Tresca vs. von Mises: Which failure criterion is more conservative in a probabilistic context?

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This tutorial examines the failure theories of Tresca and von Mises, both of which are crucial for designing metallic structures. Conventionally, Tresca is regarded as more conservative than von Mises from a deterministic perspective. This tutorial, however, introduces a different viewpoint, presenting a scenario where von Mises' theory may appear more conservative when variability in the mechanical system parameters is considered. This often overlooked aspect is not extensively addressed in standard textbooks on solid mechanics and the strength of materials. The tutorial aims to shed light on the non-negligible probability where von Mises' criterion yields a smaller equivalent stress than Tresca, thus being more conservative. It underscores the importance of integrating probabilistic considerations into stress analyses of solids, offering valuable insights for the education of structural mechanics.

Keywords: Probabilistic Design; Material Failure Theories; von Mises Criterion; Tresca Criterion; Uncertainty Quantification

1 Introduction

Material failure theory is central to understanding the conditions under which a solid body (e.g., beams, plates, shells, machine parts, etc.) fails in response to external loads [1–6]. There are several failure criteria, each with specific applicability depending on the mechanical problem characteristics. For instance, the Tresca and von Mises criteria are often applied in the context of metallic materials, while others like Mohr-Coulomb and Bresler-Pister are more commonly associated with concrete-made solids and Drucker–Prager is suited for contexts of plasticity [7–11].

Among these, Tresca and von Mises criteria hold considerable relevance in the context of polycrystalline isotropic materials, as reflected in their frequent discussion in solid mechanics and strength of materials textbooks [12–18]. In a purely deterministic context, the Tresca criterion is more conservative than von Mises since it defines a smaller region of elastic behavior in the principal stress state space (Tresca polyhedron is inscribed in von Mises ellipsoid).

However, this perspective changes when considering the variability in loading, material properties, or geometric dimensions in a mechanical analysis [19–21]. In a structural analysis problem with uncertainties, it becomes apparent that the assertion of Tresca always being more conservative is not an absolute truth. Cases exist where von Mises can yield (in a probabilistic sense) smaller equivalent stress than Tresca, making it, in those instances, more conservative [22].

Understanding which theory results in smaller equivalent stress in certain scenarios is crucial, as it affects the material's effective usage and structural safety. This consideration seems to be overlooked in conventional textbooks, suggesting a gap in structural mechanics educational literature.

Addressing this issue, this tutorial aims to shed light on scenarios where von Mises appears more conservative when uncertainties in geometric dimensions and force conditions are considered. Our analysis compares Tresca and von Mises's failure criteria for a material point of a solid body experiencing a plane stress state, providing a deeper understanding of structural design under uncertainties.

2 Tresca and von Mises failure criteria

To understand the conditions leading to material failure, we start with the representation of the stress state at a material point in a solid body. This is described by a stress tensor, represented in matrix form as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}, \qquad (1)$$

where σ_x, σ_y, and σ_z denote the normal stresses, while τ_{xy}, τ_{xz}, and τ_{yz} represent shear stresses.

The computation of principal stresses and their respective directions involves solving the eigenvalue problem represented by

$$\left([\boldsymbol{\sigma}] - \sigma_i [\boldsymbol{I}] \right) \boldsymbol{v}_i = \boldsymbol{0} \qquad i = 1,2,3 \;, \qquad (2)$$

wherein $[\boldsymbol{I}]$ is the identity tensor and $(\sigma_i, \boldsymbol{v}_i)$ symbolizes an eigenpair that determines the i-th principal stress and its direction. Due to their invariant nature, the resulting principal stresses σ_1, σ_2, and σ_3 are integral to defining the most prevalent failure criteria [23].

Notably, the crux of a failure theory lies in representing a complex, possibly three-dimensional stress state using an equivalent, one-dimensional stress state (Fig. 1). The essence is to create an equivalent stress state that can be compared to a tensile test's uni-axial mechanical state. A failure is assumed to occur when the equivalent stress state mirrors the yielding experienced during the tensile test.

