Closed-form Generation of Paths for Motion Planning of a Convexified Reeds–Shepp Vehicle on a Sphere

Sixu Li*, Deepak Prakash Kumar†, Swaroop Darbha†, Yang Zhou*

April 17, 2025

Abstract

This paper presents an analytical derivation of the path generation process for time-optimal convexified Reeds–Shepp paths on a sphere. Specifically, a sufficient list of 23 optimal path types for this problem was derived in [1]; we focus here on the explicit construction of candidate paths using inverse kinematics. Given an initial configuration, a desired terminal configuration, and a specified maximum turning rate U_{max}, closed-form expressions for the segment angles corresponding to each path type in the sufficient list are derived. These expressions enable efficient and accurate generation of feasible time-optimal paths on the sphere. The results are used for the implementation at https://github.com/sixuli97/Optimal-Spherical-Convexified-Reeds-Shepp-Paths.

1 Derivation of Closed-Form Expressions for Paths on a Sphere

The spherical convexified Reeds-Shepp model [1]:

$$\frac{d\mathbf{X_v}}{dt} = v(t)\mathbf{T_v}(t), \tag{1}$$

$$\frac{d\mathbf{T_v}}{dt} = -v(t)\mathbf{X_v}(t) + u_g(t)\mathbf{N_v}(t), \tag{2}$$

$$\frac{d\mathbf{N_v}}{dt} = -u_g(t)\mathbf{T_v}(t), \tag{3}$$

$$\mathbf{R}(0) = \mathbf{I_3},\ \mathbf{R}(T) = R_f, \tag{4}$$

where $v \in [-1,1]$ and $u_g \in [-U_{max}, U_{max}]$, $\mathbf{R}(t) = [\mathbf{X_v}(t), \mathbf{T_v}(t), \mathbf{N_v}(t)] \in SO(3)$ and R_f is the desired terminal configuration. Note that the model is equivalent to:

$$\frac{d\mathbf{R}(t)}{dt} = \mathbf{R}(t)\underbrace{\begin{pmatrix} 0 & -v & 0 \\ v & 0 & -u_g \\ 0 & u_g & 0 \end{pmatrix}}_{\Omega}. \tag{5}$$

Since v and u_g remain constant on each segment, the solution of (5) on each segment is

$$\mathbf{R}(t) = \mathbf{R}(t_i)e^{(t-t_i)\Omega}, \tag{6}$$

where t_i denotes the initial time of the i^{th} segment. It is simpler to deal with arc angles instead of time; hence, we define $\phi = \omega(t-t_i) = \sqrt{v^2+u_g^2}(t-t_i)$, where ϕ represents the arc angle, and ω denotes the angular frequency. Let $\hat{\Omega} = \frac{1}{\sqrt{v^2+u_g^2}}\Omega$. We define $\mathbf{M}(\phi) := e^{\phi\hat{\Omega}} = e^{(t-t_i)\Omega}$. Substituting specific values of v and u_g, $\mathbf{M}(\phi)$ for each type of segment can be calculated using the Euler-Rodriguez formula. Hence, we obtain

$$\mathbf{M}_{G^+}(\phi) = \begin{pmatrix} c(\phi) & -s(\phi) & 0 \\ s(\phi) & c(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7}$$

$$\mathbf{M}_{L^+}(r,\phi) = \begin{pmatrix} \eta_{11} & -rs(\phi) & \eta_{13} \\ rs(\phi) & c(\phi) & -\eta_{23} \\ \eta_{13} & \eta_{23} & \eta_{33} \end{pmatrix}, \tag{8}$$

$$\mathbf{M}_{R^+}(r,\phi) = \begin{pmatrix} \eta_{11} & -rs(\phi) & -\eta_{13} \\ rs(\phi) & c(\phi) & \eta_{23} \\ -\eta_{13} & -\eta_{23} & \eta_{33} \end{pmatrix}, \tag{9}$$

$$\mathbf{M}_{L^0}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{pmatrix}, \tag{10}$$

$$\mathbf{M}_{G^-}(\phi) = \mathbf{M}_{G^+}^T(\phi), \tag{11}$$

$$\mathbf{M}_{L^-}(r,\phi) = \mathbf{M}_{R^+}^T(r,\phi), \tag{12}$$

$$\mathbf{M}_{R^-}(r,\phi) = \mathbf{M}_{L^+}^T(r,\phi), \tag{13}$$

*Zachry Department of Civil & Environmental Engineering, Texas A&M University, College Station, TX 77843, USA (e-mail: sixuli@tamu.edu; yangzhou295@tamu.edu).

†Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843 USA (e-mail: deepakprakash1997@gmail.com; dswaroop@tamu.edu).

$$\mathbf{M}_{R^0}(\phi) = \mathbf{M}_{L^0}^T(\phi), \tag{14}$$

where $\eta_{11} = 1 - (1 - c(\phi))r^2$, $\eta_{13} = (1 - c(\phi))r\sqrt{1 - r^2}, \eta_{23} = s(\phi)\sqrt{1 - r^2}$, $\eta_{33} = c(\phi) + (1 - c(\phi))r^2, c(\phi) = \cos(\phi)$, and $s(\phi) = \sin(\phi)$.
The corresponding axial vectors are $\mathbf{u}_{G^+} := [0, 0, 1]^T$, $\mathbf{u}_{L^+} := [\sqrt{1 - r^2}, 0, r]^T$, $\mathbf{u}_{R^+} := [-\sqrt{1 - r^2}, 0, r]^T$, $\mathbf{u}_{L^0} := [1, 0, 0]^T$, $\mathbf{u}_{G^-} := [0, 0, -1]^T$, $\mathbf{u}_{L^-} := [-\sqrt{1 - r^2}, 0, -r]^T$, $\mathbf{u}_{R^-} := [\sqrt{1 - r^2}, 0, -r]^T$, $\mathbf{u}_{R^0} := [-1, 0, 0]^T$.
The sufficient list of optimal paths is characterized as follows [1]:

Theorem 1. *For $U_{max} \geq 1$ (or $r \leq \frac{1}{\sqrt{2}}$), the optimal path may be restricted to the following types, together with their symmetric forms:*
C, G, T, CC, GC, $C|C$, TC,
$CC_\psi|C$, CGC, $C|C_\beta G$, CTC,
$C|C_\psi C_\psi|C$, $CGC_\beta|C$, $CC_\mu|C_\mu C$,
$C|C_\beta GC_\beta|C$, $C|C_\mu C_\mu|C_\mu C$, $CC_\mu|C_\mu C_\mu|C_\mu C$,
where $0 < \psi \leq \arctan(\frac{1}{\sqrt{U_{max}^4 - 1}}) + \frac{\pi}{2}$, $\beta = \arctan(\frac{1}{\sqrt{U_{max}^4 - 1}}) + \frac{\pi}{2}$, and $0 < \mu < \arctan(\frac{1}{\sqrt{U_{max}^4 - 1}}) + \frac{\pi}{2}$.

