

Linear programming with vector coefficients in the constraints

Somdeb Lahiri

(Former Professor) PD Energy University, Gandhinagar (EU-G), India.

ORCID: <https://orcid.org/0000-0002-5247-3497>

somdeb.lahiri@gmail.com

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Abstract

We provide a model of linear programming in which all the parameters of the constraints are vectors. We define the dual of the problem and obtain a necessary and sufficient condition for an optimal solution. We also prove the analogous version of Farkas' lemma in this more general framework.

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1. Framework of analysis

In what follows we consider a generalization of the standard model of linear programming in Dorfman, Samuelson and Solow (1958) (i.e., DOSSO) and considerably more concisely in chapter 3 of Lancaster (1968) and chapters 5 and 22 of Mote and Madhavan (2016).

For positive integers r, s and \mathbb{S} a non-empty subset of \mathbb{R} , let $\mathbb{S}^{r \times s}$ denote the set of all $r \times s$ matrices with entries in \mathbb{S} .

Given positive integers m, n an $m \times n$ matrix A and $j \in \{1, \dots, n\}$ let A^j denote the m -dimensional j^{th} column vector of A .

For a positive integer n , let $E^{(n,i)}$ denote the n -dimensional column unit vector, i.e., the n -dimensional column vector whose i^{th} coordinate is 1 and all other coordinates are 0.

Given a positive integer n and a square matrix, A of size 'n' (i.e., $n \times n$ matrix A) the trace of A denoted $\text{trace}(A) = \sum_{i=1}^n E^{(n,i)T} A E^{(n,i)}$, i.e., the sum of the diagonal elements of A .

Given positive integers m, n, K , let $\langle A^{(k)} | k = 1, \dots, K \rangle$ be an array of $m \times n$ matrices let B be an $m \times n$ matrix and p be a K -dimensional column vector.

For each $i \in \{1, \dots, m\}$, let B_i denote the i^{th} row of B and $A_i^{(k)}$ the i^{th} row of $A^{(k)}$ for $k = 1, \dots, K$,

For $x \in \mathbb{R}^K$ let x_k denote the k^{th} coordinate of x .

The problem that we are concerned with here denoted (P1) is the following:

Maximize $p^T x$, subject to $\sum_{k=1}^K A_i^{(k)} x_k = B_i, i = 1, \dots, m, x \in \mathbb{R}_+^K$.

Such a problem is referred to as **linear programming problem with vector coefficients (LP-VC)**. The reason for such a nomenclature is that for each equation in the ‘m’ linear constraints, for all $k \in \{1, \dots, K\}$, the coefficient of the variable x_k is a row vector and the right-hand side of each equation is a row vector too.

We will refer to a system of linear equations such as $\sum_{k=1}^K A_i^{(k)} x_k = B_i, i = 1, \dots, m$, as **linear equations with vector coefficients (LE-VC)**.

An equivalent way of stating (P1) is the following:

Maximize $p^T x$, subject to $\sum_{k=1}^K A^{(k)} x_k = B, x \in \mathbb{R}_+^K$.

Given positive integers m, n an $m \times n$ matrix A can be expressed as an $m \times n$ dimension column vector $\mathcal{A}(A)$ such that for each $j \in \{1, \dots, n\}$, its coordinates numbered $(j-1)m + 1, \dots, jm$ form the column vector A^j .

Thus (P1) is equivalent to the following linear programming problem denoted ($\wp 1$).

Maximize $p^T x$, subject to $\sum_{k=1}^K \mathcal{A}(A^{(k)}) x_k = \mathcal{A}(B), x \in \mathbb{R}_+^K$.

It is easily verified that if $C \in \mathbb{R}^{r \times s}$ and $D \in \mathbb{R}^{s \times r}$ then $\text{trace}(CD) = \mathcal{A}(C^T)^T \mathcal{A}(D)$.

Thus, the dual of ($\wp 1$) denoted (Dual- $\wp 1$) is the following linear programming problem.

Minimize $\mathcal{A}(Y^T)^T \mathcal{A}(B)$ subject to $\mathcal{A}(Y^T)^T \mathcal{A}(A^{(k)}) \geq p_k$ for all $k = 1, \dots, K, Y \in \mathbb{R}^{n \times m}$.

An equivalent way of stating (Dual- $\wp 1$) is the following problem denoted (Dual-P1).

Maximize $\text{trace}(YB)$ subject to $\text{trace}(YA^{(k)}) \geq p_k$ for all $k = 1, \dots, K, Y \in \mathbb{R}^{n \times m}$.

