Obtaining Ellipsoid Dimensions from Circumferential Measurements

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Ellipsoids are geometric shapes that extend the concept of an ellipse to three dimensions. They have numerous applications in various fields due to the simplicity of their mathematical representation, unique geometric properties, and their natural occurrence across scales—from cosmic bodies to the atomic level. While a perfect sphere has only one radius, an ellipsoid has three radii. The general equation of an ellipsoid centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

where a, b, and c represent the radii, known as the major axis, semi-major-axis, and semi-minor axis, respectively.

The dimensions of a rectangular solid can be simply measured through linear measurements. In the case of a sphere where accurate linear measurements may not be practical, its circumference can be measured and divided by 2π to determine its radius. The radius can then be used to calculate the sphere's properties including its surface area and volume. However, this known circumference-radius relationship only exists for a perfect sphere, not an ellipsoid. Consequently, the dimensions, surface area, and volume of an ellipsoid can only be determined by directly measuring its axes or diameters, a process that may sometimes be challenging. We therefore developed a mathematical formula for obtaining the dimensions of an ellipsoid based on its circumferential measurements.

Just as an ellipsoid has three axes (radii), it also has three circumferences, known in planetary science as the equatorial, meridional, and polar circumference. Importantly, each of these circumferences is a two-dimensional ellipse that shares the same axes as the parent ellipsoid (Figure 1 A, B). Calculating the axes of the whole ellipsoid is therefore possible by calculating the axes of each independent ellipse.

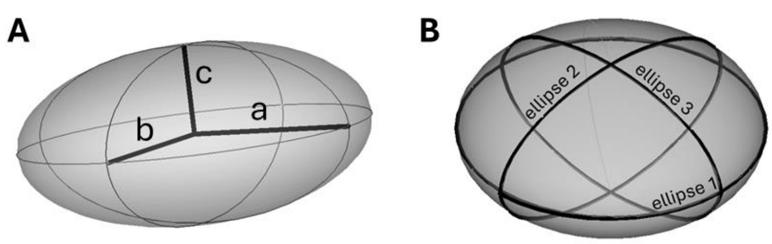

Figure 1: A- An ellipsoid with radii (axes) a, b, and c. **B-** The three orthogonal circumferences of an ellipsoid are each an ellipse with the same axes as the parent ellipsoid.

Several methods have been proposed to approximate the circumference of an ellipse. Among these, the well-established formula by Ramanujan is widely regarded as one of the most accurate approximations [1]. However, its complexity makes it impractical to extend into three dimensions. To address this limitation, we opted for a simpler formula recently introduced by Nguyen, which offers a more computationally efficient approach [2].

For an ellipse with a semi-major axis a, semi-minor axis b, and circumference (perimeter) P, Nguyen's equation states

$$P = 4a\sqrt{1 + \left(\frac{b}{a}\right)^2 \left(\left(\frac{\pi}{2}\right)^2 - 1\right)}. \quad (2)$$

Using Nguyen's equation, we can construct a system of three equations to solve the three radii, one equation for each circumference.

$$P_1 = 4a\sqrt{1 + \left(\frac{b}{a}\right)^2 \left(\left(\frac{\pi}{2}\right)^2 - 1\right)}, \quad (3a)$$

$$P_2 = 4a\sqrt{1 + \left(\frac{c}{a}\right)^2 \left(\left(\frac{\pi}{2}\right)^2 - 1\right)}, \quad (3b)$$

$$P_3 = 4b\sqrt{1 + \left(\frac{c}{b}\right)^2 \left(\left(\frac{\pi}{2}\right)^2 - 1\right)}. \quad (3c)$$

where $a \geq b \geq c$.

This system of three equations can be solved in respect to a, b, and c, using the following algebraic manipulations

We can begin by substituting $\left(\frac{\pi}{2}\right)^2 - 1$ with 'k' in Equation 3a as follows:

$$P_1 = 4a\sqrt{1 + \left(\frac{b}{a}\right)^2 k}$$

$$\left(\frac{P_1}{4a}\right)^2 = 1 + \left(\frac{b}{a}\right)^2 k$$

$$\frac{{P_1}^2}{16a^2} - 1 = \frac{b^2}{a^2}k$$

$$\frac{{P_1}^2 - 16a^2}{16a^2} = \frac{b^2}{a^2}k$$

$$\frac{{P_1}^2 - 16a^2}{16} = b^2 k$$

$$b^2 = \frac{{P_1}^2 - 16a^2}{16k} \quad (4)$$

We can manipulate Equation 3b in a similar fashion

$$P_2 = 4a\sqrt{1 + \left(\frac{c}{a}\right)^2 k}$$

$$\left(\frac{P_2}{4a}\right)^2 = 1 + \left(\frac{c}{a}\right)^2 k$$

$$\frac{{P_2}^2}{16a^2} - 1 = \frac{c^2}{a^2}k$$

$$\frac{{P_2}^2 - 16a^2}{16a^2} = \frac{c^2}{a^2}k$$

$$\frac{{P_2}^2 - 16a^2}{16} = c^2 k$$

$$c^2 = \frac{{P_2}^2 - 16a^2}{16k} \quad (5)$$

Performing similar operations to Equation 3c we obtain the following:

$$P_3 = 4b\sqrt{1 + \left(\frac{c}{b}\right)^2 k}$$

$$\left(\frac{P_3}{4b}\right)^2 = 1 + \left(\frac{c}{b}\right)^2 k$$

$$\frac{{P_3}^2}{16b^2} - 1 = \frac{c^2}{b^2}k$$

$$\frac{{P_3}^2 - 16b^2}{16b^2} = \frac{c^2}{b^2}k$$

$$\frac{{P_3}^2 - 16b^2}{16} = c^2 k$$

$$c^2 = \frac{{P_3}^2 - 16b^2}{16k} \quad (6)$$

Inserting Equation 6 into Equation 5, we obtain the following:

$$\frac{{P_2}^2 - 16a^2}{16k} = \frac{{P_3}^2 - 16b^2}{16k}$$

$${P_2}^2 - 16a^2 = {P_3}^2 - 16b^2$$

$$a^2 - b^2 = \frac{{P_2}^2 - {P_3}^2}{16}$$

$$b^2 = a^2 - \frac{{P_2}^2 - {P_3}^2}{16} \quad (7)$$

Inserting Equation 7 into Equation 4, we obtain the following:

$$\frac{{P_1}^2-16a^2}{16k} = a^2 - \frac{{P_2}^2-{P_3}^2}{16}$$

$${P_1}^2 - 16a^2 = 16ka^2 - k({P_2}^2 - {P_3}^2)$$

$${P_1}^2 + k\left({P_2}^2 - {P_3}^2\right) = 16ka^2 + 16a^2$$

$$\frac{{P_1}^2+k\left({P_2}^2-{P_3}^2\right)}{16(k+1)} = a^2$$

$$a^2 = \frac{{P_1}^2+\left(\left(\frac{\pi}{2}\right)^2-1\right)\left({P_2}^2-{P_3}^2\right)}{16\left(\frac{\pi}{2}\right)^2}$$

$$a^2 = \frac{{P_1}^2+\left(\frac{\pi^2-4}{4}\right)\left({P_2}^2-{P_3}^2\right)}{4\pi^2}$$

$$a^2 = \frac{4{P_1}^2+(\pi^2-4)\left({P_2}^2-{P_3}^2\right)}{16\pi^2} \qquad (8a)$$

The value of a^2 can now be substituted into Equation 3d to solve for b^2

$$b^2 = \frac{{P_1}^2-16\frac{4{P_1}^2+(\pi^2-4)\left({P_2}^2-{P_3}^2\right)}{16\pi^2}}{16\left(\left(\frac{\pi}{2}\right)^2-1\right)}$$

$$b^2 = \frac{\pi^2{P_1}^2-4{P_1}^2-(\pi^2-4)\left({P_2}^2-{P_3}^2\right)}{4\pi^2(\pi^2-4)}$$

$$b^2 = \frac{(\pi^2-4)\left({P_1}^2-{P_2}^2+{P_3}^2\right)}{4\pi^2(\pi^2-4)}$$

$$b^2 = \frac{{P_1}^2-{P_2}^2+{P_3}^2}{4\pi^2} \qquad (8b)$$

Similarly, the value of a^2 can now be substituted into Equation 5 to solve for c^2

$$c^2 = \frac{{P_3}^2-16\frac{4{P_1}^2+(\pi^2-4)\left({P_2}^2-{P_3}^2\right)}{16\pi^2}}{16\left(\left(\frac{\pi}{2}\right)^2-1\right)}$$

$$c^2 = \frac{\pi^2{P_3}^2-4{P_1}^2-(\pi^2-4)\left({P_2}^2-{P_3}^2\right)}{4\pi^2(\pi^2-4)}$$

$$c^2 = \frac{\pi^2{P_2}^2-4{P_1}^2-\pi^2{P_2}^2+\pi^2{P_3}^2+4{P_2}^2-4{P_3}^2}{4\pi^2(\pi^2-4)}$$

$$c^2 = \frac{(\pi^2-4){P_3}^2}{4\pi^2(\pi^2-4)} - \frac{4\left({P_1}^2-{P_2}^2\right)}{4\pi^2(\pi^2-4)}$$

$$c^2 = \frac{{P_3}^2}{4\pi^2} - \frac{{P_1}^2-{P_2}^2}{\pi^2(\pi^2-4)} \qquad (8c)$$

The general ellipsoid equation can then be written in respect to its three circumferences as

$$\frac{x^2 16\pi^2}{4{P_1}^2+(\pi^2-4)\left({P_2}^2-{P_3}^2\right)} + \frac{y^2 4\pi^2}{{P_1}^2-{P_2}^2+{P_3}^2} + \frac{z^2}{\frac{{P_3}^2}{4\pi^2}-\frac{{P_1}^2-{P_2}^2}{\pi^2(\pi^2-4)}} = 1 \qquad \text{where } P_1 \geq P_2 \geq P_3. \quad (9)$$

It should be noted that this equation in its current form assumes that a (major axis) lies on the x-axis, b (semi-major axis) lies on the y-axis, and c (semi-minor axis) lies on the z-axis. When this is not the case, the placements of x, y, and z in the equation should be swapped.

Once the equation of an ellipsoid is known, it is then possible to compute its surface area, volume and other geometric properties using known methods explained elsewhere [3-6].

References

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