Here, C represents a tight turn with radius $r = \frac{1}{\sqrt{1 + U_{max}^2}}$, G represents a great circular arc, and T represents a turn-in-place motion.
Given the sufficient list above, for each path, candidate solutions must be generated using inverse kinematics, based on an initial configuration, a desired terminal configuration, and a U_{max} (or r). In this note, we employ rotation matrices and their associated axial vectors to derive closed-form expressions for the angles of each path in the sufficient list.

1.1 C Paths

1.1.1 L^+ Paths

For a $L_{\phi_1}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{15}$$

Pre-multiplying (15) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2 - 1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{16}$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \tag{17}$$

and yields two solutions of ϕ_1.

1.1.2 R^+ Paths

For a $R_{\phi_1}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{18}$$

Pre-multiplying (18) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2 - 1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{19}$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \tag{20}$$

and yields two solutions of ϕ_1.

1.1.3 R^- Paths

For a $R_{\phi_1}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{21}$$

Pre-multiplying (21) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2 - 1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{22}$$

which gives

$$\cos(\phi_1) = \frac{r^2 - \alpha_{33}}{r^2 - 1}, \tag{23}$$

and yields two solutions of ϕ_1.

1.1.4 L^- Paths

For a $L^-_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{24}$$

Pre-multiplying (24) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2-1\right)\cos(\phi_1) - r^2 = -\alpha_{33}, \tag{25}$$

which gives

$$\cos(\phi_1) = \frac{r^2-\alpha_{33}}{r^2-1}, \tag{26}$$

and yields two solutions of ϕ_1.

1.2 G Paths

1.2.1 G^+ Paths

For a $G^+_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{27}$$

Pre-multiplying (27) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(1-r^2\right)\cos(\phi_1) - r^2 = -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{28}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + (\alpha_{33}-1)\,r\right)}{r^2-1}, \tag{29}$$

and yields two solutions of ϕ_1.

1.2.2 G^- Paths

For a $G^-_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{30}$$

Pre-multiplying (30) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(1-r^2\right)\cos(\phi_1) - r^2 = -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{31}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + (\alpha_{33}-1)\,r\right)}{r^2-1}, \tag{32}$$

and yields two solutions of ϕ_1.

1.3 T Paths

1.3.1 L^0 Paths

For a $L^0_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{33}$$

Pre-multiplying (33) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\cos(\phi_1) = \alpha_{33}, \tag{34}$$

and yields two solutions of ϕ_1.

1.3.2 R^0 Paths

For a $R^0_{\phi_1}$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{35}$$

Pre-multiplying (35) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\cos(\phi_1) = \alpha_{33}, \tag{36}$$

and yields two solutions of ϕ_1.

1.4 CC Paths

1.4.1 L^+R^+ Paths

For a $L^+_{\phi_1} R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r, \phi_1)\mathbf{M}_{R^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{37}$$

Pre-multiplying (37) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$-2r^3 + 2\left(r^2 - 1\right) r \cos(\phi_1) + r = \alpha_{31}\sqrt{1 - r^2} - \alpha_{33} r, \tag{38}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33} r + r}{2r - 2r^3}, \tag{39}$$

and yields two solutions of ϕ_1.
Pre-multiplying (37) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$-2r^3 + 2\left(r^2 - 1\right) r \cos(\phi_2) + r = \alpha_{13}\left(-\sqrt{1 - r^2}\right) - \alpha_{33} r, \tag{40}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1 - r^2} + \alpha_{33} r + r}{2r - 2r^3}, \tag{41}$$

and yields two solutions of ϕ_2.

1.4.2 R^+L^+ Paths

For a $R^+_{\phi_1} L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r, \phi_1)\mathbf{M}_{L^+}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{42}$$

Pre-multiplying (42) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$-2r^3 + 2\left(r^2 - 1\right) r \cos(\phi_1) + r = \alpha_{31}\left(-\sqrt{1 - r^2}\right) - \alpha_{33} r, \tag{43}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1 - r^2} + \alpha_{33} r + r}{2r - 2r^3}, \tag{44}$$

and yields two solutions of ϕ_1.
Pre-multiplying (42) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$-2r^3 + 2\left(r^2 - 1\right) r \cos(\phi_2) + r = \alpha_{13}\sqrt{1 - r^2} - \alpha_{33} r, \tag{45}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1 - r^2} + \alpha_{33} r + r}{2r - 2r^3}, \tag{46}$$

and yields two solutions of ϕ_2.

1.4.3 R^-L^- Paths

For a $R^-_{\phi_1} L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r, \phi_1)\mathbf{M}_{L^-}(r, \phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{47}$$

Pre-multiplying (47) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$-2r^3 + 2\left(r^2 - 1\right) r \cos(\phi_1) + r = \alpha_{31}\sqrt{1 - r^2} - \alpha_{33} r, \tag{48}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1 - r^2} + \alpha_{33} r + r}{2r - 2r^3}, \tag{49}$$

and yields two solutions of ϕ_1.
Pre-multiplying (47) with $\mathbf{u}^T_{R^-}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$-2r^3 + 2\left(r^2 - 1\right) r \cos(\phi_2) + r = \alpha_{13}\left(-\sqrt{1 - r^2}\right) - \alpha_{33} r, \tag{50}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1 - r^2} + \alpha_{33} r + r}{2r - 2r^3}, \tag{51}$$

and yields two solutions of ϕ_2.

1.4.4 L^-R^- Paths

For a $L^-_{\phi_1}R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{52}$$

Pre-multiplying (52) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$-2r^3 + 2\left(r^2-1\right) r\cos(\phi_1) + r = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{53}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{54}$$

and yields two solutions of ϕ_1.
Pre-multiplying (52) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$-2r^3 + 2\left(r^2-1\right) r\cos(\phi_2) + r = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r, \tag{55}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{56}$$

and yields two solutions of ϕ_2.

1.5 GC Paths

1.5.1 G^+L^+ Paths

For a $G^+_{\phi_1}L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{57}$$

Pre-multiplying (57) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\left(1-r^2\right)\cos(\phi_1) - r^2 = -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{58}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + (\alpha_{33}-1)r\right)}{r^2-1}, \tag{59}$$

and yields two solutions of ϕ_1.
Pre-multiplying (57) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2-1\right)\cos(\phi_2) - r^2 = -\alpha_{33}, \tag{60}$$

which gives

$$\cos(\phi_2) = \frac{r^2-\alpha_{33}}{r^2-1}, \tag{61}$$

and yields two solutions of ϕ_2.