For instance, the Tresca equivalent stress σ_T, is defined as half the maximum absolute difference among the principal stresses

$$\sigma_T = \frac{1}{2} \max \left\{ |\sigma_1 - \sigma_2|, \, |\sigma_1 - \sigma_3|, \, |\sigma_2 - \sigma_3| \right\}, \qquad (3)$$

while the von Mises equivalent stress σ_M, is defined as the root of half the sum of squares of differences among principal stresses

$$\sigma_M = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}} \,. \qquad (4)$$

The yielding criteria are then established based on these equivalent stresses

$$\sigma_T < \frac{S_Y}{2} \;\text{ or }\; \sigma_M < S_Y \,, \qquad (5)$$

with the condition that the material remains elastic when the equivalent stress is less than the material yield stress S_Y.

Upon simultaneous application of both the Tresca and von Mises criteria, four theoretical scenarios (Fig. 2) can arise:

I $\sigma_T/S_Y < 1/2$ (elastic) and $\sigma_M/S_Y < 1$ (elastic);
II $\sigma_T/S_Y \geq 1/2$ (plastic) and $\sigma_M/S_Y < 1$ (elastic);
III $\sigma_T/S_Y \geq 1/2$ (plastic) and $\sigma_M/S_Y \geq 1$ (plastic);
IV $\sigma_T/S_Y < 1/2$ (elastic) and $\sigma_M/S_Y \geq 1$ (plastic).

They include combinations of elastic and plastic behaviors based on the equivalent stresses and yield stress ratio. The common interpretation of these scenarios is that while areas I and III represent agreement between the two criteria, regions II and IV are zones of discrepancy.

In traditional textbooks, discussions generally focus on the region II scenario, where the Tresca criterion is seen as more conservative due to its smaller elastic region. Region IV, representing the theoretical possibility of von Mises' criterion being more conservative, is overlooked as it is an empty set in a deterministic context due to mechanical constraints.

However, this tutorial highlights that once parametric uncertainties are accounted for, the possibility of a region IV scenario arises probabilistically. This often-overlooked scenario will be demonstrated using two examples (Fig. 3).

3 Stress state under uncertainties

We examine two distinct mechanical systems: a shaft and a cylindrical pressure vessel. The shaft has a circular cross-sectional area, with a diameter $d = 80$ *mm*, and it undergoes combined axial-torsional loading (axial load

