Turbulent theory of velocity-profile-induced jet breakup

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Abstract

Velocity profile relaxation is commonly believed to be a cause of jet breakup. This claim is critically reevaluated in this work. Contrary to common belief, laminar liquid jets with parabolic velocity profiles are actually more stable than laminar jets with flatter velocity profiles. This is shown using prior theory and experiments. For turbulent jets, the influence of the velocity profile is more difficult to determine. Previous experimentalists claimed to show that the velocity profile has an effect by varying the nozzle length. The claim is that the boundary layer thickness grows with nozzle length, and that the larger the boundary layer, the less stable the jet. In this work, nozzle length is shown to be a poor proxy for velocity profile effects because the turbulence intensity also increases as the nozzle length increases. Studies with this confounding were ignored in this work. Thinner boundary layers have greater shear, yet experiments have shown that if the boundary layer were made thinner (all else constant), the jet often is more stable. This is termed the "shear paradox". A potential resolution to the shear paradox is developed by considering that the area with shear also decreases as the boundary layer thickness is decreased, and by non-dimensionalizing the turbulent production rate by the dissipation. This theory shows an interaction between the integral scale and velocity profile relaxation which has not been previously discussed. The theoretical prediction that a smaller integral scale leads to more stable jets (due to increased turbulent dissipation) is shown to be somewhat consistent with the limited experimental and DNS data available.

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Introduction

Velocity profile relaxation has been proposed as a cause of jet breakup since the early work of Schweitzer [1] in 1937, and independently a few years later by Littaye [2] in 1942. The present explanation is that non-uniform velocity profiles have excess energy, which can accelerate the breakup process [3–7], though the details of the physical mechanism remain vague. This effect could alternatively be called a boundary layer or shear instability [8, 9].

McCarthy and Molloy [5, p. 7] proposed the velocity profile kinetic energy coefficient as a measure of the tendency for velocity profile relaxation to cause breakup:

$$\alpha \equiv \frac{\int_{A_0} U_0^3 \, \mathrm{d}A}{\overline{U}_0^3 A_0}. \tag{1}$$

Above, the 0 subscript refers to a quantity located at the nozzle exit plane, U_0 is the (local) mean axial velocity at the nozzle exit, A_0 is the nozzle exit area ($A_0 = \pi d_0^2/4$, where d_0 is the nozzle diameter), and $\overline{U}_0$ is the mean plane-averaged axial velocity at the nozzle exit.

The larger the kinetic energy coefficient is, the stronger breakup due to velocity profile relaxation is according to McCarthy and Molloy. The kinetic energy coefficient α is smallest ($\alpha = 1$) for a uniform/flat velocity profile, and largest for a parabolic profile ($\alpha = 2$). For a fully developed turbulent pipe velocity profile $\alpha \approx 1.06$ [10, p. 219][1], which, if the criteria is true, suggests that fully developed turbulent pipe flows are not likely to see major velocity profile relaxation effects. (Note that this does mean that turbulent flows in general won't see velocity profile relaxation effects.) McCarthy and Molloy justified this with figure 1, which shows jets which are identical aside from changing the nozzle length, which changes the velocity profile from relatively flat to more parabolic as the flow develops. The flow is initially laminar in each case shown, though it likely would be turbulent for the fully developed case as the Reynolds number is 4750. In this work, the Reynolds number is denoted $\mathrm{Re}_{\ell 0} \equiv \overline{U}_0 d_0/\nu_\ell$, where d_0 is the nozzle diameter and ν_ℓ is the liquid viscosity. Similarly, the Weber number is $\mathrm{We}_{\ell 0} \equiv \rho_\ell \overline{U}_0^2 d_0/\sigma$, where ρ_ℓ is the liquid density and σ is the surface tension.

To avoid velocity profile relaxation and consequently enhance the stability of turbulent liquid jets, special nozzles have been designed to produce flat velocity profiles [8, 11], and these efforts appear to produce more stable jets. Additionally, the largest breakup lengths I am aware of, $\langle x_\mathrm{b} \rangle = O(1000)$, were obtained with short nozzles (nozzle orifice length $L_0 = d_0$) [12, fig. 4] which produce relatively flat velocity profiles [13, fig. 4, p. 283]. However, there are flaws in this narrative.