1.5.2 G^+R^+ Paths

For a $G^+_{\phi_1}R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^+}(\phi_1)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{62}$$

Pre-multiplying (62) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\left(r^2-1\right)\cos(\phi_1) - r^2 = \alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r\right), \tag{63}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r\right)}{r^2-1}, \tag{64}$$

and yields two solutions of ϕ_1.
Pre-multiplying (62) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$\left(r^2-1\right)\cos(\phi_2) - r^2 = -\alpha_{33}, \tag{65}$$

which gives

$$\cos(\phi_2) = \frac{r^2-\alpha_{33}}{r^2-1}, \tag{66}$$

and yields two solutions of ϕ_2.

1.5.3 G^-R^- Paths

For a $G^-_{\phi_1}R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{67}$$

Pre-multiplying (67) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\left(r^2-1\right)\cos(\phi_1) - r^2 = \alpha_{11}\left(r^2-1\right) - r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{68}$$

which gives

$$\cos(\phi_1) = \alpha_{11} - \frac{r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + (\alpha_{33}-1)\,r\right)}{r^2-1}, \tag{69}$$

and yields two solutions of ϕ_1.
Pre-multiplying (67) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$\left(r^2-1\right)\cos(\phi_2) - r^2 = -\alpha_{33}, \tag{70}$$

which gives

$$\cos(\phi_2) = \frac{r^2-\alpha_{33}}{r^2-1}, \tag{71}$$

and yields two solutions of ϕ_2.

1.5.4 G^-L^- Paths

For a $G^-_{\phi_1}L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{G^-}(\phi_1)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{72}$$

Pre-multiplying (72) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(1-r^2\right)\cos(\phi_1) - r^2 = -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{73}$$

which gives

$$\cos(\phi_1) = \alpha_{11} + \frac{r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + (\alpha_{33}-1)\,r\right)}{r^2-1}, \tag{74}$$

and yields two solutions of ϕ_1.
Pre-multiplying (72) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$\left(r^2-1\right)\cos(\phi_2) - r^2 = -\alpha_{33}, \tag{75}$$

which gives

$$\cos(\phi_2) = \frac{r^2-\alpha_{33}}{r^2-1}, \tag{76}$$

and yields two solutions of ϕ_2.

1.6 $C|C$ Paths

1.6.1 $L^+|L^-$ Paths

For a $L^+_{\phi_1}|L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{77}$$

Pre-multiplying (77) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \tag{78}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{79}$$

and yields two solutions of ϕ_1.
Pre-multiplying (77) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\left(-2\left(r^2-1\right)\cos(\phi_2) + 2r^2 - 1\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r, \tag{80}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{81}$$

and yields two solutions of ϕ_2.

1.6.2 $R^+|R^-$ Paths

For a $R^+_{\phi_1}|R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{82}$$

Pre-multiplying (82) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_1) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r, \tag{83}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{84}$$

and yields two solutions of ϕ_1.
Pre-multiplying (82) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_2) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}, \tag{85}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{86}$$

and yields two solutions of ϕ_2.

1.6.3 $R^-|R^+$ Paths

For a $R^-_{\phi_1}|R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{87}$$

Pre-multiplying (87) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \tag{88}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{89}$$

and yields two solutions of ϕ_1.
Pre-multiplying (87) with $\mathbf{u}^T_{R^-}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_2) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r, \tag{90}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{91}$$

and yields two solutions of ϕ_2.

1.6.4 $L^-|L^+$ Paths

For a $L^-_{\phi_1}|L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{92}$$

Pre-multiplying (92) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_1) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r, \tag{93}$$

which gives

$$\cos(\phi_1) = \frac{-2r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{94}$$

and yields two solutions of ϕ_1.
Pre-multiplying (92) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\left(2r^2-1\right) - 2r\left(r^2-1\right)\cos(\phi_2) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}, \tag{95}$$

which gives

$$\cos(\phi_2) = \frac{-2r^3 - \alpha_{13}\sqrt{1-r^2} + \alpha_{33}r + r}{2r-2r^3}, \tag{96}$$

and yields two solutions of ϕ_2.

1.7 TC Paths

1.7.1 L^0L^+ Paths

For a $L^0_{\phi_1}L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{97}$$

Pre-multiplying (97) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$-r\cos(\phi_1) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{98}$$

which gives

$$\cos(\phi_1) = \alpha_{33} + \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{99}$$

and yields two solutions of ϕ_1.
Pre-multiplying (97) with $\mathbf{u}^T_{L^0}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\sqrt{1-r^2}(\cos(\phi_2)-1) = -\alpha_{13}, \tag{100}$$

which gives

$$\cos(\phi_2) = 1 - \frac{\alpha_{13}}{r\sqrt{1-r^2}}, \tag{101}$$

and yields two solutions of ϕ_2.

1.7.2 L^0L^- Paths

For a $L^0_{\phi_1}L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^0}(\phi_1)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{102}$$

Pre-multiplying (102) with $\mathbf{u}^T_{G^-}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r\cos(\phi_1) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}, \tag{103}$$

which gives

$$\cos(\phi_1) = \alpha_{33} - \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{104}$$

and yields two solutions of ϕ_1.
Pre-multiplying (102) with $\mathbf{u}^T_{L^0}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\sqrt{1-r^2}(\cos(\phi_2)-1) = \alpha_{13}, \tag{105}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{13}}{r\sqrt{1-r^2}} + 1, \tag{106}$$

and yields two solutions of ϕ_2.

1.7.3 R^0R^- Paths

For a $R^0_{\phi_1}R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{107}$$

Pre-multiplying (107) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$-r\cos(\phi_1) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r, \tag{108}$$

which gives

$$\cos(\phi_1) = \alpha_{33} + \frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{109}$$

and yields two solutions of ϕ_1.
Pre-multiplying (107) with $\mathbf{u}^T_{R^0}$ and post-multiplying $\mathbf{u}_{G^+}$:

$$r\sqrt{1-r^2}(\cos(\phi_2)-1) = -\alpha_{13}, \tag{110}$$

which gives

$$\cos(\phi_2) = 1 - \frac{\alpha_{13}}{r\sqrt{1-r^2}}, \tag{111}$$

and yields two solutions of ϕ_2.

1.7.4 R^0R^+ Paths

For a $R^0_{\phi_1}R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^0}(\phi_1)\mathbf{M}_{R^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{112}$$

Pre-multiplying (112) with $\mathbf{u}^T_{G^+}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r\cos(\phi_1)=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}, \tag{113}$$

which gives

$$\cos(\phi_1)=\alpha_{33}-\frac{\alpha_{31}\sqrt{1-r^2}}{r}, \tag{114}$$

and yields two solutions of ϕ_1.
Pre-multiplying (112) with $\mathbf{u}^T_{R^0}$ and post-multiplying $\mathbf{u}_{G^-}$:

$$r\sqrt{1-r^2}(\cos(\phi_2)-1)=\alpha_{13}, \tag{115}$$

which gives

$$\cos(\phi_2)=\frac{\alpha_{13}}{r\sqrt{1-r^2}}+1, \tag{116}$$

and yields two solutions of ϕ_2.

