \sum_{j=1}^{L-1} (\sum_{k=1}^j p_k) (u_j x_j - u_{j+1}x_{j+1}) + (\sum_{k=1}^L p_k)u_Lx_L$.

Proposition 1: Let (x, p) and (x, q) be two poras satisfying $x_j < x_{j+1}$ for all $j \in \{1, \dots, L-1\}$. Then $(x, p) >_{\text{FSD}} (x, q)$ if and only if $[Eu(x, p) > Eu(x, q)]$ for all linear utility profile u satisfying $u_j x_j < u_{j+1}x_{j+1}$ for all $j \in \{1, \dots, L-1\}$.

5. Mean-preserving Spread and Increasing-Concave Linear Utility Profiles:

For this section assume $L \geq 3$.

Given a return vector x satisfying $x_j < x_{j+1}$ for all $j \in \{1, \dots, L-1\}$, a linear utility profile u is said to be **increasing-concave with respect to** x , if for all $j \in \{1, \dots, L-1\}$, $u_j x_j < u_{j+1}x_{j+1}$ and for all $i, j, k \in \{1, 2, \dots, L\}$ with $i < j < k$, $u_j x_j > (1-\delta)u_i x_i + \delta u_k x_k$ where $\delta \in (0, 1)$ satisfies $x_j = (1-\delta)x_i + \delta x_k$.

Clearly, $\delta = \frac{x_j - x_i}{x_k - x_i}$ and $0 < x_j - x_i < x_k - x_i$.

Given a return vector x satisfying $x_j < x_{j+1}$ for all $j \in \{1, \dots, L-1\}$, pora (x, q) is said to be **obtained by a mean-preserving spread from** pora (x, p) , denoted $(x, p) \rightarrow_{\text{MSP}} (x, q)$, if $E(x, p) = E(x, q)$ and there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i$, $q_j < p_j$, $q_k > p_k$ and $p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$.

$[E(x, p) = E(x, q)$ and there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i, q_j < p_j, q_k > p_k$ and $p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$] if and only if [there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i, q_j < p_j, q_k > p_k, p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$ and $(p_j - q_j)x_j = (q_i - p_i)x_i + (q_k - p_k)x_k]$

[there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i, q_j < p_j, q_k > p_k$ and $p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$ and $(p_j - q_j)x_j = (q_i - p_i)x_i + (q_k - p_k)x_k]$ is equivalent to [there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i, q_j < p_j, q_k > p_k, p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$ and $x_j = \frac{q_i - p_i}{p_j - q_j} x_i + \frac{q_k - p_k}{p_j - q_j} x_k]$.

Thus, $(x, p) \rightarrow_{MSP} (x, q)$ if and only if [there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i, q_j < p_j, q_k > p_k, p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$ and $x_j = \frac{q_i - p_i}{p_j - q_j} x_i + \frac{q_k - p_k}{p_j - q_j} x_k]$.

Proposition 2: Let (x, p) and (x, q) be two poras satisfying $x_j < x_{j+1}$ for all $j \in \{1, \dots, L-1\}$.

- (a) If $(x, p) \rightarrow_{MSP} (x, q)$ then $[Eu(x, p) > Eu(x, q)]$ for all linear utility profile u which is increasing-concave with respect to x].
- (b) If $L = 3, p_2 \neq q_2$ and $[Eu(x, p) > Eu(x, q)]$ for all linear utility profile u which is increasing-concave with respect to x then $(x, p) \rightarrow_{MSP} (x, q)$.

6. Risk Aversion:

Given a pora (x, p) , an agent with linear utility profile u is said to be:

- (i) **Risk Averse relative to** (x, p) if $E(x, p) > CE(u, x, p)$;
- (ii) **Risk Neutral relative to** (x, p) if $E(x, p) = CE(u, x, p)$;
- (iii) **Risk Loving/Seeking relative to** (x, p) if $E(x, p) < CE(u, x, p)$.

Example 1: Let $L = 2, u_1 = 1$ let $u_2 = 2$.

Let $(x, p) = ((2, 0), (\frac{1}{2}, \frac{1}{2}))$. Thus, $E(x, p) = 1$.

In this case, $Eu(x, p) = 1$ and $p^T u = \frac{3}{2}$, so that $CE(u, x, p) = \frac{2}{3} < 1 = E(x, p)$.

Thus, the agent is risk averse relative to $((2, 0), (\frac{1}{2}, \frac{1}{2}))$.

Now let $(x, p) = ((0, 2), (\frac{1}{2}, \frac{1}{2}))$. Once again, $E(x, p) = 1$.

Now, $Eu(x, p) = 2$ and since $p^T u = \frac{3}{2}$, we have $CE(u, x, p) = \frac{4}{3} > 1 = E(x, p)$.

Thus, the same agent is risk loving/seeking relative to $((0, 2), (\frac{1}{2}, \frac{1}{2}))$.

Now suppose $(x, p) = ((1, 1), (\frac{1}{2}, \frac{1}{2}))$. Once again, $E(x, p) = 1$.