McCarthy and Molloy's photographs are qualitative at best. Objective measurements of the breakup length are necessary to validate the theory. The apparent worst-case-scenario, a parabolic velocity profile, appears to be more stable than previously thought. Linear stability analysis for laminar jets shows that parabolic velocity profiles are actually *more stable than uniform* [14–16]. Indeed, the experimental data shows that for laminar jets, as the orifice (development) length increases (again, changing the velocity profile from roughly flat to parabolic), the breakup length increases by 30%[2], qualitatively consistent with linear stability theory. See figure 2. For this plot I compiled data from Arai et al. [12] for breakup length as a function of nozzle length, and interpolated the data to get lines of constant Weber number. Each line is also a constant, but different, Reynolds number[3]. The trend is not as expected for the two laminar cases — the breakup length is roughly constant as the nozzle aspect ratio L_0/d_0 increases, and slightly increasing for the lowest Reynolds number case. Here, L_0 is the length of the nozzle orifice. A longer nozzle aspect ratio for the laminar case would indicate that the velocity profile is closer to parabolic. If velocity profile relaxation were so catastrophic for laminar jets as McCarthy and Molloy [5] claim, then why is the this not observed in this experiment?

Clues come from the nozzle aspect ratios and hydrodynamic regime in the example used by McCarthy and Molloy [5, fig. 5]. The example of McCarthy and Molloy is for an initially laminar but transitional jet at high Weber number ($\mathrm{We}_{\ell 0} \approx 1.5 \cdot 10^4$), which is close to the $\mathrm{We}_{\ell 0} = 1 \cdot 10^4$ case in figure 2[4]. For this case, as the nozzle aspect ratio increased, the breakup length decreases through $L_0/d_0 \approx 10$, which is the longest aspect ratio considered by McCarthy and Molloy. The trend McCarthy and Molloy observed applies only for high Weber number liquid jets with $L_0/d_0 \lesssim 10$. The breakup length trend at higher nozzle aspect ratios is slightly increasing, contrary to what one might expect from reading McCarthy and Molloy. Indeed, Debler and Yu [15] examine only $L_0/d_0 \gtrsim 10$ and come to the opposite conclusion as McCarthy and Molloy at lower Reynolds numbers.

An alternative mechanism can explain the observed trends. The transitional and turbulent data of McCarthy and Molloy [5] and Arai et al. [12] both suffer from *con-*

[1]McCarthy and Molloy [5, p. 7L] incorrectly suggest that α is in the range of 1.1 to 1.2, but direct computation indicates lower. This suggests that α barely changes at all with nozzle length in the turbulent case.

[2]The difference appears to be statistically significant. Arai et al. [12] provide no uncertainty estimates, but assuming the statistical uncertainty is negligible for electrical conductivity measurements (as they are essentially a large number of pulses), the main source of uncertainty is the precision of the length measurement. If the experimental setup of Arai et al. was similar to that of Phinney and Humphries [17, p. 9], the measurement was within 2 mm. Then for $L_0/d_0 = 1$, $\langle x_\mathrm{b} \rangle/d_0 = 72.4 \pm 6.7$, and for $L_0/d_0 = 50$, $\langle x_\mathrm{b} \rangle/d_0 = 93.8 \pm 6.7$.

[3]See the caption of figure 2 for the precise Reynolds numbers.

[4]Also compare the Reynolds numbers: $\mathrm{Re}_{\ell 0} = 4750$ for McCarthy and Molloy [5, fig. 5], 16 000 in figure 2.

founding between the turbulence intensity and velocity profile[5]. (A "plane-averaged" turbulence intensity in this work is defined as $\overline{\mathrm{Tu}}_0 \equiv \sqrt{2\overline{k}_0/(3\overline{U}_0^2)}$ where $\overline{k}_0$ is a plane-averaged turbulent kinetic energy.) I discuss confounding in detail in another paper at this conference [18]. Essentially, both the turbulence intensity and velocity profile are changing when the nozzle length changes [19]. McCarthy and Molloy [5, p. 10] do not seem to be aware of this given that they state that "these jets, on issuing from the nozzles, differ from each other *only* in the value of α" (the emphasis is my own). An increase in disturbances in the flow as measured by turbulence intensity would obviously affect the stability of the jet and its transition to turbulence. Others agree on this point — at higher Reynolds numbers Debler and Yu attribute McCarthy and Molloy's observations to the onset of turbulence. To be clear, both the turbulence intensity and velocity profile are factors, but the relative contributions of each can not be determined based on the experiments of McCarthy and Molloy and Arai et al.

The case where the breakup length $\langle x_\mathrm{b} \rangle$ in Arai et al. [12] was $O(1000)$ likely had low turbulence intensity as the diameter contraction ratio was $d_\mathrm{in}/d_0 = 10$. Large contractions tend to reduce the turbulence intensity [20]. As the nozzle length increases, the turbulence intensity at the nozzle exit increases due to the effects of the shear at the walls[6]. The increase in breakup length as the nozzle length increases from $L_0/d_0 = 10$ to 50 could be explained by turbulence transition moving to inside the nozzle as nozzle becomes basically a fully developed pipe flow. Turbulence transition inside the nozzle appears to stabilize jets, as will be discussed more below.