1.8 $CC_\psi|C$ Paths

1.8.1 $L^+R^+_\psi|R^-$ Paths

For a $L^+_{\phi_1}R^+_\psi|R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{R^+}(r,\psi)\mathbf{M}_{R^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{117}$$

Pre-multiplying (117) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$4r^2\left(r^2-1\right)\cos(\psi)-\left(1-2r^2\right)^2=\alpha_{11}\left(r^2-1\right)-r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{118}$$

which gives

$$\cos(\psi)=\frac{-\alpha_{11}+4r^4+\alpha_{11}r^2-\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-4r^2+1}{4r^2\left(r^2-1\right)}, \tag{119}$$

and yields two solutions of ψ.
Pre-multiplying (117) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&r\sin(\phi_1)\left(2\sin(\psi)-2r^2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1)\\ =&\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{120}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$.
It is obtained that

$$\cos(\gamma+\phi_1)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{121}$$

$$\implies\phi_1=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)-\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{122}$$

which yields two solutions for each value of ψ.
Pre-multiplying (117) with $\mathbf{u}^T_{L^+}$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&r\sin(\phi_2)\left(2r^2\sin(\psi)-2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2)\\ =&\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{123}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{124}$$

$$\implies \phi_2=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{125}$$

which yields two solutions for each value of ψ.

1.8.2 $L^-R^-_\psi|R^+$ Paths

For a $L^-_{\phi_1}R^-_\psi|R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\psi)\mathbf{M}_{R^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{126}$$

Pre-multiplying (126) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$4r^2\left(r^2-1\right)\cos(\psi)-\left(1-2r^2\right)^2=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \tag{127}$$

which gives

$$\cos(\psi)=\frac{-\alpha_{11}+4r^4+\alpha_{11}r^2-\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-4r^2+1}{4r^2\left(r^2-1\right)}, \tag{128}$$

and yields two solutions of ψ.
Pre-multiplying (117) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&r\sin(\phi_1)\left(2\sin(\psi)-2r^2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1)\\&=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}.\end{aligned} \tag{129}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$.
It is obtained that

$$\cos(\gamma+\phi_1)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{130}$$

$$\implies \phi_1=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)-\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{131}$$

which yields two solutions for each value of ψ.
Pre-multiplying (117) with $\mathbf{u}^T_{L^-}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&r\sin(\phi_2)\left(2r^2\sin(\psi)-2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2)\\&=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{132}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{133}$$

$$\implies \phi_2=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{134}$$

which yields two solutions for each value of ψ.

1.8.3 $R^-L^-_\psi|L^+$ Paths

For a $R^-_{\phi_1}L^-_\psi|L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{L^-}(r,\psi)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{135}$$

Pre-multiplying (135) with $\mathbf{u}^T_{R^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$4r^2\left(r^2-1\right)\cos(\psi)-\left(1-2r^2\right)^2=\alpha_{11}\left(r^2-1\right)-r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{136}$$

which gives

$$\cos(\psi)=\frac{-\alpha_{11}+4r^4+\alpha_{11}r^2-\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-4r^2+1}{4r^2\left(r^2-1\right)}, \tag{137}$$

and yields two solutions of ψ.
Pre-multiplying (135) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}&r\sin(\phi_1)\left(2\sin(\psi)-2r^2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1)\\ =&\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{138}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$.
It is obtained that

$$\cos(\gamma+\phi_1)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{139}$$

$$\implies \phi_1=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)-\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{140}$$

which yields two solutions for each value of ψ.
Pre-multiplying (135) with $\mathbf{u}^T_{R^-}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&r\sin(\phi_2)\left(2r^2\sin(\psi)-2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2)\\ =&\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{141}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}} \tag{142}$$

$$\implies \phi_2=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)+\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right), \tag{143}$$

which yields two solutions for each value of ψ.

1.8.4 $R^+L^+_\psi|L^-$ Paths

For a $R^+_{\phi_1}L^+_\psi|L^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{L^+}(r,\psi)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{144}$$

Pre-multiplying (144) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$4r^2\left(r^2-1\right)\cos(\psi)-\left(1-2r^2\right)^2=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \tag{145}$$

which gives

$$\cos(\psi)=\frac{-\alpha_{11}+4r^4+\alpha_{11}r^2-\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-4r^2+1}{4r^2\left(r^2-1\right)}, \tag{146}$$

and yields two solutions of ψ.
Pre-multiplying (144) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&r\sin(\phi_1)\left(2\sin(\psi)-2r^2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_1)\\&=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}.\end{aligned}\tag{147}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r\left(2\sin(\psi)-2r^2\sin(\psi)\right)}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\gamma+\phi_1)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\tag{148}$$

$$\implies\phi_1=\cos^{-1}\left(\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{31}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)-\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right),\tag{149}$$

which yields two solutions for each value of ψ.
Pre-multiplying (144) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&r\sin(\phi_2)\left(2r^2\sin(\psi)-2\sin(\psi)\right)+r\left(4r^4-4\left(r^2-1\right)r^2\cos(\psi)-4r^2+1\right)-4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)\cos(\phi_2)\\&=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned}\tag{150}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(2\sin(\psi)-2r^2\sin(\psi))^2+\left(4r(2r^4-3r^2+1)\sin^2\left(\frac{\psi}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\tag{151}$$

$$\implies\phi_2=\cos^{-1}\left(-\frac{r\left(\alpha_{33}+4\left(r^2-1\right)r^2\cos(\psi)-\left(1-2r^2\right)^2\right)-\alpha_{13}\sqrt{1-r^2}}{4\sqrt{-r^2\left(r^2-1\right)^2\sin^2\left(\frac{\psi}{2}\right)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\psi)+2r^2-1\right)}}\right)+\tan^{-1}\left(\frac{2r(\sin(\psi)-r^2\sin(\psi))}{4r\left(2r^4-3r^2+1\right)\sin^2\left(\frac{\psi}{2}\right)}\right),\tag{152}$$

which yields two solutions for each value of ψ.

1.9 CGC Paths

1.9.1 $L^+G^+L^+$ Paths

For a $L^+_{\phi_1}G^+_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{153}$$

Pre-multiplying (153) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)=r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),\tag{154}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2-\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+r^2}{r^2-1},\tag{155}$$

and yields two solutions of ϕ_2.
Pre-multiplying (153) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$-r^3+\sin(\phi_1)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.\tag{156}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\tag{157}$$

$$\implies\phi_1=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right),\tag{158}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (153) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3+\sin(\phi_3)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{159}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{160}$$

$$\implies\phi_3=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{161}$$

which yields two solutions for each value of ϕ_2.

