Obtaining Ellipsoid Dimensions from Circumferential Measurements

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Ellipsoids are geometric shapes that extend the concept of an ellipse to three dimensions. They have numerous applications in various fields due to the simplicity of their mathematical representation, unique geometric properties, and their natural occurrence across scales—from cosmic bodies to the atomic level. While a perfect sphere has only one radius, an ellipsoid has three radii. The general equation of an ellipsoid centered at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{(Equation 1)}$$

where a, b, and c represent the radii, known as the major axis, semi-major-axis, and semi-minor axis, respectively.

The dimensions of a rectangular solid can be simply measured through linear measurements. In the case of a sphere where accurate linear measurements may not be practical, its circumference can be measured and divided by 2π to determine its radius. The radius can then be used to calculate the sphere's properties including its surface area and volume. However, this known circumference-radius relationship only exists for a perfect sphere, not an ellipsoid. Consequently, the dimensions, surface area, and volume of an ellipsoid can only be determined by directly measuring its axes or diameters. This can be impractical due to the inherent challenge of inserting a tape measure through a solid object. We therefore developed a mathematical formula for obtaining the dimensions of an ellipsoid based on its circumferential measurements.

Just as an ellipsoid has three axes (radii), it also has three circumferences, known in planetary science as the equatorial, meridional, and polar circumference. These circumferences can be obtained by inserting 0 into the ellipsoid equation for either x, y, or z. The ellipsoid equation then simplifies into either of the following three equations:

$$When\ x = 0;\ \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad When\ y = 0;\ \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 \quad When\ z = 0;\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

These equations are the three orthogonal ellipses that serve as the circumferences of the ellipsoid. Importantly, each of these circumferences share the same axes as the parent ellipsoid **(Figure 1 A, B)**. Calculating the axes of the whole ellipsoid is therefore possible by calculating the axes of each independent ellipse.

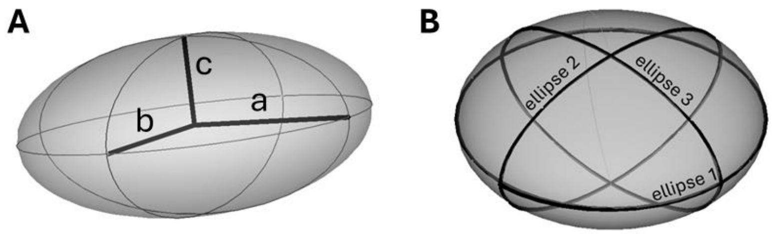
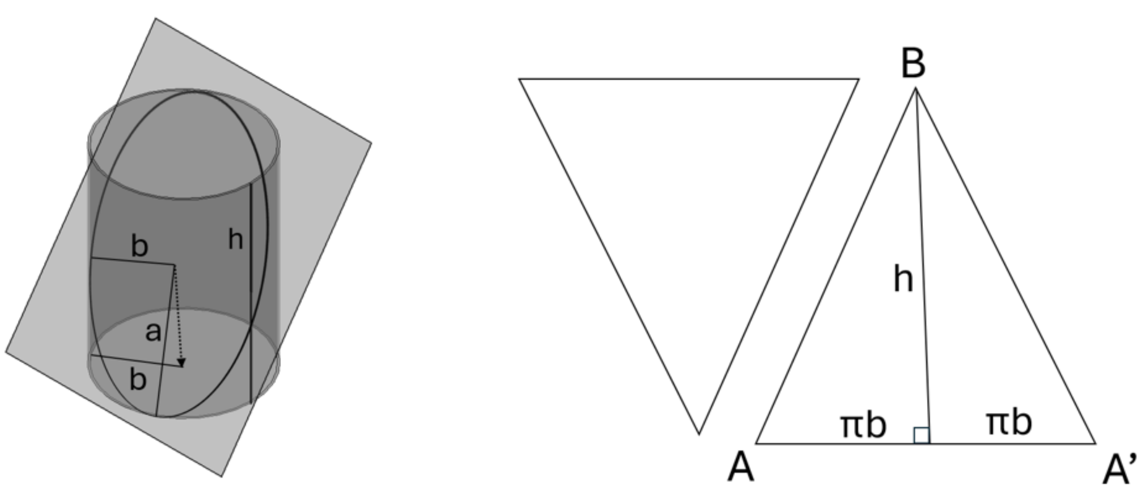


Figure 1: A- An ellipsoid with radii (axes) a, b, and c. **B-** The three orthogonal circumferences of an ellipsoid are each an ellipse with the same axes as the parent ellipsoid.