Now, $Eu(x, p) = \frac{3}{2}$ and since $p^T u = \frac{3}{2}$, we have $CE(u, x, p) = 1 = E(x, p)$.

Thus, the same agent is now risk neutral relative to $((1,1), (\frac{1}{2}, \frac{1}{2}))$.

Given a pora (x,p) and a linear utility profile u , the **risk premium relative to** (x,p) denoted $R(u, x, p) = E(x, p) - CE(u, x, p)$.

Thus, $\sum_{j=1}^L p_j u_j (E(x, p) - R(u, x, p)) = \sum_{j=1}^L p_j u_j CE(u, x, p) = Eu(x, p)$.

If the agent is:

- (i) Risk Averse relative to (x, p) , then $R(u, x, p) > 0$;
- (ii) Risk Loving/Seeking relative to (x, p) , then $R(u, x, p) < 0$;
- (iii) Risk Neutral relative to (x, p) , then $R(u, x, p) = 0$.

Given two linear utility profiles u, v and two poras $(x, p), (y, q)$ we say that u relative to (x, p) **is more risk averse than** v relative to (y, q) if $R(u, x, p) > R(v, y, q)$.

7. Insurance contracts with the possibility of partial coverage:

As before consider a situation with 2 states of nature 1,2, where an agent with initial wealth $w > 0$ may face a loss of $L \in (0, w)$ units of money in the second son. Let $p \in (0,1)$ be the probability of loss. Suppose that the agent's linear utility profile is (u_1, u_2) with $0 < u_1 < u_2$.

The expected value of the "risk" is $-pL$

In the absence of an insurance policy the expected utility of the agent is $-pu_2L$.

If CE_1 is the certainty equivalent in the absence of any insurance policy, then $[(1-p)u_1 + pu_2]CE_1 = -pu_2L$.

$$\text{Thus, } CE_1 = \frac{-pu_2L}{(1-p)u_1 + pu_2} = -pL \frac{u_2}{(1-p)u_1 + pu_2}$$

An insurance policy with a **deductible** $d \in [0,L)$ (i.e., in case of loss, the insurer pays $L-d$ to the agent) is available for a premium π .

Hence the expected profit of the insurer is $\pi - p(L-d)$.

For the insurer to voluntarily sell the insurance, it must be "**profitable**", i.e., $\pi - p(L-d) \geq 0$.

Thus, profitability is equivalent to the condition $-pL \geq -(\pi + pd)$.

The expected value of this policy to the agent is $-(\pi + pd)$.

The expected utility of the agent from buying this policy is $-(1-p)u_1\pi - pu_2(\pi + d) = -[(1-p)u_1 + pu_2]\pi - pu_2d$.

For the agent to voluntarily buy the insurance, it must be the case that $-[(1-p)u_1 + pu_2]\pi - pu_2d \geq -pu_2L$, i.e., $-\pi - \frac{u_2}{(1-p)u_1 + pu_2} pd \geq CE_1$.

$$-\pi - \frac{u_2}{(1-p)u_1 + pu_2} pd = -(\pi + pd) + pd \left[1 - \frac{u_2}{(1-p)u_1 + pu_2} \right].$$

Thus the agent will voluntarily buy the insurance policy if and only if $-(\pi + pd) + pd[1 - \frac{u_2}{(1-p)u_1 + pu_2}] \geq CE_1$.

A profit maximizing insurer will choose an **insurance contract**, i.e., a pair (π, d) that maximizes $\pi - p(L-d)$ subject to $\pi - p(L-d) \geq 0$, $-[(1-p)u_1 + pu_2]\pi - pu_2d \geq -pu_2L$ and $d \in [0, L]$.

The above problem is equivalent to choosing a pair (π, d) that maximizes $\pi + pd$ subject to $\pi + pd \geq pL$, $[(1-p)u_1 + pu_2]\pi + pu_2d \leq pu_2L$ and $d \in [0, L]$.

It is easy to see that at an optimal solution, $[(1-p)u_1 + pu_2]\pi + pu_2d = pu_2L$.

$$\text{Thus, } \pi = \frac{pu_2(L-d)}{(1-p)u_1 + pu_2}.$$

$$\text{Thus, } \pi + pd = p[\frac{u_2(L-d)}{(1-p)u_1 + pu_2} + d] = pd[1 - \frac{u_2}{(1-p)u_1 + pu_2}] + \frac{pu_2L}{(1-p)u_1 + pu_2}.$$

Since $u_2 > u_1$, we have $\frac{u_2}{(1-p)u_1 + pu_2} > 1$ and hence $1 - \frac{u_2}{(1-p)u_1 + pu_2} < 0$.

Thus, $pd[1 - \frac{u_2}{(1-p)u_1 + pu_2}] + \frac{pu_2L}{(1-p)u_1 + pu_2}$ is maximized at $d = 0$, thereby implying $\pi = \frac{pu_2L}{(1-p)u_1 + pu_2}$.

