A note on the Buckingham equation*

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Abstract

Friction losses of the laminar flow of a Bingham plastic fluid through a pipe is characterised by a friction factor, f, which depends on two dimensionless parameters, the Reynolds and the Bingham numbers. The relationship is a fourth degree equation for f, named after Buckingham (1921). In this short communication, we present a novel expression for its only physically meaningful solution, which is easier to handle than those presented previously by other authors.

Keywords: Bingham fluid, Buckingham equation, friction factor, laminar flow, analytic solution

Pressure losses in pipeline flow of a Bingham plastic (Bingham, 1922; Bird et al., 1987) of Bingham viscosity μ_B and yield stress τ_0, are normally obtained in terms of the nondimensional Fanning friction factor $f = 2\tau_w/\rho v^2$, where τ_w, ρ and v is the wall stress, bulk density and mean velocity, respectively. In laminar flow, f corresponds to the solution of the Buckingham equation (Buckingham, 1921):

$$\frac{1}{Re} = \frac{f}{16} - \frac{1}{6}\frac{B}{Re} + \frac{1}{3}\frac{B^4}{f^3 Re^4} \tag{1}$$

$Re = \rho v D/\mu_B$ and $B = \tau_0 D/v\mu_B$ are the Reynolds and Bingham numbers, respectively. D is the internal diameter of the pipe. An alternative notation for (1) is to use the Hedström number, $He = ReB$ (Wasp et al., 1977). This fourth degree equation for f admits two complex conjugate and two real roots. The complete analytic solution was reported by Hedström (1952), and remarks the solution is "too troublesome for technical calculations." In his paper, the corresponding analytic solution was not further analysed and, in turn, a simplified version, valid for high wall shear stress was considered (see also Caldwell and Babbitt (1941); McMillen (1948); Wasp et al. (1977)). Imposing the physical condition that the wall shear stress τ_w must exceed the yield stress to have a sheared layer and thus a flow inside the pipe ($f \gg 2B/Re$), it results that the only physically meaningful solution corresponds to the largest real root of (1). More recently, an explicit formulation for the friction factor, obtained using a computer algebra system, was reported by Sablani et al. (2003).

Solutions for the quartic equation where found by Ferrari and Descartes in the XVI and XVII century, respectively (Dickson, 1922). To apply those results to the Buckingham equation involves a long and tedious algebraic work, even when the solution is obtained through a computer package. However, the final expression for the largest root of the Buckingham equation is greatly simplified

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when the following parameters P and Q are defined:

$$P = \frac{B}{Re}; \quad Q = \frac{2}{Re}\left(\frac{B}{3} + 2\right), \tag{2}$$

This results in Equation (3), being the solution for f:

$$f = Q + \left(Q^2 + S\right)^{1/2} + \left\{2Q^2\left[1 + \left(1 + \frac{S}{Q^2}\right)^{-1/2}\right] - S\right\}^{1/2} \tag{3}$$

with $S = S_1 + S_2$, where

$$S_n = \frac{2}{3}P^{4/3}\left\{\left(\frac{3}{2}Q\right)^2 + (-1)^n\left[\left(\frac{3}{2}Q\right)^4 - P^4\right]^{1/2}\right\}^{1/3} \tag{4}$$

In particular, when $B = 0$, the friction factor reduces to that of a Newtonian fluid, that is, $f = 16/Re$. Equation (3) can be expanded around $B = 0$ for fixed Reynolds number to get Equation (5), an approximate solution for f:

$$f \approx 4Q - \frac{1}{12}\frac{P^4}{Q^3} - \frac{1}{192}\frac{P^8}{Q^7} - \frac{5}{9216}\frac{P^{12}}{Q^{11}} - \dots \tag{5}$$

The first term on the right hand side of (5) corresponds to the expression resulting from dropping the fourth degree term in (1).

For small values of B, the first two terms of (5) are a very good approximation of (3):

$$f \approx \frac{1}{Re}\left[16 + \frac{8}{3}B - \frac{9}{32}\frac{B^4}{(B+6)^3}\right] \tag{6}$$

The asymptotic case of large Bingham number reduces to the lowest feasible solution of (1), given by Equation (7):

$$f \approx 2\frac{B}{Re} \tag{7}$$

Table 1 shows a reference for the relative errors corresponding to the different approximations of Equation (3). Relative error is practically independent of Reynolds number.

Table 1: Relative percent error between the largest root of the Buckingham equation and the different approximations proposed, corresponding to Eqs. (5)-(7).

f	B				
	1	10	100	1000	10000
Equation (5)	3.8×10^{-14}	7.0×10^{-4}	1.3	6.7	10.4
Equation (6)	5.8×10^{-7}	8.6×10^{-2}	4.7	12.8	17.0
Equation (7)	−89.3	−52.3	−19.2	−6.3	−2.0

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