Continuous and discrete operation of potential flow networks with dissimilar sink potentials

Tarak Das [*] **Varghese Kurian** [**]
Sridharakumar Narasimhan [*,***]

[*] *Department of Chemical Engineering, IIT Madras, Chennai, India*
[**] *Department of Chemical and Biomolecular Engineering, University of Delaware, Newark, DE 19716 USA*
[***] *Department of Chemical Engineering & Robert Bosch Center for Data Science and Artifical Intelligence, Indian Institute of Technology Madras, Chennai, TN 600036 India.*
Correspondence: sridharkrn@iitm.ac.in (SN)

Abstract: Nodal demand satisfaction in potential flow networks often requires a scheduled manipulation of edge resistances. The actuators used to implement the resistance variations can broadly be classified into (i) discrete (edge fully OPEN or CLOSED) and (ii) continuous (continuously vary edge resistance). Previously, we presented theoretical bounds on the ratio of time required to transport a given quantum of material with either type of actuators assuming that that all sinks are at the same potential. In the present work, we extend the results to a class of networks with demand nodes at different potentials. Based on our findings, we conjecture that the bounds remain the same irrespective of whether the sinks are at the same potentials or not.

Keywords: Potential flow networks, continuous and discrete actuators, network control MSC 2020: 90B10, Deterministic network models in operations research; 76B75, Flow control and optimization

1. INTRODUCTION

Potential flow networks are common in both natural and engineered systems (Hendrickson and Janson, 1984; Raghunathan, 2013; Ronellenfitsch and Katifori, 2016; Sankar EM and Rengaswamy, 2022; Klimm et al., 2022). In these networks, the potential difference between the upstream and downstream nodes–whether it be mechanical, electrical, or chemical–drives the flow through the edges. The equilibrium flow rates in these networks, computed by simultaneously solving the flow conservation equation at the nodes and the potential drop across the edges, may not be in accordance with the demand at the sinks. To effectively meet the demand, networks are usually engineered to allow variations in edge resistances or node potentials. For an agent modifying the edge resistances, the ability to do so can either be discrete (OPEN/CLOSE an edge) or continuous (continuously vary the edge resistance). For example, turning an electrical switch ON or OFF is a discrete operation. On the other hand, vasodilation–the process of widening a blood vessel to increase the blood flow to a part of the body–is an example of continuous control. Several systems, such as water and gas networks, could have actuators that may either allow discrete or continuous operation depending on the type of valve used.

In prior studies, we compared the minimum time required to transport a given quantum of material with discrete and continuous actuators. The problem was presented for the first time in Velmurugan et al. (2023). Here, for a basic network, we derived theoretical bounds on the ratio of minimum operational times with either type of actuation. The results were later generalized to branched networks of arbitrary size where we showed that the upper bound on the ratio of operational times scaled sub-linearly with the maximum depth (m of the network (Kurian and Narasimhan, 2023), and the flow head relationship is of the form $\delta h = kq^n$. In both studies mentioned above, we assumed that all demand nodes in the system are at the same potential. While this is a fair assumption in some systems (e.g. electrical networks discharging to a common ground), there are systems with the sinks located at different potentials (e.g. water networks transporting water to consumers at different elevations). Therefore, in the present work, by solving the appropriate mathematical programs, we analyze the effect of sink node potential on the ratio of minimal transportation times.

In the rest of the document, we limit our discussion to water distribution networks, an important class of potential flow networks. However, the results are applicable to any other system satisfying the network assumptions given in Section 2.

2. SYSTEM AND MATHEMATICAL MODEL

We consider a water network with two demand nodes supplied from a single source through a system of three pipes as shown in Fig. 1. The source is maintained at a constant potential (referred to as hydraulic head in water networks) of one unit. Without loss of generality, we consider demand node 3 to be at the lowest head of zero units. The other sink, i.e., node 2, is located a head of h_2, where $0 \leq h_2 \leq 1$.

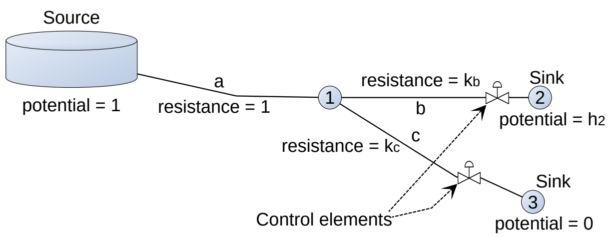

Fig. 1. A schematic of the three pipe network considered in the present work.

