Distributed leaderless formation maneuvers over directed graphs

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***Abstract*—This paper investigates fully distributed leaderless formation maneuver control over a directed sensing topology. For formation maneuvering, generally, a leader-follower approach is adopted. Leaderless formation maneuvers have no designated leaders, unlike traditional approaches. Some of the leaderless approaches proposed recently in literature considered undirected sensing graphs, which means the agents need to sense in both directions. Also, in these approaches, although the control law implementation is distributed in nature, the design of the control laws requires a stabilizing diagonal matrix design, which makes the design of the control law centralized and knowledge of exact interaction among the agents is required for this. When an agent's sensing capabilities are limited, directed sensing topology becomes crucial. First, we propose control laws for leaderless formation maneuvers in 2-D by manipulating the weights of complex-laplacian for single integrators with directed sensing graphs, and GAS (Global Asymptotic Stability) is established for it. In this approach, only translational, scale, and rotational formation maneuvers are possible. Then, we propose control laws for leaderless affine formation maneuvering for single integrators with directed sensing graphs for 2-D and 3-D, where translational, rotational, scale, and sheer formation maneuvering is possible. Also, an extension for the higher-order integrators is provided using a back-stepping-based design. Constant disturbances are also considered in the system dynamics. Finally, simulations are provided to validate the results.**

Index terms: Formation control, affine formation, leaderless maneuvers, cooperative control

I. Introduction

Controlling robotic swarms has emerged as a prominent area of research within the control and robotics community [1]. Multi-agent formations have a wide range of applications. These include satellite formation flying, search and rescue operations, source seeking, underwater ocean data retrieval, and estimating the gradient of a field, etc. [2], [3], [4], [5].

Achieving and maintaining a specific geometric pattern using distributed control laws, and performing maneuvers maintaining the desired formation shape are two important aspects of formation control. Traditional formation control techniques rely on position measurements of agents, but distributed formation control aims to achieve formation control using onboard sensors of the agents and relative measurements of their neighbors. This approach allows for the establishment of formations even in environments where GPS signals are unavailable [6], [7], [8]. Different types of distributed formation control approaches are classified as displacement-based, distance-based, and bearing-based depending on whether the desired formation shape is specified in terms of displacement, distance, or bearing constraints [7], [9], [10]. With displacement-based control, only translational maneuvers were possible because the displacement constraints exhibit translation invariance [11], [12]. Distance-based formation control enables the execution of both translational and rotational maneuvers [13]–[15], while a bearing-based approach allows for translation and scaling formation maneuvers [9]. [16]–[18] proposed complex laplacian-based formation control where the constraints specifying the formation are translational, rotational, and scale-invariant, which enabled translational, rotational, and scale formation maneuvers. However, this approach is limited to 2-D. Affine formation control is another noteworthy approach to formation control [19]–[23], where the primary goal is to acquire a geometric shape that is an affine transformation of the intended reference shape. In this methodology, the interactions between agents are encapsulated through the utilization of a stress matrix, commonly referred to as a weighted Laplacian or signed Laplacian. Specifically, every agent within the system endeavors to ensure that the weighted aggregate of their neighboring agents' relative positions remains equal to zero. In [19] the necessary and sufficient conditions for affine formation stabilization in $\mathbb{R}^d$ were established. It is possible to achieve translational, scaling, shear, and rotational maneuvers through the control of leaders within the framework of the affine formation control paradigm, as all of these alterations can be regarded as affine transformations [20]–[23]. [20] focused on investigating affine maneuvering strategies for the double integrator and unicycle agents operating within an undirected interaction topology. [21] considered higher-order integrator agents with a directed interaction topology. In [22] the authors put forth a layered affine formation control strategy, wherein various layers delineate the transmission of information among the leaders and the followers. [23], also proposed a layered affine formation strategy for higher order integrator systems, which did not require any acceleration information (control input for double integrator systems) which were necessitated in [20].

However, to accomplish formation maneuvers, past methodologies have entailed the independent control of a small number of leaders, with the followers employing distributed control laws to fulfill the objectives of formation control. This particular approach necessitated an additional control layer and required the leaders to access their position measurements via GPS for autonomous navigation or to be remotely controlled by human operators. Consequently, it is imperative to explore strategies that facilitate formation maneuvering without the presence of a leader, thereby eliminating the requirement for GPS control or position measurements. [24] proposed a method for performing leaderless maneuvers with rigid formations, based on distance measurements. The design parameter used was a mismatch in distance measurements and ensured a constant translational velocity of the formation and rotation

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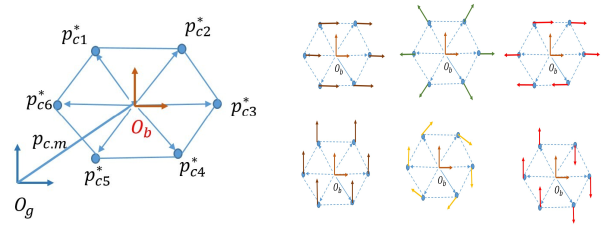
about its centroid. In [25], the positions of the agents were represented using complex numbers from the set $\mathbb{C}$, and by perturbing the weights of the complex Laplacian maneuvering forces were generated to achieve translational, scaling, and rotational leaderless maneuvers for planar formations. In [26], the weights of the stress matrix were perturbed to generate affine formation maneuvers in $\mathbb{R}^d$, $d = 2, 3$. However, the control algorithms proposed in [24], [26], and [25] were for single integrators, and an undirected sensing graph was considered. A directed sensing graph among agents is of paramount importance in order to accommodate the constraints imposed by an agent's sensing capabilities. Also, although the control laws proposed in [25], [26] can be implemented in a distributed manner, the design of the laws requires a stabilizing diagonal matrix whose computation requires information on the overall interaction graph topology, hence making the design of the control laws centralized.

Motivated by the aforementioned factors, we put forth a control strategy aimed at achieving leaderless formation maneuvers completely distributed way without the need for any stabilizing diagonal matrix for its implementation. Also, this strategy takes into account a directed sensing topology between agents, a topic that has yet to be thoroughly investigated in the existing literature on leaderless formation maneuvers. First, we propose control laws for translational, scale, and rotational formation maneuvering by manipulating the weights of a complex Laplacian for planar, i.e., 2-D, agents taking into account a directed interaction topology among the agents. Then control laws for affine formation maneuvering are proposed for $\mathbb{R}^d$, $d = 2, 3$, agents with directed sensing graphs. Unlike [25], [26], we do not need any stabilizing diagonal matrix in our laws, making the design and implementation both decentralized. Also, our analysis takes into account the existence of persistent constant disturbances in the dynamics of agents, in contrast to the setups examined in previous works such as [24]–[26]. Such disturbances can arise from uncertainties in modeling or the presence of external disturbances within the environment. Furthermore, we expand upon the findings obtained for single integrator dynamics to encompass the inclusion of agents represented as higher-order integrators. Although [20] did consider double integrators and unicycles, and [22] addressed higher-order systems, the topology examined in both studies was undirected. In addition, [20] required $d+1$ leaders for formations in $\mathbb{R}^d$, while [22] required at least one leader to possess absolute position information for maneuvering. In contrast, our approach is leaderless. [21] did consider a directed graph topology with higher-order integrator systems, but it also necessitated $d+1$ leaders in $\mathbb{R}^d$. In contrast, our approach considers leaderless digraphs. In [20]–[22] a stabilizing diagonal matrix is needed which will require centralized information while our algorithm can be implemented without such need.

Remark 1. *Although a preliminary version of this work is published in [27] which considered directed graphs, the control laws in [27] also required a stabilizing diagonal matrix, and hence the design of the laws required centralized computation. In contrast, the laws proposed in this article have both design and implementation distributed in nature. Also, we have included the complex laplacian-based leaderless formation maneuvering, which was not there in [27]. In addition, for the design of higher order systems to avoid computational and noise amplification issues, we have adopted a command filter-based back-stepping design here, whereas in [27], a direct derivative of the virtual control laws was considered.*



(a) Reference Shape (b) Affine motions in 2-D

Fig. 1. (a) Example of a reference shape in 2-D. $p^* = (\mathbf{1}_6 \otimes p_{c.m.}) + [p_{c1}^{*\,T}\ p_{c2}^{*\,T} \dots p_{c6}^{*\,T}]^T$. (b) Different affine motions are possible in 2-D. Brown arrows represent 2 translations, green arrows represent scale change, yellow arrows are for rotation and red arrows represent 2 types of shear motion.

Notations: I is used to represent the identity matrix. $\mathbf{1_n}$ represents a vector composed entirely of ones. $||.||$ is employed to indicate the induced 2-norm for a matrix and the 2-norm for a vector. The symbol $|.|$ is used to denote the absolute value of a real or complex number. The spaces $\mathcal{L}_2$ and $\mathcal{L}_\infty$ refer to the space of square-integrable functions and the space of bounded functions, respectively. A^H, $A \in \mathbb{C}^{n\times n}$, represents the transpose of complex conjugate of matrix A. Ker$\{A\}$ and Im$\{A\}$ represents the null space and image of matrix A, respectively. The symbol ι represents the unit complex number. $\mathbb{N}$ is the set of natural numbers.

II. Preliminaries and Problem Formulation

A. Graph Theory

The interaction between the agents is represented through a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where the node set is denoted as $\mathcal{V} = \{1, 2, \dots, n\}$, and the edge set is denoted as $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. In the case of a directed graph, the edge set consists of directed edges (j, i), where node j is the tail node and node i is the head node. Node j is referred to as the in-neighbor of node i, while node i is referred to as the out-neighbor of node j. The in-neighbor set for agent i is denoted as $\mathcal{N}_i := \{j | (j, i) \in \mathcal{E}\}$. It should be noted that in a directed graph, the presence of $(j, i) \in \mathcal{E}$ does not necessarily imply the presence of $(i, j) \in \mathcal{E}$. For a directed graph, the definitions of κ-*reachable*, κ-*rooted*, and κ-*spanning tree* are taken from [19].

Definition 1.1: (κ-*reachable*) A node v is called κ-*reachable* from a non-singleton set of nodes, $\mathcal{U}$, if there exist κ disjoint paths from a node in $\mathcal{U}$ to node v.

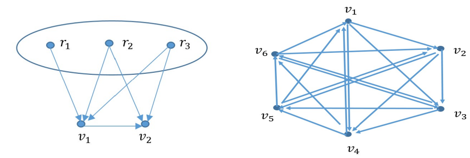
Alternately stated, the above definition implies that we can find a path from a node in $\mathcal{U}$ to node v even after removing any $\kappa - 1$ nodes from the graph, except node v.

Definition 1.2: (κ-*rooted*) A digraph is called κ-rooted if there exists a subset of nodes, called root nodes, from which all other nodes are κ reachable.

Definition 1.3: (κ*-spanning tree*) For a digraph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with a set $\mathcal{U}=\{u_1,u_2,\ldots,u_k\}\subset\mathcal{V}$, a spanning κ-tree is a subgraph $\mathcal{T}(\mathcal{V},\mathcal{E}_0)$, such that
(1) Every node $u\in\mathcal{U}$ has no in-neighbours,
(2) Every node $v\notin\mathcal{U}$ has κ in-neighbours,
(3) Every node $v\notin\mathcal{U}$ is κ-reachable from $\mathcal{U}$.

We now state an important existing result.

Lemma 1. *[19, Lemma 2.1] A digraph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is κ-rooted if and only if it has a κ-spanning tree.*



(a) 3-rooted graph (b) 3-rooted directed regular graph

Fig. 2. (a) Example of a 3-rooted graph. All nodes are 3-reachable from root set: $\mathcal{R}=\{r_1,r_2,r_3\}$. (b) Example of a directed graph which is 3-rooted. Any three nodes can be considered as root nodes, and every node has 3 in-neighbours

B. Preliminaries for Complex Laplacian-based control

We have n planar agents for complex Laplacian-based formation control, i.e., agents are in 2-D.

1) Complex Laplacian matrix: A Complex Laplacian matrix $\mathcal{L}\in\mathbb{C}^{n\times n}$ is a matrix whose entries are complex numbers. Let $w_{ij}\in\mathbb{C}\neq 0$, be the weight associated with the edge $(i,j)\in\mathcal{E}$, then we have the complex laplacian $\mathcal{L}$ whose entries are given by $\mathcal{L}(i,j)=-w_{ij}$ for $(i,j)\in\mathcal{E}$, otherwise $\mathcal{L}(i,j)=0$ for $(i,j)\notin\mathcal{E}$, and $\mathcal{L}(i,i)=\sum_{k\in\mathcal{N}_i}w_{ik}$. Also, note that $\mathcal{L}\mathbf{1}_n=0$.

2) Desired shape: The 2-D position of every agent $i\in\mathcal{V}$ is a complex number $p_i\in\mathbb{C}$, where the real and imaginary parts describe the two coordinates of the Euclidean plane. All the positions p_i of the agents are stacked in a single vector $p\in\mathbb{C}^n$, which is called a *configuration*. The pair $(\mathcal{G},p)$, is called a *framework* $\mathcal{F}$, where the position of each agent p_i is attached with the node $i\in\mathcal{V}$.