An unambiguous test would try to maintain the turbulence intensity as close as possible between the two velocity profiles tested to isolate the effect of the velocity profile. Likely the breakup lengths were much longer for shorter nozzle aspect ratios in the data of Arai et al. due to the low turbulence intensity for their short nozzles. Including other studies, the data for short nozzle aspect ratios seems to vary greatly, likely due to the turbulence intensity being a strong function of the inflow. See, for example, that the breakup lengths reported by Chen and Davis [22] for their short nozzles are much lower than those of Arai et al. [12].

A qualitative unambiguous test was conducted by Wu et al. [9]. The turbulence intensity was low in this case, but not quantified. While breakup lengths were not measured, the photographs show that breakup was suppressed when the boundary layer was thinner. The DNS study of Sander and Weigand [23, fig. 11], and it makes clear that breakup at least qualitatively is influenced by the velocity profile.

The early studies of Eisenklam and Hooper [3] (at $\mathrm{Re}_{\ell 0} \approx 3000$ to 12 000) and Rupe [4, fig. 4, p. 12] (at $\mathrm{Re}_{\ell 0} \approx 2100$) have photos showing initially laminar jets with near complete breakup occurring over a short distance. While these researchers suggested that the parabolic velocity profile was responsible for this observation, as was previously detailed, laminar jets with parabolic velocity profiles show no particular inclination towards instability at Reynolds numbers below the critical Reynolds number. Instead, I hypothesize that turbulence transition outside of the nozzle causes the strong breakup, and that turbulence transition is sensitive to the velocity profile. The Reynolds numbers where this form of breakup occurs coincides well with where transition to turbulence occurs in fully developed pipe flows[7]. This idea is consistent with the observations of Hoyt and Taylor [8, p. 96L] who suggest that when a liquid jet transitions to turbulence after it has exited the nozzle, the transition is more violent than if the jet were already turbulent. Further, Hoyt and Taylor propose that reducing boundary layer thickness at the nozzle exit can reduce these effects, which again, is consistent with the idea that the velocity profile influences turbulence transition. The earlier mentioned DNS study of Sander and Weigand [23] showed a clear sensitivity to the velocity profile. Sander and Weigand used low Reynolds numbers from 3000 to 7000, which are roughly in the transitional range for fully developed pipe flows, ultimately consistent with the idea that turbulence transition can cause strong breakup and also can be influenced by the velocity profile.

The reader is referred to Portillo et al. [25] and Umemura [26] for the latest experimental and theoretical research on transitional liquid jets, including detailed explanations of the transition mechanism which may be useful for nozzle design in this regime. More research is needed to explain why transitional liquid jets appear so unstable. The criteria developed later in this paper may explain why transition can produce such violent breakup, but the fit with the transitional breakup data available at present (which is largely qualitative) is mixed.

Note that many of the trends discussed previously seem to apply only at low ambient densities, i.e., density

[5]Jets which will remain laminar even in the fully developed state, e.g., the cases for which $\mathrm{Re}_{\ell 0} < 2000$ in figure 2 do not suffer from this confounding problem because the disturbances introduced from the wall are damped by viscosity. Hence the low Reynolds number (always laminar regardless of L_0/d_0) case mentioned earlier does not suffer from this confounding.

[6]The nozzle design approach of Theobald [11] came from Whitehead et al. [21], and suffers from a similar problem. In addition to having a flat velocity profile, Whitehead et al. also avoid boundary layer separation to keep the turbulence intensity low. Consequently, the apparent success of Theobald's nozzle design is not necessarily due to the velocity profile relaxation effects discussed in Theobald [11].

[7]Some readers may believe that a Reynolds number of 12 000 is far too high to be transitional, but laminar fully developed pipe flows have been maintained at Reynolds numbers as high as 100 000 depending on the quality of the experimental setup [24, p. 7]. Indeed, Reynolds himself was able to establish laminar pipe flow at a Reynolds number of 13 000.

ratios $\rho_\ell/\rho_g > 500$. The data of Arai et al. [12, fig. 7] seems to suggest that at high ambient densities and moderate jet velocities, nozzle length has little effect, but at higher jet velocities, longer nozzles have much longer breakup lengths than shorter nozzles. At present I can not explain these trends. This paper focuses only on the low ambient density case.

Physics of turbulent velocity-profile-induced breakup

The mechanism by which the velocity profile contributes to turbulent breakup has so far not been detailed. For turbulent jets, if turbulent fluctuations are indeed a major cause of breakup, then velocity profile relaxation destabilizes jets by increasing the transverse RMS turbulent velocity v' ($\equiv \sqrt{\langle v^2 \rangle}$) at the surface[8]. The most natural mechanism is turbulent production. Consider a jet where $\langle U \rangle$ is the mean convective velocity, $\langle V \rangle$ is the mean radial velocity, and $\langle W \rangle = 0$ (no swirl). The production terms for the u RMS velocity, u', and v RMS velocity, v', are [28, appendix 5][9]:

$$\mathcal{P}_u \equiv -2\langle u^2 \rangle \frac{\partial \langle U \rangle}{\partial x} - 2\langle uv \rangle \frac{\partial \langle U \rangle}{\partial r} - 2\frac{\langle uw \rangle}{r} \frac{\partial \langle U \rangle}{\partial \theta}, \tag{2}$$

$$\mathcal{P}_v \equiv -2\langle uv \rangle \frac{\partial \langle V \rangle}{\partial x} - 2\langle v^2 \rangle \frac{\partial \langle V \rangle}{\partial r} - 2\frac{\langle vw \rangle}{r} \left(\frac{\partial \langle V \rangle}{\partial \theta} - \langle W \rangle \right), \tag{3}$$

where the subscript on $\mathcal{P}$ refers to the direction.

Velocity profile relaxation is often claimed to work through the creation of radial velocity components [4, p. 13L, 29, p. 3385R, 6, p. 512L], which is possible, however, the process is not direct. The production of v' term, $\mathcal{P}_v$, is not a direct function of $\partial \langle U \rangle / \partial r|_{0s}$. No other terms allow for energy transfer from the mean axial velocity $\langle U \rangle$ to the turbulent radial velocity v' through variation in r. Consequently, in turbulent velocity profile relaxation, u' is produced first and then the energy is redistributed to v', leading to breakup.

Rather than model the redistribution process, I will solely examine the amount of production of u' for simplicity. To determine an appropriate dimensionless group for the velocity gradient, I normalized the production rate for u' by the dissipation rate, as production will need to exceed dissipation for turbulence to be generated from shear. If I use the common dissipation model $\varepsilon \propto {v'_0}^3/\Lambda_0$, the expression for the production of u' (equation 2) neglecting all terms other than the mean radial gradient term, and assume that $\langle uv \rangle \propto {v'_0}^2$ (a common turbulence modeling approximation [30, p. 121]), then I obtain:

$$\frac{\mathcal{P}}{\varepsilon} \underset{\sim}{\propto} \frac{\langle uv \rangle}{\varepsilon} \frac{\partial \langle U \rangle}{\partial r} \propto \frac{{v'_0}^2}{{v'_0}^3/\Lambda_0} \frac{\partial \langle U \rangle}{\partial r} = \frac{\Lambda_0}{v'_0} \frac{\partial \langle U \rangle}{\partial r}. \tag{4}$$

The u subscript on $\mathcal{P}$ has been dropped for brevity. Instead of thinking in terms of velocity gradients, it is common in the jet breakup literature to think in terms of boundary layer thickness. Let's use the notation $\partial \langle U \rangle / \partial r|_{0s}$ to refer to the liquid velocity gradient located at the nozzle exit plane (0) and free surface (s). If I assume that $\partial \langle U \rangle / \partial r|_{0s} \approx \overline{U}_0/\delta_0$ where δ_0 is a measure of the boundary layer thickness at the nozzle exit then

$$\frac{\Lambda_0}{v'_0} \frac{\partial \langle U \rangle}{\partial r}\bigg|_{0s} \approx \frac{\Lambda_0 \overline{U}_0}{v'_0 \delta_0} = \frac{\Lambda_0}{\delta_0} \mathrm{Tu}_0^{-1} = \left(\mathrm{Tu}_0 \frac{\delta_0}{\Lambda_0} \right)^{-1}. \tag{5}$$

Consequently, $(\mathrm{Tu}_0 \delta_0/\Lambda_0)^{-1}$ would presumably need to be minimized to prevent velocity profile relaxation effects. However, this criteria leads to a paradox.

Shear instability paradox

Shear can cause instability in fluid flows and generate turbulent kinetic energy. The larger the velocity gradient, the stronger the production. Equation 5 agrees with this view. Or equivalently: the thinner the boundary layer, the stronger the production. This seems to be in direct contradiction with the common suggestion in the literature to reduce the boundary layer thickness to improve stability. The experiments of Wu et al. [9] are clear: a nozzle designed to have thinner boundary layers while changing nothing else does seem to result in more stable liquid jets. So why does the theory suggest the opposite?

One possibly resolution of this paradox is that the *overall* level of production is reduced with thinner boundary layers because the total area with gradients is smaller. One might expect the two effects to roughly cancel each other out. That would seem to lead to another problem about why thinner boundary layers would be better.

I attempt to resolve the paradox by estimating the ratio of the plane-averaged production to the plane-averaged dissipation. This ratio is motivated by the fact that production needs to exceed dissipation for turbulent kinetic energy to increase. A plane average is used to take into account the fact that the region of high production shrinks as the boundary layer shrinks, ultimately reducing the total amount of production. The plane-averaged production is relevant because the turbulence naturally becomes more homogeneous downstream, distributing the production over the entire jet cross section. Unfortunately, this approach

[8] One goal of this work is to put all of turbulent breakup theory into a more consistent framework. As turbulent velocity fluctuations are the generally accepted cause of turbulent breakup, I prefer explanations using turbulent fluctuations rather than stability theory or wave-based arguments, even if the alternatives are in some sense equivalent. See Trettel [27] for my earlier theoretical work on turbulent breakup.