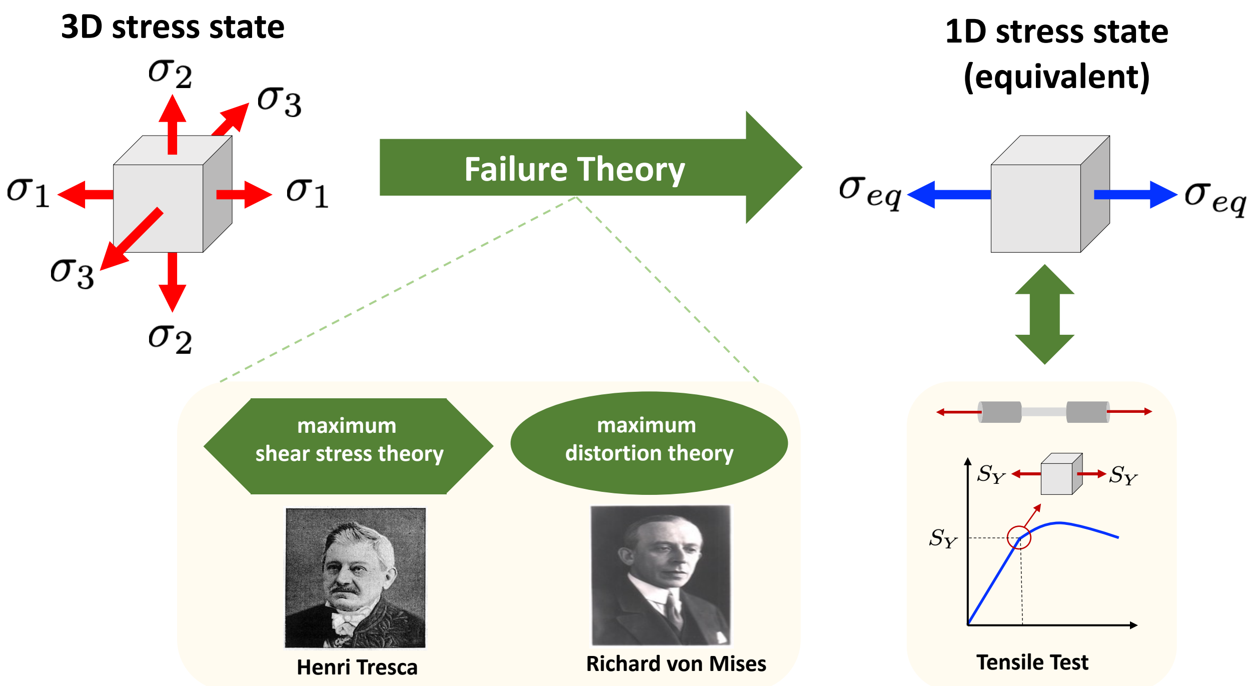

Fig. 1. Schematic representation of the conceptual idea of a failure theory: it seeks to represent a three-dimensional stress state by a uniaxial stress state equivalent to the original state. The failure event is assumed if this equivalent state corresponds to the material's yield stress in a tensile test.

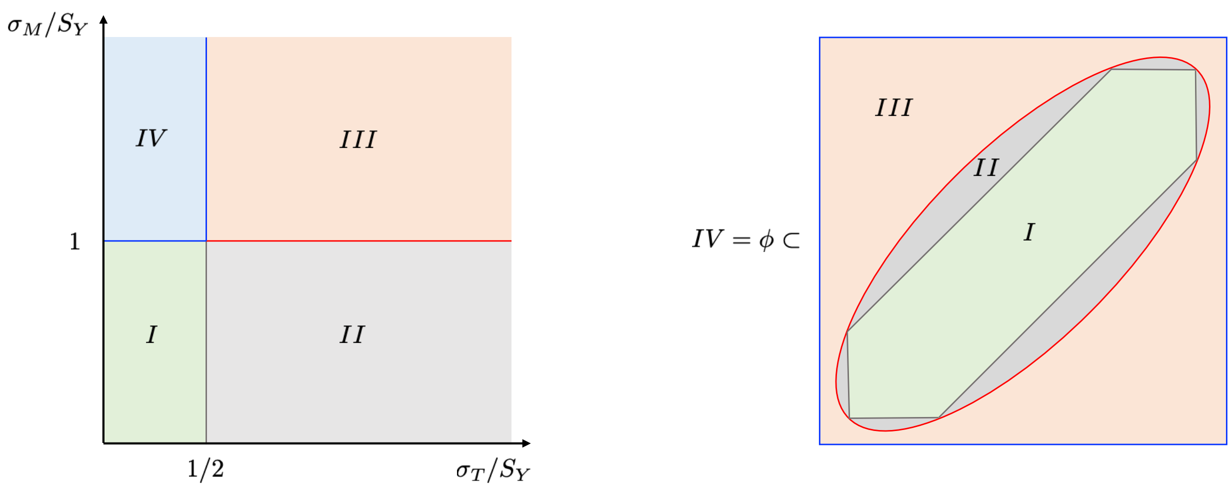

Fig. 2. Illustration of the four hypothetical scenarios when the Tresca and von Mises criteria are used together.

$P = 25\ kN$ and torsional load $T = 8\ kNm$). This loading induces a plane stress state at a specific material point in the shaft, defined by $\sigma_x = P/A$, $\tau_{xy} = (Td)/(2J)$, and $\sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$. The cross-sectional area is $A = \pi d^2/4$, and the polar moment of inertia is $J = \pi d^4/32$. Simultaneously, we investigate a cylindrical pressure vessel characterized by a radius $r = 0.5\ m$ and a wall thickness $t = 25\ mm$. It is subjected to an internal pressure $p = 10\ MPa$. This component provides a contrasting study to the shaft due to its different geometry and stress conditions, specifically, the stresses induced by internal pressure. Both the shaft and the pressure vessel are constructed from materials with a yield strength $S_Y = 210\ MPa$.

This parallel examination allows a comprehensive exploration into the stress behaviors and failure criteria of different mechanical components under varied loading conditions and geometries, contributing to our understanding of the comparative conservativeness between the von Mises and Tresca criteria under uncertainties in system parameters. Notably, in the case of the vessel, the nominal von Mises stress is observed to be exceedingly close to the yield limit, enhancing the relevance of a meticulous and thorough examination of stress conditions and failure criteria in design processes.