The reference shape p^* is an arbitrary configuration of interest, which is given by

$$p^*=p_{c.m.}\mathbf{1}_n+p_c^*, \tag{1}$$

where $p_{c.m.}\in\mathbb{C}$ is the position of the center of mass of the configuration, and p_c^* gives the shape/ appearance to the formation starting from $p_{c.m.}$. For simplicity and without loss of generality, we have $p_{c.m.}=0$, and hence $p^*=p_c^*$.

Now, we define the desired formation shape in the case of Complex Laplacian-based formation control.

Definition 1. *(Desired formation shape) The formation or the framework is at the desired shape when we have*

$$p\in\mathcal{M}:=\{p=k_1\mathbf{1}_n+k_2p^*|k_1,k_2\in\mathbb{C}\}. \tag{2}$$

Here, k_1 accounts for the translation, and k_2 accounts for the scaling and rotation of the reference shape p^*. However, in our case, we are not interested in a static formation shape as in [16]. Rather, we are interested in formation maneuvering, which refers to a trajectory in $\mathcal{M}$.

Remark 2. *For the static complex Laplacian-based formation control reported in [16], the weights w_{ij} of the complex laplacian must satisfy $\mathcal{L}p^*=0$, making p^* to be in the null space of $\mathcal{L}$. Note that this condition is satisfied if and only if the graph is 2-rooted [16]. As $\mathbf{1}_n$ and p^* are in the null space of $\mathcal{L}$, we have $\mathcal{L}$ has two zero eigenvalues and the corresponding eigenvectors are given by $\mathbf{1}_n$ and p^*.*

C. Preliminaries for Affine formation control

1) Signed-laplacian: A signed-Laplacian matrix (or stress matrix) associated with graph $\mathcal{G}$ is denoted by $\mathcal{L}\in\mathbb{R}^{n\times n}$ whose entries can be positive and negative real numbers. If $w_{ij}\neq 0$ is the weight associated with an edge (j,i), the (i,j)-th entry of the signed-Laplacian $\mathcal{L}$ is given by $\mathcal{L}(i,j)=-w_{ij}$ if $j\in\mathcal{N}_i$, otherwise it is 0. The diagonal entries are given by $\mathcal{L}(i,i)=\sum_{k\in\mathcal{N}_i}w_{ik}$. For a digraph, $\mathcal{L}$ is usually not a symmetric matrix, but it still satisfies the condition $\mathcal{L}\mathbf{1}_n=0$, like its undirected counterpart.

2) Desired affine trajectory: Let us consider a set of n agents in the Euclidean space $\mathbb{R}^d$, where d can be either 2 or 3. The position of each agent $i\in\mathcal{V}$ is denoted as $p_i\in\mathbb{R}^d$. We can collect all the positions p_i in a vector $p\in\mathbb{R}^{nd}$, which is referred to as the configuration. A framework $\mathcal{F}$ is defined as the pair $(\mathcal{G},p)$, where $\mathcal{G}$ represents the underlying control/sensing topology among the agents. A shape of interest or a reference shape $p^*\in\mathbb{R}^{nd}$ (as depicted in Figure 1(a)) can be arbitrarily given:

$$p^*=(\mathbf{1}_n\otimes p_{c.m.})+p_c^*$$

where the center of mass is denoted as $p_{c.m.}\in\mathbb{R}^d$, and the shape of the formation is denoted as $p_c^*\in\mathbb{R}^{nd}$, with respect to $p_{c.m.}$. The desired shape is constructed based on the reference shape p^*, as described by the following definition:

Definition 2. *(Desired affine trajectory) The desired shape is achieved when $p\in\mathcal{S}:=\{p=(I_n\otimes A)p^*+(\mathbf{1}_n\otimes b)|A\in\mathbb{R}^{d\times d},b\in\mathbb{R}^d\}$, i.e., the desired shape is an affine transformation of the reference shape p^*, hence it is called affine formation control. Affine maneuver indicates the desired shape is not a static one as in [19]; rather, it can be a trajectory in $\mathcal{S}$.*

Definition 3. *(Generic Configuration) A configuration, p, is generic if there does not exist a non-zero polynomial $g(y_1,\ldots,x_{nd})$, having integer coefficients, such that $g(p_1^1,\ldots,p_1^d,\ldots,p_n^1,\ldots,p_n^d)=0$ where p_k^l is the l-th element of p_k [19]. Physically interpreted, generic configurations do not have any degeneracy, i.e., points on a line in $\mathbb{R}^2$ and planar formations in $\mathbb{R}^3$ are examples of non-generic configurations.*

Remark 3. *For static affine formation control as reported in [19], the weights w_{ij} of the signed Laplacian, $\mathcal{L}$, are designed* *in a way so that $(\mathcal{L}\otimes I_d)p^* = 0$, thereby making the reference shape p^* an eigenvector of $\mathcal{L}\otimes I_d$ corresponding to the zero eigenvalue. When p^* is an eigenvector, the desired affine shape $p \in \mathcal{S}$ becomes an eigenvector of $\mathcal{L}\otimes I_d$ with zero eigenvalue, implying that $\mathcal{S}$ is the null space/kernel of $\mathcal{L}\otimes I_d$ [19].*

D. Problem formulation

1) Complex Laplacian-based formation maneuvering: Single integrator kinematics of the agents is given by:

$$\dot{p}_i = u_i + \delta_i, \quad i \in \mathcal{V}, \tag{3}$$

where $\delta_i \in \mathbb{C}$ is the constant disturbance input to agent i, and $u_i \in \mathbb{C}$ is its control input.

Assumption 1:

(a) The desired configuration (p) is generic.

(b) The directed sensing graph ($\mathcal{G}(\mathcal{V},\mathcal{E})$) among the agents is 2-rooted with each root node having at least 2 in-neighbors.

Problem 1: With agents whose dynamics are represented by (3), sensing over a directed graph, and subject to Assumption 1, the objective is to design a control input u_i for agent $i \in \mathcal{V}$ such that a desired trajectory can be achieved asymptotically. In other words, we aim for $p(t)$ and $\dot{p}(t)$ to converge to $\mathcal{M}$ as t approaches infinity.

2) Affine formation maneuvering: Single integrator kinematics of the agents is given by:

$$\dot{p}_i = u_i + \delta_i, \quad i \in \mathcal{V}, \tag{4}$$

where $\delta_i \in \mathbb{R}^d$ is the constant disturbance input to agent i, and $u_i \in \mathbb{R}^d$ is its control input, $d = 2$ or 3 is the dimension of ambient space.

Dynamics of ρ-th order integrator is given by:

$$\begin{aligned} \dot{y}_{i,l} &= y_{i,l+1} + \delta_{i,l}, \quad l \leq \rho - 1 \\ \dot{y}_{i,\rho} &= u_i + \delta_{i,\rho}, \quad i \in \mathcal{V} \end{aligned} \tag{5}$$

where $u_i \in \mathbb{R}^d$ is the control input for i-th agent, and $\delta_{i,l} \in \mathbb{R}^d$ are the disturbances terms. We are now well placed to state the main problem addressed in this paper, along with the key assumptions.

Assumption 2:

(a) The desired configuration (p) is generic.

(b) The directed sensing graph ($\mathcal{G}(\mathcal{V},\mathcal{E})$) among the agents is $d+1$-rooted with each root node having at least $d+1$ in-neighbours where d is the dimension of the ambient space.

(c) The number of agents, n, satisfies $n \geq d+2$.

Remark 4. *For static affine formation control over a digraph, the necessary and sufficient condition was that the digraph should be $d+1$-rooted [19]. It is important to note that for a $d+1$-rooted graph, the root nodes need not have any in-neighbours. The extra condition of $d+1$ in-neighbors of the root nodes is imposed to solve the problem of affine maneuvering over a directed topology. Part (c) of the assumption is used so that part (b) can be satisfied.*

Remark 5. *Note that the additional condition does not change the d+1- rooted structure of the graph because from Definition 1.3 and Lemma 1, it is clear that if graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is $d+1$-rooted, it contains a $d+1$-spanning tree $\mathcal{T}(\mathcal{V},\mathcal{E}_0)$ and after addition of extra edges to the root nodes, the new graph $\mathcal{G}(\mathcal{V},\bar{\mathcal{E}})$ will also contain the same $d+1$-spanning tree $\mathcal{T}(\mathcal{V},\mathcal{E}_0)$, thereby ensuring that it is $d+1$-rooted.*

Remark 6. *The assumption used here is weaker than that used in [26] for solving the undirected case, i.e., the assumption of a universally rigid interaction topology. Universally rigid configurations ensure $d+1$ connectedness and hence are always $d+1$-rooted [19, Remark 2.1], while the converse is not always true.*

Problem 2: With agents whose dynamics are represented by (4) or (5), sensing over a directed graph and subject to Assumption 2, the objective is to design control input u_i for agent i to achieve an asymptotic convergence of the desired affine trajectory, i.e., $p(t) \to \mathcal{S}$ and $\dot{p}(t) \to \mathcal{S}$ as $t \to \infty$.

III. Main results

In this section, we first provide the key idea used that makes the leaderless formation maneuvers for directed graphs tractable. Then, we propose the control laws to solve Problem 1 and Problem 2 and analyze the stability.

A. Key idea used for the directed leaderless maneuvers

Here, we first provide an intuitive idea of why the control laws presented in [25], [26] for agents sensing over undirected graphs will not work for the directed sensing graph. In [25], the formation maneuvers are generated by perturbing the edge weights for complex Laplacian. Similarly, in [26], the signed Laplacian/stress matrix weights were perturbed in order to induce affine motions. The assumption of only a d-rooted graph, which is the necessary and sufficient condition for static formation control based on complex-laplacian [16], [18], and the assumption of only a $d+1$-rooted graph, which is the necessary and sufficient condition for static affine formation control [19] is not sufficient for generating formation maneuvers adopting a similar approach of perturbing the edge weights as in [25], [26]. This is because the alteration of weights leads to changes in the forces acting on an agent, which are in equilibrium, thereby causing it to move in a particular direction [25], [26]. However, in a directed graph, sensing is unidirectional, resulting in some nodes not having any in-neighbors. Unlike the case for a d-rooted undirected graph considered in [25], for a directed d-rooted graph, the root nodes do not have any in-neighbors, so they will not experience any force by modification of the edge weights in case of complex-laplacian. Hence, we need to modify the assumption, and we have added another condition to the d-rooted graph, i.e., each root node will also need to have at least d in neighbors. Similarly, for affine formation maneuvering, a $d+1$-rooted graph where the root node must have at least $d+1$ in-neighbors assumption is used. This condition entails the addition of extra edges to the sensing graph, thus enabling a solution to the problem of directed sensing topology. In Fig.2(a), the root nodes do not possess any in-neighbors. Consequently, perturbing edge weights will not impact the force exerted on these root nodes, leading to a lack of movement. Directed 3-regular graphs, exemplified in Fig.2(b), inherently satisfy these requirements as each agent can access measurements from three neighbors. The topology in Fig.2(a) does not facilitate affine maneuvers, whereas the one in Fig.2(b) does. Importantly, the stipulated condition does not restrict the graph to a regular digraph like the one depicted in Fig.2(b); any arbitrary $d+1$-rooted digraph meeting the additional conditions specified by Assumption 1 will suffice.

B. Leaderless formation maneuvers by manipulating complex laplacian

In this section, we solve Problem 1, i.e., we discuss how to generate only translational, rotational, and scale leaderless formation maneuvers by manipulating weights of complex Laplacian for directed graphs. Unlike the case in [25], where such motions can be generated by manipulating weights of complex Laplacian for undirected graphs, directed graphs are considered here. Also, we do not need to know global interaction topology among the agents as our proposed law does not use the stabilizing diagonal matrix K as used in [25].

1) Control strategy for single-integrators: The following control law is proposed for complex laplacian-based formation maneuvering ($\forall i \in \mathcal{V}, j \in \mathcal{N}_i$)

$$u_i = -\sum_{j\in\mathcal{N}_i} \bar{w}_{ij}\zeta_i + \sum_{i\in\mathcal{N}_j} \bar{w}_{ji}\zeta_j - \frac{1}{h}\sum_{j\in\mathcal{N}_i} \hat{w}_{ij}(p_j - p_i) - \hat{\delta}_i,$$

$$\dot{\zeta}_i = -a\zeta_i + \sum_{j\in\mathcal{N}_i} w_{ij}(p_j - p_i) \tag{6a}$$

$$\dot{\hat{\delta}}_i = \sum_{j\in\mathcal{N}_i} \bar{w}_{ij}\zeta_i - \sum_{i\in\mathcal{N}_j} \bar{w}_{ji}\zeta_j + \frac{1}{h}\sum_{j\in\mathcal{N}_i} \hat{w}_{ij}(p_j - p_i) + v_i, \tag{6b}$$

where $\bar{w}_{ij}$ is the complex conjugate of w_{ij}. It is important to observe that here we have $\zeta_i, p_i \in \mathbb{C},\ \forall i \in \mathcal{V}$, and $\zeta_i \in \mathbb{C}$ is an auxiliary state variable. Also, the information $w_{ji}\zeta_j$ has to be transmitted from the out-neighbors of agent i, which are sensing the agent i. The velocity information, i.e., v_i, can be obtained by integrating acceleration measurements obtained by accelerometer sensors.