1.9.2 $R^+G^+R^+$ Paths

For a $R^+_{\phi_1}G^+_{\phi_2}R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{162}$$

Pre-multiplying (162) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)=r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \tag{163}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2-\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+r^2}{r^2-1}, \tag{164}$$

and yields two solutions of ϕ_2.
Pre-multiplying (162) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$-r^3+\sin(\phi_1)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \tag{165}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{166}$$

$$\implies\phi_1=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2)))+r^2+\cos(\phi_2)\right)-\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{167}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (162) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3+\sin(\phi_3)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \tag{168}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{169}$$

$$\implies\phi_3=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{170}$$

which yields two solutions for each value of ϕ_2.

1.9.3 $R^- G^- R^-$ Paths

For a $R^-_{\phi_1} G^-_{\phi_2} R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{171}$$

Pre-multiplying (171) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{172}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 - \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2}{r^2-1}, \tag{173}$$

and yields two solutions of ϕ_2.
Pre-multiplying (171) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$-r^3 + \sin(\phi_1)\left(r^2\sin(\phi_2) - \sin(\phi_2)\right) + 2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + \left(r^2-1\right)r\cos(\phi_2) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \tag{174}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2 + \left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2 + \left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2 + \left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{175}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{r^2\sin(\phi_2) - \sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{176}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (171) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3 + \sin(\phi_3)\left(r^2\sin(\phi_2) - \sin(\phi_2)\right) + 2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right) + \left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \tag{177}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2 + \left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3 - \gamma\right) = \frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{178}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r\left(-\alpha_{33} + r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2)\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{r^2\sin(\phi_2) - \sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{179}$$

which yields two solutions for each value of ϕ_2.

1.9.4 $L^- G^- L^-$ Paths

For a $L^-_{\phi_1} G^-_{\phi_2} L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{180}$$

Pre-multiplying (180) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r^2(-\cos(\phi_2)) + r^2 + \cos(\phi_2) = r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{181}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 - \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2}{r^2-1}, \tag{182}$$

and yields two solutions of ϕ_2.
Pre-multiplying (180) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$-r^3 + \sin(\phi_1)\left(r^2\sin(\phi_2) - \sin(\phi_2)\right) + 2\left(r^2-1\right)r\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + \left(r^2-1\right)r\cos(\phi_2) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \tag{183}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r^2\sin(\phi_2)-\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{184}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{185}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (180) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3+\sin(\phi_3)\left(r^2\sin(\phi_2)-\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_3)\sin^2\left(\frac{\phi_2}{2}\right)+\left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \tag{186}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos(\phi_3-\gamma) = \frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{187}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\sin^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{r^2\sin(\phi_2)-\sin(\phi_2)}{2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{188}$$

which yields two solutions for each value of ϕ_2.

1.9.5 $L^+G^+R^+$ Paths

For a $L^+_{\phi_1}G^+_{\phi_2}R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{189}$$

Pre-multiplying (189) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\left(r^2-1\right)\cos(\phi_2)+r^2 = \alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{190}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-r^2}{r^2-1}, \tag{191}$$

and yields two solutions of ϕ_2.
Pre-multiplying (189) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\left(r-r^3\right)\cos(\phi_2)-r^3+\sin(\phi_1)\left(\sin(\phi_2)-r^2\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \tag{192}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3\cos(\phi_2)+r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{193}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{194}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (189) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3-\left(r^2-1\right)\sin(\phi_2)\sin(\phi_3)+2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)-\left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{195}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{196}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r\left(-\alpha_{33}+r^2(-\cos(\phi_2))+r^2+\cos(\phi_2)\right)-\alpha_{13}\sqrt{1-r^2}}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{197}$$

which yields two solutions for each value of ϕ_2.

1.9.6 $R^+G^+L^+$ Paths

For a $R^+_{\phi_1}G^+_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{198}$$

Pre-multiplying (198) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\left(r^2-1\right)\cos(\phi_2)+r^2=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{199}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-r^2}{r^2-1}, \tag{200}$$

and yields two solutions of ϕ_2.
Pre-multiplying (198) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\left(r-r^3\right)\cos(\phi_2)-r^3+\sin(\phi_1)\left(\sin(\phi_2)-r^2\sin(\phi_2)\right)+2\left(r^2-1\right)r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{201}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{202}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{203}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (198) with $\mathbf{u}^T_{R^+}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$-r^3-\left(r^2-1\right)\sin(\phi_2)\sin(\phi_3)+2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)-\left(r^2-1\right)r\cos(\phi_2)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \tag{204}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{205}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{206}$$

which yields two solutions for each value of ϕ_2.

1.9.7 $R^- G^- L^-$ Paths

For a $R^-_{\phi_1} G^-_{\phi_2} L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{207}$$

Pre-multiplying (207) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\left(r^2-1\right)\cos(\phi_2) + r^2 = \alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r\right), \tag{208}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11} r^2 + \alpha_{33} r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r - r^2}{r^2-1}, \tag{209}$$

and yields two solutions of ϕ_2.
Pre-multiplying (207) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\left(r-r^3\right)\cos(\phi_2) - r^3 + \sin(\phi_1)\left(\sin(\phi_2) - r^2\sin(\phi_2)\right) + 2\left(r^2-1\right) r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33} r. \tag{210}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3\cos(\phi_2) + r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33} r - r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{211}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) + r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33} r - r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{\sin(\phi_2) - r^2\sin(\phi_2)}{2\left(r^2-1\right) r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{212}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (207) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3 - \left(r^2-1\right)\sin(\phi_2)\sin(\phi_3) + 2\left(r^2-1\right) r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) - \left(r^2-1\right) r\cos(\phi_2) = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33} r. \tag{213}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3\cos(\phi_2) + r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{33} r - r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}} \tag{214}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) + r^3 - \alpha_{13}\sqrt{1-r^2} - \alpha_{33} r - r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right) + \sin^2(\phi_2)\right)}}\right) + \tan^{-1}\left(\frac{\sin(\phi_2) - r^2\sin(\phi_2)}{2\left(r^2-1\right) r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{215}$$

which yields two solutions for each value of ϕ_2.