We extend our exploration to discern potential outcomes when the distinct parameters of a shaft and a pressure vessel are imbued with uncertainties. For the shaft, we scrutinize

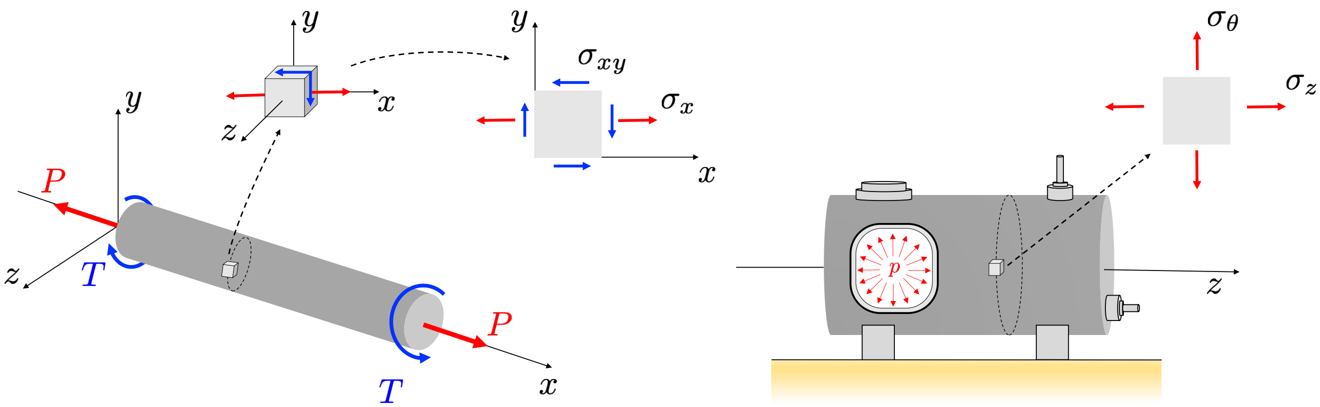

Fig. 3. A solid shaft under combined axial-torsional loading (left), resulting in a plane stress state at a selected material point, juxtaposed with a cylindrical pressure vessel subjected to internal pressure (right), illustrating the differing stress states induced in each component.

the uncertainties in diameter d, axial load P, and torque T. We assume d to follow a uniform distribution between 78 and 82 mm, while P and T adhere to Gaussian distributions centered around their nominal values. These parameters can exhibit a coefficient of variation of 5% or 25%. Concurrently, we assess the uncertainties in the pressure vessel, focusing on the internal pressure p and the wall thickness t. The internal pressure p is presumed to follow a Gaussian distribution, fluctuating around its nominal value. Conversely, the wall thickness t is ascribed a uniform distribution, ranging from $t_{min} = 24$ mm to $t_{max} = 26$ mm, symbolizing the plausible variations during the manufacturing process. Similar to the shaft, the pressure loading is subject to coefficients of variation of 5% or 25%.

In the interest of simplicity and to streamline the comparative analysis, we consider all these random parameters as statistically independent within both mechanical components. In a consequence of the variability, the induced equivalent stresses in the shaft and the pressure vessel materialize as random variables. Their distributions and consequential statistical moments are estimated using Monte Carlo simulations [24, 25] with 1024 samples. For this comparative analysis, samples of the uncertain parameters are generated, and subsequent equivalent stress samples are calculated using Eqs. (3) and (4), providing a probabilistic perspective on the relative conservativeness of the von Mises and Tresca criteria in diverse mechanical setups (Fig 4).

Figure 5 delineates a comprehensive statistical portrayal of the stress states and the subsequent equivalent stresses under both Tresca and von Mises criteria for a shaft and a pressure vessel. The figure is structured so that the left column represents the shaft, and the right column elucidates the pressure vessel under a coefficient of variation of 25%. The first row displays the failure curves of both theories, juxtaposed with the stress state samples, providing a comparative insight into the inherent stresses in the respective components. The second row showcases the marginal probability distributions for the non-zero principal stresses, maintaining a uniform approach to the comparative analysis between the shaft and the pressure vessel. Transitioning to the third row, the probability distributions for the equivalent stresses are demonstrated, offering a visual representation of the variations and dispersions in the equivalent stresses due to uncertainties. The nominal (deterministic) values are marked by vertical lines, accentuating the deviations induced by the uncertainties. Finally, the fourth row illustrates the corresponding probability distributions for the design safety factors, calculated as the yielding ratio to equivalent stresses. This row highlights the variations in the safety margins due to the propagated uncertainties in the mechanical systems.