Since $\frac{pu_2L}{(1-p)u_1 + pu_2} = (\frac{u_2}{(1-p)u_1 + pu_2})pL$ and $\frac{u_2}{(1-p)u_1 + pu_2} > 1$, we have $\pi > pL$. Since $d = 0$, $\pi + pd > pL$.

Hence, the optimal contract is the pair $(\frac{pu_2L}{(1-p)u_1 + pu_2}, 0)$, with the “expected profit of the insurer” being $\frac{pu_2L}{(1-p)u_1 + pu_2} - pL = pL(\frac{u_2 - (1-p)u_1 - pu_2}{(1-p)u_1 + pu_2}) = \frac{p(1-p)(u_2 - u_1)L}{(1-p)u_1 + pu_2} > 0$.

Note: $\pi = \frac{pu_2(L-d)}{(1-p)u_1 + pu_2}$ implies $-\pi - \frac{u_2}{(1-p)u_1 + pu_2}pd = -\frac{pu_2L}{(1-p)u_1 + pu_2} = CE_1$.

We know that $-\pi - \frac{u_2}{(1-p)u_1 + pu_2}pd = -(\pi + pd) + pd[1 - \frac{u_2}{(1-p)u_1 + pu_2}]$.

Thus, at an optimal solution $-(\pi + pd) + pd[1 - \frac{u_2}{(1-p)u_1 + pu_2}] = CE_1$.

“**Strict Profitability**” is equivalent to the condition $-pL > -(\pi + pd)$ which now reduces to

$$-pL + pd[1 - \frac{u_2}{(1-p)u_1 + pu_2}] > CE_1 = -\frac{pu_2L}{(1-p)u_1 + pu_2}.$$

Thus **strict profitability** is equivalent to $-pL[1 - \frac{u_2}{(1-p)u_1 + pu_2}] + pd[1 - \frac{u_2}{(1-p)u_1 + pu_2}] > 0$, i.e.

$$p(d-L)[1 - \frac{u_2}{(1-p)u_1 + pu_2}] \geq 0.$$

Since $d \in [0, L]$, this is possible if and only if $1 - \frac{u_2}{(1-p)u_1 + pu_2} < 0$, i.e. $1 < \frac{u_2}{(1-p)u_1 + pu_2}$

Multiplying throughout by pL which is strictly positive, we get $1 < \frac{u_2}{(1-p)u_1 + pu_2}$ if and only if

$$pL < \frac{u_2}{(1-p)u_1 + pu_2}pL, \text{ the latter being equivalent to } -\frac{u_2}{(1-p)u_1 + pu_2}pL < -pL.$$

Since $CE_1 = -\frac{u_2}{(1-p)u_1+pu_2}pL$ and $-pL$ is the expected value of the “risk”.

Thus “**Strict Profitability**” is equivalent to the requirement that the agent is risk averse relative to $((-L, 0), (p, 1-p))$.

Let us now consider the somewhat more realistic situation with three sons: 1-where there is no loss, 2- where there is a loss and the agent “has not” bought the insurance policy and 3- where there is a loss and the agent “has” bought the insurance policy, with $u_2 > u_3 > u_1 > 0$.

Then, the expected utility of the agent from buying this policy is $-(1-p)u_1\pi - pu_3(\pi + d) = -[(1-p)u_1 + pu_3]\pi - pu_3d$.

Since $u_2 > u_3$, $-(1-p)u_1\pi - pu_3(\pi + d) > -(1-p)u_1\pi - pu_2(\pi + d)$.

A profit maximizing insurer will choose an **insurance contract**, i.e., a pair (π, d) that maximizes $\pi - p(L-d)$, subject to $\pi - p(L-d) \geq 0$, $-[(1-p)u_1 + pu_3]\pi - pu_3d \geq -pu_2L$ and $d \in [0, L]$.

The above problem is equivalent to choosing a pair (π, d) that maximizes $\pi + pd$, subject to $\pi + pd \geq pL$, $[(1-p)u_1 + pu_3]\pi + pu_3d \leq pu_2L$ and $d \in [0, L]$.

It is easy to see that at an optimal solution, $[(1-p)u_1 + pu_3]\pi + pu_3d = pu_2L$.

Thus, $\pi = \frac{p(u_2L - u_3d)}{(1-p)u_1 + pu_3}$.

Thus, $\pi + pd = p\left[\frac{u_2L - u_3d}{(1-p)u_1 + pu_3} + d\right] = pd\left[1 - \frac{u_3}{(1-p)u_1 + pu_3}\right] + \frac{pu_2L}{(1-p)u_1 + pu_3}$.

Since $u_3 > u_1$, we have $\frac{u_3}{(1-p)u_1 + pu_3} > 1$ and hence $1 - \frac{u_3}{(1-p)u_1 + pu_3} < 0$.

Thus, $pd\left[1 - \frac{u_3}{(1-p)u_1 + pu_3}\right] + \frac{pu_2L}{(1-p)u_1 + pu_3}$ is maximized at $d = 0$, thereby implying $\pi = \frac{pu_2L}{(1-p)u_1 + pu_3} > \frac{pu_2L}{(1-p)u_1 + pu_2}$, since $u_3 < u_2$.