1.9.8 $L^- G^- R^-$ Paths

For a $L^-_{\phi_1} G^-_{\phi_2} R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{216}$$

Pre-multiplying (216) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\left(r^2-1\right)\cos(\phi_2) + r^2 = \alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r\right), \tag{217}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11} r^2 + \alpha_{33} r^2 - \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r - r^2}{r^2-1}, \tag{218}$$

and yields two solutions of ϕ_2.
Pre-multiplying (216) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\left(r-r^3\right)\cos(\phi_2) - r^3 + \sin(\phi_1)\left(\sin(\phi_2) - r^2\sin(\phi_2)\right) + 2\left(r^2-1\right) r\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33} r. \tag{219}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{\sin(\phi_2)-r^2\sin(\phi_2)}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{220}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{221}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (216) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$-r^3-\left(r^2-1\right)\sin(\phi_2)\sin(\phi_3)+2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)-\left(r^2-1\right)r\cos(\phi_2) = \alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{222}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\phi_2)-\sin(\phi_2))^2+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}} \tag{223}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2)+r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r-r\cos(\phi_2)}{\sqrt{\left(r^2-1\right)^2\left(4r^2\cos^4\left(\frac{\phi_2}{2}\right)+\sin^2(\phi_2)\right)}}\right)+\tan^{-1}\left(\frac{\sin(\phi_2)-r^2\sin(\phi_2)}{2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{224}$$

which yields two solutions for each value of ϕ_2.

1.10 $C|C_\beta G$ Paths

1.10.1 $L^+|L^-_\beta G^-$ Paths

For a $L^+_{\phi_1}|L^-_\beta G^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{225}$$

Pre-multiplying (225) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$2r^4+\sin(\phi_1)\left(r^2\sin(\beta)-\sin(\beta)\right)-r^2+\cos(\phi_1)\left(-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)\right)+\left(2r^2-2r^4\right)\cos(\beta) = \alpha_{33}, \tag{226}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}} \tag{227}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}}\right) + \tan^{-1}\left(\frac{r^2\sin(\beta)-\sin(\beta)}{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}\right), \tag{228}$$

which yields two solutions.
Pre-multiplying (225) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned} &2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)-2\left(r^2-1\right)r^2\cos(\beta)+\left(2r^2-1\right)r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\ =&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right). \end{aligned} \tag{229}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining $\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$,

it is obtained that

$$\cos\left(\phi_2-\theta\right)=\frac{r\left(2r^3\cos(\beta)-2r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}} \tag{230}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{r\left(2r^3\cos(\beta)-2r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}}\right) \tag{231}$$

$$+\tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}\right), \tag{232}$$

which yields two solutions.

1.10.2 $R^+|R^-_\beta G^-$ Paths

For a $R^+_{\phi_1}|R^-_\beta G^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{233}$$

Pre-multiplying (233) with $\mathbf{u}_{G^-}^T$ and post-multiplying $\mathbf{u}_{G^-}$:

$$2r^4+\sin(\phi_1)\left(r^2\sin(\beta)-\sin(\beta)\right)-r^2+\cos(\phi_1)\left(-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)\right)+\left(2r^2-2r^4\right)\cos(\beta)=\alpha_{33}, \tag{234}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}} \tag{235}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{\alpha_{33}+r^2\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4+2\left(r^2-1\right)r^2\cos(2\beta)-2r^2+\left(4r^2-8r^4\right)\cos(\beta)+1\right)}}\right)$$

$$+\tan^{-1}\left(\frac{r^2\sin(\beta)-\sin(\beta)}{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}\right), \tag{236}$$

which yields two solutions.
Pre-multiplying (233) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)-2\left(r^2-1\right)r^2\cos(\beta)+\left(2r^2-1\right)r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\ =&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right).\end{aligned} \tag{237}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining
$\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$,
it is obtained that

$$\cos\left(\phi_2-\theta\right)=\frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}} \tag{238}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+\left(1-2r^2\right)^2+\left(4r^2-8r^4\right)\cos(\beta)\right)}}\right) \tag{239}$$

$$+\tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}\right), \tag{240}$$

which yields two solutions.

1.10.3 $R^-|R^+_\beta G^+$ Paths

For a $R^-_{\phi_1}|R^+_\beta G^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{241}$$

Pre-multiplying (241) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$2r^4 + \sin(\phi_1)\left(r^2\sin(\beta) - \sin(\beta)\right) - r^2 + \cos(\phi_1)\left(-2r^4 + 2r^2 + \left(2r^4 - 3r^2 + 1\right)\cos(\beta)\right) + \left(2r^2 - 2r^4\right)\cos(\beta) = \alpha_{33}, \tag{242}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{33} + r^2\left(2\left(r^2-1\right)\cos(\beta) - 2r^2 + 1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4 + 2\left(r^2-1\right)r^2\cos(2\beta) - 2r^2 + \left(4r^2 - 8r^4\right)\cos(\beta) + 1\right)}} \tag{243}$$

$$\begin{aligned}\implies \phi_1 =& \cos^{-1}\left(\frac{\alpha_{33} + r^2\left(2\left(r^2-1\right)\cos(\beta) - 2r^2 + 1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4 + 2\left(r^2-1\right)r^2\cos(2\beta) - 2r^2 + \left(4r^2 - 8r^4\right)\cos(\beta) + 1\right)}}\right)\\ &+ \tan^{-1}\left(\frac{r^2\sin(\beta) - \sin(\beta)}{-2r^4 + 2r^2 + \left(2r^4 - 3r^2 + 1\right)\cos(\beta)}\right),\end{aligned} \tag{244}$$

which yields two solutions.
Pre-multiplying (241) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2) - 2\left(r^2-1\right)r^2\cos(\beta) + \left(2r^2-1\right)r^2 + \cos(\phi_2)\left(2r^4 - 2\left(r^2-1\right)r^2\cos(\beta) - 3r^2 + 1\right)\\ =& r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right).\end{aligned} \tag{245}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining
$\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$,
it is obtained that

$$\cos\left(\phi_2 - \theta\right) = \frac{r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right) - \alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta) + 4r^2\sin^2(\beta) + \left(1-2r^2\right)^2 + \left(4r^2 - 8r^4\right)\cos(\beta)\right)}} \tag{246}$$

$$\implies \phi_2 = \cos^{-1}\left(\frac{r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right) - \alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta) + 4r^2\sin^2(\beta) + \left(1-2r^2\right)^2 + \left(4r^2 - 8r^4\right)\cos(\beta)\right)}}\right) \tag{247}$$

$$+ \tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4 - 2\left(r^2-1\right)r^2\cos(\beta) - 3r^2 + 1}\right), \tag{248}$$

which yields two solutions.