Design safety factors are pivotal in understanding the resilience of the mechanical components under operational loads. Under nominal conditions, the component's ability to withstand operational loading varies depending on the criterion applied, with discrepancies becoming more evident as parameter uncertainties are considered. The varied dispersions in equivalent stress values and safety factors, depicted in Fig. 5, are manifestations of the uncertainties introduced during manufacturing and loading.

For a dispersion level of 25%, it is observed that some stress states transition into the plastic regime, introducing an overlap of samples and consequently highlighting situations where the classical relation, where Tresca is more conservative than von Mises, is challenged. This refined approach to analyzing stress states under uncertainties provides enhanced insights into the reliability and robustness of design methodologies in structural mechanics.

4 Discussion

The observation of von Mises samples surpassing the Tresca line and the consequent overlapping of distributions reveal instances in an environment fraught with variability where von Mises is demonstrably more conservative than the deterministic value given by the Tresca criterion. While this occurs infrequently—approximately 4% of the time in the provided examples—it bears critical implications, especially in design scenarios where such instances, though rare, can have serious repercussions (e.g., component failure in aerospace or nuclear systems).

This illustration is not intended to portray von Mises as ubiquitously more conservative but seeks to highlight the po-

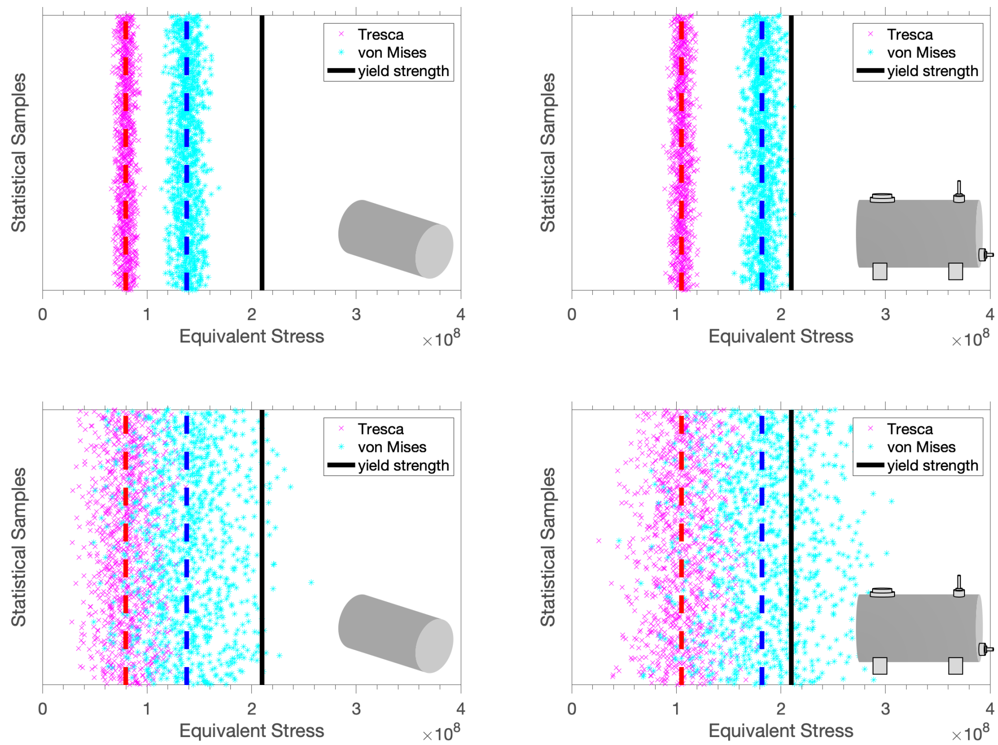

Fig. 4. Statistical samples of equivalent stresses, derived from Tresca and von Mises criteria, for both the shaft (left) and the pressure vessel (right), under coefficients of variation of 5% (top) and 25% (bottom). These illustrations provide insights into stress distributions and variabilities under differing uncertainty levels in mechanical components.

tential pitfalls of an uncritical, blanket application of classical failure theories. In designs, particularly those prone to high variability and uncertainty, structural designers might conventionally favor the Tresca criterion to achieve a more conservative design. However, in such variable conditions, reliance on the von Mises criterion could offer enhanced robustness in system specifications.