The complex weights $w_{ij},\ (i,j) \in \mathcal{E}$ are calculated from the following equation [16],which was required for static formation control,

$$\sum_{j\in\mathcal{N}_i} w_{ij}(p_i^* - p_j^*) = 0. \tag{7}$$

For calculating $\hat{w}_{ij}$ the following equation is used

$$\sum_{j\in\mathcal{N}_i} \hat{w}_{ij}(p_i^* - p_j^*) = v_i^*, \tag{8}$$

where v_i^* is the desired velocity of the i-th agent. The perturbation weight $\hat{w}_{ij}$ is responsible for generating maneuvering force. Note that as we have at least 2 neighbors for each agent i, the solution of (8) always exists. The desired velocity (v^*) can be split into translation, rotation, and scale components, i.e., $v^* = v_t^*\mathbf{1}_n + \gamma p^* + iwp^*$, where $v_t^* \in \mathbb{C}$ is the common translational velocity, $\gamma \in \mathbb{R} > 0$ keeps track of whether the formation grows or contracts, i.e., it controls the scale, and $w \in \mathbb{R}$ controls the angular speed and hence controls the rotational motion of the formation. All agents' desired reference velocities are stacked together to obtain $v^* := [v_1^* \ \dots \ v_n^*]^T$.

Now, we define a new state variable $x := [p^T \ \zeta^T]^T \in \mathbb{C}^{2n}$. In the compact form (6) can be written as:

$$\dot{x} = \begin{bmatrix}\dot{p}\\ \dot{\zeta}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{0} & -\mathcal{L}^H\\ \mathcal{L} & -aI_n\end{bmatrix}}_{\mathcal{A}}\begin{bmatrix}p\\ \zeta\end{bmatrix} - \frac{1}{h}\underbrace{\begin{bmatrix}\hat{\mathcal{L}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}}_{\hat{\mathcal{A}}}\begin{bmatrix}p\\ \zeta\end{bmatrix} + \underbrace{\begin{bmatrix}\tilde{\delta}\\ \mathbf{0}\end{bmatrix}}_{\tilde{\Delta}},$$

$$\implies \dot{x} = \mathcal{A}x - \frac{1}{h}\hat{\mathcal{A}}x + \tilde{\Delta}, \tag{9a}$$

$$\dot{\hat{\delta}} = \mathcal{L}^H\zeta + \frac{1}{h}\hat{\mathcal{L}}p + v, \tag{9b}$$

where $\hat{\delta} = [\hat{\delta}_1^T \ \hat{\delta}_2^T \ \dots \ \hat{\delta}_n^T]^T, \delta = [\delta_1^T \ \delta_2^T \ \dots \ \delta_n^T]^T, \tilde{\delta} = \delta - \hat{\delta}, u = [u_1^T \ u_2^T \ \dots \ u_n^T]^T, v = [v_1^T \ v_2^T \ \dots \ v_n^T]^T$, and $\mathcal{A} := \begin{bmatrix}\mathbf{0} & -\mathcal{L}^H\\ \mathcal{L} & -aI_n\end{bmatrix}$ and $\hat{\mathcal{A}} := \begin{bmatrix}\hat{\mathcal{L}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$, and $\tilde{\Delta} := \begin{bmatrix}\tilde{\delta}\\ \mathbf{0}\end{bmatrix}$. Here, $\mathcal{L}^H$ denotes the transpose of complex conjugate of the matrix $\mathcal{L}$. We may rewrite $\hat{\mathcal{L}} = \hat{\mathcal{L}}_t + \hat{\mathcal{L}}_r + \hat{\mathcal{L}}_s$, where $\hat{\mathcal{L}}_t, \hat{\mathcal{L}}_r$, and $\hat{\mathcal{L}}_s$ denotes the translational, rotational, and scale component of $\hat{\mathcal{L}}$.

Lemma 2. *The eigenvalues of $\mathcal{A}$ are given by $\lambda = \frac{-a\pm\sqrt{a^2-4\sigma}}{2}$, where σ is an eigenvalue of $\mathcal{L}^H\mathcal{L}$, and $\mathcal{A}$ has two zero eigenvalues whose corresponding eigenvectors are $[\Omega^T \ \mathbf{0}^T]^T \in \mathbb{C}^{2n}$ where $\Omega \in \mathbb{C}^n$ is the eigenvector corresponding to the zero eigenvalue of $\mathcal{L}^H\mathcal{L}$.*

Proof. Let $[\Omega^T \ \bar{\Omega}^T]^T \in \mathbb{C}^{2n}$ be an eigenvector of $\mathcal{A}$, and λ be the corresponding eigenvalue. Hence, $\mathcal{A}[\Omega^T \ \bar{\Omega}^T]^T = \begin{bmatrix}\mathbf{0} & -\mathcal{L}^H\\ \mathcal{L} & -aI_n\end{bmatrix}\begin{bmatrix}\Omega\\ \bar{\Omega}\end{bmatrix} = \lambda\begin{bmatrix}\Omega\\ \bar{\Omega}\end{bmatrix}$. Hence, $-\mathcal{L}^H\bar{\Omega} = \lambda\Omega$, and $\mathcal{L}\Omega - a\bar{\Omega} = \lambda\bar{\Omega}$, implies $-\mathcal{L}^H\mathcal{L}\Omega = -\lambda(\lambda + a)\Omega$, implies $\Omega \in \mathbb{C}^n$ is an eigenvector of $\mathcal{L}^H\mathcal{L}$ and the corresponding eigenvalue is $\sigma := -\lambda(\lambda+a)$. Hence, $\lambda = \frac{-a\pm\sqrt{a^2-4\sigma}}{2}$. Note that eigenvalues of $\mathcal{L}^H\mathcal{L}$ are real and non negative. As, the nullity and null space of $\mathcal{L}^H\mathcal{L}$ is same as that of $\mathcal{L}$, we have 2 zero eigenvalues of $\mathcal{A}$. Also, when $\lambda = 0$, $\mathcal{L}\Omega = 0$, implying $\bar{\Omega} = 0$. □

2) Stability analysis: Here, we present our main theorem regarding the leaderless translation, scale, and rotation formation maneuvering. From Lemma 2, we have $[\Omega^T \ \mathbf{0}^T]^T \in \mathbb{C}^{2n}$ is an eigenvector corresponding to the zero eigenvalues of $\mathcal{A}$, and we have $\Omega \in \mathcal{M} = \text{span}\{\mathbf{1}_n, p^*\}$. The following Lemma will be useful in our future analysis.

Lemma 3. *For a vector $v = [v_1^T \ v_2^T]^T \in \mathbb{C}^{2n}$, when $v_1 \in \mathcal{M}$, we have $\hat{\mathcal{A}}v = [v_3^T \ 0^T]^T$ where $v_3 \in \mathcal{M}$.*

Proof. As we have $v_1 \in \mathcal{M}$, we can write $v_1 = c_1\mathbf{1}_n + c_2p^*$, where $c_1, c_2 \in \mathbb{C}$. Now we have

$$\hat{\mathcal{A}}v = \begin{bmatrix}\hat{\mathcal{L}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}\hat{\mathcal{L}}v_1\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}v_3\\ \mathbf{0}\end{bmatrix}. \tag{10}$$

Now we have $\hat{\mathcal{L}}v_1 = c_1\hat{\mathcal{L}}\mathbf{1}_n + c_2\hat{\mathcal{L}}p^* = 0 + c_2(\hat{\mathcal{L}}_t p^* + \hat{\mathcal{L}}_r p^* + \hat{\mathcal{L}}_s p^*) = c_2(v_t^*\mathbf{1}_n + \gamma p^* + \iota\omega p^*) = c_2 v_t^*\mathbf{1}_n + c_2(\gamma + \iota\omega)p^* := v_3 \in \mathcal{M}$. □

Theorem 1. *For a group of single integrators described by* (3) *and the control law* (6), *subject to Assumption 1, when the control gain parameter, h, is sufficiently high, then* $p(t) \to \mathcal{M}$ *and* $\dot{p}(t) \to -\frac{1}{h}\hat{\mathcal{L}}p(t) \to \mathcal{M}$ *as* $t \to \infty$.

Proof. Let us consider the subspace $\mathcal{M}^{\perp}$ to be the orthogonal complement of $\mathcal{M}$. The dimensions of $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are 2 and $n-2$, respectively, over the complex field. Here, $\mathcal{M} = \text{Ker}\{\mathcal{L}\}$. Also, we have $\text{Ker}\{\mathcal{L}\} = \text{Ker}\{\mathcal{L}^H\mathcal{L}\}$, and hence, $\mathcal{M} = \text{Ker}\{\mathcal{L}^H\mathcal{L}\}$. Now, we add zeros to these subspaces to extend them and get

$$\hat{\mathcal{M}} := \text{Im}\left(\begin{bmatrix}\mathcal{B}_{\mathcal{M}} \\ \mathbf{0}_{n\times 2}\end{bmatrix}\right),\ \hat{\mathcal{M}}^{\perp} := \text{Im}\left(\begin{bmatrix}\mathcal{B}_{\mathcal{M}^{\perp}} \\ \mathcal{B}_{\mathcal{W}}\end{bmatrix}\right).$$

where $\{\mathcal{B}_{\mathcal{M}}\}$, $\{\mathcal{B}_{\mathcal{M}^{\perp}}\}$, and $\{\mathcal{B}_{\mathcal{W}}\}$ are a set of basis for $\mathcal{M}$, $\mathcal{M}^{\perp}$, and $\mathcal{W}$, respectively. Here, $\mathcal{W} \subset \mathbb{C}$ having a dimension of $n-2$. Hence, $\hat{\mathcal{M}}, \hat{\mathcal{M}}^{\perp} \subset \mathbb{C}^{2n}$, having dimension of 2 and $2n-2$, respectively. We may observe that these two subspaces are mutually orthogonal, i.e., for any vector $s \in \hat{\mathcal{M}}$ and $\bar{s} \in \hat{\mathcal{M}}^{\perp}$ we have $\langle s, \bar{s}\rangle = 0$. Also, $\hat{\mathcal{M}} \oplus \hat{\mathcal{M}}^{\perp} = \mathbb{C}^{2n}$. Now the projection matrix for a vector $v \in \mathbb{C}^{2n}$ to the subspace $\hat{\mathcal{M}}$ is denoted by $P_{\hat{\mathcal{M}}} := \begin{bmatrix}P_{\mathcal{M}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$, where $P_{\mathcal{M}}$ is the projection for a vector $v' \in \mathbb{C}^n$ to the subspace $\mathcal{M}$. Also, from Lemma 2, it is clear that $\hat{\mathcal{M}}$ is the kernel of $\mathcal{A}$. Now, we can rewrite $x = [p^T\ \zeta^T]^T \in \mathbb{C}^{2n}$ and $\tilde{\Delta}$ as

$$x = P_{\hat{\mathcal{M}}}x + P_{\hat{\mathcal{M}}^{\perp}}x = x_{\parallel} + x_{\perp},$$
$$\tilde{\Delta} = P_{\hat{\mathcal{M}}}\tilde{\Delta} + P_{\hat{\mathcal{M}}^{\perp}}\tilde{\Delta} = \tilde{\Delta}_{\parallel} + \tilde{\Delta}_{\perp},$$

where $x_{\parallel}$ and $x_{\perp}$ are the components of $x \in \mathbb{C}^{2n}$ along $\hat{\mathcal{M}}$ and $\hat{\mathcal{M}}^{\perp}$, respectively. Hence, p and $\hat{\delta}$ are written as

$$p = P_{\mathcal{M}}p + P_{\mathcal{M}^{\perp}}p = p_{\parallel} + p_{\perp} \quad (11a)$$
$$\hat{\delta} = P_S\hat{\delta} + P_{S^{\perp}}\hat{\delta} = \hat{\delta}_{\parallel} + \hat{\delta}_{\perp}, \quad (11b)$$

where $p_{\parallel}, \hat{\delta}_{\parallel}, p_{\perp}, \hat{\delta}_{\perp}$ are the components of $p, \hat{\delta} \in \mathbb{C}^n$ along $\mathcal{M}$ and $\mathcal{M}^{\perp}$, respectively. Observe that $x_{\parallel} = [p_{\parallel}^T\ \mathbf{0}^T]^T \in \mathbb{C}^{2n}$, and $x_{\perp} = [p_{\perp}^T\ \zeta^T]^T \in \mathbb{C}^{2n}$. From (9), the closed loop dynamics is

$$\dot{x} = \mathcal{A}x - \frac{1}{h}\hat{\mathcal{A}}x + \tilde{\Delta}, \quad (12a)$$
$$\dot{\hat{\delta}} = -\mathcal{L}^H\zeta + \frac{1}{h}\hat{\mathcal{L}}p + v. \quad (12b)$$

