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Employing 0-1 test for chaos to characterize the chaotic dynamics of a generalized Gauss iterated map

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Abstract. This work deals with the numerical analysis of the nonlinear dynamics of a Gauss iterated map in a generalized form, obtained with the inclusion of two extra parameters, which makes the map dynamic behavior quite different from the original system. For the exploration of such dynamics, the 0-1 test for chaos is employed to identify intervals of parameters associated with chaotic and regular behavior. The validity of the chaoticity-regularity classification obtained with 0-1 test is confirmed by comparison with bifurcation diagrams. Contour maps reveal regions where chaotic behavior could be observed when two parameters are considered for the analysis.

Palavras-chave. nonlinear dynamics, chaotic system, generalized Gauss iterated map, 0-1 test for chaos

1 Introduction

Nonlinear systems are constantly studied in the scientific literature as attempts to reproduce complex phenomena with greater reliability. Several of these systems are highly dependent on the parameters and/or initial conditions and may exhibit chaotic behavior for some configurations [3,6]. Since chaotic behavior may or may not be useful in applications of interest, it is crucial to identify under what conditions such behavior can manifest in a dynamical system. Such knowledge is also important because, depending on the system response sensitivity to minor disturbances, considerations should be given to the measurement errors effects on the nominal values of the system parameters or initial

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conditions, in order to ensure the consistency between theoretical and computational model predictions for the real system behavior [9, 10].

When dealing with a system for which the dynamics is unknown, the exponents of Lyapunov [3, 6] are often used for classification of the dynamic regime of operation in regular or chaotic. However, the computational cost associated with this method is very high, due to the required eigenvalue calculations. So, when in possession of a large system or costly computing system, the proposed 0-1 test for chaos by Gottwald and Melbourne [4] shows up a good balance between theoretical certification and fast processing. The 0-1 test for chaos has been widely used to identify areas of interest from system parameters and initial conditions, as well as discrimination of attractors [2, 7]. In this sense, this paper intend to use the 0-1 test for chaos to identify regions of chaos or regularity for the parameters of a generalized Gauss iterated map, in an attempt to obtain to get a better understanding about its qualitative behavior in comparison with the standard Gauss map.

2 Gauss iterated map

2.1 Map description

The standard Gauss iterated map uses a zero-mean unit-amplitude Gaussian as evolution law, having only two parameters: (i) the width α, interpreted as the variance inverse in a probabilistic contexts; and (ii) the offset β [8]. Therefore, the map evolution law is written as

$$x_{n+1} = exp\left(-\alpha x_n^2\right) + \beta \,. \tag{1}$$

Here this map generalized by adding two other parameters, the scale γ, and the mean δ. Supplementing this map with an initial condition $x_0 = a$, the evolution law of this new generalized Gauss map is described by the following (discrete) initial value problem

$$\begin{cases} x_{n+1} &= \gamma\, exp\left(-\alpha(x_n - \delta)^2\right) + \beta \,, \\ x_0 &= a \,. \end{cases} \tag{2}$$

2.2 Presence of chaos and regularity

Although regions of chaotic behavior are well known for the standard Gauss map, the addition of two new parameters open a range of possible new scenarios. To demonstrate this greater versatility, Figure 1 shows a bifurcation diagram of the dynamical system described (2), where it can be observed a qualitative variation of the response as γ assumes 3001 values in the range $-10 \le \gamma \le 10$, with $\alpha = 4.9$, $\beta = 0.5$, $\delta = 0$, $x_0 = 0.2$, for a temporal evolution of the map with 5000 steps. In this figure, three scenarios can be highlighted: (i) regular dynamics of period 1 at $\gamma = 1$; (ii) regular dynamics of period 10 at $\gamma = -2.9$; (iii) achaotic dynamics at $\gamma = -1.05$. Also remember that, with $\delta = 0$ and $\gamma = 1$, the response is equivalent to that of the standard Gauss map.