Since $\frac{pu_2L}{(1-p)u_1 + pu_3} = \left(\frac{u_2}{(1-p)u_1 + pu_3}\right)pL$ and $\frac{u_2}{(1-p)u_1 + pu_3} > \frac{u_2}{(1-p)u_1 + pu_2} > 1$, we have $\pi > \frac{pu_2L}{(1-p)u_1 + pu_2} pL$. Since $d = 0$, $\pi + pd > \frac{pu_2L}{(1-p)u_1 + pu_2} > pL$.

Hence, the optimal contract is the pair $\left(\frac{pu_2L}{(1-p)u_1 + pu_3}, 0\right)$, with the “expected profit of the insurer” being $\frac{pu_2L}{(1-p)u_1 + pu_3} - pL = pL\left(\frac{u_2 - (1-p)u_1 - pu_3}{(1-p)u_1 + pu_2}\right) > \frac{p(1-p)(u_2 - u_1)L}{(1-p)u_1 + pu_2} > 0$.

Thus, the expected profit of the insurer is higher in this more realistic situation than in the earlier situation.

8. Almost Linear Utility Function for Monetary Returns:

While compatibility of state-dependent linear utility function for “monetary gains and losses” with loss aversion, is easily taken care of by defining $u(x) = u^+\max\{x, 0\} + u^-\min\{x, 0\}$ with $u^+ \geq u^- > 0$, for all $x \geq -w$, where $w > 0$ is the initial wealth of the decision maker, compatibility of a state-dependent linear utility function with the “Friedman and Savage” hypothesis, is somewhat more problematic.

Suppose $w > 0$ is the initial wealth. For a positive integer n , let $\langle x_{-j}(w) \mid j \in \{1, \dots, n\} \rangle$ be a strictly “decreasing” and finite sequence of negative real numbers satisfying $x_{-n}(w) = -w$. Let $\langle u_{-j}(w) \mid j \in \{1, \dots, n\} \rangle$ be a strictly “increasing” (with respect to ‘j’) and finite sequence of positive real numbers.

Let $x_0(w) = 0$.

Let $u: [-w, +\infty) \rightarrow \mathbb{R}$ be a function such that:

(i) $u(0) = 0$.

(ii) For all $x \in [-w, x_{-(n-1)})$, $u(x) = u_{-n}x$.

(ii) If $n > 1$, then for all $j \in \{n-1, \dots, 1\}$ and $x \in (x_{-j}, x_{-(j+1)})$, $u(x) = u_{-j}x$ and $u(x_{-j}) \in \{u_j x_{-j}, u_{j+1} x_{-j}\}$.

(iii) There exists a strictly positive integer m , a strictly “increasing” and finite sequence of positive real numbers $\langle x_{j-1}(w) \mid j \in \{1, \dots, m\} \rangle$, a strictly “increasing” (with respect to j) and finite sequence of positive real numbers $\langle u_j(w) \mid j \in \{1, \dots, m\} \rangle$ such that if $m = 1$, then $u(x) = u_1 x$ for all $x > 0$, and if $m > 1$ $u(x) = u_j x$ if $x \in (x_{j-1}, x_j)$ and $j \in \{1, \dots, m-1\}$, $u(x) = u_m x$ for all $x > x_{m-1}$ and $u(x_j) \in \{u_j x_j, u_{j+1} x_j\}$ for $j \in \{1, \dots, m\}$.

It is easy to see that u is strictly increasing in its domain.

Clearly there are discontinuities at x_{-j} for $j \in \{1, \dots, n-1\}$ if $n > 1$ and at x_j for $j \in \{1, \dots, m\}$ if $m > 0$.

The parameters defined above are explicitly dependent on w and it is generally assumed that for $j \in \{-1, +1\}$, $u_j(w)$ is “*non-increasing*” in w .

Since in what follows $w > 0$ is fixed, for the purpose of economy of notation, in what follows, not omit w in the definition of the parameters determining u .

We will refer a function u which is of the type defined above as an **Almost Linear utility function for monetary returns**.

In the case of an Almost Linear utility function u , given any **pora** (x, p) : (i) $u(E(x, p)) < Eu(x, p)$ could mean **risk aversion** relative to (x, p) , (ii) $u(E(x, p)) > Eu(x, p)$ could mean **risk seeking** relative to (x, p) , and (iii) $u(E(x, p)) = Eu(x, p)$ could mean **risk neutrality** relative to (x, p) ,

Consider a pora that at some point of discontinuity x_k (i.e., $k \in \{-1, \dots, -n\}$ if $m = 1$ and $k \in \{-1, \dots, -n\} \cup \{0, \dots, m-1\}$ if $m > 1$) of u returns $x_k - \delta$ with probability with probability $\frac{1}{2}$ and $x_k + \delta$ with probability $\frac{1}{2}$, where $x_k - \delta > x_{k-1}$ and $x_k + \delta < x_{k+1}$.