[9] The single-phase production is used here because I am assuming that when velocity profile relaxation is occurring, breakup is not yet significant, so the two phases are separate. Favre averaged equations would be more general.

appears to be sensitive to the modeling approximations used. Most approximations lead to infinite production as the boundary layer thickness decreases[10]. Those approximations were rejected as implausible as empirically, thinner boundary layers do not appear to cause extreme instability in these jets. The simplest specification which is stable uses the Reynolds stress model $\langle uv \rangle = C_{uv} k$ (which Durbin and Pettersson Reif [30, p. 121] notes is acceptable in the log law region of the boundary layer) and the dissipation model $\overline{\varepsilon} = C_\varepsilon \overline{v_0'}^3 / \overline{\Lambda}_0$. I also approximate the velocity profile as linear in the boundary layer, with velocity U_c in the flat center region:

$$\langle U \rangle(r) = \begin{cases} U_\mathrm{c} & \text{if } r \leq \dfrac{d_0}{2} - \delta_0 \\ \dfrac{U_\mathrm{c}}{\delta_0} \left(\dfrac{d_0}{2} - r \right) & \text{if } r \geq \dfrac{d_0}{2} - \delta_0. \end{cases} \tag{6}$$

This form is particularly convenient as the production will be zero in the center region of the flow, so that only one integral is required in the computation of the production. The plane-averaged velocity $\overline{U}_0$ can be computed as

$$\overline{U}_0 = U_\mathrm{c} \left[1 - 2 \left(\frac{\delta_0}{d_0} \right) + \frac{4}{3} \left(\frac{\delta_0}{d_0} \right)^2 \right]. \tag{7}$$

Considering only the r gradient of $\langle U \rangle$ for simplicity, applying the $\langle uv \rangle$ model, and assuming that k_0 equals its plane-averaged value $\overline{k}_0$ everywhere, the plane-averaged production is

$$\begin{aligned} \overline{\mathcal{P}}_0 &= \frac{\int_A \langle uv \rangle_0 \dfrac{\partial \langle U \rangle}{\partial r} \, \mathrm{d}A}{\pi d_0^2 / 4} \\ &= \frac{8 C_{uv} \overline{k}_0}{d_0^2} \int_{d_0/2 - \delta_0}^{d_0/2} \frac{U_\mathrm{c}}{\delta} r \, \mathrm{d}r \\ &= \frac{4 C_{uv} \overline{k}_0 U_\mathrm{c}}{d_0} \left(1 - \frac{\delta_0}{d_0} \right). \end{aligned} \tag{8}$$

Applying the $\overline{\varepsilon}$ model, the overall production-dissipation ratio is

$$\frac{\overline{\mathcal{P}}_0}{\overline{\varepsilon}_0} = \frac{6 C_{uv}}{C_\varepsilon} \frac{U_\mathrm{c}}{\overline{v_0'}} \frac{\overline{\Lambda}_0}{d_0} \left(1 - \frac{\delta_0}{d_0} \right) \tag{9}$$

$$\propto \frac{\dfrac{\overline{U}_0}{\overline{v_0'}} \dfrac{\overline{\Lambda}_0}{d_0} \left(1 - \dfrac{\delta_0}{d_0} \right)}{1 - 2 \left(\dfrac{\delta_0}{d_0} \right) + \dfrac{4}{3} \left(\dfrac{\delta_0}{d_0} \right)^2}, \tag{10}$$

so

$$\frac{\overline{\mathcal{P}}_0}{\overline{\varepsilon}_0} \underset{\sim}{\propto} \overline{\mathrm{Tu}}_0^{-1} \frac{\overline{\Lambda}_0}{d_0} \left(1 + \frac{\delta_0}{d_0} \right) \quad \text{for small } \frac{\delta_0}{d_0}. \tag{11}$$

Equation 11 will be termed the "overall production-dissipation ratio". Contrary to equation 5, this criteria does suggest that thinner boundary layers have less production and consequently lead to liquid jets which are more stable. However, this analysis would suggest that in the limit as the boundary layer thickness approaches zero, production would not go to zero. This seems unusual and may be the result of unrealistic modeling assumptions. It also suggests that velocity profile relaxation is relatively weak in turbulent jets as the 1 term is always essentially much larger than δ_0/d_0. This conclusion is consistent with the experiments of Durbin et al. [31], who found that having a thin boundary layer is less important than having low turbulence for reducing jet breakup. Additionally, the overall production-dissipation ratio is controlled by more than the boundary layer thickness (or equivalently, the velocity profile), as will be discussed in the next sections.

