

Volumetric Strain Energy Theory Of Failure

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Abstract

The Volumetric Strain Energy Theory (VSET) of Failure provides a theoretical basis to predict material failure under complex stress conditions. This theory proposes that failure occurs when the volumetric strain energy within the material reaches a specific threshold, identified by the energy stored in a standard tensile test sample at failure. By concentrating on volumetric energy alterations rather than stresses or strains alone, VSET offers an enriched perspective on material failure.

Keywords: Volumetric Strain Energy

1. Introduction

This research paper presents a comprehensive analysis of the Volumetric Strain Energy Theory (VSET) of failure, an advanced theoretical framework for predicting material failure under complex loading conditions. The VSET is introduced as a robust alternative, emphasizing its foundation on the volumetric strain energy density, which accounts for hydrostatic component of the stress state.

The theoretical development of VSET is detailed, including the derivation of failure criteria and the formulation of corresponding constitutive models. The paper elaborates on the mathematical representation of volumetric strain energy and its correlation with material failure.

Furthermore, the paper explores the practical implications of VSET in engineering design and material selection, highlighting its potential to enhance the reliability and performance of structural components.

This study concludes with recommendations for future research, emphasizing the necessity for experimental validation and the advancement of computational tools to promote VSET application in industry.

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Theorem 1. “Failure of a component subjected to a complex state of stress occurs when the Volumetric Strain Energy stored in it becomes equal to the Volumetric Strain Energy stored in a simple Tensile Test Specimen”

Failure criteria are defined as $\sigma_1 + \sigma_2 + \sigma_3 = \pm (\sigma_y/FOS)$, where the primary stresses reach a yield threshold divided by the safety factor. This ensures the design maintains a robust margin from the failure point under operational loads.

$$\sigma_1 + \sigma_2 + \sigma_3 < \pm (\sigma_y/FOS) \text{ (This is design criteria)}$$

Where:

$\sigma_1, \sigma_2, \sigma_3$ are the principle stresses

σ_y is the failure stress in simple tensile testing specimen

2. General Definition:

2.1 Stress Tensor (in 3 dimensions)

It is a three dimensional matrix whose diagonal elements represents the normal stresses (σ_{xx} ; σ_{yy} ; σ_{zz}) acting in all three directions (x, y, z) and remaining elements represents the shear stresses (σ_{xy} ; σ_{yz} ; σ_{zx} ; σ_{yx} ; σ_{zy} ; σ_{xz}) on all the three planes (xy, yz, zx) as shown below:

$$\begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{matrix}$$

2.2 Principal Stresses[2]

$\sigma_1, \sigma_2, \sigma_3$ are the principle stresses (sum of hydrostatic stress and distortional stresses) in the directions x, y and z respectively these stresses are combination of both the hydrostatic and distortional stresses as shown in the figure. These can be calculated as the roots of the following cubic equation:

$$S^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})S^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \sigma_{yz}^2 - \sigma_{zx}^2 - \sigma_{xy}^2)S - (\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{yz}\sigma_{zx}\sigma_{xy} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{xz}^2 - \sigma_{zz}\sigma_{xy}^2) = 0 \quad [1]$$

2.3 Hydrostatic Stress

The hydrostatic stress is responsible for “CHANGE IN SIZE” and this results in volumetric strain energy which forms the base of this theory.

2.4 Distortional Stress

The distortional stresses are responsible for “CHANGE IN SHAPE” and results in distortion energy.

2.5 Total Strain Energy

Both the volumetric strain energy and distortion energy gets stored in the body as total strain energy.

2.6 Theories Of Failure[5]

Failure theories are essential concepts in the field of strength of materials, focusing on understanding and predicting how engineering materials fail under various loading conditions. These theories offer engineers and designers crucial insights into the safety and reliability of structures and components, helping them make well-informed decisions during the design and analysis stages.