The expected value of this pora is x_k , whereas its expected utility is $\frac{1}{2} u_k(x_k - \delta) + \frac{1}{2} u_{k+1}(x_k + \delta)$ if $k > 0$ and $\frac{1}{2} u_{k-1}(x_k - \delta) + \frac{1}{2} u_k(x_k + \delta)$ if $k < 0$.

If $k > 0$ and $u(x_k) = u_k x_k$, then $\frac{1}{2} u_k(x_k - \delta) + \frac{1}{2} u_{k+1}(x_k + \delta) - u_k x_k = \frac{1}{2}(x_k + \delta)(u_{k+1} - u_k) > 0$, since $u_{k+1} > u_k$ and $x_k + \delta > 0$. Thus, if $k > 0$ and $u(x_k) = u_k x_k$, then the agent is risk loving/seeking relative to the pora,

If $k > 0$ and $u(x_k) = u_{k+1}x_k$, then $\frac{1}{2}u_k(x_k - \delta) + \frac{1}{2}u_{k+1}(x_k + \delta) - u_{k+1}x_k = \frac{1}{2}(x_k - \delta)(u_k - u_{k+1}) < 0$, since $u_{k+1} > u_k$ and $x_k - \delta > 0$. Thus, if $k > 0$ and $u(x_k) = u_{k+1}x_k$, then the agent is risk averse relative to the pora.

If $k < 0$ and $u(x_k) = u_{k-1}x_k$, then $\frac{1}{2}u_{k-1}(x_k - \delta) + \frac{1}{2}u_k(x_k + \delta) - u_{k-1}x_k = \frac{1}{2}(u_k - u_{k-1})(x_k + \delta) > 0$, since $u_k < u_{k-1}$ and $x_k + \delta < 0$. Thus, if $k < 0$ and $u(x_k) = u_{k-1}x_k$, then the agent is risk loving/seeking relative to the pora,

If $k < 0$ and $u(x_k) = u_kx_k$, then $\frac{1}{2}u_{k-1}(x_k - \delta) + \frac{1}{2}u_k(x_k + \delta) - u_kx_k = \frac{1}{2}(u_{k-1} - u_k)(x_k - \delta) < 0$, since $u_k < u_{k-1}$ and $x_k - \delta < 0$. Thus, if $k < 0$ and $u(x_k) = u_kx_k$, then the agent is risk averse relative to the pora,

Hence, the attitude of the agent towards risk depends on the utility of the agent at a point of discontinuity x_k . If it is the greater of the two values it is permitted, then the agent is risk averse relative to the pora. Otherwise, the agent is risk loving/seeking relative to the pora.

Note: An Almost Linear utility function is compatible with both “loss aversion” and the “Friedman-Savage” hypothesis. Preference generated by expected utility with such utility functions, with a suitable upper bound on monetary gains, are compatible with the five assumptions of expected utility theory, in Proposition 1 of Lahiri (2023a).

9. State-dependent Linear Utility Functions:

In the context of an Almost Linear utility function of the first type defined in the previous section, let $X = [-w, +\infty)$ be the sample space.

For $k \in \{-1, \dots, -n\}$, let E_k be the interval in which average utility is u_k .

If $m = 1$, then let $E_0 = [0, +\infty)$ with average utility u_1 .

If $m > 1$, then for $k \in \{1, \dots, m-1\}$ let E_{k-1} be the interval in which average utility is u_k and let $E_m = \mathbb{R}_+ \setminus (\cup_{k=1}^{m-1} E_{k-1})$ be the interval in which average utility is u_m .

$\{E_{-n}, \dots, E_{-1}\} \cup \{E_0, \dots, E_m\}$ can be considered to be a collection of mutually exclusive events (or states of nature), on each of which the average utility of money is a constant.

For $E_k \in \{E_{-n}, \dots, E_{-1}\} \cup \{E_0, \dots, E_m\}$, let $u(\cdot|E_k): E_k \rightarrow \mathbb{R}$ be the function such that for all $x \in E_k$, $u(x|E_k) = u_kx$ if $k < 0$ and let $u(\cdot|E_k): E_k \rightarrow \mathbb{R}$ be the function such that for all $x \in E_k$, $u(x|E_k) = u_{k+1}x$ if $k \geq 0$.

The ordered array $(u(\cdot|E_{-n}), u(\cdot|E_{-1}), u(\cdot|E_0), u(\cdot|E_1), \dots, u(\cdot|E_m))$ is a **profile of state-dependent linear utility functions** for monetary returns.

Consider the pora discussed in the previous section that at some point x_k returns $x_k - \delta$ with probability $\frac{1}{2}$ and $x_k + \delta$ with probability $\frac{1}{2}$, where $x_k - \delta > x_{k-1}$ and $x_k + \delta < x_{k+1}$.

As in the previous section, the expected value of the pora is x_k and its expected utility is $\frac{1}{2}u_k(x_k - \delta) + \frac{1}{2}u_{k+1}(x_k + \delta)$ if $k > 0$ and $\frac{1}{2}u_{k-1}(x_k - \delta) + \frac{1}{2}u_k(x_k + \delta)$ if $k < 0$.