We limit our discussion to networks with the potential drop along the edges (Δh) given by the equation:

$$\Delta h = kq^n, \quad n \geq 1 \tag{1}$$

where, k is the edge resistance and q is the flow rate. In water networks, n is usually assumed to be equal to 1.85, and in gas transportation networks, $1.8 \leq n \leq 2$ (Velmurugan et al., 2023; Shiono and Suzuki, 2016). Solving the potential loss equations for the pipes along with the nodal mass conservation equations gives the flow rates along the edges and the potentials at the intermediate nodes of a given network. It may be noted that, even though the head at two nodes and the resistance across one edge is fixed, the system still has enough degrees of freedom to represent any three pipe network with suitable choice of units.

The demand, i.e., the quantum of water that has to be transported to either demand node is given. On operating the network, the flow rates in pipes b and c need not be proportional to the demands at nodes 2 and 3. Hence, control elements (valves) are provided immediately upstream of either demand points to regulate the flow in the network. The network, operating under a setting of the valves, is referred to as a *state of the network*. Identifying the appropriate sequence of such valve operations to meet the consumer demands is referred to as the scheduling problem in water networks.

3. PROBLEM DEFINITION

The control elements in the network may be of discrete or continuous type. In the former, there are only two choices for the resistance added to an edge by a valve: 0 or ∞, representing the OPEN and CLOSED settings respectively. On the other hand, continuous control valves can be kept partially open to offer any resistance in the interval $[0, \infty)$. The time required to meet the nodal demands could vary with the type of actuators available. For a given demand, let the minimum time required to meet the demand using discrete valves be equal to t_d and that using continuous control valves be equal to t_{cv}. We define the ratio of operating times, $\mathbf{R} := \frac{t_d}{t_{cv}}$.

$\mathbf{R}$ has a lower bound of unity as the feasible space of operation using discrete valves is a subset of those using continuous control valves. For a system where the demand nodes are at the same head, the supremum of $\mathbf{R}$ was shown to be equal to $2^{(1-1/n)}$ (Velmurugan et al., 2023; Kurian and Narasimhan, 2023), where n is the exponent in Eq. 1. *The task at hand is to find the supremum of* $\mathbf{R}$ *when* h_2*, the head (potential) at demand node 2 is different from that at demand node 3.* This involves a search over all combinations of resistances along edges b and c, and all combinations of demands at nodes 2 and 3. For every choice of network resistances, scheduling problems have to be solved to identify the optimal schedules for meeting the demand using either types of actuators. Altogether, the mathematical program to identify the supremum of $\mathbf{R}$ is a complex bilevel optimization problem.

In our previous analyses, we used several results from the general behavior of potential flow networks to simplify the search space primarily relying on the observation that the set of achievable flow rates is convex (Proposition 4 in Kurian and Narasimhan (2023)). In a network with demand nodes being at different heads, the set of achievable flow rates is not guaranteed to be convex. Hence, we combine a subset of the existing analytical results with additional numerical simulations to elucidate the behaviour of the class of networks considered here.

4. THEORETICAL RESULTS AND MATHEMATICAL PROGRAM

Searching over the entire space of pipe resistances, network demands, and the respective optimal operational schedules to identify the upper bound on $\mathbf{R}$ is a laborious task. Hence, we simplify the search space using two theoretical results. Following this, we formulate mathematical problem and solve it numerically to find the supremum of $\mathbf{R}$.

Proposition 1: $\mathbf{R}$ *approaches its supremum when the supply using continuous control valves has only one state active.*

A proof of the proposition is available in Velmurugan et al. (2023). The argument is on the basis that, for any scheduled operation of continuous control valves ($\mathcal{P}$) that involves multiple states, the resultant $\mathbf{R}$ would be a convex function of $\mathbf{R}_p$'s, where $\mathbf{R}_p$ denote the ratio of operational times when considering only the demand supplied by individual states p, $p \in \mathcal{P}$. The proposition narrows down the search space to only those demands that can be met using a constant setting of the continuous control valves.

Proposition 2: *In order to supply a desired amount of water in the minimal time using continuous control valves at a constant setting, at optimum, one valve is completely open.*

The proof (available in Velmurugan et al. (2023)) explains that if all valves are partially closed, one could simultaneously increase the flow rate through all of them to meet the given demand in a lower time a contradiction to the claim that the given valve setting is optimal. This exercise can be continued until one valve is completely open.

Combining Propositions 1 & 2, we are able to reduce our search space to demands that can be met by continuous control valves operated at a constant setting with one valve completely open. Given these simplifications, we could formulate a mathematical program to identify the supremum of $\mathbf{R}$ as described below.