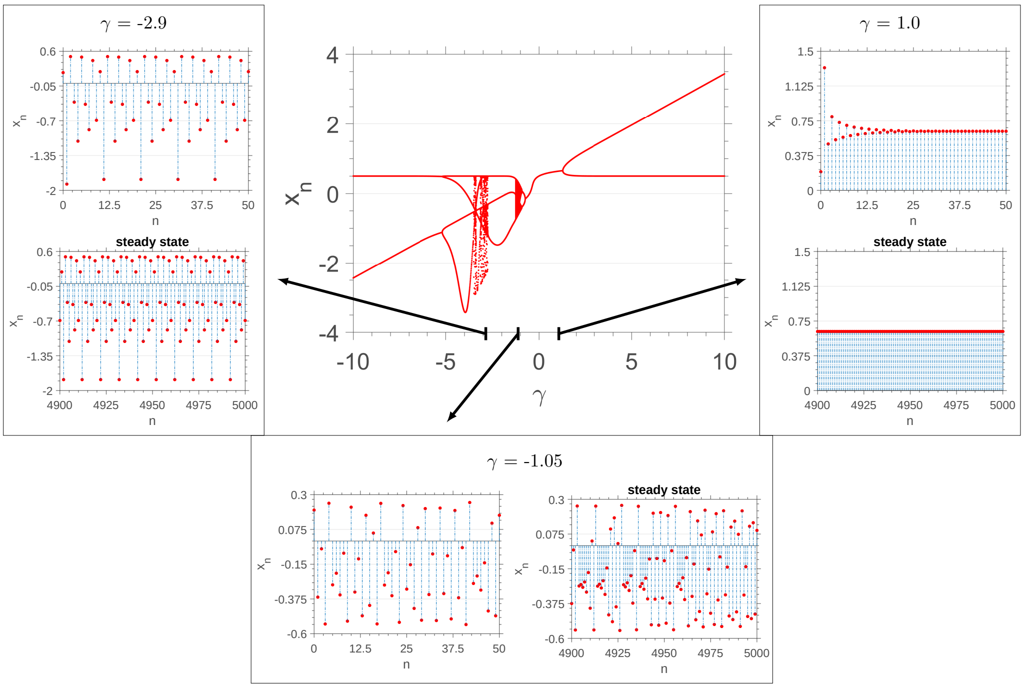

Figure 1: Bifurcation diagram of the generalized Gauss iterated map with $\alpha = 4.9$, $\beta = 0.5$, $\delta = 0$, $x_0 = 0.2$ and $-10 \leq \gamma \leq 10$.

3 The 0-1 test for chaos

As can be seen in Figure 1, varying γ one can find complex behavior, with regions of chaos and regularity switching faster. Thus, it is complicated to use only bifurcation diagrams to identify them. To address this kind of problem more conclusively, classification tests are valuable allies. In general, the most widespread method in the literature is the classification by Lyapunov exponents [3,6], which is based on measuring sensitivity of the response by means of a construction made with the eigenvalues of the gradient of the system evolution law, from which a negative value is a discriminator in cases of chaotic dynamics. However, the processing associated with calculating eigenvalues for a high number of problems is very costly. So lower computational cost alternatives have been sought in the literature. One that has been well used is the 0-1 test for chaos developed by Gottwald and Melbourne [4].

This binary test for chaos identification applies a change of coordinates from physical coordinates $(\phi(t), \dot{\phi}(t))$ to a pair of extened coordinates $(p_c,\ q_c)$, parameterized by an angle c (see [1] for details). From this new coordinates, a time-averaged mean square displacement M_c is extracted, and the classifier K_c can be calculated. The test is repeated for several values of c, and $K = \texttt{median}\{K_c\}$ is employed as the classifier for the dynamic regime of the time-series $\phi(t)$. It can be demonstrated that, theoretically that $K_c \in \{0, 1\}$, so that, numerically $K \approx 0$ for regular dynamics, and $K \approx 1$ in the chaotic case [4].

Algorithmically, the method depends on the information from three input information, the time-series $\phi(t)$, the number of time-steps N_{dt}, and number of c values N_c. The output is the classifier K. The algorithm steps can be seen in the pseudo code shown in Figure 2.

