The St. Petersburg Paradox with State Dependent Linear Utility Functions for Monetary Returns: A Note

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Abstract

In the experiment underlying the St. Petersburg paradox, we use state-dependent linear utility functions for money with a countably infinite set of states of nature to show that a potential participant will be willing to pay no more than a finite sum of money to participate in the experiment.

Keywords: St. Petersburg paradox, countably infinite states of nature, state-dependent linear utility function of money

1. Introduction

The conventional statement of the St. Petersburg paradox (see for instance page 4 in Chapter 1 of Biswas (1997)) goes something like the following:

In an experiment consisting of independent tosses of an unbiased coin, a participant receives 2^n units of money if the first head occurs on the n^{th} toss. How much should one be willing to pay for participating in the experiment?

"The problem was invented by Nicolas Bernoulli (1687- 1759) who stated it in a letter to Pierre Raymond de Montmort (1678- 1719) on September 9, 1713. However, the paradox takes its name from its analysis by Nicolas' cousin Daniel Bernoulli (1700-1782), one-time resident of Saint Petersburg, who in 1738 published his thoughts about the problem in the *Commentaries of the Imperial Academy of Science of Saint Petersburg"*. (https://en.wikipedia.org/wiki/St._Petersburg_paradox). It is important to note that the genesis of the paradox, suggests that none of the three mathematicians mentioned above were familiar with the work of the statistician and philosopher Thomas Bayes (1701- 1761) at the time of their correspondence.

The standard conclusion based on the above experiment is that if the utility function for monetary returns was strictly increasing and linear, then since $\sum_{n=1}^{\infty} 2^n (\frac{1}{2})^n = +\infty$, there should be no upper bound on what a participant would be willing to pay to participate in the experiment, contrary to what would be expected in reality. In reality, no one would be willing to pay more than some finite amount of money to participate in the experiment. Hence, this could be construed as an argument against strictly increasing and linear utility function for money (i.e., individuals being risk neutral) and therefore an argument in favour of a utility function for money which is non-linear, such as the natural logarithm of money. In 1728, a mathematician from Geneva- Gabriel Cramer (1704- 1752)- in a private communication suggested that there is a saturation level for the utility money, beyond which its utility or use value remains constant. This could be the idea that motivated the use of utility functions for wealth that are strictly concave, strictly increasing but bounded above (e.g. $u(x) = a - ae^{-x}$ for some positive real number 'a'). Much work and discussion has taken place on this topic and a comprehensive survey of the same may be found in Gasparian, Kiseleva, Korneev, Lebedev and Lebedev (2018), Peterson (2023) and references therein. Feller (1968) suggested a solution based on sampling, and argued that the "worth" of the experiment be based on the total amount that the participants in the sample would be willing to pay. The fallacy in this suggestion is similar to the one we address in section 4 of this paper. Feller (1968) allows for the sample size to be infinite. Samuelson (1960) offers a "clever" explanation of the paradox, i.e., no one would in reality offer such an experiment since it requires an expected payment of an infinite amount of money. Of particular relevance to our approach in this note is the survey by Gasparian etc. (2018), as we will see towards the end of this note. Another attempt at resolving the paradox, that is not considered in the survey, is available in Vivian (2013), wherein it is argued that a different perspective on the paradox would lead to a different expected value and further that an alternative methodology would show that there is no paradox arising out of the experiment.

In Lahiri (2023), we suggest a critique of the way participants would calculate probabilities when faced with such a problem. We argued that it is unrealistic to assume that the subjective probability assigned by a participant to the occurrence of a head on the n^{th} toss conditional on all preceding tosses resulting in tails is $\frac{1}{2}$, regardless of how large n is. For instance, it is not unreasonable to assume that the probability assigned by a participant to the occurrence of a head on the n^{th} toss conditional on all preceding tosses resulting in tails is $\frac{1}{2}$ for n = 1,2, $\frac{1}{4}$ for n = 3, $\frac{1}{8}$ for n = 4 and 0 for $n \geq 5$. In this case the expected value from participating in the experiment is $1 + 1 + \frac{1}{2} + \frac{1}{8} = 2\frac{5}{8}$ and not $+\infty$.

In this note, we take a different route that reconciles probabilities that were initially proposed for the experiment with linear utility functions for money- albeit state dependent ones.