Addressing these nuanced aspects of design methodology, recent advances in codes governing structures in high seismic zones indicate a progressive shift. They represent a movement from allowable stress-based methods toward semi-probabilistic approaches focusing on "Limit States" over the past decades [26, 27]. This transition underscores a recognition of the significance of incorporating probabilistic assessments in structural design procedures, particularly in scenarios where the consequences of failure are severe.

Integrating probabilistic design tools [19, 28] is crucial in instances marked by high parametric variability. Doing so in conjunction with conservative design approaches, such as limit analysis [29, 30], is paramount to navigating the intricacies of variable environments effectively and avoiding the potential hazards associated with singular reliance on deterministic methods. Unfortunately, incorporating these probabilistic tools and methodologies is often underemphasized in most undergraduate textbooks on solid mechanics or the strength of materials. This lack of emphasis could lead to a diminished appreciation of the necessity to integrate such tools within the foundational frameworks for structural analysis.

While we acknowledge the existence of an undergraduate textbook like Popov [31] that briefly elaborates on probabilistic assessments, we observe a general trend where the majority do not delve deeply into the comparative analysis of failure theories from a probabilistic viewpoint. The scarcity of in-depth treatment of probabilistic tools in undergraduate educational resources underscores a missed opportunity to foster a comprehensive understanding of design principles among structural engineers.

In conclusion, our discourse emphasizes the invaluable role of a balanced approach to design, one that marries deterministic methods with probabilistic considerations, reflecting a holistic understanding of design principles. It is not just about weighing the probabilities but about comprehending the profound implications in real-world scenarios and cultivating an approach to design that is both robust and nuanced, ensuring both the safety and efficiency of structures in variable environments.

5 Concluding Remarks

The role of failure theories in structural mechanics is undeniable, and they represent well-established methodologies within the field. However, their application in a probabilistic context—where variability is crucial—still appears to be underrepresented in standard textbooks. To bridge this gap, our discussion highlighted scenarios where von Mises'