We consider (12a) as a nominal system with input with $\tilde{\Delta}$ being the input. First, we show that the error term, $\tilde{\Delta}$, goes to zero exponentially. From (12b) and (12a) , $\tilde{\delta}$ has dynamics:

$$\dot{\tilde{\delta}} = \dot{\delta} - \dot{\hat{\delta}} = -\dot{\hat{\delta}} = -\mathcal{L}^H\zeta - \frac{1}{h}\hat{\mathcal{L}}p - v$$
$$= -\tilde{\delta} \implies \tilde{\delta}(t) = \exp(-I_{n\times n}t)\tilde{\delta}(t_0). \quad (13)$$

From (13), it is clear that $\tilde{\delta}$ goes to zero exponentially, and hence $\tilde{\Delta}$ goes to zero exponentially. Now, the dynamics of $x_{\perp}$ is given by:

$$\dot{x}_{\perp} = P_{\hat{\mathcal{M}}^{\perp}}\mathcal{A}x - \frac{1}{h}P_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}x + \tilde{\Delta}_{\perp}$$
$$= P_{\hat{\mathcal{M}}^{\perp}}\mathcal{A}(x_{\parallel} + x_{\perp}) - \frac{1}{h}P_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}(x_{\parallel} + x_{\perp}) + \tilde{\Delta}_{\perp}. \quad (14)$$

We will now consider only the nominal system (without input $\tilde{\Delta}_{\perp}$) and show that $x_{\perp}(t) \to 0$ and as $t \to \infty$. As $x_{\parallel}$ is in the Null space of $\mathcal{A}$, we have $\mathcal{A}x_{\parallel} = 0$. Also,

$$P_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}x_{\parallel} = \begin{bmatrix}P_{\mathcal{M}^{\perp}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\hat{\mathcal{L}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}p_{\parallel} \\ \zeta_{\parallel}\end{bmatrix} = \begin{bmatrix}P_{\mathcal{M}^{\perp}}\hat{\mathcal{L}}p_{\parallel} \\ \mathbf{0}\end{bmatrix}.$$

Also, from Lemma 3 we have $\hat{\mathcal{L}}p_{\parallel} \in \mathcal{M}$. Hence, $P_{\mathcal{M}^{\perp}}\hat{\mathcal{L}}p_{\parallel} = 0$, implying $P_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}x_{\parallel} = 0$. Hence, (14) becomes

$$\dot{x}_{\perp} = P_{\hat{\mathcal{M}}^{\perp}}\mathcal{A}x_{\perp} - \frac{1}{h}P_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}x_{\perp}. \quad (15)$$

Now, let a basis for $\hat{\mathcal{M}}$ be given by the columns of $\mathcal{B}_{\hat{\mathcal{M}}} := [v_1\ v_2] \in \mathbb{C}^{2n\times 2}$. We can obtain a set of vectors orthogonal to the columns of $\mathcal{B}_{\hat{\mathcal{M}}}$, which form a basis for the space $\hat{\mathcal{M}}^{\perp}$ and let these vectors be given by the columns of $\mathcal{B}_{\hat{\mathcal{M}}^{\perp}} = [u_1\ u_2\ \dots\ u_{2n-2}]$. Now we define $T^{-1} = [\mathcal{B}_{\hat{\mathcal{M}}}\ \mathcal{B}_{\hat{\mathcal{M}}^{\perp}}] \in \mathbb{C}^{2n\times 2n}$, also these basis vectors are chosen such that $TP_{\hat{\mathcal{M}}^{\perp}}T^{-1} = \begin{bmatrix}\mathbf{0}_{2\times 2} & \mathbf{0} \\ \mathbf{0} & I_{2n-2\times 2n-2}\end{bmatrix}$. Now, we apply change of coordinates, T, to x, i.e.,

$$[z_1^T\ z_2^T]^T = Tx = T(x_{\parallel} + x_{\perp}).$$

Here, $z_1 \in \mathbb{C}^2$ and $z_2 \in \mathbb{R}^{2n-2}$. Observe that $Tx_{\parallel} = [z_1^T\ 0^T]^T$ and $Tz_{\perp} = [0^T\ z_2^T]^T$. Also, $T\mathcal{A}T^{-1} = \begin{bmatrix}\mathcal{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_{22}\end{bmatrix}$, where $\mathcal{A}_{11} \in \mathbb{C}^{2\times 2}$ sharing the zero eigenvalues of $\mathcal{A}$, and $\mathcal{A}_{22} \in \mathbb{C}^{2n-2\times 2n-2}$ which shares all eigenvalues of $\mathcal{A}$ with negative real parts. Applying the same co-ordinate change to (15) and obtain

$$\frac{\text{d}}{\text{d}t}\begin{bmatrix}0 \\ z_2\end{bmatrix} = TP_{\hat{\mathcal{M}}^{\perp}}\mathcal{A}T^{-1}\begin{bmatrix}0 \\ z_2\end{bmatrix} - \frac{1}{h}TP_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}T^{-1}\begin{bmatrix}0 \\ z_2\end{bmatrix}. \quad (16)$$

Now, form (27) and using the fact $TP_{\hat{\mathcal{M}}^{\perp}}\mathcal{A}T^{-1} = (TP_{\hat{\mathcal{M}}^{\perp}}T^{-1})(T\mathcal{A}T^{-1}) = \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_{22}\end{bmatrix}$, we have

$$\dot{z}_2 = \mathcal{A}_{22}z_2 - \frac{1}{h}(TP_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}T^{-1})^{\dagger}z_2, \quad (17)$$

where the $\dagger$ symbol indicates the last $2n-2 \times 2n-2$ block is taken because it is compatible with the size of z_2. Now, consider the Lyapunov candidate $V = z_2^TQz_2$, where Q is positive definite and satisfies the Lyapunov equation $Q\mathcal{A}_{22} + \mathcal{A}_{22}^TQ = -2I_{2n-2}$. Since all the eigenvalues of A_{22} are negative, such a matrix Q always exists. Now, taking the derivative of V along the trajectories of the dynamics of z_2 using (28), we get:

$$\dot{V} \leq -2||z_2||^2 + \frac{2}{h}||Q(TP_{\mathcal{S}^{\perp}}\hat{\mathcal{A}}T^{-1})^{\dagger}||||z_2||^2$$
$$\implies \dot{V} \leq -2||z_2||^2 + \frac{2}{h}||Q||||(TP_{\hat{\mathcal{M}}^{\perp}}\hat{\mathcal{A}}T^{-1})^{\dagger}||||z_2||^2.$$

We have $||(TP_{\hat{\mathcal{M}}^\perp}\hat{\mathcal{A}}T^{-1})|| = ||(TP_{\hat{\mathcal{M}}^\perp}\hat{\mathcal{A}}T^{-1})^\dagger||$ and $||Q|| \leq \frac{1}{||\mathcal{A}_{22}||} \leq \frac{1}{|\lambda(\mathcal{A}_{22})|_{max}} = \frac{1}{|\lambda(\mathcal{A})|_{max}} \leq \frac{1}{a}$, because from Lemma 2, we have $a \leq |\lambda(\mathcal{A})|_{max}$. Hence,

$$\dot{V} \leq -2||z_2||^2 + \frac{2||\hat{\mathcal{A}}||}{ha}||z_2||^2 \ (\because \ ||P_{\mathcal{S}^\perp}|| \leq 1)$$
$$\implies \dot{V} \leq -2\frac{||z_2||^2}{h}\left(h - \frac{||\hat{\mathcal{L}}||}{a}\right) \ (\because \ ||\hat{\mathcal{A}}|| = ||\hat{\mathcal{L}}||). \quad (18)$$

Also, from (18) we obtain $\dot{V} \leq -kV$ where $k := \frac{h-\alpha}{h\lambda_{max}(Q)}, \alpha := \frac{||\hat{\mathcal{L}}||}{a}$. Hence, with sufficiently large h, the nominal system is exponentially stable. Now, the system (14) is globally Lipschitz with respect to $x_\perp$ and input $\tilde{\Delta}$ and hence is input to state stable [28, Lemma 4.6]. Using (13), as the input $\tilde{\Delta}$ goes to zero exponentially, $x_\perp \to 0$ exponentially. Hence, from (12a), we obtain

$$x(t) \to x_\parallel(t) = P_{\hat{\mathcal{M}}}x = \begin{bmatrix} P_{\mathcal{M}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} p \\ \zeta \end{bmatrix}$$
$$\implies p(t) \to p_\parallel \in \mathcal{M} \quad t \to \infty.$$

As, $\dot{\zeta} = \mathcal{L}p - a\zeta$, $\dot{\zeta} \to -a\zeta$, as $t \to \infty$, hence $\zeta \to 0$, as $t \to \infty$. Hence, $\dot{p}(t) \to -\frac{1}{h}\hat{\mathcal{L}}p(t) \to -\frac{1}{h}\hat{\mathcal{L}}p(t)_\parallel \in \mathcal{M}, \quad t \to \infty$. □

Remark 7. *Note that we have to choose $h > \alpha := \frac{||\hat{\mathcal{L}}||}{a}$, and by choosing a sufficiently large h, global information on the interaction topology can be avoided. In [25], [26] also a sufficiently large or sufficiently small perturbation parameter was chosen, however, in addition, [25], [26] required a stabilizing diagonal matrix whose design requires the knowledge of exact interaction among the agents, which is not required here.*

Remark 8. *From Theorem 1, we know that using the control law* (6)*, we have $\dot{p}(t) \to -\frac{1}{h}\hat{\mathcal{L}}p(t) \to \mathcal{M}$. We provide an analytical solution for $p(t)$ in the next Theorem.*

Theorem 2. *The following two mutually exclusive cases hold when h is sufficiently high,*
(a) (i) When we have at least one of the speeds ω or γ is nonzero, then the matrix $\tilde{\mathcal{A}} := \mathcal{A} - \frac{1}{h}\hat{\mathcal{A}}$ has an eigenvalue 0, and the corresponding eigenvector is $[\mathbf{1}_n^T \ \mathbf{0}_n^T]^T$, one eigenvalue is $-\frac{1}{h}(\gamma + \iota\omega)$ whose corresponding eigenvector is $[(\frac{v_t^*}{\gamma+\iota\omega}\mathbf{1}_n + p^*)^T \ \mathbf{0}_n^T]^T$, where v_t^* is the designed translational velocity. The other eigenvalues have a negative real part. (ii) Also, $p(t) \to l_1\mathbf{1}_n + l_2\left(\frac{v_t^*}{\gamma+\iota\omega}\right) \in \mathcal{M}$ as $t \to \infty$, where $l_{\{1,2\}} \in \mathbb{C}$ are determined by the initial condition $p(0)$.
(b) (i) When $\omega = \gamma = 0$ and $v_t^* \neq 0$, then the matrix has a zero eigenvalue with algebraic multiplicity two, but the geometric multiplicity equals 1, and the chain of generalized (complex) associated with the zero eigenvalues are $\{[-\frac{1}{h}v_t^*\mathbf{1}_n^T \ \mathbf{0}_n^T]^T, [(p^*)^T \ \mathbf{0}_n^T]^T\}$ and the other eigenvalues have negative real part. (ii) Also, $\dot{p}_i(t) \to -\frac{1}{h}l_2v_t^*, \ \forall i \in \mathcal{V}$, as $t \to \infty$, where $l_2 \in \mathbb{C}$, is determined by the initial condition, i.e., $p(0)$.

Proof. From Lemma 2 and Remark 2, we have $\mathcal{A}$ has two zero eigenvalues having algebraic and geometric multiplicity equals to two. The eigenvectors are $[\mathbf{1}_n^T \ \mathbf{0}^T]^T$ and $[(p^*)^T \ \mathbf{0}^T]^T$, respectively. Also, from Theorem 1, we already know that $\tilde{\mathcal{A}}$ has $2n-2$ eigenvalues with negative real parts. Now, we analyze the first case, i.e., case (i). For $\mathcal{A} - \frac{1}{h}\hat{\mathcal{A}}$, we have

$$(\mathcal{A} - \frac{1}{h}\hat{\mathcal{A}})[\mathbf{1}_n^T \ \mathbf{0}_n^T]^T = 0 - \frac{1}{h}\begin{bmatrix} \hat{\mathcal{L}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} -\frac{1}{h}\hat{\mathcal{L}}\mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} = 0.$$

Also,

$$(\mathcal{A} - \frac{1}{h}\hat{\mathcal{A}})\begin{bmatrix} \frac{v_t^*}{\gamma+\iota\omega}\mathbf{1}_n + p^* \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} -\frac{1}{h}\hat{\mathcal{L}}p^* \\ \mathbf{0}_n \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{h}\hat{\mathcal{L}}_tp^* - \frac{1}{h}\hat{\mathcal{L}}_rp^* - \frac{1}{h}\hat{\mathcal{L}}_sp^* \\ \mathbf{0}_n \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{h}v_t^*\mathbf{1}_n - \frac{1}{h}(\gamma + \iota\omega) \\ \mathbf{0}_n \end{bmatrix} = -\frac{1}{h}(\gamma + \iota\omega)\begin{bmatrix} \frac{v_t^*}{(\gamma+\iota\omega)}\mathbf{1}_n + p^* \\ \mathbf{0}_n \end{bmatrix}.$$

Hence, (i) of (a) is established. Hence, we can write the solution of $\dot{x} = \tilde{\mathcal{A}}x$ in the following form $x(t) = l_1\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} + l_2\begin{bmatrix} \frac{v_t^*}{(\gamma+\iota\omega)}\mathbf{1}_n + p^* \\ \mathbf{0}_n \end{bmatrix}e^{-\frac{1}{h}(\gamma+\iota\omega)t} + \sum_{m=3}^{2n} f_me^{\lambda_mt}$, where the eigenvalues λ_m are such that $\text{Re}(\lambda_m) < 0, m \geq 3$, and the functions $f_m(t, l_m, g_m, g_{m-1}, \ldots, g_{m-\varrho}), \ \varrho \in \mathbb{N}$, depending on the algebraic and geometric multiplicity of λ_m and takes the form of a linear combination, i.e., $l_m(g_m + g_{m-1}t + g_{m-2}\frac{t^2}{2!} + \cdots + g_{m-\varrho}\frac{t^\varrho}{\varrho!})$. Now, as f_m will be a polynomial function in t, hence $f_me^{\lambda_m} \to 0$ as $t \to \infty$, and $p(t) \to l_1\mathbf{1}_n + l_2\left(\frac{v_t^*}{\gamma+\iota\omega}\right)$.