Discretize $\phi(t)$ to $\phi_j = \phi(t_j)$, $j = 1 : N_{dt}$
Data: ϕ_j, N_{dt}, N_c
for m = *1:N_c* **do**
    Chosen a value for $c \in [0, 2\pi)$
    for n = *1:N_{dt}* **do**
        $p_n(c) \leftarrow \sum_{j=1}^{n} \phi(t_j) cos(jc)$ // Apply the change
        $q_n(c) \leftarrow \sum_{j=1}^{n} \phi(t_j) sin(jc)$ // of coordinates
        $M_n(c) \leftarrow \frac{1}{N} \sum_{j=1}^{n} \left([p_{j+n}(c) - p_j(c)]^2 + \ldots\right.$ // Compute the time-averaged
        $\left. [q_{j+n}(c) - q_j(c)]^2\right)$ // mean square displacement
    end
    $\mathbf{M} \leftarrow (M_1, M_2, \ldots, M_{N_{dt}})$ // Assign mean square displacements vector
    $\mathbf{t} \leftarrow (t_1, t_2, \ldots, t_{N_{dt}})$ // Assign the discrete time vector
    $K_m \leftarrow \frac{cov(\mathbf{t},\mathbf{M})}{\sqrt{var(\mathbf{t})\, var(\mathbf{M})}}$ // Assign m-th value of classification
end
$\mathbf{K} \leftarrow (K_1, K_2, \ldots, K_{N_c})$ // Assign classifiers vector
$K \leftarrow median(\mathbf{K})$ // Compute final classifier
Result: K

Figure 2: Pseudo code for 0-1 test for chaos algorithm.

4 Numerical experiments

4.1 Validation of 0-1 test classifier

The theoretical certification that the chaos classification method used in this work is guaranteed when the N_{dt} and N_c are well chosen. Small or large values of these parameters can compromise the convergence of the estimator. Thus, a balance between these quantities must be found for classification to be reliable. Obviously, the optimal values for then is problem dependent.

A large dataset for c values is used for test the accuracy of the 0-1 test for chaos in the generalized Gauss map. Some results are shown in Figures 3 and 4, where one can see the extended dynamics in (p, q) coordinates, the evolution of the square distance MSD, and the classifier samples K_c as function of the values of c. In Figure 3 one can see that, for $\alpha = 4.9$, $\beta = 0.5$, $\gamma = -1.05$ and $\delta = 0$, the system behavior is chaotic, since the extended dynamics in (p, q) coordinates is diffusive [1, 4]. Note from Figures 3b and 3c that 0-1 test for chaos correctly detect chaotic behavior. On other hand, regularity can observed in Figure 4, once the extended dynamics is limited [1, 4].

4.2 Contour maps

With the classification test properly calibrated, more extensive results about the dynamics of the generalized Gauss map with respect to the values of γ and δ can be extracted. A broad analysis is indicated by the contour maps in Figure 5. The regular behavior is

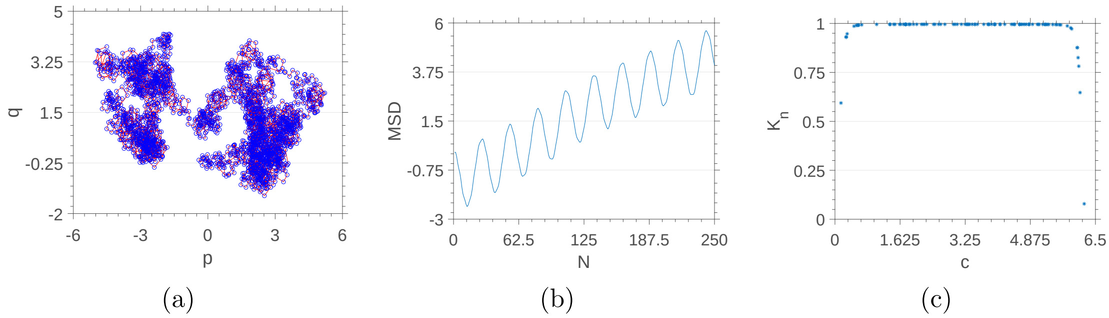

Figure 3: Test 0-1 for chaos with $\alpha = 4.9$, $\beta = 0.5$, $\gamma = -1.05$ and $\delta = 0$. (a) extended dynamics in (p, q) coordinates, (b) the evolution of the square distance MSD, and (c) the classifier samples K_c as function of the values of c.