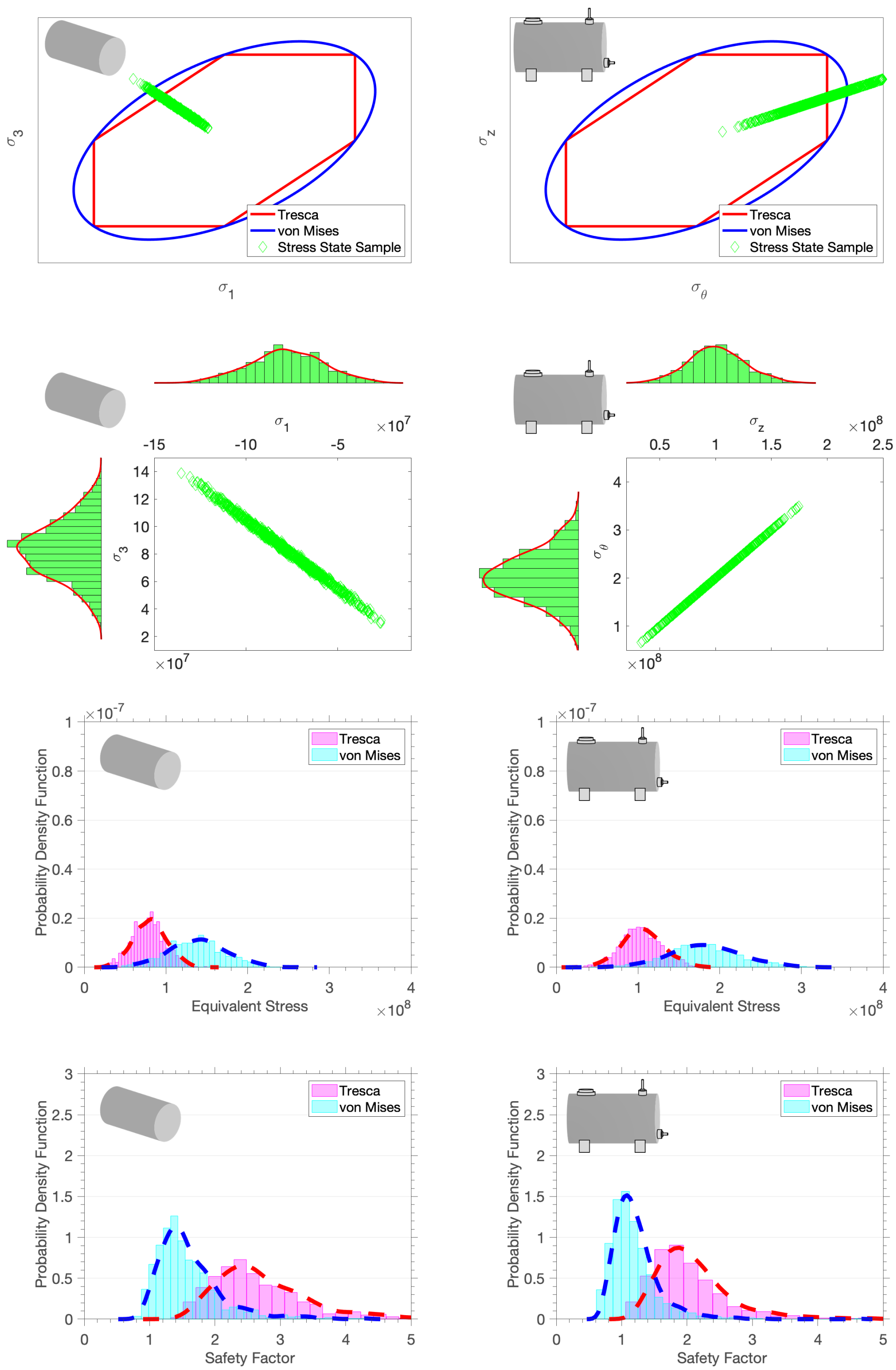

Fig. 5. Comprehensive statistical characterization of the stress states (first and second lines), equivalent stresses (third line), and safety factors (fourth line) associated with the Tresca and von Mises criteria for a shaft (left) and a pressure vessel (right) under a coefficient of variation of 25%. Monte Carlo simulation employed 1024 statistical samples.

criterion could be more conservative probabilistically than the Tresca criterion. We demonstrated this through practical structural systems in which the probability of von Mises stress surpassing the nominal design value obtained via the Tresca criterion is not zero. This underlines the essential role that uncertainty analysis plays in ensuring the robustness of structural projects.

Moreover, the gap observed in the literature regarding this topic suggests the need for a thorough discussion and potential reevaluation of pedagogical programs concerning solid mechanics and the strength of materials. We hope to stimulate this dialogue and contribute to its progression by underscoring the relevance of probabilistic considerations. We anticipate that this discussion will prove helpful in informing future revisions of academic curricula and encourage a broader appreciation for the role of uncertainty analysis in structural design.

Code availability

The simulations of this work used a Matlab code dubbed **FAILURE**. To facilitate the reproduction of the results, this code is available for free on GitHub [32].

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Disclaimer

This manuscript has undergone extensive refinement leveraging artificial intelligence-enabled resources, including Grammarly and ChatGPT, to optimize its linguistic precision and readability. Nevertheless, the authors retain complete accountability for the initial language and expressions employed in the manuscript.

References

[1] Williams, J. F., and Svensson, N. L., 1971. "A rationally based yield criterion for work hardening materials". *Meccanica,* **6**, pp. 104–114.

[2] Clausmeyer, H., Kussmaul, K., and Roos, E., 1991. "Influence of stress state on the failure behavior of cracked components made of steel". *Applied Mechanics Review,* **44**(2), pp. 77–92.

[3] Yu, M.-h., 2002. "Advances in strength theories for materials under complex stress state in the 20th century". *Applied Mechanics Review,* **55**(3), pp. 169–218.

[4] Avilés, J., and Pérez-Rocha, L. E., 2005. "Design concepts for yielding structures on flexible foundation". *Engineering Structures,* **27**, pp. 443–454.

[5] Christensen, R. M., 2013. *The Theory of Materials Failure*. Oxford University Press.

[6] Farzampour, A., and Eatherton, M. R., 2019. "Yielding and lateral torsional buckling limit states for butterfly-shaped shear links". *Engineering Structures,* **180**, pp. 442–451.

[7] Prager, W., 1953. "On the use of singular yield conditions and associated flow rules". *Journal of Applied Mechanics,* **20**(3), pp. 317–320. Published Online: June 4, 2021.