Now, we analyse the case (b), which corresponds to $\gamma = 0$ and $\omega = 0$, hence we have,

$$(\mathcal{A} - \frac{1}{h}\hat{\mathcal{A}})p^* = -\frac{1}{h}\hat{\mathcal{A}}p^* = \begin{bmatrix} -\frac{1}{h}\hat{\mathcal{L}}p^* \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} -\frac{1}{h}v_t^*\mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix}$$

and

$$(\mathcal{A} - \frac{1}{h}\hat{\mathcal{A}})\begin{bmatrix} -\frac{1}{h}v_t^*\mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} = -\frac{1}{h}v_t^*\tilde{\mathcal{A}}\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} = 0.$$

Hence, we have $(\tilde{\mathcal{A}})^2[p^{*T} \ 0^T]^T = 0$, but $(\tilde{\mathcal{A}})[p^{*T} \ 0^T]^T \neq 0$, which proofs (i) if part (b). Now, we can write the solution $x(t)$ in the following form

$$x(t) = -\frac{l_1v_t^*}{h}\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} + l_2\begin{bmatrix} (p^* - \frac{l_1v_t^*}{h}\mathbf{1}_nt) \\ \mathbf{0}_n \end{bmatrix} + \sum_{m=3}^{2n} f_me^{\lambda_mt},$$

where the eigenvalues λ_m are such that $\text{Re}(\lambda_m) < 0, m \geq 3$, and the functions $f_m(t, l_m, g_m, g_{m-1}, \ldots, g_{m-\varrho}), \ \varrho \in \mathbb{N}$, depending on the algebraic and geometric multiplicity of λ_m. Hence, $p(t) \to -\frac{1}{h}v_t^*(l_1 + l_2t)\mathbf{1}_n + l_2p^*$ and $\dot{p}(t) \to -\frac{l_2}{h}v_t^*\mathbf{1}_n$ as $t \to \infty$. □

Remark 9. *Different from the affine formation maneuver case, which will be discussed in Section III-C, for the complex laplacian-based maneuvering discussed in Section III-B although affine formation maneuvers are not possible, i.e., the sheer maneuvering is not possible, the assumption of $d+1$ rooted graph is not required. Only a d-rooted graph assumption is required here. Hence, each agent only needs to sense a minimum of d in-neighbors instead of $d+1$ in-neighbors, reducing the sensing requirements for each agent compared to the affine formation maneuvering case.*

C. Leaderless Affine formation maneuvers

In this subsection, we present the proposed control laws for solving the Problem 2.

1) Control strategy for single integrators: We propose the following control law for affine maneuvering of single integrators ($\forall i \in \mathcal{V}, j \in \mathcal{N}_i,$)

$$u_i = -\sum_{j\in\mathcal{N}_i} w_{ij}\zeta_i + \sum_{i\in\mathcal{N}_j} w_{ji}\zeta_j - \frac{1}{h}\sum_{j\in\mathcal{N}_i}\hat{w}_{ij}(p_j - p_i) - \hat{\delta}_i,$$

$$\dot{\zeta}_i = -a\zeta_i + \sum_{j\in\mathcal{N}_i} w_{ij}(p_j - p_i), \tag{19a}$$

$$\dot{\hat{\delta}}_i = \sum_{j\in\mathcal{N}_i} w_{ij}\zeta_i - \sum_{i\in\mathcal{N}_j} w_{ji}\zeta_j + \frac{1}{h}\sum_{j\in\mathcal{N}_i}\hat{w}_{ij}(p_j - p_i) + v_i, \tag{19b}$$

where $\zeta_i \in \mathbb{R}^d$ is an auxiliary state variable, $a > 0$ is a constant, and v_i is the measured velocity of the i-th agent. In the control law (19), the information $w_{ji}\zeta_j$ has to be transmitted from the out-neighbors of i; hence inter-agent communication is required. The term, $\hat{\delta}_i$, in (19) is an integral term that is used for tackling constant disturbance affecting the dynamics.

The weights w_{ij} and $\hat{w}_{ij}$ are calculated from the following equations:

$$\sum_{j\in\mathcal{N}_i} w_{ij}(p_i^* - p_j^*) = 0 \quad \forall i \in \mathcal{V}, \tag{20a}$$

$$\sum_{j\in\mathcal{N}_i} \hat{w}_{ij}(p_i^* - p_j^*) = v_i^*, \tag{20b}$$

where the desired velocity of the i-th agent is denoted as v_i^*. Note that as we have $d+1$ unknowns with d equations in (20b) for each agent i, the solution of (20b) will be the summation of a particular solution and any vector in the null space, i.e., the solution of eqn. (20a) which is one-dimensional vector space in $\mathbb{R}^{d+1}$. Based on Assumption 1, it is evident that each agent has $d+1$ in-neighbors. Additionally, we have a generic desired configuration. Therefore, for each agent with index i, it is always possible to find d vectors in $\mathbb{R}^d$ of the form $p_{ij}^*,\ j \in \mathcal{N}_i$, which are linearly independent. (According to Lemma 1 and Definition 1.3, in a $d+1$ rooted digraph, all nodes except the root nodes have $d+1$ in-neighbors, and our additional assumption on root nodes ensures that the root nodes also have $d+1$ in-neighbors.) Consequently, solutions (which are not unique) to (20a) and (20b) always exist. It is essential to observe that the solution to (20b) is the perturbation weights that are responsible for the generation of maneuvering force, and as the modified interaction topology ensures each root-node also has $d+1$ in-neighbors, the solution for (20b) for root nodes also exists.

Now, define new state variable $x := [p^T \ \ \zeta^T]^T \in \mathbb{R}^{2nd}$. In compact form (19) can be written as:

$$\dot{x} = \begin{bmatrix}\dot{p}\\ \dot{\zeta}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{0} & -\bar{\mathcal{L}}^T\\ \bar{\mathcal{L}} & -aI_{nd}\end{bmatrix}}_{\bar{\mathcal{A}}:=\mathcal{A}\otimes I_d}\begin{bmatrix}p\\ \zeta\end{bmatrix} - \frac{1}{h}\underbrace{\begin{bmatrix}\bar{\hat{\mathcal{L}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}}_{\bar{\hat{\mathcal{A}}}:=\hat{\mathcal{A}}\otimes I_d}\begin{bmatrix}p\\ \zeta\end{bmatrix} + \underbrace{\begin{bmatrix}\tilde{\delta}\\ \mathbf{0}\end{bmatrix}}_{\tilde{\Delta}},$$

$$\implies \dot{x} = \bar{\mathcal{A}}x - \frac{1}{h}\bar{\hat{\mathcal{A}}}x + \tilde{\Delta}, \tag{21a}$$

$$\dot{\hat{\delta}} = \bar{\mathcal{L}}^T\zeta + \frac{1}{h}\bar{\hat{\mathcal{L}}}p + v, \tag{21b}$$

where $\hat{\delta} = [\hat{\delta}_1^T \ \hat{\delta}_2^T \ \dots \ \hat{\delta}_n^T]^T, \delta = [\delta_1^T \ \delta_2^T \ \dots \ \delta_n^T]^T, \tilde{\delta} = \delta - \hat{\delta}, u = [u_1^T \ u_2^T \ \dots \ u_n^T]^T, v = [v_1^T \ v_2^T \ \dots \ v_n^T]^T$, and $\bar{\mathcal{L}} := \mathcal{L}\otimes I_d, \bar{\hat{\mathcal{L}}} := \hat{\mathcal{L}}\otimes I_d, \bar{\mathcal{A}} := \mathcal{A}\otimes I_d, \bar{\hat{\mathcal{A}}} := \hat{\mathcal{A}}\otimes I_d$ with $\mathcal{A} := \begin{bmatrix}\mathbf{0} & -\mathcal{L}^T\\ \mathcal{L} & -aI_n\end{bmatrix}$ and $\hat{\mathcal{A}} := \begin{bmatrix}\hat{\mathcal{L}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$, and $\tilde{\Delta} := \begin{bmatrix}\tilde{\delta}\\ \mathbf{0}\end{bmatrix}$. For designing $\hat{\mathcal{L}}$, the desired velocity (v^*) can be split into translation, rotation, scale and shear components, as shown in Fig.1(b), i.e., $v^* = v_{t1}^* + v_{t2}^* + v_r^* + v_s^* + v_{s1}^* + v_{s2}^*$, and weights can be designed for each component by solving (20b) independently i.e., for $\mathbb{R}^2$ $\hat{\mathcal{L}} = \hat{\mathcal{L}}_{t1} + \hat{\mathcal{L}}_{t2} + \hat{\mathcal{L}}_r + \hat{\mathcal{L}}_s + \hat{\mathcal{L}}_{s_1} + \hat{\mathcal{L}}_{s_2}$, where the subscripts $\{t_1, t_2\}$ stand stands for two translational directions, $\{r\}$ for rotational, $\{s\}$ for scale and $\{s_1, s_2\}$ for shear components which form a basis for all affine transformations in $\mathbb{R}^2$.

The following lemma is about the eigenvalues and eigenvectors of the matrix $\bar{\mathcal{A}}$.

Lemma 4. *The eigenvalues of $\mathcal{A}$ are given by $\lambda = \frac{-a\pm\sqrt{a^2-4\sigma}}{2}$, where σ is an eigenvalue of $\mathcal{L}^T\mathcal{L}$. $\bar{\mathcal{A}}$ has $d(d+1)$ zero eigenvalues and the corresponding eigenvectors are given by $[\Omega^T \ \ \mathbf{0}^T]^T \in \mathbb{R}^{2nd}$, where then $\Omega = \omega\otimes I_d$, and $\omega \in \mathbb{R}^n$ lies in the null space of $\mathcal{L}^T\mathcal{L}$.*

Proof. The proof is similar to the proof of Lemma 2. □

2) Stability analysis for single integrator: Next, stability analysis with formation control law (19) for single integrators is carried out. The following lemma will be useful in our analysis.

Lemma 5. *For an affine transformation $\hat{p} = (I_n\otimes A)p^* + (\mathbf{1}_n\otimes b)$, with arbitrary $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$, we have $\bar{\hat{\mathcal{L}}}\hat{p} = (I_n\otimes A)v^* \in \mathcal{S}$, where $\bar{\hat{\mathcal{L}}} = \hat{\mathcal{L}}\otimes I_d$ and $v^* = [v_1^{*T} \ v_2^{*T} \ \dots \ v_n^{*T}]^T$ is the vector containing desired velocities of the agents.*

Proof. The following identities are true for Kronecker products of matrices: $(M_1\otimes M_2)(M_3\otimes M_4) = (M_1M_3\otimes M_2M_4)$, implies $(M_1\otimes I)(I\otimes M_2) = (M_1\otimes M_2) = (I\otimes M_2)(M_1\otimes I)$, with appropriate sizes of identity matrices. It then follows that

$$\begin{aligned}\bar{\hat{\mathcal{L}}}\hat{p} &= (\hat{\mathcal{L}}\otimes I_d)((I_n\otimes A)p^* + (\mathbf{1}_n\otimes b))\\ &= (\hat{\mathcal{L}}\otimes I_d)(I_n\otimes A)p^* + (\hat{\mathcal{L}}\otimes I_d)(\mathbf{1}_n\otimes b)\\ &= (\hat{\mathcal{L}}\otimes A)p^* + (\hat{\mathcal{L}}\mathbf{1}_n\otimes b)\\ &= (I_n\otimes A)(\hat{\mathcal{L}}\otimes I_d)p^* \quad (\because \hat{\mathcal{L}}\mathbf{1}_n = 0)\\ &= (I_n\otimes A)v^* \quad (\text{from (20b), we have } v^* = (\hat{\mathcal{L}}\otimes I_d)p^*)\end{aligned}$$

As, we have chosen $v^* \in \mathcal{S}$ so $(I_n\otimes A)v^* \in \mathcal{S}$, and $\bar{\hat{\mathcal{L}}}\hat{p} \in \mathcal{S}$. □

Next, we state the main theorem for single integrators.