Several methods have been proposed to approximate the circumference of an ellipse. Among these, the well-established formula by Ramanujan is widely regarded as one of the most accurate approximations [1]. However, its complexity makes it impractical to extend into three dimensions. To address this limitation, we opted for a simpler formula recently introduced by Nguyen, which offers a more computationally efficient approach [2]. Due to the difficulty in comprehending Nguyen's proof, we present a summary here:

If we cut a cylinder in half diagonally with a plane, the intersection of the plane and cylinder is an ellipse (**Figure 2A**). While Nguyen presents a complex proof to this assumption, it can be intuitively understood considering that an ellipse is a conical section and that a cone whose height approaches infinity becomes a cylinder.

Figure 2A Figure 2B



The radius of the cylinder equals one of the ellipse's axes (b) while the height of the cylinder can be obtained using the Pythagorean theorem as follows:

$$(2a)^2 = h^2 + (2b)^2$$

$$h^2 = 4a^2 + 4b^2 \quad \text{(Equation 2)}$$

After splitting the cylinder in half, it can be unrolled to form two equal isosceles triangles (**Figure 2B**) Since the cylinder was cut symmetrically in half, the triangles must be equal. Therefore, their sides must be straight, since convexity of one triangle's side would result in concavity of the other triangle's side, making the triangles incongruent and unequal.

The ellipse's perimeter (P) now becomes the sum of the two equal sides of an isosceles triangle (AB+BA' or 2AB) and can be obtained using the Pythagorean theorem as follows:

$$P = 2\sqrt{(b\pi)^2 + h^2}$$

Plugging in Equation 2, we get

$$P = 2\sqrt{b^2\pi^2 + 4a^2 + 4b^2}$$

$$P = 2\sqrt{4a^2 + b^2(\pi^2 - 4)}$$

$$P = 2\sqrt{4a^2\left(1 + \left(\left(\frac{b^2}{a^2}\right)\frac{\pi^2 - 4}{4}\right)\right)}$$

$$P = 4a\sqrt{1 + \left(\frac{b}{a}\right)^2\left(\left(\frac{\pi}{2}\right)^2 - 1\right)} \qquad \text{(Equation 3)}$$

Using Nguyen's equation, we can construct a system of three equations to solve the three radii, one equation for each circumference.

$$P_1 = 4a\sqrt{1 + \left(\frac{b}{a}\right)^2\left(\left(\frac{\pi}{2}\right)^2 - 1\right)}, \quad \text{(Equation 3a)}$$

$$P_2 = 4a\sqrt{1 + \left(\frac{c}{a}\right)^2\left(\left(\frac{\pi}{2}\right)^2 - 1\right)}, \quad \text{(Equation 3b)}$$

$$P_3 = 4b\sqrt{1 + \left(\frac{c}{b}\right)^2\left(\left(\frac{\pi}{2}\right)^2 - 1\right)}. \quad \text{(Equation 3c)}$$

This system of three equations can be solved in respect to a, b, and c, using the following algebraic manipulations

We can begin by substituting $\left(\frac{\pi}{2}\right)^2 - 1$ with 'k' in Equation 3a as follows:

$$P_1 = 4a\sqrt{1 + \left(\frac{b}{a}\right)^2 k}$$

$$\left(\frac{P_1}{4a}\right)^2 = 1 + \left(\frac{b}{a}\right)^2 k$$

$$\frac{{P_1}^2}{16a^2} - 1 = \frac{b^2}{a^2}k$$

$$\frac{P_1^2-16a^2}{16a^2}=\frac{b^2}{a^2}k$$

$$\frac{P_1^2-16a^2}{16}=b^2k$$

$$b^2=\frac{P_1^2-16a^2}{16k} \qquad \text{(Equation 4)}$$

We can manipulate Equation 3b in a similar fashion

$$P_2=4a\sqrt{1+\left(\frac{c}{a}\right)^2k}$$

$$\left(\frac{P_2}{4a}\right)^2=1+\left(\frac{c}{a}\right)^2k$$

$$\frac{P_2^2}{16a^2}-1=\frac{c^2}{a^2}k$$

$$\frac{P_2^2-16a^2}{16a^2}=\frac{c^2}{a^2}k$$

$$\frac{P_2^2-16a^2}{16}=c^2k$$

$$c^2=\frac{P_2^2-16a^2}{16k} \qquad \text{(Equation 5)}$$

Performing similar operations to Equation 3c we obtain the following:

$$P_3=4b\sqrt{1+\left(\frac{c}{b}\right)^2k}$$

$$\left(\frac{P_3}{4b}\right)^2=1+\left(\frac{c}{b}\right)^2k$$

$$\frac{P_3^2}{16b^2}-1=\frac{c^2}{b^2}k$$

$$\frac{P_3^2-16b^2}{16b^2}=\frac{c^2}{b^2}k$$

$$\frac{P_3^2-16b^2}{16}=c^2k$$

$$c^2=\frac{P_3^2-16b^2}{16k} \qquad \text{(Equation 6)}$$

Inserting Equation 6 into Equation 5, we obtain the following:

$$\frac{P_2^2-16a^2}{16k}=\frac{P_3^2-16b^2}{16k}$$

$$P_2{}^2 - 16a^2 = P_3{}^2 - 16b^2$$

$$a^2 - b^2 = \frac{P_2{}^2 - P_3{}^2}{16}$$

$$b^2 = a^2 - \frac{P_2{}^2 - P_3{}^2}{16} \qquad \text{(Equation 7)}$$

Inserting Equation 7 into Equation 4, we obtain the following:

$$\frac{P_1{}^2 - 16a^2}{16k} = a^2 - \frac{P_2{}^2 - P_3{}^2}{16}$$

$$P_1{}^2 - 16a^2 = 16ka^2 - k(P_2{}^2 - P_3{}^2)$$

$$P_1{}^2 + k\left(P_2{}^2 - P_3{}^2\right) = 16ka^2 + 16a^2$$

$$\frac{P_1{}^2 + k\left(P_2{}^2 - P_3{}^2\right)}{16(k+1)} = a^2$$

$$a^2 = \frac{P_1{}^2 + \left(\left(\frac{\pi}{2}\right)^2 - 1\right)\left(P_2{}^2 - P_3{}^2\right)}{16\left(\frac{\pi}{2}\right)^2}$$

$$a^2 = \frac{P_1{}^2 + \left(\frac{\pi^2 - 4}{4}\right)\left(P_2{}^2 - P_3{}^2\right)}{4\pi^2}$$

$$a^2 = \frac{4P_1{}^2 + (\pi^2 - 4)\left(P_2{}^2 - P_3{}^2\right)}{16\pi^2} \qquad \text{(Equation 8)}$$

The value of a^2 can now be substituted into Equation 3d to solve for b^2

$$b^2 = \frac{P_1{}^2 - 16\frac{4P_1{}^2 + (\pi^2 - 4)\left(P_2{}^2 - P_3{}^2\right)}{16\pi^2}}{16\left(\left(\frac{\pi}{2}\right)^2 - 1\right)}$$

$$b^2 = \frac{\pi^2 P_1{}^2 - 4P_1{}^2 - (\pi^2 - 4)\left(P_2{}^2 - P_3{}^2\right)}{4\pi^2(\pi^2 - 4)}$$

$$b^2 = \frac{(\pi^2 - 4)\left(P_1{}^2 - P_2{}^2 + P_3{}^2\right)}{4\pi^2(\pi^2 - 4)}$$

$$b^2 = \frac{P_1{}^2 - P_2{}^2 + P_3{}^2}{4\pi^2} \qquad \text{(Equation 9a)}$$

Given that Equations 3a, 3b, and 3c are essentially identical except for swapping the axes and P values, we can perform the same algebraic manipulations to obtain similar equations for a^2 and c^2, resulting in the following equations:

$$a^2 = \frac{P_1{}^2 - P_3{}^2 + P_2{}^2}{4\pi^2} \qquad \text{(Equation 9b)}$$

$$c^2 = \frac{P_2{}^2 - P_1{}^2 + P_3{}^2}{4\pi^2} \qquad \text{(Equation 9c)}$$

ONotaby, in the special case of a sphere where $a = b = c$ and $P_1 = P_2 = P_3$, Equations 9a, 9b, and 9c each simplify to the equation for the circumference of a circle.

We can now write the general ellipsoid equation in respect to its perimeters as follows:

$$x^2\left({P_1}^2 - {P_3}^2 + {P_2}^2\right) + y^2\left({P_1}^2 - {P_2}^2 + {P_3}^2\right) + z^2\left({P_2}^2 - {P_1}^2 + {P_3}^2\right) = 4\pi^2 \qquad \text{(Equation 10)}$$

Once the equation of an ellipsoid is known, it is then possible to compute its surface area, volume and other geometric properties using known methods explained elsewhere [3-6].

References

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