1.10.4 $L^-|L_\beta^+G^+$ Paths

For a $L_{\phi_1}^-|L_\beta^+G_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{249}$$

Pre-multiplying (249) with $\mathbf{u}_{G^+}^T$ and post-multiplying $\mathbf{u}_{G^+}$:

$$2r^4 + \sin(\phi_1)\left(r^2\sin(\beta) - \sin(\beta)\right) - r^2 + \cos(\phi_1)\left(-2r^4 + 2r^2 + \left(2r^4 - 3r^2 + 1\right)\cos(\beta)\right) + \left(2r^2 - 2r^4\right)\cos(\beta) = \alpha_{33}, \tag{250}$$

This equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$ and defining $\cos\gamma := \frac{-2r^4+2r^2+\left(2r^4-3r^2+1\right)\cos(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$, $\sin\gamma := \frac{r^2\sin(\beta)-\sin(\beta)}{\sqrt{(r^2\sin(\beta)-\sin(\beta))^2+(-2r^4+2r^2+(2r^4-3r^2+1)\cos(\beta))^2}}$. It is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{33} + r^2\left(2\left(r^2-1\right)\cos(\beta) - 2r^2 + 1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4 + 2\left(r^2-1\right)r^2\cos(2\beta) - 2r^2 + \left(4r^2 - 8r^4\right)\cos(\beta) + 1\right)}} \tag{251}$$

$$\begin{aligned}\implies \phi_1 =& \cos^{-1}\left(\frac{\alpha_{33} + r^2\left(2\left(r^2-1\right)\cos(\beta) - 2r^2 + 1\right)}{\sqrt{\left(r^2-1\right)^2\left(6r^4 + 2\left(r^2-1\right)r^2\cos(2\beta) - 2r^2 + \left(4r^2 - 8r^4\right)\cos(\beta) + 1\right)}}\right)\\ &+ \tan^{-1}\left(\frac{r^2\sin(\beta) - \sin(\beta)}{-2r^4 + 2r^2 + \left(2r^4 - 3r^2 + 1\right)\cos(\beta)}\right),\end{aligned} \tag{252}$$

which yields two solutions.

Pre-multiplying (249) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)-2\left(r^2-1\right)r^2\cos(\beta)+\left(2r^2-1\right)r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\&=r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right).\end{aligned} \tag{253}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, and defining $\cos\theta := \frac{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, $\sin\theta := \frac{2\left(r^2-1\right)r\sin(\beta)}{\sqrt{(2(r^2-1)r\sin(\beta))^2+(2r^4-2(r^2-1)r^2\cos(\beta)-3r^2+1)^2}}$, it is obtained that

$$\cos\left(\phi_2-\theta\right)=\frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+(1-2r^2)^2+(4r^2-8r^4)\cos(\beta)\right)}} \tag{254}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{\left(r^2-1\right)^2\left(4r^4\cos^2(\beta)+4r^2\sin^2(\beta)+(1-2r^2)^2+(4r^2-8r^4)\cos(\beta)\right)}}\right) \tag{255}$$

$$+\tan^{-1}\left(\frac{2\left(r^2-1\right)r\sin(\beta)}{2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1}\right), \tag{256}$$

which yields two solutions.

1.11 CTC Paths

1.11.1 $L^+L^0L^-$ Paths

For a $L^+_{\phi_1}L^0_{\phi_2}L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{257}$$

Pre-multiplying (257) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r^2(-\cos(\phi_2))-r^2+1=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{258}$$

which gives

$$\cos(\phi_2)=\frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r-r^2+1}{r^2}, \tag{259}$$

and yields two solutions of ϕ_2.
Pre-multiplying (257) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$-r\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)+r\cos(\phi_3)\left(r^2(-\cos(\phi_2))-r^2+\cos(\phi_2)+1\right)+r\left(r^2\cos(\phi_2)+r^2-1\right)=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \tag{260}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{261}$$

$$\implies \phi_1=\cos^{-1}\left(-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{262}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (257) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2\cos(\phi_2)+r^2-1\right)=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \tag{263}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{264}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{265}$$

which yields two solutions for each value of ϕ_2.

1.11.2 $R^+R^0R^-$ Paths

For a $R^+_{\phi_1}R^0_{\phi_2}R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{266}$$

Pre-multiplying (266) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r^2(-\cos(\phi_2)) - r^2 + 1 = -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{267}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r - r^2 + 1}{r^2}, \tag{268}$$

and yields two solutions of ϕ_2.
Pre-multiplying (266) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r\left(r^2-1\right)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) + r\left(r^2\cos(\phi_2) + r^2 - 1\right) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r. \tag{269}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{270}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3(-\cos(\phi_2)) - r^3 + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r + r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right) + \tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{271}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (266) with $\mathbf{u}^T_{R^+}$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2\cos(\phi_2) + r^2 - 1\right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \tag{272}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos(\phi_3-\gamma) = -\frac{r\left(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{273}$$

$$\implies \phi_3 = \cos^{-1}\left(-\frac{r\left(-\alpha_{33} + r^2\cos(\phi_2) + r^2 - 1\right) + \alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right) + \tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{274}$$

which yields two solutions for each value of ϕ_2.

1.11.3 $R^-R^0R^+$ Paths

For a $R^-_{\phi_1}R^0_{\phi_2}R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{275}$$

Pre-multiplying (275) with $\mathbf{u}^T_{R^-}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r^2(-\cos(\phi_2)) - r^2 + 1 = -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \tag{276}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11} + \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r - r^2 + 1}{r^2}, \tag{277}$$

and yields two solutions of ϕ_2.
Pre-multiplying (275) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r\left(r^2-1\right)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right) + r\left(r^2\cos(\phi_2) + r^2 - 1\right) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}. \tag{278}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = -\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{279}$$

$$\Longrightarrow \phi_1 = \cos^{-1}\left(-\frac{r\left(-\alpha_{33}+r^2\cos(\phi_2)+r^2-1\right)+\alpha_{31}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{280}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (275) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2\cos(\phi_2)+r^2-1\right)=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \tag{281}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{282}$$

$$\Longrightarrow \phi_3 = \cos^{-1}\left(\frac{r^3(-\cos(\phi_2))-r^3+\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{283}$$

which yields two solutions for each value of ϕ_2.

1.11.4 $L^-L^0L^+$ Paths

For a $L^-_{\phi_1}L^0_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{284}$$

Pre-multiplying (284) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r^2(-\cos(\phi_2))-r^2+1 = -\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{285}$$

which gives

$$\cos(\phi_2) = \frac{-\alpha_{11}+\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r-r^2+1}{r^2}, \tag{286}$$

and yields two solutions of ϕ_2.
Pre-multiplying (284) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$-r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\cos^2\left(\frac{\phi_2}{2}\right)+r\left(r^2\cos(\phi_2)+r^2-1\right)=\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r. \tag{287}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{-r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3(-\cos(\phi|2))-r^3+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi|2}{2}\right)\left(r^2\cos(\phi|2)+r^2-2\right)}} \tag{288}$$

$$\Longrightarrow \phi_1 = \cos^{-1}\left(\frac{r^3(-\cos(\phi|2))-r^3+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r+r}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi|2}{2}\right)\left(r^2\cos(\phi|2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{289}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (284) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(-\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2\cos(\phi_2)+r^2-1\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \tag{290}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\cos^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=-\frac{r\left(-\alpha_{33}+r^2\cos(\phi|2)+r^2-1\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}} \tag{291}$$

$$\implies \phi_3=\cos^{-1}\left(-\frac{r\left(-\alpha_{33}+r^2\cos(\phi|2)+r^2-1\right)+\alpha_{13}\sqrt{1-r^2}}{\sqrt{2}\sqrt{r^2\left(r^2-1\right)\cos^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)+r^2-2\right)}}\right)+\tan^{-1}\left(\frac{-r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\cos^2\left(\frac{\phi_2}{2}\right)}\right), \tag{292}$$

which yields two solutions for each value of ϕ_2.