Theorem 3. *For a group of single integrators described by* (4) *and the control law* (19)*, subject to Assumption 2, if the control gain parameter, h, is sufficiently large then* $p(t) \to \mathcal{S}$ *and* $\dot{p}(t) \to -\frac{1}{h}\bar{\hat{\mathcal{L}}}p(t) \to \mathcal{S}$ *as* $t \to \infty$*.*

Proof. Let us consider the subspace $\mathcal{S}^{\perp}$ to be the orthogonal complement of $\mathcal{S}$. The dimensions of $\mathcal{S}$ and $\mathcal{S}^{\perp}$ are $d(d+1)$ and $d(n-d-1)$ [19, Theorem 4.1,Theorem 4.2], respectively. Here, $\mathcal{S} = \text{Ker}\{\bar{\mathcal{L}}\}$. Also, we have $\text{Ker}\{\bar{\mathcal{L}}\} = \text{Ker}\{\bar{\mathcal{L}}^T\bar{\mathcal{L}}\}$, and hence, $\mathcal{S} = \text{Ker}\{\bar{\mathcal{L}}^T\bar{\mathcal{L}}\}$. Now, we add zeros to this subspaces to extend it to get

$$\hat{\mathcal{S}} := \text{Im}\left(\begin{bmatrix}\mathcal{B}_{\mathcal{S}}\\ \mathbf{0}_{nd\times d(d+1)}\end{bmatrix}\right),\ \hat{\mathcal{S}}^{\perp} := \text{Im}\left(\begin{bmatrix}\mathcal{B}_{\mathcal{S}^{\perp}}\\ \mathcal{B}_{\mathcal{W}}\end{bmatrix}\right),$$

where $\{\mathcal{B}_{\mathcal{S}}\}, \{\mathcal{B}_{\mathcal{S}^{\perp}}\}$, and $\{\mathcal{B}_{\mathcal{W}}$ are a set of basis for $\mathcal{S}, \mathcal{S}^{\perp}$, and $\mathcal{W}$, respectively. Hence, $\mathcal{W} \subset \mathbb{R}^{nd}$ having dimension of $nd - d(d+1)$. We may observe that the sub-spaces $\hat{\mathcal{S}}, \hat{\mathcal{S}}^{\perp}$, are mutually orthogonal, i.e., for any vector $s \in \hat{\mathcal{S}}$ and $\bar{s} \in \hat{\mathcal{S}}^{\perp}$ we have $\langle s, \bar{s}\rangle = 0$. Also, $\hat{\mathcal{S}} \oplus \hat{\mathcal{S}}^{\perp} = \mathbb{R}^{2nd}$. Now the projection matrix for a vector $v \in \mathbb{R}^{2nd}$ to the subspace $\hat{\mathcal{S}}$ is denoted by $P_{\hat{\mathcal{S}}} := \begin{bmatrix}P_{\mathcal{S}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}$, where $P_{\mathcal{S}}$ is the projection for a vector $v^{'} \in \mathbb{R}^{nd}$ to the subspaces $\mathcal{S}$. Also, from Lemma 4, it is clear that $\hat{\mathcal{S}}$ is the kernel of $\bar{\mathcal{A}}$. Now, we can rewrite $x = [p^T\ \zeta^T]^T \in \mathbb{R}^{2nd}$ and $\tilde{\Delta}$ as

$$\begin{aligned}x &= P_{\hat{\mathcal{S}}}x + P_{\hat{\mathcal{S}}^{\perp}}x = x_{\parallel} + x_{\perp},\\ \tilde{\Delta} &= P_{\hat{\mathcal{S}}}\tilde{\Delta} + P_{\hat{\mathcal{S}}^{\perp}}\tilde{\Delta} = \tilde{\Delta}_{\parallel} + \tilde{\Delta}_{\perp},\end{aligned}$$

where $x_{\parallel}$ and $x_{\perp}$ are the components of $x \in \mathbb{R}^{2nd}$ along $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}^{\perp}$, respectively. Therefore, p and $\hat{\delta}$ are written as

$$p = P_{\mathcal{S}}p + P_{\mathcal{S}^{\perp}}p = p_{\parallel} + p_{\perp}, \tag{22a}$$
$$\hat{\delta} = P_{\mathcal{S}}\hat{\delta} + P_{\mathcal{S}^{\perp}}\hat{\delta} = \hat{\delta}_{\parallel} + \hat{\delta}_{\perp}, \tag{22b}$$

where $p_{\parallel}, \hat{\delta}_{\parallel}, p_{\perp}, \hat{\delta}_{\perp}$ are the components of $p, \hat{\delta} \in \mathbb{R}^{nd}$ along $\mathcal{S}$ and $\mathcal{S}^{\perp}$, respectively. Observe that $x_{\parallel} = [p_{\parallel}^T\ \mathbf{0}^T]^T \in \mathbb{R}^{2nd}$, and $x_{\perp} = [p_{\perp}^T\ \zeta^T]^T \in \mathbb{R}^{2nd}$. From (21), the closed loop dynamics is

$$\dot{x} = (\mathcal{A} \otimes I_d)x - \frac{1}{h}(\hat{\mathcal{A}} \otimes I_d)x + \tilde{\Delta} \tag{23a}$$
$$\dot{\hat{\delta}} = -(\mathcal{L}^T \otimes I_d)\zeta + \frac{1}{h}(\hat{\mathcal{L}} \otimes I_d)p + v. \tag{23b}$$

We consider (23a) as a nominal system with input with $\tilde{\Delta}$ being the input. First, we show that the error term, $\tilde{\Delta}$, goes to zero exponentially. From (23b) and (23a) , $\tilde{\delta}$ has dynamics:

$$\begin{aligned}\dot{\tilde{\delta}} &= \dot{\delta} - \dot{\hat{\delta}} = -\dot{\hat{\delta}} = (\mathcal{L}^T \otimes I_d)\zeta - \frac{1}{h}(\hat{\mathcal{L}} \otimes I_d)p - v\\ &= -\tilde{\delta} \implies \tilde{\delta}(t) = \exp(-I_{dn\times dn}t)\tilde{\delta}(t_0).\end{aligned} \tag{24}$$

From (24), it is clear that $\tilde{\delta}$ goes to zero exponentially, and hence $\tilde{\Delta}$ goes to zero exponentially.

Now, the dynamics of $x_{\perp}$ is given by:

$$\begin{aligned}\dot{x}_{\perp} &= P_{\hat{\mathcal{S}}^{\perp}}(\mathcal{A} \otimes I_d)x - \frac{1}{h}P_{\hat{\mathcal{S}}^{\perp}}(\hat{\mathcal{A}} \otimes I_d)x + \tilde{\Delta}_{\perp}\\ &= P_{\hat{\mathcal{S}}^{\perp}}\bar{\mathcal{A}}(x_{\parallel} + x_{\perp}) - \frac{1}{h}P_{\hat{\mathcal{S}}^{\perp}}\bar{\hat{\mathcal{A}}}(x_{\parallel} + x_{\perp}) + \tilde{\Delta}_{\perp}.\end{aligned} \tag{25}$$

We will now consider only the nominal system (without input $\tilde{\Delta}_{\perp}$) and show that $x_{\perp}(t) \to 0$ and as $t \to \infty$. As $x_{\parallel}$ is in the Null space of $\bar{\mathcal{A}}$, we have $\bar{\mathcal{A}}x_{\parallel} = 0$. Also,

$$P_{\hat{\mathcal{S}}^{\perp}}\bar{\hat{\mathcal{A}}}x_{\parallel} = \begin{bmatrix}P_{\mathcal{S}^{\perp}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\hat{\mathcal{L}} \otimes I_d & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}p_{\parallel}\\ \zeta_{\parallel}\end{bmatrix} = \begin{bmatrix}P_{\mathcal{S}^{\perp}}\bar{\hat{\mathcal{L}}}p_{\parallel}\\ \mathbf{0}\end{bmatrix},$$

and from Lemma 5 we have $\bar{\hat{\mathcal{L}}}p_{\parallel} \in \mathcal{S}$. Hence, $P_{\mathcal{S}^{\perp}}\bar{\hat{\mathcal{L}}}p_{\parallel} = 0$, implying $P_{\hat{\mathcal{S}}^{\perp}}\bar{\hat{\mathcal{A}}}x_{\parallel} = 0$. Hence, (25) becomes

$$\dot{x}_{\perp} = P_{\hat{\mathcal{S}}^{\perp}}\bar{\mathcal{A}}x_{\perp} - \frac{1}{h}P_{\hat{\mathcal{S}}^{\perp}}\bar{\hat{\mathcal{A}}}x_{\perp}. \tag{26}$$

Now, let a basis for $\hat{\mathcal{S}}$ be given by the columns of $\mathcal{B}_{\hat{\mathcal{S}}} := [v_1\ v_2\ \dots\ v_{d(d+1)}] \in \mathbb{R}^{2nd\times d(d+1)}$. We can obtain a set of vectors orthogonal to the columns of $\mathcal{B}_{\hat{\mathcal{S}}}$, which form a basis for the space $\hat{\mathcal{S}}^{\perp}$ and let these vectors be given by the columns of $\mathcal{B}_{\hat{\mathcal{S}}^{\perp}} = [u_1\ u_2\ \dots\ u_{2dn-d(d+1)}]$. Now we define $T^{-1} = [\mathcal{B}_{\hat{\mathcal{S}}}\ \mathcal{B}_{\hat{\mathcal{S}}^{\perp}}] \in \mathbb{R}^{2nd\times 2nd}$, also these basis vectors are chosen such that $TP_{\hat{\mathcal{S}}^{\perp}}T^{-1} = \begin{bmatrix}\mathbf{0}_{d(d+1)} & \mathbf{0}\\ \mathbf{0} & I_{2dn-d(d+1)}\end{bmatrix}$. Now, we apply change of coordinates, T, to x, i.e.,

$$[z_1^T\ z_2^T]^T = Tx = T(x_{\parallel} + x_{\perp}).$$

Here, $z_1 \in \mathbb{R}^{d(d+1)}$ and $z_2 \in \mathbb{R}^{2dn-d(d+1)}$. Observe that $Tx_{\parallel} = [z_1^T\ 0^T]^T$ and $Tz_{\perp} = [0^T\ z_2^T]^T$. Also, $T\bar{\mathcal{A}}T^{-1} = \begin{bmatrix}\mathcal{A}_{11} & \mathbf{0}\\ \mathbf{0} & \mathcal{A}_{22}\end{bmatrix}$, where $\mathcal{A}_{11} \in \mathbb{R}^{(d^2+d)\times(d^2+d)}$ sharing the zero eigenvalues of $\bar{\mathcal{A}}$, and $\mathcal{A}_{22} \in \mathbb{R}^{2dn-d(d+1)\times 2dn-d(d+1)}$ which shares all eigenvalues of $\bar{\mathcal{A}}$ with negative real parts. Applying the same co-ordinate change to (26) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}0\\ z_2\end{bmatrix} = TP_{\hat{\mathcal{S}}^{\perp}}\bar{\mathcal{A}}T^{-1}\begin{bmatrix}0\\ z_2\end{bmatrix} - \frac{1}{h}TP_{\hat{\mathcal{S}}^{\perp}}\bar{\hat{\mathcal{A}}}T^{-1}\begin{bmatrix}0\\ z_2\end{bmatrix}. \tag{27}$$

Now, form (27) and using the fact

$$TP_{\hat{\mathcal{S}}^{\perp}}\bar{\mathcal{A}}T^{-1} = (TP_{\hat{\mathcal{S}}^{\perp}}T^{-1})(T\bar{\mathcal{A}}T^{-1}) = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathcal{A}_{22}\end{bmatrix},$$

$$\dot{z}_2 = \mathcal{A}_{22}z_2 - \frac{1}{h}(TP_{\hat{\mathcal{S}}^{\perp}}\bar{\hat{\mathcal{A}}}T^{-1})^{\dagger}z_2, \tag{28}$$

where the $\dagger$ symbol indicates the last $2dn - d(d+1) \times 2dn - d(d+1)$ block is taken because it is compatible with the size of z_2. Now, consider the Lyapunov candidate $V = z_2^TQz_2$, where Q is positive definite and satisfies the Lyapunov equation $Q\mathcal{A}_{22} + \mathcal{A}_{22}^TQ = -2I_{2dn-d(d+1)}$. Since all the eigenvalues of A_{22} are negative, such a matrix Q always exists. Now, taking the derivative of V along the trajectories of the dynamics of z_2 using (28), we get:

$$\begin{aligned}&\dot{V} \le -2||z_2||^2 + \frac{2}{h}||Q(TP_{\mathcal{S}^{\perp}}\bar{\hat{\mathcal{A}}}T^{-1})^{\dagger}||||z_2||^2\\ \implies &\dot{V} \le -2||z_2||^2 + \frac{2}{h}||Q||||(TP_{\hat{\mathcal{S}}^{\perp}}\bar{\hat{\mathcal{A}}}T^{-1})^{\dagger}||||z_2||^2.\end{aligned}$$