Thus, if CE denotes the Certainty Equivalent of the pora, then for $k > 0$, $\frac{1}{2}(u_k + u_{k+1}) CE = \frac{1}{2} u_k(x_k - \delta) + \frac{1}{2} u_{k+1}(x_k + \delta) = \frac{1}{2}(u_k + u_{k+1})x_k - \frac{1}{2} \delta(u_k - u_{k+1})$ and for $k < 0$, $\frac{1}{2}(u_{k-1} + u_k) CE = \frac{1}{2} u_{k-1}(x_k - \delta) + \frac{1}{2} u_k(x_k + \delta) = \frac{1}{2}(u_{k-1} + u_k)x_k - \frac{1}{2} \delta(u_{k-1} - u_k)$.

If $k > 0$, then $u_k - u_{k+1} < 0$ and if $k < 0$ then $u_{k-1} - u_k > 0$.

Thus, if $k > 0$, then $\frac{1}{2}(u_k + u_{k+1})(CE - x_k) = -\frac{1}{2} \delta(u_k - u_{k+1}) > 0$ and if $k < 0$, then $\frac{1}{2}(u_{k-1} + u_k)(CE - x_k) = -\frac{1}{2} \delta(u_{k-1} - u_k) < 0$.

Hence, $CE > x_k$ if $x_k > 0$ and $CE < x_k$ if $x_k < 0$.

Thus, the agent is risk-loving/seeking relative to the pora if at x_k the pora concerns small perturbations in “gains” and risk-averse relative to the pora if at x_k the pora concerns small perturbations in “losses”.

Acknowledgment: I would like to thank Itzhak Gilboa for comments and queries about an earlier version of sections 8 and 9 of this paper, where we had proposed a continuous utility function “for wealth” with adjacent segments in the domain, alternating between concavity and convexity. As correctly observed by him an Almost Linear utility function, now, as well as in the previous version is “piece-wise” linear. However, in order to partition the domain of the utility function into segments in each of which average utility of monetary gains or losses is constant, it is necessary to allow for some discontinuity in the utility function. I would also like to apologise to R. Chandrasekar (Sekar) for claiming his valuable respite from “real-world finance” during weekends, to glance through the work of “dreamers(?)”. I hope practitioners of finance will find less dreaming here than in the eternal truth here: “<https://www.youtube.com/watch?v=hWWvTTbCGTI>”.

Reference

1. Bonanno, G. (2019): Uncertainty Risk and Information. Available at: https://faculty.econ.ucdavis.edu/faculty/bonanno/PDF/URI_book.pdf
2. Friedman, M. and L.J. Savage (1948): The Utility Analysis of Choices Involving Risk. *Journal of Political Economy*, Volume 56, No:4, pages 279–304.
3. Gilboa, I., and Schmeidler, D. (1989): Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, Volume 18, Issue 2, pages 141-153.
4. Kahneman, D. and Tversky, A. (1979): Prospect theory: An analysis of decision under risk. *Econometrica*, Volume 47, Number 2, Pages 263-291.
5. Karni, E. (1985): Decision Making Under Uncertainty: The Case of State-Dependent Preferences. Harvard University Press.
6. Lahiri, S. (2022): Utility function for monetary gains (and losses) satisfying Friedman-Savage hypothesis. *Journal of Mathematical Economics and Finance*, [S.I.], Volume 8, No. 1, Pages 23 – 30. Available at: <<https://journals.aserspublishing.eu/jmef/article/view/7165>>
7. Lahiri, S. (2023a): Expected Utility For Probabilistic Prospects And The Common Ratio Property. *Yugoslav Journal of Operations Research*, Volume 33, No. 4, Pages 531-548.

8. Lahiri, S. (2023b): Understanding Expected Utility for Decision Making. Management Studies, Volume 11, Number 2, Pages 93-104.

9. Lahiri, S. (2023c): Understanding Ramsey-de Finetti Probabilities and the St. Petersburg Paradox. Economics and Management Information, Volume 2, Number 2.

Appendix

Proof of Proposition 1: $Eu(x, p) - Eu(x, q) = [\sum_{j=1}^{L-1} (\sum_{k=1}^j p_k) (u_j x_j - u_{j+1} x_{j+1}) + (\sum_{k=1}^L p_k) u_{LXL}] - [\sum_{j=1}^{L-1} (\sum_{k=1}^j q_k) (u_j x_j - u_{j+1} x_{j+1}) + (\sum_{k=1}^L q_k) u_{LXL}] = \sum_{j=1}^{L-1} (\sum_{k=1}^j p_k - \sum_{k=1}^j q_k) (u_j x_j - u_{j+1} x_{j+1}) + (\sum_{k=1}^L p_k - \sum_{k=1}^L q_k) u_{LXL} = \sum_{j=1}^{L-1} (\sum_{k=1}^j p_k - \sum_{k=1}^j q_k) (u_j x_j - u_{j+1} x_{j+1})$, since $\sum_{k=1}^L p_k = 1 = \sum_{k=1}^L q_k$.