3. The St. Petersburg experiment with a set of countably infinite states of nature and state dependent linear utility of monetary gains

Let $\mathbb{N}$ - the set of natural numbers- denote the countably infinite set of state<u>s</u> of nature (son<u>s</u>), where state of nature (son) n denotes the event that in independent tosses of an unbiased coin, the first head occurs on the n^{th} toss, the probability of which is $(\frac{1}{2})^n$. For son n, the constant average utility of money $u_n > 0$ is defined as follows: there exists a subjective discount factor $\delta \in (0, 1)$ and positive integers r, s such that for n = 1, …, r, $u_n = 1$ and for all non-negative integers t, if n = r + ts + 1, …, r + (t+1)s, $u_n = \delta^{t+1}$. Clearly, this defines for us a profile of (state-dependent) linear utility functions for money given by the infinite sequence $\langle u_n | n \in \mathbb{N} \rangle$.

It is worth noting that since a "state of nature" is an "event"- and hence a subset of a sample space, it is relevant and meaningful along with all other concepts related to it, <u>only</u> in the context of the experiment for which it is defined. Thus, if $n \in \mathbb{N}$ is a son in the context of a performance of the experiment being discussed here, then $u_n > 0$ is the constant average utility money in son 'n' only in the context of the performance of the experiment being discussed here, and not universally. For instance, if son n occurs simultaneously with the totally unrelated event that a free lottery ticket wins a large sum of money, then for a participant in the experiment who also owns a free lottery ticket of this type, it is quite possible that u_n is "different" from the same participant's constant average utility of money if the "free lottery ticket wins a large sum of money". The two events occur in two different experiments, and decision making for the two by the participant may be totally unrelated in time, space and with regard to other relevant qualifiers.

The expected utility of participating in the experiment to a participant with the linear utility profile defined above is $r + \sum_{t=0}^{\infty} \delta^{t+1} s = r + \frac{s\delta}{1-\delta}$.

Thus, $\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n 2^n = \sum_{n=1}^{\infty} u_n = r + \frac{s\delta}{1-\delta}$ which is a strictly positive scalar.

If CE denotes the "certainty equivalent" of the expected utility $\sum_{n=1}^{\infty} u_n$ to a participant in the experiment with the linear utility profile $\langle u_n | n \in \mathbb{N} \rangle$ then CE satisfies the equation:
CE $\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n = \sum_{n=1}^{\infty}(\frac{1}{2})^n u_n 2^n = \sum_{n=1}^{\infty} u_n$.

Now $\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n = \sum_{n=1}^{r}(\frac{1}{2})^n + \sum_{t=0}^{\infty} \delta^{t+1} \sum_{n=r+ts+1}^{r+(t+1)s}(\frac{1}{2})^n = \frac{1}{2}\times 2\times(1- (\frac{1}{2})^r)$
$+\sum_{t=0}^{\infty} \delta^{t+1}(\frac{1}{2})^{r+ts+1} \sum_{j=0}^{s-1}(\frac{1}{2})^j = 1- (\frac{1}{2})^r + \sum_{t=0}^{\infty} \delta^{t+1}(\frac{1}{2})^{r+ts+1} 2(1 - (\frac{1}{2})^s) = 1- (\frac{1}{2})^r + 2(1 - \left(\frac{1}{2}\right)^s)(\frac{1}{2})^{r+1} \delta \sum_{t=0}^{\infty} \delta^t \left(\frac{1}{2}\right)^{ts} = 1- (\frac{1}{2})^r + (1 - \left(\frac{1}{2}\right)^s)(\frac{1}{2})^r \delta \sum_{t=0}^{\infty}(\frac{\delta}{2^s})^t = 1- (\frac{1}{2})^r + (1 - \left(\frac{1}{2}\right)^s)(\frac{1}{2})^r \frac{\delta}{1-\frac{\delta}{2^s}}$.

Since $\delta \in (0, 1)$, $\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n = 1- (\frac{1}{2})^r + (1 - \left(\frac{1}{2}\right)^s)(\frac{1}{2})^r \frac{\delta}{1-\frac{\delta}{2^s}}$ is a strictly positive scalar.

Hence, $CE = \frac{\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n 2^n}{\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n} = \frac{r + \frac{s\delta}{1-\delta}}{1- (\frac{1}{2})^r + (1- \left(\frac{1}{2}\right)^s)(\frac{1}{2})^r \frac{\delta}{1-\frac{\delta}{2^s}}}$, which is a strictly positive scalar.

Thus, an individual with a linear utility profile given by $\langle u_n | n \in \mathbb{N} \rangle$ will be willing to pay no more than $\frac{r + \frac{s\delta}{1-\delta}}{1- (\frac{1}{2})^r + (1- \left(\frac{1}{2}\right)^s)(\frac{1}{2})^r \frac{\delta}{1-\frac{\delta}{2^s}}}$ to participate in the experiment.

