Obtaining Ellipsoid Dimensions from Circumferential Measurements

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Ellipsoids are geometric shapes that extend the concept of an ellipse to three dimensions. They have numerous applications in various fields due to the simplicity of their mathematical representation, unique geometric properties, and their natural occurrence across scales—from cosmic bodies to the atomic level. While a perfect sphere has only one radius, an ellipsoid has three radii. The general equation of an ellipsoid centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a, b, and c represent the radii, known as the major axis, semi-major-axis, and semi-minor axis, respectively.

An ellipsoid also has three circumferences, known in planetary science as the equatorial, meridional, and polar circumference. These circumferences are obtained by setting x, y, or z to zero, which simplifies the ellipsoid equation down to two dimensions.

$$When\ x = 0;\ \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad When\ y = 0;\ \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad When\ z = 0;\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

These equations are the three orthogonal ellipses that circumscribe the ellipsoid. Importantly, each of these circumferences share the same axes (a, b, or c) as the parent ellipsoid (Figure 1 A, B).

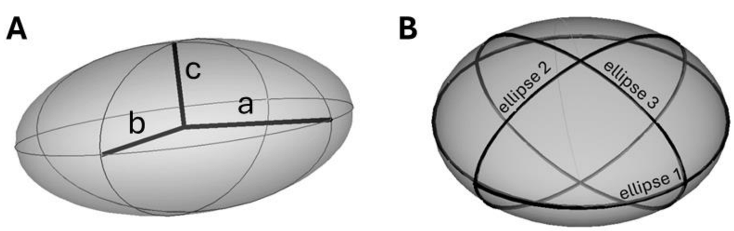

Figure 1: A- An ellipsoid with radii (axes) a, b, and c. **B-** The three orthogonal circumferences of an ellipsoid are each an ellipse with the same axes as the parent ellipsoid.

While the dimensions of a rectangular solid can be directly measured using linear methods, in the case of a sphere where precise linear measurements may be impractical, its circumference can be measured and divided by 2π to determine its radius. This radius can then be used to calculate the sphere's properties, such as surface area and volume. However, this known circumference-radius relationship is exclusive to perfect spheres and does not apply to ellipsoids. Nevertheless, the radii of an ellipsoid can be determined by taking advantage of the fact that the circumferential ellipses share the same axes as the parent ellipsoid. Calculating the axes of an ellipsoid is therefore possible by calculating the axes of each independent ellipse, which can be done as follows:

Several methods have been proposed to approximate the perimeter of an ellipse. Among these, Ramanujan's equation is widely regarded as a very accurate approximation [1]

$$p \approx \pi(a+b)\left(1+\frac{3h}{10+\sqrt{4-3h}}\right) \qquad \text{where } h=\frac{(a-b)^2}{(a+b)^2}$$

We can therefore construct a system of three equations, one equation for each ellipse. The three equations are:

$$p_1 \approx \pi(a+b)\left(1+\frac{3h_1}{10+\sqrt{4-3h_1}}\right) \quad \text{where } h_1=\frac{(a-b)^2}{(a+b)^2}$$

$$p_2 \approx \pi(a+c)\left(1+\frac{3h_2}{10+\sqrt{4-3h_2}}\right) \quad \text{where } h_2=\frac{(a-c)^2}{(a+c)^2}$$

$$p_3 \approx \pi(b+c)\left(1+\frac{3h_3}{10+\sqrt{4-3h_3}}\right) \quad \text{where } h_3=\frac{(b-c)^2}{(b+c)^2}$$

When $p_1, p_2,$ and p_3 are known, this system of non-linear equations can be solved using numerical methods to approximate a, b, and c.

Once the axes of the ellipsoid are known, it is then possible to compute its surface area, volume and other geometric properties using known methods explained elsewhere [e.g., 2-5].

Note: In a previous version of this preprint we relied on the ellipse perimeter equation presented by [6]. It later came to our attention that the equation developed in that paper is incorrect. It mistakenly assumes that unrolling a half-cylinder results in an isosceles triangle with straight edges, which is mathematically incorrect. We therefore now use Ramanujan's well-validated approximation equation.

References

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