The effect of the turbulence intensity on velocity profile relaxation

The production-dissipation ratio theory would suggest that flows with low turbulent RMS velocities $\overline{v_0'}$ (i.e., laminar or nearly laminar flows) would tend to have proportionally worse production of turbulence than flows with higher turbulence intensities. This is consistent with the observation that velocity profile relaxation seems strongest for transitional flows (which have low turbulence intensity) and is weakened for completely turbulent flows. However, the trend shown by McCarthy and Molloy [5, fig. 5] does not seem to obviously confirm the theory. Typically the turbulence intensity increases as nozzle length increases (unless the inflow was particularly turbulent). The theory would predict that a slight increase in turbulence intensity should result in less, not dramatically more, breakup. Possibly the turbulence intensity increase is small if even measurable as the nozzle length increases from $L_0/d_0 = 0$ to 10 and the flow remains laminar. In that case, the boundary layer thickness increase may indeed explain the dramatic increase in breakup, amplified by the fact that the turbulence intensity is low. This hypothesis needs experimental validation.

Some researchers have reported that when the turbulence intensity is low, increasing turbulence intensity can lead to increased stability of liquid jets, contrary to the suggestion that increased turbulence intensity only decreases jet stability. This has been shown experimentally for both circular [32] and flat [33–35] nozzle geometries. The researchers have hypothesized that velocity profile relaxation occurs more quickly in these instances because turbulent diffusion would smooth out the velocity profile,

[10] In particular, an eddy viscosity model was unstable in all cases I tried, including one where k increased quadratically from zero at the wall to take into account the no-slip condition. I had thought that would kill any instability, but it did not.

in turn reducing total production. The overall production-dissipation ratio theory developed here also can explain how increasing turbulence intensity can stabilize liquid jets in special circumstances using a different mechanism (turbulent dissipation). Both mechanisms are present in liquid jet breakup.

In future work I will examine the evolution of turbulence in the jet to develop a criteria taking into account turbulent diffusion as well. It seems most reasonable to develop an estimate for the maximum turbulence kinetic energy in the jet and use that as a measure of the tendency for velocity profile relaxation to cause breakup. Turbulence production is countered by not only dissipation but also turbulent diffusion.

The effect of the integral scale on velocity profile relaxation

Previous researchers have speculated about integral scale effects, most often in the form of using the integral scale Λ_0 in place of the nozzle diameter d_0 in an equation [36, 37]. While likely part of the picture, that assumption misses the role of the integral scale in dissipation. In the common dissipation model $\varepsilon \propto k^{3/2}/\Lambda$, it's clear that smaller integral scales would lead to greater dissipation. The greater dissipation would reduce the overall production dissipation ratio, $\overline{\mathcal{P}}/\overline{\varepsilon}$, as can be seen as the integral scale is reduced in equation 11.

This has not been recognized in the turbulent jet breakup literature, but it is consistent with the experiments of Durbin et al. [31]. Durbin et al. compared the breakup of turbulent liquid (sheet) jets produced by nozzles with and without a screen placed immediately upstream. The addition of a screen reduced surface fluctuations [31, fig. 11, p. 317], which presumably are a proxy for breakup. They concluded was that the screen reduced turbulence intensity, leading to reduced breakup. However, the integral scale also changes appreciably as the screen is removed. To first-order, the integral scale will be proportional to the mesh size of the screen. Durbin et al. [31, p. 310] state that the honeycomb in their nozzle immediately upstream of the screen has 0.32 cm diameter openings. The screens have openings of width 0.51 mm, which would suggest that the integral scale immediately after the screen is about 6 times smaller than without the screen. Unfortunately this is unlikely to represent the change in integral scale at the nozzle as approximately 15 nozzle thicknesses pass before entering the contraction. The integral scale presumably also increases over this distance, so this experiment is not entirely conclusive.

Fortunately, the DNS study of Sander and Weigand [23, fig. 11], where the integral scale was specified precisely, qualitatively shows the stabilizing effect of reducing the integral scale. However, the effect appears to be weak. The later DNS study of Salvador et al. [38] looks further into the effect of the integral scale. Salvador et al. [38, fig. 7] shows qualitatively that a reduced integral scale leads to a more stable jet. A plot of axial mass concentration Salvador et al. [38, fig. 7], which is analogous to the breakup length, also shows that a smaller integral scale is more stable. The number of droplets generated from the jet (a measure of how much the jet has broken up) increases as the integral scale increases [38, fig. 11a]. The droplet size distribution changes relatively little as the integral scale is increased from 0 to $0.17d_0$ with other variables held constant or approximately so [38, fig. 11a], though this may not be clear in the original work. As an example, using the droplet size histograms given by Salvador et al., I computed that for $\Lambda_0/d_0 = 0$, $D_{32} = 9.17\,\mu\text{m}$, and for $\Lambda_0/d_0 = 0.17d_0$, $D_{32} = 9.45\,\mu\text{m}$. This contradicts the hypothesis that characteristic droplet sizes (e.g., D_{32}) are proportional to the integral scale in turbulent breakup, as stated by Huh et al. [37]. The turbulence intensity in these cases was set to that of a fully developed pipe flow, which is not low, so the destabilizing effects of low turbulence intensity are not seen in these cases.