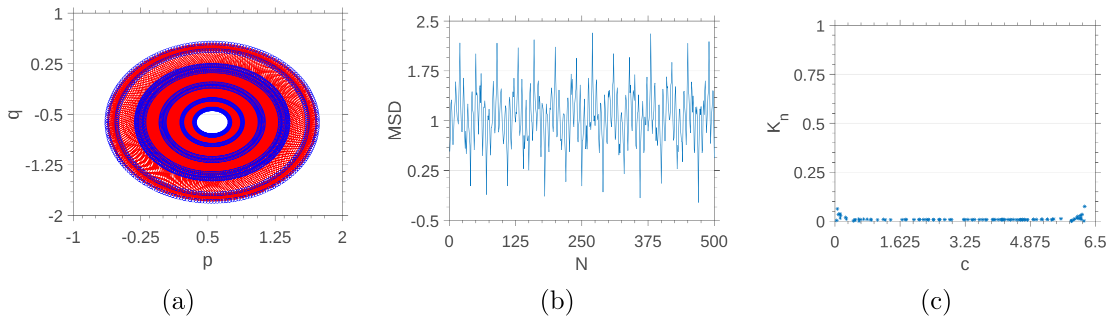

Figure 4: Test 0-1 for chaos with $\alpha = 4.9$, $\beta = 0.5$, $\gamma = 1$ and $\delta = 0$. (a) extended dynamics in (p, q) coordinates, (b) the evolution of the square distance MSD, and (c) the classifier samples K_c as function of the values of c.

indicated by the darkest blue points and chaos by the red ones. The parameters used are $\alpha = 4.9$, $\beta = 0.5$, $\gamma \in [-5, 5]$, $\delta \in [-1, 1]$, $x_0 = 0.2$, with 5000 iterations, $N_c = 300$, and a resolution of 100×100 points in (γ, δ) space. The result revels two regions where chaos is more present: $(\gamma, \delta) \in [-3, 1] \times [-1, 0]$ and $(\gamma, \delta) \in [4, 5] \times [0, 0.5]$.

5 Final remarks

In this work a generalization for the Gauss iterated map is proposed. As the dynamics becomes more complex in this generalized form, a careful analysis of the possible dynamic regimes is necessary. The 0-1 test for chaos is explored to classify the system dynamics and used to generate contour maps that revel chaotic regions in the parameters space. These results allowed conclude that two regions with a huge presence of chaos exist.

In future works the authors intend to use the 0-1 test to generate basins of attraction with respect to the parameters in order to study the system sensitivity to small chances on their nominal values.

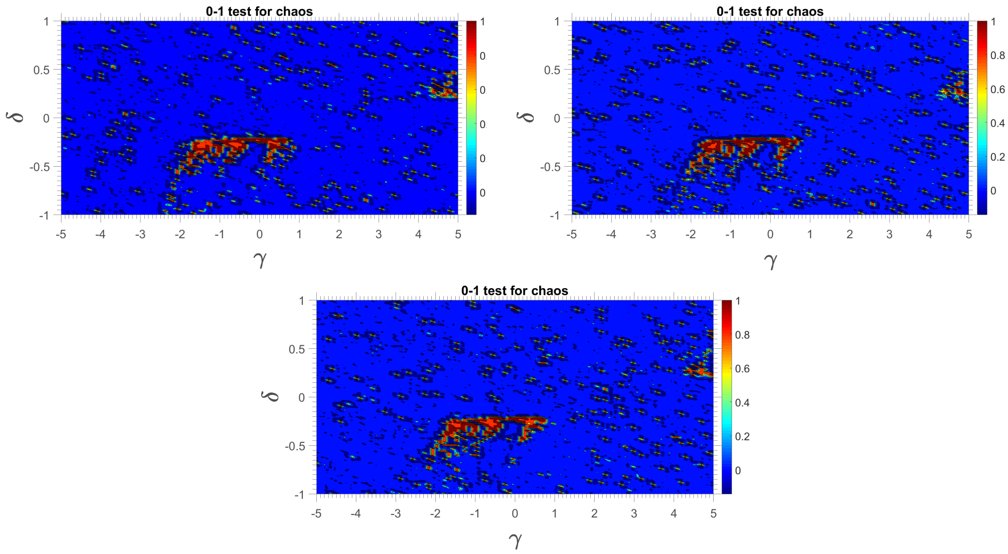

Figure 5: Contour maps for $-5 \leq \gamma \leq 5$, $-5 \leq \delta \leq 5$, with $\alpha = 4.9$, $\beta = 0.5$, and $x_0 = 0.2$.

Acknowledgments

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