Suppose $(x, p) >_{\text{FSD}} (x, q)$. Then, $\sum_{k=1}^j p_k - \sum_{k=1}^j q_k \leq 0$ for all $j \in \{1, \dots, L\}$, with strict inequality for at least one $j \in \{1, \dots, L-1\}$, since $\sum_{k=1}^L p_k = 1 = \sum_{k=1}^L q_k$.

If u is a linear utility profile satisfying $u_j x_j < u_{j+1} x_{j+1}$ for all $j \in \{1, \dots, L-1\}$, then $\sum_{j=1}^{L-1} (\sum_{k=1}^j p_k - \sum_{k=1}^j q_k) (u_j x_j - u_{j+1} x_{j+1}) > 0$.

Thus, $Eu(x, p) - Eu(x, q) > 0$, i.e., $Eu(x, p) > Eu(x, q)$.

Now suppose that it is not the case that $(x, p) >_{\text{FSD}} (x, q)$.

Thus, $\{j \in \{1, \dots, L-1\} \mid \sum_{k=1}^j p_k - \sum_{k=1}^j q_k > 0\} \neq \emptyset$. Let $\eta = \min\{\sum_{k=1}^j p_k - \sum_{k=1}^j q_k \mid \sum_{k=1}^j p_k - \sum_{k=1}^j q_k > 0\}$.

Let $u_1 = 1$. Having defined $u_j > 0$, let $u_{j+1} > 0$ be such that $u_{j+1} x_{j+1} - u_j x_j = \frac{2}{\eta}$ if $\sum_{k=1}^j p_k - \sum_{k=1}^j q_k > 0$ and $\frac{1}{2L} > u_{j+1} x_{j+1} - u_j x_j > 0$, otherwise. This is possible, since $x_{j+1} > x_j$ implies that it is not possible for both x_{j+1} and x_j to be zero.

Thus, $Eu(x, p) - Eu(x, q) = -\frac{2}{\eta} \sum_{h \in \{\{j \in \{1, \dots, L-1\} \mid \sum_{k=1}^j p_k - \sum_{k=1}^j q_k > 0\}\}} \sum_{k=1}^h p_k - \sum_{k=1}^h q_k + \sum_{h \in \{\{j \in \{1, \dots, L-1\} \mid \sum_{k=1}^j p_k - \sum_{k=1}^j q_k \leq 0\}\}} (\sum_{k=1}^h p_k - \sum_{k=1}^h q_k) (u_h x_h - u_{h+1} x_{h+1}) = -\frac{2}{\eta} \sum_{h \in \{\{j \in \{1, \dots, L-1\} \mid \sum_{k=1}^j p_k - \sum_{k=1}^j q_k > 0\}\}} \sum_{k=1}^h p_k - \sum_{k=1}^h q_k + \sum_{h \in \{\{j \in \{1, \dots, L-1\} \mid \sum_{k=1}^j p_k - \sum_{k=1}^j q_k \leq 0\}\}} (\sum_{k=1}^h q_k - \sum_{k=1}^h p_k) (u_{h+1} x_{h+1} - u_h x_h) \leq -2 + (L-1) \frac{1}{2L} \leq -2 + \frac{1}{2} = -\frac{3}{2} < 0$.

Thus, $[Eu(x, p) > Eu(x, q)]$ for all linear utility profile u satisfying $u_j x_j < u_{j+1} x_{j+1}$ for all $j \in \{1, \dots, L-1\}$ implies $(x, p) >_{\text{FSD}} (x, q)$. Q.E.D.

Proof of Proposition 2: (a) Suppose $(x, p) \rightarrow_{\text{MSP}} (x, q)$ and let u be an increasing-concave linear utility profile with respect to x .

Hence, there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i, q_j < p_j, q_k > p_k$ and $p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$ and $(p_j - q_j)x_j = (q_i - p_i)x_i + (q_k - p_k)x_k$ is equivalent to

[there exists $i, j, k \in \{1, 2, \dots, L\}$ satisfying $i < j < k$ such that $q_i > p_i$, $q_j < p_j$, $q_k > p_k$, $p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$ and $x_j = \frac{q_i - p_i}{p_j - q_j} x_i + \frac{q_k - p_k}{p_j - q_j} x_k$.

However, $x_j = (1 - \delta)x_i + \delta x_k$ where $\delta = \frac{x_j - x_i}{x_k - x_i} \in (0, 1)$.

Further $p_i + p_j + p_k = q_i + q_j + q_k$ implies $p_j - q_j = (q_i - p_i) + (q_k - p_k)$.

Thus, $\frac{q_i - p_i}{p_j - q_j} + \frac{q_k - p_k}{p_j - q_j} = 1$ with $\frac{q_i - p_i}{p_j - q_j} > 0$ and $\frac{q_k - p_k}{p_j - q_j} > 0$.