Consider the system with continuous control valves operated with one valve partially closed. For example, Fig. 2 (a) shows a schematic of the system with the valve on pipe c partially closed. Due to the positive resistance offered by the valve, the net resistance of pipe c increases from k_c to $k_{c,cv}$. Let the water supplied in one unit of time ($t_{cv} = 1$) be equal to the total demand of the system. Now the same water has to be supplied using discrete valves in the minimal time ($= t_d$). For this, we solve a mathematical program, $P1$. $P1$ involves (i) enumerating the states of the system as shown in Fig. 3; we ignore the trivial state when both valves are closed as the objective is to minimize the transportation time, (ii) finding the flow rates in each of of these sates, and (iii) solving a linear programming problem (for details of the algorithm, see Kurian et al. (2018)). The ratio of operational times, $\frac{t_d}{t_{cv}}$ gives the value of $\mathbf{R}$. To find the supremum of $\mathbf{R}$, we need to solve an outer optimization problem, P, that identifies the values of h_2, k_b, k_c, and $k_{c,cv}$ at optimum.

In the above description of the mathematical program P, we started the exercise with the valve along pipe c partially closed (Fig. 2 (a)). We must also consider the alternative–valve along pipe b partially closed (Fig. 2 (b)). Therefore, to find the supremum of $\mathbf{R}$, the problem P has to be solved for both cases.

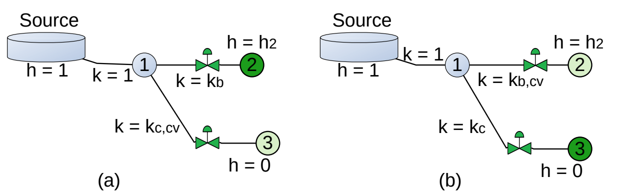

Fig. 2. System operated with with one of the valves partially closed and the other fully open: (a) valve along pipe c is partially closed (b) valve along pipe b is partially closed.

The bilevel optimization problem to identify the supremum of R can be formulated as follows:

$$(P) \max_{h_2, k_b, k_c, k_{c,cv}, \mathbf{t} \geq 0} \frac{||\mathbf{t}||_\mathbf{1}}{1} \tag{2}$$

$$\text{s.t } k_{c,cv} \geq k_c \tag{3}$$

$$\mathbf{t} \in \arg\min_{\tau >= 0} \{||\tau||_1 : \mathbf{A}\tau = \mathbf{d}(k_b, k_{c,cv})\} \tag{4}$$

where, $t_{cv} = 1$. $\mathbf{d}(k_b, k_{c,cv})$ is the vector of cumulative demands. By definition, $\mathbf{d}$ equals the water supplied to each demand node when the system is operated for 1 unit of time with pipe resistances k_b and $k_{c,cv}$. $\mathbf{t}$ is the vector of time duration for which the system is operated in each state under discrete operation, $t_d = ||\mathbf{t}||_\mathbf{1}$. Eq. 4 is the inner optimization problem $P1$. The parameter matrix $\mathbf{A}_{3\times 2}$ stores the flow rates to the two demand nodes in each of the three states shown in Fig. 3.

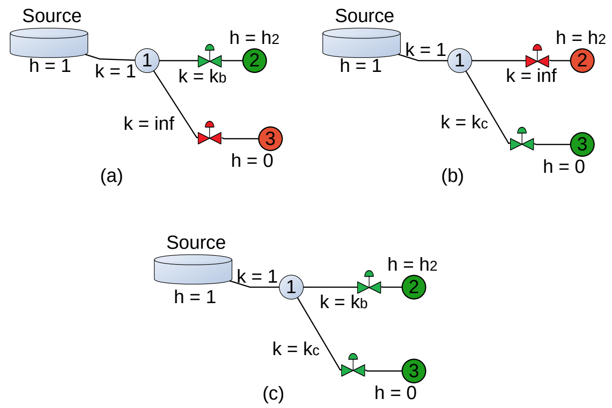

Fig. 3. Operational states of the system when discrete valves are available: (a) valve along pipe c is closed (b) valve along pipe b is closed (c) both valves are open. The trivial state with both valves closed is ignored as there is no flow.

In its current form, the problem assumes that the valve along pipe c is partially closed during supply using continuous control valves. To find the supremum of $\mathbf{R}$, we need to repeat the problem under the assumption that the valve along pipe b increases from k_b to $k_{c,cv}$ while during supply with continuous control valves (and the resistance along pipe c is k_c).

5. RESULTS AND DISCUSSION

We solved the optimization problem P numerically. Rather than solving a single problem to find the global optimum of $\mathbf{R}$, we solved the problem several times under different constraints on the choice of h_2 and the ratio of pipe resistances k_b/k_c. The objective behind imposing these constraints were two fold: (i) to make the solution of the optimization problem easy, and (ii) to shed light on the effect of the variations of these parameters on $\mathbf{R}$.