2. An application example

Consider the problem of a ‘multi-product conglomerate’ allocating inputs to factories with the objective of minimizing operating costs. Suppose that for positive integers m, n, K, there are ‘m’ inputs with known aggregate amounts pre-allocated for the production of each of ‘n’ outputs and K factories among which the inputs have to be allocated. For all $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, let b_{ij} denote the quantity of input ‘i’ pre-allocated for output j and let B be the $m \times n$ matrix whose $(i, j)^{\text{th}}$ entry is b_{ij} .

Suppose the units of measurement of the outputs are such that for n outputs, the average output per level of operation of a factory is 1 at all levels of operation.

For $k \in \{1, \dots, K\}$ and $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, let $a_{ij}^{(k)}$ be the amount of input i required to produce 1 unit of output j in factory k and let $A^{(k)}$ be the $m \times n$ matrix whose $(i, j)^{\text{th}}$ entry is $a_{ij}^{(k)}$.

For $k \in \{1, \dots, K\}$, let $x_k \geq 0$ be the operation level of factory k.

If for $k \in \{1, \dots, K\}$, c_k denotes the constant unit cost of operating factory and $p_k = -c_k$, then operating cost minimization problem faced by the multi-product conglomerate is (P1).

3. Duality theory for LP-VC

From Topic 2 of Lahiri (2020) we know that x^* solves (P1) if and only if there exists $Y^* \in \mathbb{R}^{n \times m}$ such that the following is satisfied:

(i) $\sum_{k=1}^K A^{(k)} x_k^* = B$ and $x^* \in \mathbb{R}_+^K$.

(ii) $\text{trace}(Y^* A^{(k)}) \geq p_k$ and $(\text{trace}(Y^* A^{(k)}) - p_k)x_k^* = 0$ for all $k = 1, \dots, K$.

From (i) and (ii) it follows that $p^T x^* = \sum_{k=1}^K p_k x_k^* = \sum_{k=1}^K \text{trace}(Y^* A^{(k)}) x_k^* = \sum_{k=1}^K \mathcal{A}(Y^{*T}) \mathcal{A}(A^{(k)}) x_k^* = \mathcal{A}(Y^{*T}) \sum_{k=1}^K \mathcal{A}(A^{(k)}) x_k^* = \mathcal{A}(Y^{*T}) \mathcal{A}(B) = \text{trace}(Y^* B)$.

4. Farkas' Lemma for LE-VC

We provide below a statement and proof of Farkas' lemma for linear equations with vector coefficients.

Theorem 1: Either [there exists $x \in \mathbb{R}_+^K$ such that $\sum_{k=1}^K A^{(k)} x^{(k)} = B$] or [there exists a $n \times m$ matrix Y , such that $\text{trace}(YA^{(k)}) \leq 0$ for all $k = 1, \dots, K$ and $\text{trace}(YB) > 0$], but never both.

Proof: $x^* \in \mathbb{R}_+^K$ solves $\sum_{k=1}^K A^{(k)} x^{(k)} = B$ if and only if it solves $\sum_{k=1}^K \mathcal{A}(A^{(k)}) x^{(k)} = \mathcal{A}(B)$.

By Farkas' lemma (see Topic 3 in Lahiri (2020)), either [there exists $x \in \mathbb{R}_+^K$ such that $\sum_{k=1}^K \mathcal{A}(A^{(k)}) x^{(k)} = \mathcal{A}(B)$] or [there exists an $m \times n$ dimensional column vector y whose coordinates numbered $(j-1)m + 1, \dots, jm$ is denoted by the m dimensional column vector y^j such that $y^T \mathcal{A}(A^{(k)}) \leq 0$ for all $k = 1, \dots, K$ and $y^T \mathcal{A}(B) > 0$] but never both.

$$y^T \mathcal{A}(A^{(k)}) = \sum_{j=1}^n y^j{}^T A^{(k)j} \text{ for all } k = 1, \dots, K \text{ and } y^T \mathcal{A}(B) = \sum_{j=1}^n y^j{}^T B^j.$$

Let Y be the $n \times m$ matrix whose j^{th} row is $y^j{}^T$. For all $j = 1, \dots, n$, $y^j{}^T B^j$ is the j^{th} diagonal element of YB and $y^j{}^T A^{(k)j}$ is the j^{th} diagonal element of $YA^{(k)}$ for $k \in \{1, \dots, K\}$.

Thus, $y^T \mathcal{A}(B) = \text{trace}(YB)$ and $y^T \mathcal{A}(A^{(k)}) = \text{trace}(YA^{(k)})$. for $k \in \{1, \dots, K\}$.

This proves the theorem. Q.E.D.

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