Linear programming with vector coefficients in the constraints

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Abstract

We provide a model of linear programming in which all the parameters of the constraints are vectors. We define the dual of the problem and obtain a necessary and sufficient condition for an optimal solution. We also prove the analogous version of Farkas' lemma in this more general framework.

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1. Framework of analysis

In what follows we consider a generalization of the standard model of linear programming in Dorfman, Samuelson and Solow (1958) (i.e., DOSSO) and considerably more concisely in chapter 3 of Lancaster (1968) and chapters 5 and 22 of Mote and Madhavan (2016).

For positive integers r, s and $\mathbb{S}$ a non-empty subset of $\mathbb{R}$, let $\mathbb{S}^{r\times s}$ denote the set of all r×s matrices with entries in $\mathbb{S}$.

Given positive integers m, n an m×n matrix A and j∈{1, …, n} let A^j denote the m-dimensional j^{th} column vector of A.

For a positive integer n, let $E^{(n,i)}$ denote the n-dimensional column unit vector, i.e., the n-dimensional column vector whose i^{th} coordinate is 1 and all other coordinates are 0.

Given a positive integer n and a square matrix, A of size 'n' (i.e., n× n matrix A) the trace of A denoted trace (A) = $\sum_{i=1}^{n} E^{(n,i)^T} A E^{(n,i)}$, i.e., the sum of the diagonal elements of A.

Given positive integers m, n, K, let $<A^{(k)}|k = 1, \dots , K>$ be an array of m×n matrices let B be an m×n matrix and p be a K-dimensional column vector.

For each i∈{1, …, m}, let B_i denote the i^{th} row of B and $A_i^{(k)}$ the i^{th} row of $A^{(k)}$ for k = 1, …, K,

For $x \in \mathbb{R}^K$ let x_k denote the k^{th} coordinate of x.

The problem that we are concerned with here denoted (P1) is the following:

Maximize $p^T x$, subject to $\sum_{k=1}^{K} A_i^{(k)} x_k = B_i$, i = 1, …, m, $x \in \mathbb{R}_+^K$.

Such a problem is referred to as **linear programming problem with vector coefficients (LP-VC)**. The reason for such a nomenclature is that for each equation in the 'm' linear constraints, for all $k \in \{1, …, K\}$, the coefficient of the variable x_k is a row vector and the right-hand side of each equation is a row vector too.

We will refer to a system of linear equations such as $\sum_{k=1}^{K} A_i^{(k)} x_k = B_i$, i = 1, …, m, as **linear equations with vector coefficients (LE- VC)**.

An equivalent way of stating (P1) is the following:

Maximize $p^T x$, subject to $\sum_{k=1}^{K} A^{(k)} x_k = B$, $x \in \mathbb{R}_+^K$.

Given positive integers m, n an m×n matrix A can be expressed as an m×n dimension column vector $\mathcal{A}(A)$ such the for each $j \in \{1, …, n\}$, its coordinates numbered (j-1)m + 1, …, jm form the column vector A^j.

Thus (P1) is equivalent to the following linear programming problem denoted ($\wp$1).

Maximize $p^T x$, subject to $\sum_{k=1}^{K} \mathcal{A}(A^{(k)})\, x_k = \mathcal{A}(B)$, $x \in \mathbb{R}_+^K$.

Thus, the dual of ($\wp$1) denoted (Dual-$\wp$1) is the following linear programming problem.

Minimize $\mathcal{A}(Y^T)^T \mathcal{A}(B)$ subject to $\mathcal{A}(Y^T)^T \mathcal{A}(A^{(k)}) \geq p_k$ for all k = 1, ..., K, $Y \in \mathbb{R}^{n \times m}$.