Ultimately, these integral scale results are preliminary. Further detailed computations and experiments are required to make clear conclusions.

Conclusions

When strongly stable liquid jets are desired, one frequently reads recommendations to make the velocity profile as flat as possible. This design criteria should be reevaluated.

For laminar liquid jets, parabolic velocity profiles are more stable than flat velocity profiles, contrary to common belief.

For liquid jets which are laminar at the nozzle exit but transition to a turbulent state downstream, the transition can be particularly violent, leading to strong breakup. Hoyt and Taylor [8] recommend that if stable jets are desired, it is better to bring the transition point into the nozzle than to allow the jet to transition outside of the nozzle. The transition process appears to be sensitive to the boundary layer thickness. The turbulent theory I developed may help explain certain aspects of the transitional breakup process including why transitional breakup is so strong, but it does not explain whether or where a jet will transition.

For liquid jets which are turbulent at the nozzle exit, the following criteria may characterize how much velocity profile relaxation contributes to breakup (higher means more breakup, all else equal):

$$\overline{\mathrm{Tu}}_0^{-1} \frac{\overline{\Lambda}_0}{d_0} \left(1 + \frac{\delta_0}{d_0}\right). \tag{11}$$

The overall production-dissipation ratio, equation 11, appears to explain the qualitative trends observed in the velocity profile relaxation literature (why thinner boundary layers are more stable, why transitional breakup is so strong), including some lesser known effects (stabilization at moderate turbulence intensities, integral scale effects). However, the theory is still only a hypothesis as no quantitative data is available to fully validate the theory at present. New experiments and detailed computations directly testing this hypothesis are encouraged. The sensitivity to the overall production-dissipation ratio is at present unknown, but I believe for most breakup quantities the Weber number and turbulence intensity are more important.

The theory in this paper is preliminary. Future work will improve the theory with more accurate turbulence modeling approximations. As much of the present data is qualitative or vague, new experimental and computational studies are also needed to better examine the role of the velocity profile and its interactions with the integral scale in turbulent jet breakup.

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Nomenclature

A	area
D_{32}	Sauter mean diameter
d_0	nozzle exit diameter
L_0	nozzle orifice length
k	$\equiv \frac{1}{2}(u'^2 + v'^2 + w'^2)$, turb. kinetic energy
$\overline{k}$	plane-averaged turbulent kinetic energy
$\mathcal{P}$	turbulent production
$\mathrm{Re}_{\ell 0}$	$\equiv \overline{U}_0 d_0/\nu_\ell$, liquid Reynolds number
r	radial coordinate
Tu	$\equiv u'/U$, local turbulence intensity
$\overline{\mathrm{Tu}}_0$	$\equiv \sqrt{2\overline{k}_0/(3\overline{U}_0^2)}$, turbulence intensity using plane-averaged k
U	velocity in the x direction
u	$\equiv U - \langle U \rangle$, U velocity fluctuation
u'	$\equiv \sqrt{\langle u^2 \rangle}$, u RMS velocity
V	velocity in the r direction
v	$\equiv V - \langle V \rangle$, V velocity fluctuation
v'	$\equiv \sqrt{\langle v^2 \rangle}$, v RMS velocity
W	velocity in the ϕ direction
w	$\equiv W - \langle W \rangle$, W velocity fluctuation
w'	$\equiv \sqrt{\langle w^2 \rangle}$, w RMS velocity
$\mathrm{We}_{\ell 0}$	$\equiv \rho_\ell \overline{U}_0^2 d_0/\sigma$, liquid Weber number
x	axial coordinate
$\langle x_\mathrm{b} \rangle$	average breakup length
α	kinetic energy coefficient, equation 1
δ	boundary layer thickness
ε	turbulent dissipation rate
Λ	integral scale of turbulence
ν_ℓ	liquid kinematic viscosity
ϕ	azimuth
ρ_g	gas mass density
ρ_ℓ	liquid mass density
σ	surface tension

Operators

$\overline{x}$	$\equiv (\int_A x\, \mathrm{d}A)/A$, plane average of x
$\langle x \rangle$	ensemble average of x

Subscripts

0	at nozzle exit
g	using gas properties
ℓ	using liquid properties
s	at free surface