We have $||(TP_{\hat{\mathcal{S}}^\perp}\bar{\hat{\mathcal{A}}}T^{-1})|| = ||(TP_{\hat{\mathcal{S}}^\perp}\bar{\hat{\mathcal{A}}}T^{-1})^\dagger||$ and $||Q|| \leq \frac{1}{||\mathcal{A}_{22}||} \leq \frac{1}{|\lambda(\mathcal{A}_{22})|_{max}} = \frac{1}{|\lambda(\bar{\mathcal{A}})|_{max}} \leq \frac{1}{a}$, because from Lemma 4, we have $a \leq |\lambda(\bar{\mathcal{A}})|_{max}$. Hence,

$$\dot{V} \leq -2||z_2||^2 + \frac{2||\bar{\hat{\mathcal{A}}}||}{a}||z_2||^2 \ (\because \ ||P_{\mathcal{S}^\perp}|| \leq 1)$$

$$\implies \dot{V} \leq -2\frac{||z_2||^2}{h}\left(h - \frac{||\bar{\hat{\mathcal{L}}}||}{a}\right) \ (\because \ ||\bar{\hat{\mathcal{A}}}|| = ||\bar{\hat{\mathcal{L}}}||). \quad (29)$$

Hence, if $h > \frac{||\hat{\mathcal{L}}||}{a} := \alpha$, $\dot{V}$ is negative definite. Also, from (29) we obtain

$$\dot{V} \leq -kV \quad (30)$$

where $k := \frac{h-\alpha}{h\lambda_{max}(Q)}$ and hence the nominal system is exponentially stable. Now, the system (25) is globally Lipschitz with respect to $x_\perp$ and input $\tilde{\Delta}$ and hence is input to state stable [28, Lemma 4.6]. Using (24), as the input $\tilde{\Delta}$ goes to zero exponentially, $x_\perp \to 0$ exponentially. Hence, from (23a), we obtain

$$x(t) \to x_\parallel(t) = P_{\hat{\mathcal{S}}}x = \begin{bmatrix} P_{\mathcal{S}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} p \\ \zeta \end{bmatrix}$$

$$\implies p(t) \to p_\parallel \text{ and } \zeta(t) \to 0 \quad t \to \infty.$$

Hence, from (21a) it is clear that

$$\dot{p}(t) \to -\frac{1}{h}(\hat{\mathcal{L}} \otimes I_d)p(t) = -\frac{1}{h}\bar{\hat{\mathcal{L}}}p_\parallel(t) \in \mathcal{S}, \quad t \to \infty.$$

□

Remark 10. *After subtracting the term* $\frac{1}{h}\bar{\hat{\mathcal{A}}}$ *from* $\bar{\mathcal{A}}$ *in* (21), *with* $h > \alpha := \frac{||\hat{\mathcal{L}}||}{a}$, *we end up modifying the* $d(d+1)$ *zero eigenvalues of* $\bar{\mathcal{A}}$ *that span* $\mathcal{S}$. *The eigenvectors corresponding to these modified eigenvalues now also span the space of affine transformations, i.e.,* $\mathcal{S}$. *Moreover, the non-zero eigenvalues of* $\bar{\mathcal{A}}$ *that originally resided in the left half of the complex plane are also changed so that they do not migrate to the right half of the complex plane.*

Remark 11. *In [25], [26] a diagonal matrix is required for the control design for undirected graphs, and a fully decentralized design without the need for any diagonal matrix was not addressed for undirected graphs in the literature. Although we have considered a directed interaction topology here, using our proposed laws in* (6) *and* (19) *we can solve this problem for undirected graphs as well.*

3) Affine maneuvering for higher-order integrators: We next design a control law and establish the stability for affine maneuvering of higher order integrator systems in the presence of constant disturbances as described in (5).

We adopt an adaptive back-stepping-based approach for the design. Let $y^*_{i,l+1}, \ 1 \leq l \leq \rho - 1$ be the desired input for stabilizing l-th order term for the i-th agent. Then, the error term is defined as $\tilde{y}_{i,l+1} := y_{i,l+1} - y^*_{i,l+1}$. Further, to avoid the use of the derivatives of virtual control input, i.e., $y^*_{i,l+1}, \ 1 \leq l \leq \rho - 1$, in the control law, for avoiding the amplification of noise and computational issues, motivated by design methods in [29], we use a second order command filter $H(s) := \frac{w_n^2}{s^2+2\zeta w_n s+w_n^2}$, with $\zeta = 1$. The filtered signal is denoted by $y^*_{if,l+1}, \ 1 \leq l \leq \rho - 1$, and here $y^*_{if,l+1} = H(s)y^*_{i,l+1}(s)$. In frequency domain, the derivative of $y^*_{if,l+1}$ is obtained by $\dot{y}^*_{if,l+1}(s) := sH(s)y^*_{i,l+1}$. Here, w_n is the filter's natural frequency, and the filter's dc gain and damping ratio are 1, which ensures no overshoot for tracking the signal. The dynamics of the command filter in the state space form is:

$$\dot{x}_{i,l+1} = w_n(F \otimes I_d)x_{i,l+1} + w_n(B \otimes I_d)y^*_{i,l+1}, \quad (31a)$$

$$[(y^*_{if,l+1})^T(\dot{y}^*_{if,l+1})^T]^T = (C \otimes I_d)x_{i,l+1}, \quad (31b)$$

where $x_{i,l+1} := [\eta^T_{1i,l+1} \ \eta^T_{2i,l+1}]^T$ are the state variables associated with the command filter, $y^*_{if,l+1}$ is the filtered output of $y^*_{i,l+1}$, and $F := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, B := [0 \ 1]^T, C := \begin{bmatrix} 1 & 0 \\ 0 & w_n \end{bmatrix}$. The output is considered to be $[(y^*_{if,l+1})^T(\dot{y}^*_{if,l+1})^T]^T$. The equilibrium point for the filter is calculated as $x^{eq}_{i,l+1} = -(F \otimes I_d)^{-1}(B \otimes I_d)y^*_{i,l+1} = -(F^{-1}B \otimes I_d)y^*_{i,l+1} = ([1,0]^T \otimes I_d)y^*_{i,l+1}$. The following coordinate transform for agent i is defined, $\tilde{x}_{i,l+1} := [\tilde{\eta}^T_{1i,l+1} \ \tilde{\eta}^T_{2i,l+1}]^T = x_{i,l+1} - x^{eq}_{i,l+1} = [(\eta_{1i,l+1} - y^*_{i,l+1})^T \ \eta^T_{2i,l+1}]^T$. Hence, the filter error dynamics of the filter states becomes

$$\frac{d\tilde{x}_{i,l+1}}{dt} = w_n(F \otimes I_d)\tilde{x}_{i,l+1}.$$

As the matrix F is Hurwitz, this filter error dynamics is exponentially stable.

Now, the desired state and control laws that are used for i-th agent are given by:

$$y^*_{i,2} = -\sum_{j\in\mathcal{N}_i} w_{ij}\zeta_i + \sum_{i\in\mathcal{N}_j} w_{ji}\zeta_j - \frac{1}{h}\sum_{j\in\mathcal{N}_i}\hat{w}_{ij}(y_{j,1} - y_{i,1}) - \hat{\delta}_i, \quad (32a)$$

$$\dot{\zeta}_i = -a\zeta_i + \sum_{j\in\mathcal{N}_i} w_{ij}(y_{j,1} - y_{i,1}), \quad (32b)$$

$$y^*_{i,l+1} = -k_l\tilde{y}_{i,l} + \dot{y}^*_{if,l} - \hat{\delta}_{i,2}, \quad 2 \leq l \leq \rho - 1, \quad (32c)$$

$$u_i = -k_\rho\tilde{y}_{i,\rho} + \dot{y}^*_{if,\rho} - \hat{\delta}_{i,\rho}, \quad i \in \mathcal{V}, \quad (32d)$$

$$\dot{\hat{\delta}}_{i,1} = \sum_{j\in\mathcal{N}_i} w_{ij}\zeta_i - \sum_{i\in\mathcal{N}_j} w_{ji}\zeta_j + \dot{y}_{i,1}, \quad (32e)$$

$$\dot{\hat{\delta}}_{i,l} = \tilde{y}_{i,l}, \quad 2 \leq l \leq \rho. \quad (32f)$$

Let $\bar{y}_{i,1} = [y^T_{i,1} \ \zeta^T_i]^T$ and $\bar{y}_1 = [\bar{y}^T_{1,1} \ \ldots \ \bar{y}^T_{i,1}]^T$. With the above control law (32) for the ρ-th order integrator system (5), we have the following closed loop dynamics in compact form for n agents:

$$\dot{\bar{y}}_1 = (\mathcal{A} \otimes I_d)\bar{y}_1 - \frac{1}{h}(\hat{\mathcal{A}} \otimes I_d)\bar{y}_1 + \begin{bmatrix} \tilde{\delta}_1, \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{y}_2 \\ 0 \end{bmatrix}, \quad (33a)$$

$$\dot{y}_l = -k_l\tilde{y}_l + w_n\tilde{\eta}_{2,l} + \tilde{\delta}_l + \tilde{y}_{l+1}, \quad 2 \leq l \leq \rho - 1, \quad (33b)$$

$$\dot{y}_\rho = -k_\rho\tilde{y}_\rho + w_n\tilde{\eta}_{2,\rho} + \tilde{\delta}_\rho, \quad (33c)$$

$$\dot{\hat{\delta}}_1 = (\mathcal{L}^T \otimes I_d)\zeta + \frac{1}{h}(\hat{\mathcal{L}} \otimes I_d)y_1 + \dot{y}_1, \quad (33d)$$

$$\dot{\hat{\delta}}_l = \tilde{y}_l, \quad 2 \leq l \leq \rho, \quad (33e)$$

where $y_l, \hat{\delta}_l \in \mathbb{R}^{dn}, \ 2 \leq l \leq \rho$, are the stacked state vectors of all agents, and $\tilde{\eta}_{2,l} := [\tilde{\eta}^T_{21,l} \ \ldots \ \tilde{\eta}^T_{2n,l}]^T$, i.e., the stacked filter error vector of all agents.

Theorem 4. *Using the control law in* (32) *for a group of ρ-th order systems described by* (5) *under Assumption 1 when the gain parameter $h > \alpha$ and $k_1 > 1, k_\rho > 1, k_l > 1.5\ \forall l \in \{2,\ldots,\rho-1\}$, affine maneuvering is achieved, i.e., $y_{i,1}(t) \to \mathcal{S}$ and $y_{i,2}(t) \to -\frac{1}{h}\bar{\hat{\mathcal{L}}}y_{i,1} \in \mathcal{S}\ \forall i \in \mathcal{V}$.*

Proof. We first show the error system $\tilde{y}_l,\ 2 \le l \le \rho$ goes to zero asymptotically as $t \to \infty$. Choosing the positive definite and radially unbounded Lyapunov candidate $V = \frac{1}{2}\sum_{l=2}^{\rho}\tilde{y}_l^T\tilde{y}_l + \frac{1}{2}\sum_{l=2}^{\rho}\tilde{\delta}_l^T\tilde{\delta}_l$, and taking its derivative along the trajectories of we get:

$$\dot{V} = \sum_{l=2}^{\rho}\tilde{y}_l^T(\dot{y}_l - \dot{y}_l^*) - \sum_{l=2}^{\rho}\tilde{\delta}_l^T(\dot{\hat{\delta}}_l). \tag{34}$$