Note: It is easily verified that if $C \in \mathbb{R}^{r \times s}$ and $C \in \mathbb{R}^{s \times r}$, then trace (CD) = $\mathcal{A}(C^T)^T \mathcal{A}(D)$. Further, if for some positive integer H, $<C^{(1)}, …, C^{(H)}>$ is an array of matrices in $\mathbb{R}^{r \times s}$, $F \in \mathbb{R}^{r \times s}$ and $\mathcal{B}$ is a matrix with H columns such that the j^{th} column is $\mathcal{A}(C^{(j)})$, then $\mathcal{B}^T \mathcal{B}$ is an H×H matrix whose $(i, j)^{th}$ entry for $i, j \in \{1, …, H\}$ is trace $(C^{(i)T} C^{(j)})$ and $\mathcal{B}^T \mathcal{A}(F)$ is the H-dimensional row vector whose i^{th} coordinate for $i \in \{1, …, H\}$ is trace $(C^{(i)T} F)$.

Thus, an equivalent way of stating (Dual-$\wp$1) is the following problem denoted (Dual-P1).

Minimize trace (YB) subject to trace $(YA^{(k)}) \geq p_k$ for all k = 1, ..., K, $Y \in \mathbb{R}^{n \times m}$.

2. Duality theory for LP-VC

From Topic 2 of Lahiri (2020) we know that x^* solves (P1) if and only if there exists $Y^* \in \mathbb{R}^{n \times m}$ such that the following is satisfied:

(i) $\sum_{k=1}^{K} A^{(k)} x_k^* = B$ and $x^* \in \mathbb{R}_+^K$.

(ii) trace $(Y^* A^{(k)}) \geq p_k$ and (trace $(Y^* A^{(k)}) - p_k) x_k^* = 0$ for all k = 1, …, K.

From (i) and (ii) it follows that $p^T x^* = \sum_{k=1}^{K} p_k x_k^* = \sum_{k=1}^{K}$ trace $(Y^* A^{(k)}) x_k^* = \sum_{k=1}^{K} \mathcal{A}(Y^{*T})^T \mathcal{A}(A^{(k)}) x_k^* = \mathcal{A}(Y^{*T})^T \sum_{k=1}^{K} \mathcal{A}(A^{(k)}) x_k^* = \mathcal{A}(Y^{*T})^T \mathcal{A}(B)$ = trace $(Y^* B)$.

3. Farkas' Lemma for LE-VC

We provide below a statement and proof of Farkas' lemma for linear equations with vector coefficients.

Theorem 1: Either [there exists $x \in \mathbb{R}_+^K$ such that $\sum_{k=1}^{K} A^{(k)} x^{(k)} = B$] or [there exists a n×m matrix Y, such that trace $(YA^{(k)}) \leq 0$ for all k = 1, …, K and trace $(YB) > 0$], but never both.

Proof: $x^* \in \mathbb{R}_+^K$ solves $\sum_{k=1}^{K} A^{(k)} x^{(k)} = B$ if and only if it solves $\sum_{k=1}^{K} \mathcal{A}(A^{(k)})\, x^{(k)} = \mathcal{A}(B)$.

By Farkas' lemma (see Topic 3 in Lahiri (2020)), either [there exists $x \in \mathbb{R}_+^K$ such that $\sum_{k=1}^{K} \mathcal{A}(A^{(k)})\, x^{(k)} = \mathcal{A}(B)$] or [there exists an m×n dimensional column vector y whose coordinates numbered (j-1)m + 1, …, jm is denoted by the m dimensional column vector y^j such that $y^T\mathcal{A}(A^{(k)}) \leq 0$ for all k = 1, …, K and $y^T\mathcal{A}(B) > 0$] but never both.

$y^T \mathcal{A}(A^{(k)}) = \sum_{j=1}^{n} {y^j}^T {A^{(k)}}^j$ for all k = 1, …, K and $y^T\mathcal{A}(B) = \sum_{j=1}^{n} {y^j}^T B^j$.

Let Y be the n×m matrix whose j^{th} row is ${y^j}^T$. For all j = 1, …, n, ${y^j}^T B^j$ is the j^{th} diagonal element of YB and ${y^j}^T {A^{(k)}}^j$ is the j^{th} diagonal element of $YA^{(k)}$ for $k \in \{1, …, K\}$.

Thus, $y^T\mathcal{A}(B)$ = trace (YB) and $y^T\mathcal{A}(A^{(k)})$ = trace $(YA^{(k)})$. for $k \in \{1, …, K\}$.

This proves the theorem. Q.E.D.

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