3. Equation Derivation

In The Case Of Body Subjected To Complex State Of Stress:

Volumetric strain energy $E_v = (1/2) * \bar{\sigma} * \bar{\epsilon}_v$

$$\bar{\epsilon}_v = \epsilon_x + \epsilon_y + \epsilon_z$$

Where;

$$\epsilon_x = (\sigma_1/E) - (\mu * (\sigma_2 + \sigma_3)/E) \text{ (Strain along x-direction)} \epsilon_y$$

$$= (\sigma_2/E) - (\mu * (\sigma_3 + \sigma_1)/E) \text{ (Strain along y-direction)}$$

$$\epsilon_z = (\sigma_3/E) - (\mu * (\sigma_1 + \sigma_2)/E) \text{ (Strain along z-direction)} \epsilon_v$$

: Volumetric strain in the body

$\bar{\sigma}$: Hydrostatic or Volumetric stress (As seen in Fig. 2)

As we know that the volumetric strain energy due to distortional stresses is equal to zero In the case of distortional stresses that is

$$\sigma_{d1} = \sigma_1 - \bar{\sigma} \text{ (as we can see from the Fig. 2 is the distortional stress in the x direction)}$$

$$\sigma_{d2} = \sigma_2 - \bar{\sigma} \text{ (as we can see from the Fig. 2 is the distortional stress in the y direction)}$$

$$\sigma_{d3} = \sigma_3 - \bar{\sigma} \text{ (as we can see from the Fig. 2 is the distortional stress in the z direction)}$$

μ : Poisson's ratio

E : Young's modulus

$$\epsilon_{d1} = (\sigma_{d1}/E) - ((\mu/E) * (\sigma_{d2} + \sigma_{d3})) \text{ (This is the strain along x direction due to the distortional stresses)}$$

$$\epsilon_{d2} = (\sigma_{d2}/E) - ((\mu/E) * (\sigma_{d3} + \sigma_{d1})) \text{ (This is the strain along y direction due to the distortional stresses)}$$

$$\epsilon_{d3} = (\sigma_{d3}/E) - ((\mu/E) * (\sigma_{d1} + \sigma_{d2})) \text{ (This is the strain along z direction due to the distortional stresses)}$$

$$\epsilon_{dv} = \epsilon_{d1} + \epsilon_{d2} + \epsilon_{d3} \text{ (Volumetric strain due to distortional stresses)}$$

But as we know that the Volumetric strain due to distortional stresses must be equal to zero as the distortional stresses won't cause any change in the size of the body.

As a result of which we get:

$$\epsilon_{dv} = (\sigma_{d1}/E) - ((\mu/E) * (\sigma_{d2} + \sigma_{d3})) + (\sigma_{d2}/E) - ((\mu/E) * (\sigma_{d3} + \sigma_{d1})) + (\sigma_{d3}/E) - ((\mu/E) * (\sigma_{d1} + \sigma_{d2})) = 0$$

$$(\sigma_{d1} + \sigma_{d2} + \sigma_{d3}) * ((1 - (2 * \mu))/E) = 0$$

$$(\sigma_{d1} + \sigma_{d2} + \sigma_{d3}) = 0$$

$$(\sigma_1 + \sigma_2 + \sigma_3) - (3 * \bar{\sigma}) = 0$$

$$\text{Therefore, } \bar{\sigma} = (\sigma_1 + \sigma_2 + \sigma_3)/3$$

$$\epsilon_v = (\sigma_1/E) - (\mu * (\sigma_2 + \sigma_3)/E) + (\sigma_2/E) - (\mu * (\sigma_3 + \sigma_1)/E) + (\sigma_3/E) - (\mu * (\sigma_1 + \sigma_2)/E)$$

$$\epsilon_v = (\sigma_1 + \sigma_2 + \sigma_3) * ((1 - (2 * \mu))/E)$$

$$E_v = (1/2) * (\sigma_1 + \sigma_2 + \sigma_3)^2 * ((1 - (2 * \mu))/(3 * E)) = (1/(6 * E)) * (1 - (2 * \mu)) * (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ (Eq.1)}$$

In Case Of Body Subjected To Simple Tension At The Time Of Failure:

At the time of failure we know that the body will be subjected to a stress equal to the yield stress and the principle stresses in this case will be as follows:

$$\sigma_1 = \sigma_y$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$$

$$\begin{aligned}\varepsilon_x &= (\sigma_1/E) - (\mu^*(\sigma_2 + \sigma_3)/E) = (\sigma_y/E) \text{ (Strain along x-direction)} \\ \varepsilon_y &= (\sigma_2/E) - (\mu^*(\sigma_3 + \sigma_1)/E) = -(\mu^*(\sigma_y)/E) \text{ (Strain along y-direction)} \\ \varepsilon_z &= (\sigma_3/E) - (\mu^*(\sigma_1 + \sigma_2)/E) = -(\mu^*(\sigma_y)/E) \text{ (Strain along z-direction)} \\ \varepsilon_v &= (\sigma_y/E)*(1-(2*\mu)) \\ \sigma_v &= (\sigma_1 + \sigma_2 + \sigma_3)/3 = (\sigma_y/3)\end{aligned}$$

Therefore,

$$E_v = (1/2)*(\sigma_y)^2*(1-(2*\mu))/(3*E) = (1/(6*E))*(1-(2*\mu))* (\sigma_y)^2 \text{ (Eq.2)}$$

Upon equating the equations 1 and 2 we get:

$$(1/(6*E))*(1-(2*\mu))* (\sigma_1 + \sigma_2 + \sigma_3)^2 = (1/(6*E))*(1-(2*\mu))* (\sigma_y)^2$$

Final Expression for this theory is as follows:

$$\sigma_1 + \sigma_2 + \sigma_3 = \pm \sigma_y$$

In case of planar stress $\sigma_3 = 0$

Therefore, $\sigma_1 + \sigma_2 = \pm \sigma_y$

If factor of safety (FOS) is considered then we get: (“+” for tension and “-“ for compression)

Failure criteria are defined as $\sigma_1 + \sigma_2 + \sigma_3 = \pm (\sigma_y/\text{FOS})$, where the primary stresses reach a yield threshold divided by the safety factor. This ensures the design maintains a robust margin from the failure point under operational loads.

$\sigma_1 + \sigma_2 + \sigma_3 < \pm (\sigma_y/\text{FOS})$ (This is design criteria)

4. Figures

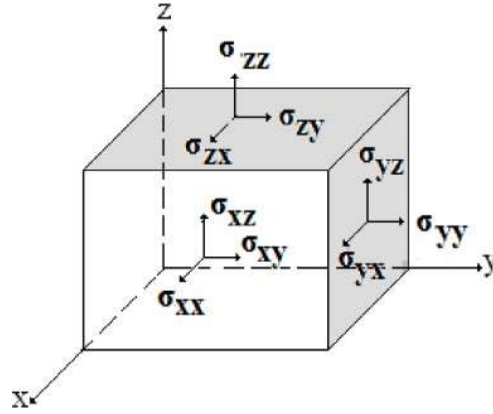


Fig. 1 Three Dimensional Stress Element

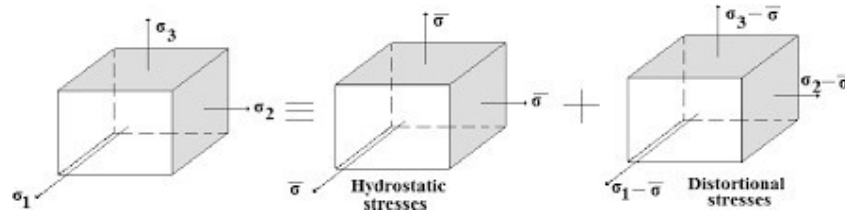


Fig.2 Stress Component

5. Conclusion

Statement : “Failure of a component subjected to a complex state of stress occurs when the Volumetric Strain Energy stored in it becomes equal to the Volumetric Strain Energy stored in a simple Tensile Test Specimen”

Application : As this theory considers normal stresses this theory is applicable for BRITTLE MATERIALS as we know that they are weak in TENSION as well as this theory considers only strain energy due to change in size so it is applicable for HYDROSTATIC STATE OF STRESS as it involves only size change.

Future Scope : The study concludes with a discussion on the experimental investigations and the development of computational tools to facilitate the widespread adoption of VSET in industry.

6. References

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Competing Interests

The authors, Kumaran T and Dr. K. MalarMohan, declare that they have no competing interests.