Using (33b),(33c),(33e) in the expression for $\dot{V}$ we get:

$$\begin{aligned}
\dot{V} &\le -\sum_{l=2}^{\rho}k_l||\tilde{y}_l||^2 + \sum_{l=2}^{\rho-1}||\tilde{y}_l||||\tilde{y}_{l+1}|| + \sum_{l=2}^{\rho}||\tilde{y}_l||||\dot{\tilde{\eta}}_{1,l}|| \\
\implies \dot{V} &\le -\sum_{l=2}^{\rho}k_l||\tilde{y}_l||^2 + \sum_{l=2}^{\rho-1}\left(\frac{||\tilde{y}_l||^2}{2} + \frac{||\tilde{y}_{l+1}||^2}{2}\right) \\
&+ \sum_{l=2}^{\rho}\left(\frac{||\tilde{y}_l||^2}{2} + \frac{||\dot{\tilde{\eta}}_{1,l}||}{2}\right) \\
\implies \dot{V} &\le -\sum_{l=2}^{\rho}(k_l-\beta_l)||\tilde{y}_l||^2 + \sum_{l=2}^{\rho}\frac{||\dot{\tilde{\eta}}_{1,l}||^2}{2} \\
\implies \dot{V} &\le -\sum_{l=2}^{\rho}(k_l-\beta_l)\left(||\tilde{y}_l||^2 - \frac{||\dot{\tilde{\eta}}_{1,l}||^2}{2(k_l-\beta_l)}\right),
\end{aligned} \tag{35}$$

where $\beta_l = 1.5,\ 3 \le l \le \rho-1$ and $\beta_2 = 1, \beta_\rho = 1$. Setting $k_l > \beta_l$, when we have $||\tilde{y}_l|| > \frac{||\dot{\tilde{\eta}}_{1,l}||}{\sqrt{2(k_l-\beta_l)}}$ we have $\dot{V} \le 0$ from (35). Hence, from [28], we have the closed loop system is input-to-state stable, where the input is given by $\frac{||\dot{\tilde{\eta}}_{1,l}||}{\sqrt{2(k_l-\beta_l)}}$. However, as the filter dynamics is exponentially stable, and we have $\frac{d^2\tilde{x}_{i,l+1}}{dt} = w_n(F\otimes I_d)\frac{d\tilde{x}_{i,l+1}}{dt}$, hence as F is Hurwitz, $\dot{\tilde{x}}_{i,l+1}$ goes to zero exponentially as $t \to \infty$, hence $\dot{\tilde{\eta}}_{1,l}$ goes to zero exponentially $t \to \infty$. Now, from 35, we have the system 33 is Lyapunov stable. Hence, $\tilde{y}_l \in \mathcal{L}_\infty$ and $\tilde{\delta} \in \mathcal{L}_\infty$. Also, taking derivative of (32e) and using (32a) , we have $\dot{y}_{i,2}^* \in \mathcal{L}_\infty$ and similarly from (32c) and (32f) we have $\dot{y}_l^* \in \mathcal{L}_\infty$. Then from (33b) we have $\dot{y}_l \in \mathcal{L}_\infty$, implies y_l is uniformly-continuous. Now, integrating both sides of (35), we get:

$$\begin{aligned}
&V(\infty) - V(0) - \sum_{l=2}^{\rho}\int_0^\infty\frac{||\dot{\tilde{\eta}}_{1,l}||^2}{2}\mathrm{d}t \\
&\le -\sum_{l=2}^{\rho}(k_l-\beta_l)\int_0^\infty||\tilde{y}_l||^2\mathrm{d}t.
\end{aligned} \tag{36}$$

As $\dot{\tilde{\eta}}_{1,l}$ goes to zero exponentially fast, we have $||\dot{\tilde{\eta}}_{1,l}|| \le \lambda_1 e^{-\lambda_2}t$ for some $\lambda_1, \lambda_2 > 0$, implying $||\dot{\tilde{\eta}}_{1,l}||^2 \le \lambda_1^2 e^{-2\lambda_2 t}$. Hence, $\int_0^\infty\frac{||\dot{\tilde{\eta}}_{1,l}||^2}{2} = \frac{\lambda_1^2}{4\lambda_2} < \infty, \forall l \in \{2,\ldots,\rho\}$. Hence, the L.H.S. of (36) is bounded and so will be the R.H.S., implying $\tilde{y}_l \in \mathcal{L}_2$. By Barbalat's lemma [28, Lemma 8.2] we have $\tilde{y}_l \to 0$ as $t \to \infty,\quad 2 \le l \le \rho$. Now, system (33a) can be thought of as a nominal system with input $[\tilde{y}_2^T\ 0^T]^T$. Here, the input $\tilde{y}_2$ goes to zero as $t \to \infty$, and using similar arguments in the Theorem 3, exponential stability of the nominal system can be established, and hence using the arguments of input-to-state stability, we have $y_{i,1}(t) \to \mathcal{S}$ and $y_{i,2}(t) \to -\frac{1}{h}\hat{\mathcal{L}}y_{i,1} \in \mathcal{S}\ \forall i \in \mathcal{V}$. □

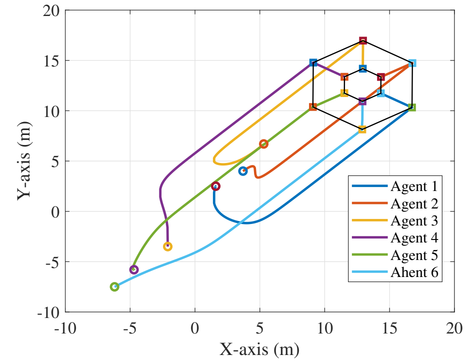


Fig. 3. Trajectory of single integrators performing translation and scale maneuvers using control law (6).

IV. Simulations

In this section, simulations are provided for illustration.

A. Complex Laplacian based maneuvering

The reference shape (p^*) is considered to be a hexagon in $\mathbb{C}$, which is defined by the the positions $p_1^* = -\cos(\pi/3) + \iota\sin(\pi/3), p_2^* = \cos(\pi/3) + \iota\sin(\pi/3), p_3^* = 1, p_4^* = \cos(\pi/3) - \iota\sin(\pi/3), p_5^* = -\cos(\pi/3) - \iota\sin(\pi/3), p_6^* = -1$. The initial position of the six agents are $p_1(0) = 1.6 + \iota 2.5, p_2(0) = 3.7 + \iota 0.4, p_3(0) = 5.3 + \iota 6.7, p_4(0) = -2.1 - \iota 3.5, p_5(0) = -4.7 - \iota 5.8, p_6(0) = -6.2 - \iota 7.5$. The constant disturbances for single integrator dynamics are taken to be $\delta_1 = 1 + \iota 2, \delta_2 = -1 + \iota 0.1, \delta_3 = 0.2 + \iota 0.1, \delta_4 = 0.9 + \iota 0.4, \delta_5 = 2.3 - \iota 0.5, \delta_6 = 0.5 + \iota 2.1$. The interaction topology among the agents is a 2-rooted graph with each agent having the previous two agents as the in-neighbors, i.e., $\mathcal{N}_1 = \{5,6\}, \mathcal{N}_2 = \{1,6\}, \mathcal{N}_3 = \{1,2\}, \mathcal{N}_4 = \{2,3\}, \mathcal{N}_5 = \{3,4\}, \mathcal{N}_6 = \{4,5\}$. Then, the perturbation weights for the control law are designed by solving (8) through the choice of appropriate velocity $v_i^*, i \in \mathcal{V}$ for translation, rotation, scale maneuvers. The parameter h is set as 10.

In Fig. 3, the agents perform a translational maneuver for the first 30 sec., followed by a scaling maneuver for 20 sec. Here, the velocities for designing the perturbation weights are $v_i^* = 1 - \iota 1\ \forall i$ for first 30 sec., followed by $v_i^* = p_i^*$ for the next 20 sec.

In Fig. 4, the agents are performing rotational motion, keeping agent 3 as their center. The desired velocities for different agents are $v_1^* = \iota(p_1^* - p_3^*), v_2^* = \iota(p_1^* - p_3^*), v_3^* = 0, v_4^* = \iota(p_4^* - p_3^*), v_5^* = \iota(p_5^* - p_3^*), v_1^* = \iota(p_6^* - p_3^*)$.

B. Affine formation maneuvering

The reference shape (p^*) is considered to be a hexagon in $\mathbb{R}^2$, which is defined by the the positions $p_1^* =$

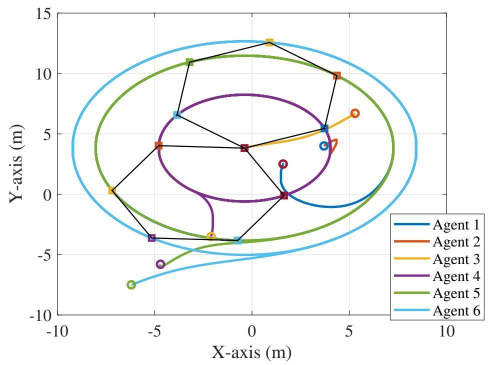


Fig. 4. Trajectory of single integrators performing a rotation maneuver making agent 3 as their center using control law (6).

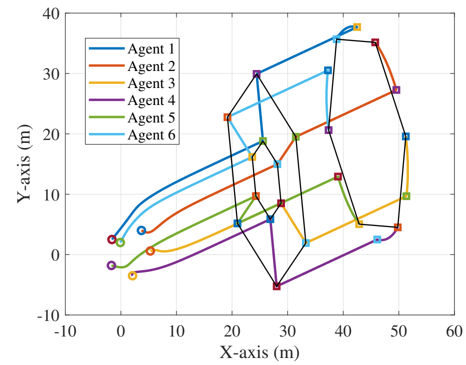


Fig. 5. Trajectory of single integrators performing translation, scale, and rotation maneuvers using control law (19).

$[-\cos(\pi/3)\ \ \sin(\pi/3)]^T, p_2^* = [\cos(\pi/3)\ \ \sin(\pi/3)]^T, p_3^* = [1\ \ 0]^T, p_4^* = [\cos(\pi/3)\ \ -\sin(\pi/3)]^T, p_5^* = [-\cos(\pi/3)\ \ -\sin(\pi/3)]^T, p_6^* = [-1\ \ 0]^T$. The initial position of the six agents are $p_1(0) = [-1.6\ 2.5]^T, p_2(0) = [3.7\ 4.0]^T, p_3(0) = [5.3\ \ 0.6]^T, p_4(0) = [2.1\ \ -3.5]^T, p_5(0) = [-1.7\ \ -1.8]^T, p_6(0) = [-0.1\ 2]^T$. The constant disturbances for single integrator dynamics are taken to be $\delta_1 = [3\ 4]^T, \delta_2 = [1\ \ -1]^T, \delta_3 = [1\ 0]^T, \delta_4 = [-1\ 0]^T, \delta_5 = [2\ \ -1]^T, \delta_6 = [1\ 2]^T$. The interaction topology among the agents is a 3-rooted graph, which is the same as in Fig. 2(b), i.e., each agent has the previous three agents as the in-neighbors. Then, the perturbation weights for the control law are designed by solving (20b) through the choice of appropriate velocity $v_i^*, i \in \mathcal{V}$ for translation, rotation, scale, and shear transformations. The parameter h is set as 10. Fig. 5 and Fig. 6 illustrate the simulations related to the single integrators. In Fig. 7, a simulation corresponding to higher-order (third-order) integrators is shown.

In Fig. 5, the agents perform a translational maneuver for the first 70 sec., followed by a scaling maneuver for 10 sec., which is further followed by a translational maneuver for 20 sec., and finally, a rotational maneuver is performed for 6 sec. Here, the velocities for designing the perturbation weights are $v_i^* = [-1\ \ -1]^T\ \forall i$ for first 70 sec., followed by $v_i^* = -p_i^*$ for the next 10 sec. Subsequently, $v_i^* = [-1\ -1]^T\ \forall i$ for next 20 sec., and finally $v_i^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} p_i^*$.

In Fig. 6, first, a translational maneuver is performed by the single integrators, followed by a shearing maneuver. Here, we have taken $v_i^* = [-0.5\ \ -0.8]^T$, and $v_i^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} p_i^* \forall i$ for the two phases, respectively.

In Fig. 7 the agents perform scaling and rotational maneuvers simultaneously which gives rise to a spiral pattern. Here, $v_i^* = -\frac{1}{2}p_i^* + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} p_i^*\ \forall i$.

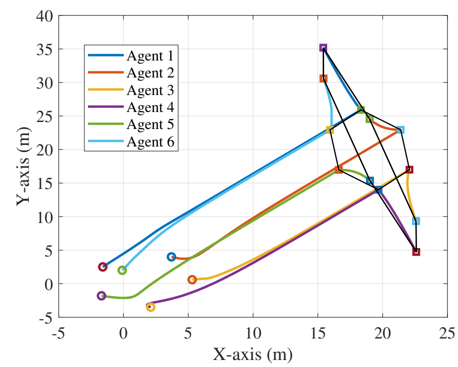


Fig. 6. Trajectory of single integrators performing translation and shear maneuvers using control law (19).

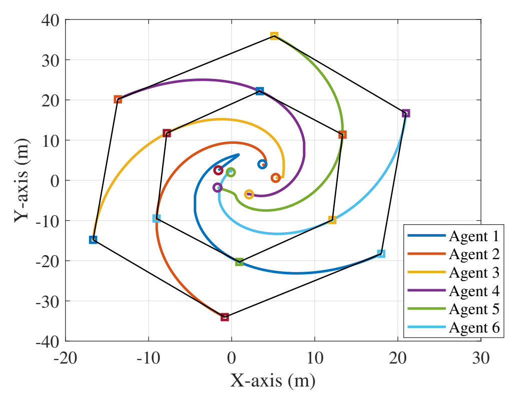


Fig. 7. Trajectory of 3-rd order integrators performing scaling and rotational maneuver yielding a spiral pattern using control law (32).

V. Conclusion

In this work, we have proposed control laws for leaderless maneuverings, considering a directed interaction topology between the agents. With a modification of the interaction graph, i.e., by adding in-neighbors to the root nodes, the maneuvering force is generated to the root nodes. The control laws are fully distributed in nature. For maneuvering no stabilizing diagonal matrix is required. Hence, the full knowledge of the interaction topology among the agents is not required, which was not the case in previously reported literature. In complex laplacian-based maneuvering, translation, rotation, and scale formation maneuvering are reported for planar agents. In affine formation maneuvering translation, rotation, scale, and sheer formation maneuverings are reported for both 2-D and 3-D cases. For higher-order systems, control laws are also proposed, and stability is analyzed.

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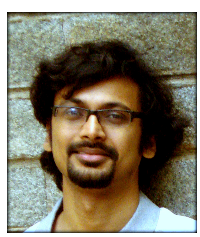

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