We solved the problem for four choices of h_2 (0.00, 0.25, 0.50, 0.75) and 200 different choices of the ratio k_b/k_c ranging from 10^{-10} to 10^{10}. Two values, viz., $n = 2$ and $n = 3$ are considered for the exponent in the potential loss equation (Eq. 1). The results are shown in Fig. 4 and Fig.5 for $n = 2$ and $n = 3$ respectively. The x-axis indicates the ratio of pipe resistances k_b/k_c and the y-axis showing the ratio of operational times under discrete and continuous operation. The markers show the limits on $\mathbf{R}$ when the valve along pipe c is partially closed during supply with continuous control valves. The dashed lines indicate the same when the valve along b is partially closed.

The blue circular markers show the limits on $\mathbf{R}$ when both demand nodes are at the same potential ($= 0$). This is same as the plot given by Velmurugan et al. (2023), where it was analytically shown that the supremum of $\mathbf{R}$ is $m^{1-1/n}$, where m is the depth of the network. Moreover, the limit is reached as $k_b \to \infty$ and $k_c \to 0$. The results being the same serves as a preliminary verification of optimization problem P and its solution. From the remaining plots, we observe that even at varying potentials of node 2 (h_2), the ratio of operational times–denoted by

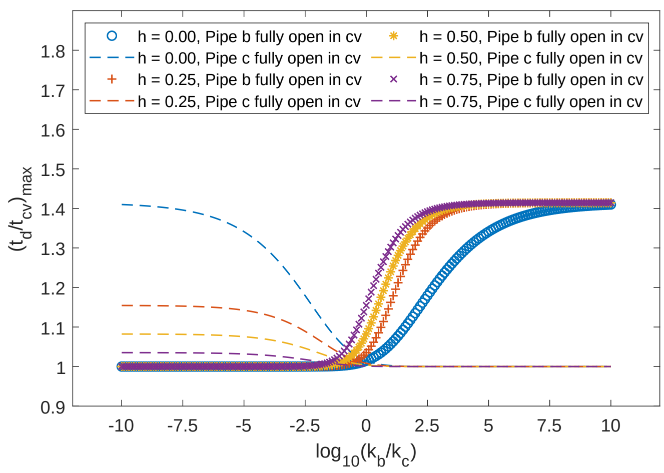

Fig. 4. The upper bound on the ratio of operational times $(\frac{t_d}{t_{cv}})$ at different choices of pipe resistances $(\frac{k_b}{k_c})$ and sink node potential h_2, $n = 2$. The markers denote scenarios where supply using continuous control valves have the valve along pipe c partially closed (Fig. 2 (a)). The dashed lines denote the same when the valve along pipe b is partially closed (Fig. 2 (b)).

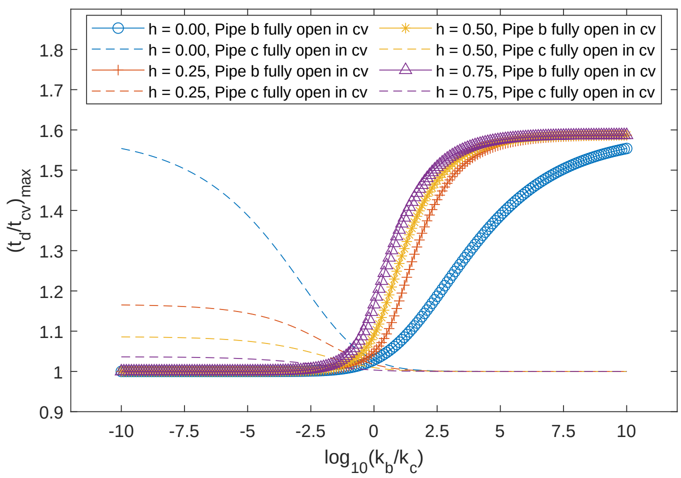

Fig. 5. The upper bound on the ratio of operational times $(\frac{t_d}{t_{cv}})$ at different choices of pipe resistances $(\frac{k_b}{k_c})$ and sink node potential h_2, $n = 3$. The markers denote scenarios where supply using continuous control valves have the valve along pipe c partially closed (Fig. 2 (a)). The dashed lines denote the same when the valve along pipe b is partially closed (Fig. 2 (b)).

the $+$, $*$, and $\times$ markers–still converge to the same upper bound. However, as h_2 increased, $\mathbf{R}$ approached high values even when k_b/k_c was relatively low. In summary, with varying node potentials, though its bounds remain the same, the ratio of operational times can approach its limits at more realistic values of pipe resistances.

