Linear programming with vector coefficients in the constraints

Somdeb Lahiri

(Former Professor) PD Energy University, Gandhinagar (EU-G), India.

ORCID: https://orcid.org/0000-0002-5247-3497

somdeb.lahiri@gmail.com

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Abstract

We provide a model of linear programming in which all the parameters of the constraints are vectors. We define the dual of the problem and obtain a necessary and sufficient condition for an optimal solution. While the primal problem is not a conic linear programming problem, its dual is a problem of this type. We also prove the analogous version of Farkas' lemma in this more general framework.

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1. Introduction

This paper explores linear programming problem with vector coefficients in the constraints and proposes a generalization of the traditional linear programming framework which results by expressing the original linear programming problem as a related linear programming problem in standard form.

2. Problem Description

The standard model of linear programming is discussed in detail in Dorfman, Samuelson and Solow (1958) (i.e., DOSSO) and considerably more concisely in chapter 3 of Lancaster (1968) and chapters 5 and 22 of Mote and Madhavan (2016). We propose an extension of this model by allowing the coefficients in the linear constraints to be vector valued.

For positive integers r, s and $\mathbb{S}$ a non-empty subset of $\mathbb{R}$, let $\mathbb{S}^{r\times s}$ denote the set of all r×s matrices with entries in $\mathbb{S}$.

Given positive integers m, n an m×n matrix A and $j\in\{1, \ldots, n\}$ let A^j denote the m-dimensional j^{th} column vector of A.

For a positive integer n, let $E^{(n,i)}$ denote the n-dimensional column unit vector, i.e., the n-dimensional column vector whose i^{th} coordinate is 1 and all other coordinates are 0.

Given a positive integer n and a square matrix, A of size 'n' (i.e., n× n matrix A) the trace of A denoted trace (A) = $\sum_{i=1}^{n} E^{(n,i)^T} A E^{(n,i)}$, i.e., the sum of the diagonal elements of A.

Given positive integers m, n, K, let $\langle A^{(k)} | k = 1, \ldots, K\rangle$ be an array of m×n matrices let B be an m×n matrix and p be a K-dimensional column vector.

For each $i \in \{1, \ldots, m\}$, let B_i denote the ith row of B and $A_i^{(k)}$ the ith row of $A^{(k)}$ for k = 1, …, K.

For each $j \in \{1, \ldots, n\}$, let B^j denote the jth column of B and $A^{(k,j)}$ the jth column of $A^{(k)}$ for k = 1, …, K.

For $x \in \mathbb{R}^K$ let x_k denote the kth coordinate of x.

The problem that we are concerned with here denoted (P1) is the following:

Maximize $p^T x$, subject to $\sum_{k=1}^{K} A_i^{(k)} x_k = B_i$, i = 1, …, m, $x \in \mathbb{R}_+^K$.

Such a problem is referred to as **linear programming problem with vector coefficients (LP-VC)**. The reason for such a nomenclature is that for each equation in the 'm' linear constraints, for all $k \in \{1, \ldots, K\}$, the coefficient of the variable x_k is a row vector and the right-hand side of each equation is a row vector too.

Note 1: (P1) is not a conic linear programming problem, although as we will see in the subsequent section, its dual is a conic linear programming problem. For a brief account of conic linear programming, see chapter 6 of Luenberger and Ye (2008).

We will refer to a system of linear equations such as $\sum_{k=1}^{K} A_i^{(k)} x_k = B_i$, i = 1, …, m, as **linear equations with vector coefficients (LE- VC)**.

A considerably simpler way of expressing a LE-VC is the following: $\sum_{k=1}^{K} A^{(k,j)} x_k = B^j$, j = 1, …, n.

Thus, an equivalent and simpler way of stating (P1) is the following:

Maximize $p^T x$, subject to $\sum_{k=1}^{K} A^{(k,j)} x_k = B^j$, j = 1, …, n, $x \in \mathbb{R}_+^K$,

which is simply a **linear programming problem in standard form**.

Thus, the dual of (P1) denoted (Dual-P1) is the following linear programming problem.

Minimize $\sum_{j=1}^{n} y^{(j)^T} B^j$ subject to $\sum_{j=1}^{n} y^{(j)^T} A^{(k,j)} \geq p_k$ for all k = 1, …, K, $y^{(j)} \in \mathbb{R}^m$ for j = 1, …, n.

Let Y be the m×n matrix whose jth column is $y^{(j)}$.

Then, Dual-P1 is equivalent to the following problem:

Minimize trace $(Y^T B)$ subject to trace $(Y^T A^{(k)}) \geq p_k$ for all k = 1, …, K, $Y \in \mathbb{R}^{m \times n}$.

Note 2: (Dual-P1) is an instance of a conic linear programming problem.

3. Duality theory for LP-VC

In this section we present the duality theory that is relevant for an LP-VC.

Theorem 1: x^* solves (P1) if and only if the following two conditions are satisfied:

(i) $\sum_{k=1}^{K} A^{(k)} x_k^* = B$ and $x^* \in \mathbb{R}_+^K$.

(ii) There exists $Y^* \in \mathbb{R}^{m \times n}$ such that trace $(Y^{*T}A^{(k)}) \geq p_k$ and (trace $(Y^{*T}A^{(k)}) - p_k)x_k^* = 0$ for all k = 1, …, K.