To give a numerical example for the suggested formulas, let $r = s = 1$ and $\delta = \frac{1}{2}$.

Then, the expected utility of participating in the experiment to a participant with the above values of the relevant parameters $\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n 2^n = r + \frac{s\delta}{1-\delta} = 2$.

Further, $CE = \frac{\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n 2^n}{\sum_{n=1}^{\infty}(\frac{1}{2})^n u_n} = \frac{r + \frac{s\delta}{1-\delta}}{1- (\frac{1}{2})^r + (1- \left(\frac{1}{2}\right)^s)(\frac{1}{2})^r \frac{\delta}{1-\frac{\delta}{2^s}}} = \frac{2}{1-\frac{1}{2} + (1-\frac{1}{2})(\frac{1}{2})\frac{\frac{1}{2}}{1-\frac{\frac{1}{2}}{2}}} = \frac{2}{\frac{1}{2} + \frac{1}{6}} = 3$.

Hence, a participant with the values of the parameters being r = s = 1 and $\delta = \frac{1}{2}$, will be willing to pay no more than 3 units of money.

4. A more realistic resolution of the paradox

The above analysis was in the context of an unrestricted "thought experiment" without any constraints that reality may impose. Hence, let us recall the immortal words in the poem entitled "Garden of Prosperine" by Algernon Charles Swinburne (https://www.poetryfoundation.org/poems/45288/the-garden-of-proserpine):

"… no life lives for ever;

That dead men rise up never …"

For this reason, and also because no potential participant would want to commit his/her entire future to such an experiment, for any state-dependent linear utility profile $<u_n|n\in \mathbb{N}>$, there is a positive integer N, such that $u_n > 0$ for all $n \leq N$ and $u_n = 0$ for all $n > N$. This is true for any participant with any state-dependent linear utility profile, regardless of how it is defined or what the experiment under consideration is. This is analogous to the point raised by Carl Menger as briefly summarized in two paragraphs of the second column in page 188 of Gasparian etc. (2018) which is reproduced below:

"The next type of practical restrictions that Menger notes is limited playing time, interrupting too long chains of the coin flips.

The limited time allowed D. Brito to interpret the St. Petersburg paradox in terms of G. Becker's theory of time allocation, linking the time and capital constraints at the optimal point of the mathematical problem of the consumer's behavior".

Vivian (2013) is related to issues discussed in the paragraph that immediately follows the two above.

Hence the expected utility of the participant from participating in the St. Petersburg experiment is $\sum_{n=1}^{N}(\frac{1}{2})^n u_n\, 2^n = \sum_{n=1}^{N} u_n$, which is a positive scalar.

Since $\sum_{n=1}^{N}(\frac{1}{2})^n u_n$ is a positive scalar, the "certainty equivalent" corresponding to the expected utility $\sum_{n=1}^{N} u_n$ is the positive scalar $\frac{\sum_{n=1}^{N} u_n}{\sum_{n=1}^{N}(\frac{1}{2})^n u_n}$, which, is also a positive scalar.

Thus, an individual with a linear utility profile given by $<u_n|n\in \mathbb{N}>$ such that for some positive integer N, it is the case that $u_n > 0$ for all $n \leq N$ and $u_n = 0$ for all $n > N$ will be willing to pay no more than $\frac{\sum_{n=1}^{N} u_n}{\sum_{n=1}^{N}(\frac{1}{2})^n u_n}$ and certainly not an unbounded sum of money, to participate in the experiment underlying the St. Petersburg paradox.

To give a simple numerical example illustrating what the formulas yield, suppose N = 6, u_n = 1, for n = 1, …, 6 and $u_n = 0$ for n > 6. The numbers seem to be quite reasonable.

Then, any participant in the experiment with these values of the parameters for the formula determining the certainty equivalent would be willing to pay no more than $\frac{6}{\sum_{n=1}^{6}(\frac{1}{2})^n} = \frac{6}{\frac{63}{64}} \cong$ (approximately) 6 units of money.

5. Conclusion

Whether this is simply the conclusion of the St. Petersburg paradox or the beginning of expected utility (decision) analysis with constant state-dependent linear utility of money as initiated in Lahiri (2024), only time can tell. At the very least, what our discussion above should have conveyed is that the St. Petersburg paradox, is crucially dependent on the choice of perspective for it to be recognized as a paradox, and by no means can the perspective that justifies its paradoxical nature, be considered robust.

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