Hence, $\frac{q_k - p_k}{p_j - q_j} = \delta$ and $\frac{q_i - p_i}{p_j - q_j} = 1 - \delta$.

Since u is increasing-concave $u_j x_j > (1 - \delta)u_i x_i + \delta u_k x_k$.

Thus, $(p_j - q_j)u_j x_j > (q_i - p_i)u_i x_i + (q_k - p_k)u_k x_k$, i.e., $p_i u_i x_i + p_j u_j x_j + p_k u_k x_k > q_i u_i x_i + q_j u_j x_j + q_k u_k x_k$.

Since, $p_h = q_h$ for $h \in \{1, 2, \dots, L\} \setminus \{i, j, k\}$, we get $Eu(x, p) > Eu(x, q)$.

(b) Now suppose $L = 3$ and $x_1 < x_2 < x_3$ and $p_2 \neq q_2$.

We have $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1$.

Suppose, $E(x, p) = E(x, q)$. Thus, $p_1 x_1 + p_2 x_2 + p_3 x_3 = q_1 x_1 + q_2 x_2 + q_3 x_3$.

Suppose, $Eu(x, p) > Eu(x, q)$ for all linear utility profiles satisfying $u_1 x_1 < u_2 x_2 < u_3 x_3$ and $u_2 x_2 > (1 - \delta)u_1 x_1 + \delta u_3 x_3$, where $x_2 = (1 - \delta)x_1 + \delta x_3$.

Since $p_2 - q_2 \neq 0$, $(p_2 - q_2)x_2 = (q_1 - p_1)x_1 + (q_3 - p_3)x_3$ implies $x_2 = \frac{q_1 - p_1}{p_2 - q_2} x_1 + \frac{q_3 - p_3}{p_2 - q_2} x_3 = \frac{q_1 - p_1}{p_2 - q_2} x_1 + \frac{(1 - q_1 - q_2) - (1 - p_1 - p_2)}{p_2 - q_2} x_3 = \frac{q_1 - p_1}{p_2 - q_2} x_1 + \frac{(p_2 - q_2) - (q_1 - p_1)}{p_2 - q_2} x_3 = x_3 - \frac{q_1 - p_1}{p_2 - q_2} (x_3 - x_1)$, i.e., $x_2 = x_3 - \frac{q_1 - p_1}{p_2 - q_2} (x_3 - x_1)$.

$x_2 < x_3$ and $x_3 > x_1$ implies $\frac{q_1 - p_1}{p_2 - q_2} > 0$.

Similarly, $x_2 = \frac{q_1 - p_1}{p_2 - q_2} x_1 + \frac{q_3 - p_3}{p_2 - q_2} x_3 = \frac{(1 - q_2 - q_3) - (1 - p_2 - p_3)}{p_2 - q_2} x_1 + \frac{q_3 - p_3}{p_2 - q_2} x_3 = \frac{(p_2 - q_2) - (q_3 - p_3)}{p_2 - q_2} x_1 + \frac{q_3 - p_3}{p_2 - q_2} x_3 = x_1 + \frac{q_3 - p_3}{p_2 - q_2} (x_3 - x_1)$.

$x_2 > x_1$ and $x_3 > x_1$ implies $\frac{q_3 - p_3}{p_2 - q_2} > 0$.

Thus, $x_2 = \frac{q_1 - p_1}{p_2 - q_2} x_1 + \frac{q_3 - p_3}{p_2 - q_2} x_3$, $x_2 = (1 - \delta)x_1 + \delta x_3$, $\delta > 0$, $1 - \delta > 0$, $\frac{q_3 - p_3}{p_2 - q_2} > 0$, $\frac{q_1 - p_1}{p_2 - q_2} > 0$ and $x_1 < x_2 < x_3$ implies $\delta = \frac{q_3 - p_3}{p_2 - q_2}$ and $1 - \delta = \frac{q_1 - p_1}{p_2 - q_2}$.

Thus, $u_2 x_2 > \frac{q_1 - p_1}{p_2 - q_2} u_1 x_1 + \frac{q_3 - p_3}{p_2 - q_2} u_3 x_3$.

If $p_2 < q_2$, then $(p_2 - q_2)u_2 x_2 < (q_1 - p_1)u_1 x_1 + (q_3 - p_3)u_3 x_3$ and thus, $Eu(x, p) = p_1 u_1 x_1 + p_2 u_2 x_2 + p_3 u_3 x_3 < q_1 u_1 x_1 + q_2 u_2 x_2 + q_3 u_3 x_3 = Eu(x, q)$, leading to a contradiction.

Thus, it must be the case that $p_2 > q_2$.

Hence, $\frac{q_1 - p_1}{p_2 - q_2} > 0$ implies $q_1 > p_1$ and $\frac{q_3 - p_3}{p_2 - q_2} > 0$ implies $q_3 > p_3$.

Thus, we have $(x, p) \rightarrow_{\text{MSP}} (x, q)$. Q.E.D.

Note: The proof of part (b) in Proposition 2, can very likely be extended to $L > 3$.