[8] Hosford, W. F., 1972. "A generalized isotropic yield criterion". *Journal of Applied Mechanics,* **39**(2), pp. 607–609.

[9] Christensen, R. M., 2016. "Perspective on materials failure theory and applications". *Journal of Applied Mechanics,* **83**(11), p. 111001. Published Online: August 10, 2016.

[10] Ding, B., and Li, X., 2017. "An eccentric ellipse failure criterion for amorphous materials". *Journal of Applied Mechanics,* **84**(8), p. 081005.

[11] Christensen, R. M., 2020. "The failure theory for isotropic materials: Proof and completion". *Journal of Applied Mechanics,* **87**(5), p. 051001. Published Online: December 11, 2019.

[12] Crandall, S. H., Dahl, N. C., Lardner, T. J., and Sivakumar, M. S., 2012. *An Introduction to Mechanics of Solids: (In SI Units)*, 3rd ed. Tata McGraw Hill Education Private Limited.

[13] Den Hartog, J. P., 1961. *Strength of Materials*, reprinted ed. Dover Publications.

[14] Heyman, J., 1998. *Structural Analysis: A Historical Approach*, illustrated ed. Cambridge University Press.

[15] Goodno, B. J., and Gere, J. M., 2017. *Mechanics of Materials*, 9th ed. Cengage Learning.

[16] Hibbeler, R., 2016. *Mechanics of Materials*, 10th ed. Pearson.

[17] Timoshenko, S., 2002. *Strength of Materials, Part 2: Advanced Theory and Problems*, 3rd ed. CBS Publishers & Distributors Pvt Ltd, India.

[18] Timoshenko, S. P., 1983. *History of Strength of Materials*. Dover Publications.

[19] Elishakoff, I., 2017. *Probabilistic Methods in the Theory of Structures*, 3rd ed. WSPC.

[20] Cunha Jr, A., 2017. "Modeling and Quantification of Physical Systems Uncertainties in a Probabilistic Framework". In *Probabilistic Prognostics and Health Management of Energy Systems*, S. Ekwaro-Osire, A. C. Goncalves, and F. M. Alemayehu, eds. Springer International Publishing, New York, pp. 127–156.

[21] Soize, C., 2017. *Uncertainty Quantification: An Accelerated Course with Advanced Applications in Computational Engineering*. Springer.

[22] Yanik, Y., da Silva, S., and Cunha Jr, A., 2018. "Uncertainty quantification in the comparison of structural criterions of failure". In X Congresso Nacional de En-

genharia Mecânica (CONEM 2018).
[23] Stokes, V. K., 1981. "On the space of stress invariants". *Journal of Applied Mechanics,* **48**(3), pp. 664–666. Published Online: September 1, 1981.
[24] Cunha Jr, A., Nasser, R., Sampaio, R., Lopes, H., and Breitman, K., 2014. "Uncertainty quantification through Monte Carlo method in a cloud computing setting". *Computer Physics Communications,* **185**, pp. 1355–1363.
[25] Kroese, D. P., Taimre, T., and Botev, Z. I., 2013. *Handbook of Monte Carlo methods*, Vol. 706. John Wiley & Sons.
[26] Institution, B. S., 2003. "Structural use of steelwork in building: code of practice for fire resistant design". *British Standards Institution*.
[27] e Trasporti, M. I., 2008. "Ntc (2008) norme tecniche per le costruzioni". *D.M. Ministero Infrastrutture e Trasporti 14 gennaio 2008, G.U.R.I. 4 Febbraio 2008, Roma (in Italian)*.
[28] Raizer, V., and Elishakoff, I., 2022. *Philosophies of Structural Safety and Reliability*. Taylor & Francis.
[29] Baker, J., and Heyman, J., 1980. *Plastic Design of Frames: Volume 1, Fundamentals*. Cambridge University Press.
[30] Heyman, J., 2008. *Plastic Design of Frames: Volume 2, Applications*. Cambridge University Press.
[31] Popov, E., 1998. *Engineering Mechanics of Solids*, 2nd edition ed. Pearson.
[32] Cunha Jr, A., Yanik, Y., Olivieri, C., and da Silva, S., 2023. FAILURE. https://americocunhajr.github.io/FAILURE.