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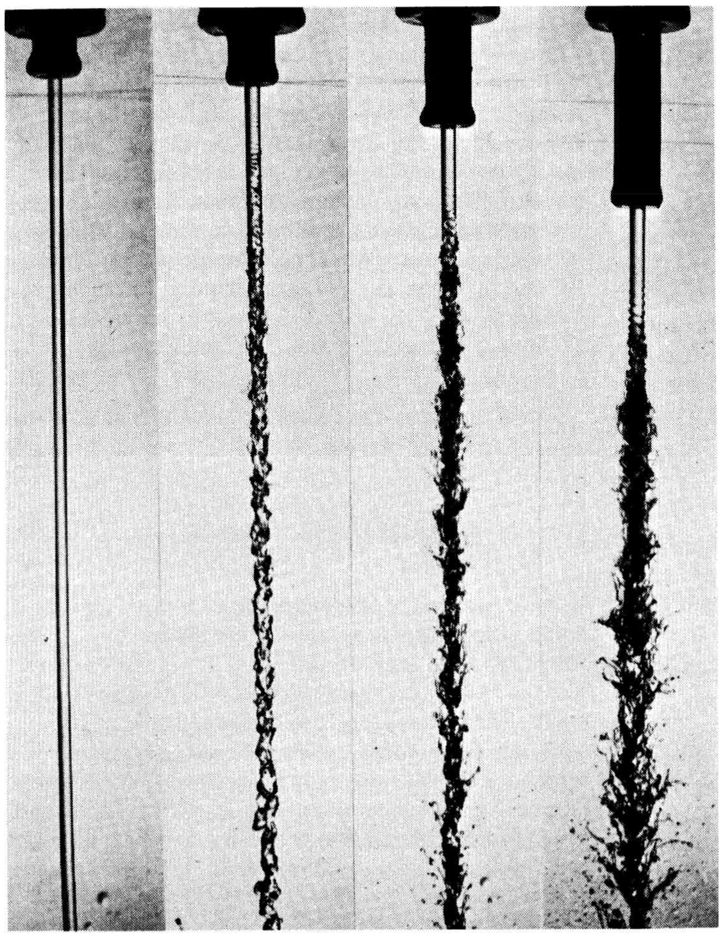

Figure 1: Figure 5 from McCarthy and Molloy [5]: Effect of nozzle design on the stability of glycerol-water jets.

Jet viscosity	11 cP
Jet velocity	$20\,\mathrm{m\,s^{-1}}$ (approx.)
Nozzle diameter	2.54 mm
Jet Reynolds no.	4750
Jet Ohnesorge no.	0.026
Exposure	30 µs
Nozzle aspect ratio	L_0/d_0 = 0, 1, 5 and 10 (left to right)

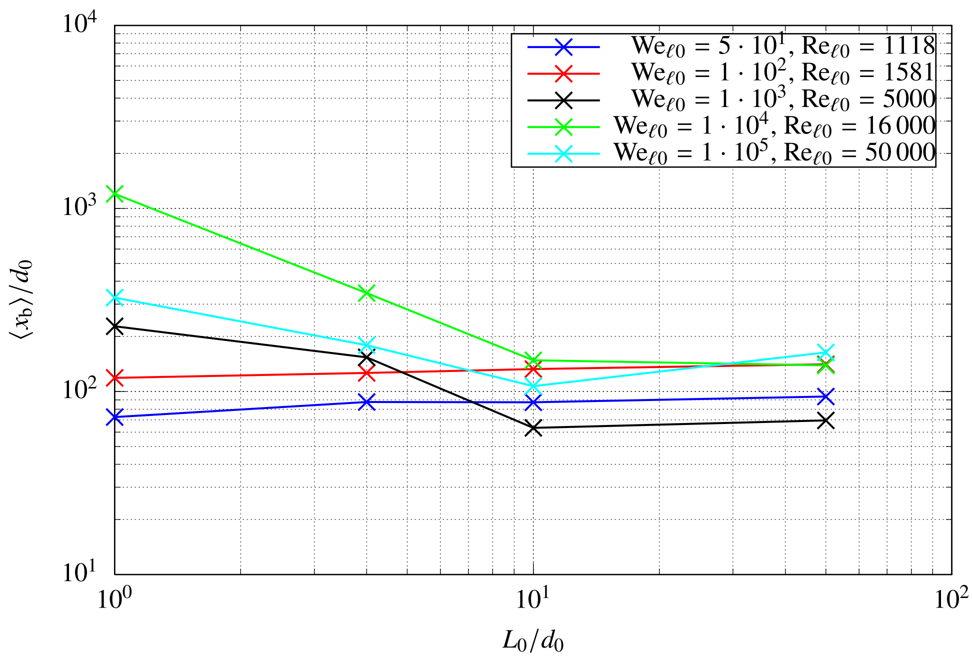

Figure 2: Effect of nozzle aspect ratio (L_0/d_0) on dimensionless breakup length ($\langle x_b \rangle / d_0$) from the data of Arai et al. [12].