When the demand nodes were at the same potential, the ratio of operational times reached it maximum as $k_b \to \infty$ and $k_c \to 0$, i.e., the transportation to node 2 became much more difficult compared with transportation to node 3. In the present work, as we increased the node potential h_2, we established the same effect–transportation to node 2 became difficult. As bringing h_2 to be closer to the source potential ($= 1$) is essentially similar to increasing the pipe resistance k_b, the upper bounds tends to be the same in either case. Moreover, as we already made the transportation to node 2 difficult by increasing its potential, the ratio of operational times can now remain high over a broader range of pipe resistances–evidenced by the steep rise of the purple, orange, and red plots relative to the blue.

The dashed lines converge to bounds that are lower than those given by the markers. If the valve along pipe b (leading to the node at higher potential) is partially closed during supply with continuous control valves, the bounds are, therefore, shown to be lower than that of the converse. This observation is also in alignment with our earlier discussion on the difficulty in transporting water to either demand nodes. If pipe b has the partially closed valve during supply with continuous control valves, then for $\mathbf{R}$ to be high, the transportation to demand node 3 has to be difficult compared with node 2. However, by placing node 3 at a lower potential than node 2, we imposed a check on this. What so ever be the choice of pipe resistances, the placement of node 3 at a lower potential does give a boost to the flow rate into the node. Consequently, $\mathbf{R}$ remains lower for these cases.

6. CONCLUSION

In this work, we considered the bounds on the ratio of operational times under discrete and continuous operation of a certain class of potential flow networks. Different from our prior works, we considered networks with varying sink potentials. Interestingly, the ratio of operational times, $\mathbf{R}$, converged to the same upper bound irrespective of whether the demand nodes were at the same potential or not. In the past, the results we generated for the basic three pipe network (Velmurugan et al., 2023) was clearly an indicator of what would later be shown for large networks (Kurian and Narasimhan, 2023). This leads us to speculate that the bounds on $\mathbf{R}$ derived for large networks with uniform sink node potentials would remain valid even when the sinks are at different potentials. A confirmation of the hypothesis would, of course, require a more detailed analysis, which is outside the scope of the current work. Such an analysis may also reveal the bounds on the ratio of operational times under decentralized and centralized operation as shown earlier in Kurian and Narasimhan (2023). Additionally, we are exploring opportunities for applying the results to reaction networks and emergency planning (Kaltenbach, 2012; Pyakurel et al., 2023).

REFERENCES

Hendrickson, C.T. and Janson, B.N. (1984). A common network flow formulation for several civil engineering problems. *Civil Engineering Systems*, 1(4), 195–203.

Kaltenbach, H.M. (2012). A unified view on bipartite species-reaction and interaction graphs for chemical reaction networks. *arXiv preprint arXiv:1210.0320*.

Klimm, M., Pfetsch, M.E., Raber, R., and Skutella, M. (2022). On the robustness of potential-based flow networks. *Mathematical Programming*, 1–38.

Kurian, V. and Narasimhan, S. (2023). Analysis of potential flow networks: Variations in transport time with

discrete, continuous, and selfish operation. *Physica A: Statistical Mechanics and its Applications*, 632, 129303.
Kurian, V., Narasimhan, S., and Narasimhan, S. (2018). Optimal scheduling of rural water supply schemes. *IFAC-PapersOnLine*, 51(1), 142–147. doi: https://doi.org/10.1016/j.ifacol.2018.05.024.
Pyakurel, U., Khanal, D.P., and Dhamala, T.N. (2023). Abstract network flow with intermediate storage for evacuation planning. *European Journal of Operational Research*, 305(3), 1178–1193.
Raghunathan, A.U. (2013). Global optimization of nonlinear network design. *SIAM Journal on Optimization*, 23(1), 268–295.
Ronellenfitsch, H. and Katifori, E. (2016). Global optimization, local adaptation, and the role of growth in distribution networks. *Physical review letters*, 117(13), 138301.
Sankar EM, A. and Rengaswamy, R. (2022). Droplet microfluidic networks as hybrid dynamical systems: Inlet spacing optimization for sorting of drops. *AIChE Journal*, 68(6), e17633.
Shiono, N. and Suzuki, H. (2016). Optimal pipe-sizing problem of tree-shaped gas distribution networks. *European Journal of Operational Research*, 252(2), 550–560.
Velmurugan, S., Kurian, V., and Narasimhan, S. (2023). Continuous and discrete operation of water distribution networks. *Optimization and Engineering*, 24, 2619–2650. doi:10.1007/s11081-022-09787-4.