Proof: x^* solves (P1) <u>if and only if</u> x^* solves **[**Maximize p^Tx, subject to $\sum_{k=1}^{K} A^{(k,j)} x_k = B^j$, j = 1, …, n, $x \in \mathbb{R}_+^K$].

From Topic 2 of Lahiri (2020) we know that x^* solves (P1) <u>if and only if</u> the following two conditions are satisfied:

(a) $x^* \in \mathbb{R}_+^K$ and $\sum_{k=1}^{K} A^{(k,j)} x_k = B^j$, j = 1, …, n.

(b) there exists $y^* \in (\mathbb{R}^m)^n$ whose transpose $y^{*T} = (y^{*1^T}, \ldots, y^{*n^T})$ with $y^{*j} \in \mathbb{R}^m$ for all j∈{1, …, n} satisfies $\sum_{j=1}^{n} y^{*j^T} A^{(k,j)} \geq p_k$ and $(\sum_{j=1}^{n} y^{*j^T} A^{(k,j)} - p_k)\, x_k^* = 0$ for all k = 1, ..., K.

Let $Y^* \in \mathbb{R}^{n \times m}$ be such that the j^th^ column of Y^* is y^{*j} for all j∈{1, …, n}.

Thus, $\sum_{j=1}^{n} y^{*j^T} A^{(k)j}$ = trace $(Y^{*T}A^{(k)})$ for all k = 1, …, K.

Thus (b) is equivalent to [there exists $Y^* \in \mathbb{R}^{n \times m}$ such that trace $(Y^{*T}A^{(k)}) \geq p_k$ and (trace $(Y^{*T}A^{(k)}) - p_k)x_k^* = 0$ for all k = 1, …, K].

This proves the theorem. Q.E.D.

4. Farkas' Lemma for LE-VC

We provide below a statement and proof of Farkas' lemma for linear equations with vector coefficients.

Theorem 2: <u>Either</u> [there exists $x \in \mathbb{R}_+^K$ such that $\sum_{k=1}^{K} A^{(k)} x^{(k)} = B$] <u>or</u> [there exists a m×n matrix Y, such that trace $(Y^TA^{(k)}) \leq 0$ for all k = 1, …, K and trace $(Y^TB) > 0$], <u>but never both</u>.

Proof: $x^* \in \mathbb{R}_+^K$ solves $\sum_{k=1}^{K} A^{(k)} x^{(k)} = B$, $x \in \mathbb{R}_+^K$ <u>if and only if</u> it solves $\sum_{k=1}^{K} A^{(k,j)} x_k = B^j$, j = 1, …, n, $x \in \mathbb{R}_+^K$.

By Farkas' lemma (see Topic 3 in Lahiri (2020)), <u>either</u> [there exists $x \in \mathbb{R}_+^K$ such that$\sum_{k=1}^{K} A^{(k,j)} x_k = B^j$, j = 1, …, n] <u>or</u> [there exists an m×n dimensional column vector y whose coordinates numbered (j-1)m + 1, …, jm is denoted by the m dimensional column vector y^j such that $\sum_{j=1}^{n} y^{j^T} A^{(k)^j} \leq 0$ for all k = 1, …, K and $\sum_{j=1}^{n} y^{j^T} B^j > 0$] <u>but never both</u>.

Let Y be the m×n matrix whose j^th^ column is y^j. For all j = 1, …, n, $y^{j^T} B^j$ is the j^th^ diagonal element of Y^TB and $y^{j^T} A^{(k)^j}$ is the j^th^ diagonal element of $Y^TA^{(k)}$ for k∈{1, …, K}.

Thus, $\sum_{j=1}^{n} y^{j^T} B^j$ = trace (Y^TB) and $\sum_{j=1}^{n} y^{j^T} A^{(k)^j}$ = trace $(Y^TA^{(k)})$. for k∈{1, …, K}.

This proves the theorem. Q.E.D.

5. Allocation of resources for production that requires time

The existing theory of input allocation in production processes is static, in the sense that the process of production is assumed to be completed within one unit of time. In reality, the duration of production may affect the use of input in each period. This is particularly so in the agricultural sector where "seasonality" plays a significant role.

Consider the production of 'K' different products, each of which requires 'n' days for its manufacture. During each day, each product requires the use of 'm' different inputs and there is a pre-assigned quantity of each input for each day. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, let $b_{ij} \geq 0$ be the total amount of input 'i' that is available on day j. For each $k \in \{1, \ldots, K\}$, let $a_{ij}^{(k)} \geq 0$ be the amount of input 'i' that on day 'j' to produce 1 unit of the product 'k'.

Note: In any application such as this for the agricultural sector, the effect of seasonality may be reflected in one or more columns of $A^{k)}$ being the 0- column vector, for each $k \in \{1, \ldots, K\}$.

Let p_k denote the net revenue per unit of product k and the objective here is to maximize net revenue.

If B denotes the m×n matrix whose $(i, j)^{th}$ entry is b_{ij}, $A^{(k)}$ the m×n matrix whose $(i, j)^{th}$ entry is $a_{ij}^{(k)}$ and let p denote the K-dimensional vector whose k^{th} coordinate is p_k. Then the net-revenue maximization problem reduces to the following: Maximize $p^T x$, subject to $\sum_{k=1}^{K} A^{(k)} x_k = B$, $x \in \mathbb{R}_+^K$.

6. Conclusion

Such a section is quite "out of place" in a paper, whose purpose is to extend an invitation for further research. We hope we have provided some food for thought to students who are interested in the mathematics and modelling aspects of linear optimization.

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