1.11.5 $L^+L^0L^+$ Paths

For a $L^+_{\phi_1}L^0_{\phi_2}L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{293}$$

Pre-multiplying (293) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$r^2\cos(\phi_2)-r^2+1=r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \tag{294}$$

which gives

$$\cos(\phi_2)=\frac{\alpha_{11}-\alpha_{11}r^2+\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+r^2-1}{r^2}, \tag{295}$$

and yields two solutions of ϕ_2.
Pre-multiplying (293) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+r\left(r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{296}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3\cos(\phi_2)-r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{297}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3-\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{298}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (293) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{299}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3\cos(\phi_2)-r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{300}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{301}$$

which yields two solutions for each value of ϕ_2.

1.11.6 $R^+R^0R^+$ Paths

For a $R^+_{\phi_1}R^0_{\phi_2}R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{302}$$

Pre-multiplying (302) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$r^2\cos(\phi_2) - r^2 + 1 = r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{303}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 - \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \tag{304}$$

and yields two solutions of ϕ_2.
Pre-multiplying (302) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + r\left(r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \tag{305}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3\cos(\phi_2) - r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}} \tag{306}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 + \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{307}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (302) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3) - 2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3) + r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r. \tag{308}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right) = \frac{r^3\cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}} \tag{309}$$

$$\implies \phi_3 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{310}$$

which yields two solutions for each value of ϕ_2.

1.11.7 $R^-R^0R^-$ Paths

For a $R^-_{\phi_1}R^0_{\phi_2}R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^0}(\phi_2)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{311}$$

Pre-multiplying (311) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$r^2\cos(\phi_2) - r^2 + 1 = r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right) - \alpha_{11}\left(r^2-1\right), \tag{312}$$

which gives

$$\cos(\phi_2) = \frac{\alpha_{11} - \alpha_{11}r^2 + \alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r + \alpha_{31}\sqrt{1-r^2}r + r^2 - 1}{r^2}, \tag{313}$$

and yields two solutions of ϕ_2.
Pre-multiplying (311) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2) - 2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right) + r\left(r^2(-\cos(\phi_2)) + r^2 - 1\right) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \tag{314}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma := \frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma := \frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma) = \frac{r^3\cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}} \tag{315}$$

$$\implies \phi_1 = \cos^{-1}\left(\frac{r^3\cos(\phi_2) - r^3 - \alpha_{31}\sqrt{1-r^2} - \alpha_{33}r + r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2) - r^2 + 2\right)}}\right) + \tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{316}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (311) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{317}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3\cos(\phi_2)-r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{318}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3-\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{319}$$

which yields two solutions for each value of ϕ_2.

1.11.8 $L^-L^0L^-$ Paths

For a $L^-_{\phi_1}L^0_{\phi_2}L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^0}(\phi_2)\mathbf{M}_{L^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{320}$$

Pre-multiplying (320) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$r^2\cos(\phi_2)-r^2+1=r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \tag{321}$$

which gives

$$\cos(\phi_2)=\frac{\alpha_{11}-\alpha_{11}r^2+\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+r^2-1}{r^2}, \tag{322}$$

and yields two solutions of ϕ_2.
Pre-multiplying (320) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$r\sqrt{1-r^2}\sin(\phi_1)\sin(\phi_2)-2r\left(r^2-1\right)\cos(\phi_1)\sin^2\left(\frac{\phi_2}{2}\right)+r\left(r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \tag{323}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$ and defining $\cos\gamma:=\frac{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, $\sin\gamma:=\frac{r\sqrt{1-r^2}\sin(\phi_2)}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$. It is obtained that

$$\cos(\phi_1-\gamma)=\frac{r^3\cos(\phi_2)-r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{324}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{325}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (320) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\left(\sqrt{1-r^2}\sin(\phi_2)\sin(\phi_3)-2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\cos(\phi_3)+r^2(-\cos(\phi_2))+r^2-1\right)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \tag{326}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{r^2(1-r^2)\sin^2(\phi_2)+\left(2(r^2-1)r\sin^2\left(\frac{\phi_2}{2}\right)\right)^2}}$, it is obtained that

$$\cos\left(\phi_3-\gamma\right)=\frac{r^3\cos(\phi_2)-r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}} \tag{327}$$

$$\implies \phi_3=\cos^{-1}\left(\frac{r^3\cos(\phi_2)-r^3+\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r+r}{\sqrt{2}\sqrt{-r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(r^2\cos(\phi_2)-r^2+2\right)}}\right)+\tan^{-1}\left(\frac{r\sqrt{1-r^2}\sin(\phi_2)}{-2\left(r^2-1\right)r\sin^2\left(\frac{\phi_2}{2}\right)}\right), \tag{328}$$

which yields two solutions for each value of ϕ_2.

1.12 $C|C_\psi C_\psi|C$ Paths

1.12.1 $L^+|L^-_\psi R^-_\psi|R^+$ Paths

For a $L^+_{\phi_1}|L^-_\psi R^-_\psi|R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\psi)\mathbf{M}_{R^-}(r,\psi)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{329}$$

Pre-multiplying (329) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned} &12r^6 - 20r^4 + 10r^2 + 8\left(r^2-1\right)r^4\cos^2(\psi) - 4\left(r^2-1\right)r^4 - 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) - 1 \\ =&\alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \end{aligned} \tag{330}$$

which gives

$$\cos(\psi) = \frac{4r^6 - 6r^4 + 2r^2 \pm \sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2 + \alpha_{13}\sqrt{1-r^2}r - \alpha_{31}\sqrt{1-r^2}r + \alpha_{11}\left(r^2-1\right) - 1\right)}}{4r^4\left(r^2-1\right)}, \tag{331}$$

and yields four solutions of ψ.
Pre-multiplying (329) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned} &-12r^7 + 20r^5 - 10r^3 + 2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi) - 2r^2 + 1\right) - 4\left(r^2-1\right)r^5\cos(2\psi) \\ &-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4 - 8r^2 + 1\right)\cos(\psi) - \left(2r^2-1\right)\left(r^2\cos(2\psi) + 3r^2 - 2\right)\right) + 8\left(2r^4 - 3r^2 + 1\right)r^3\cos(\psi) + r \\ =&\alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \end{aligned} \tag{332}$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right)r\sin(\psi)\left(2r^2\cos(\psi) - 2r^2 + 1\right) \tag{333}$$

$$B = -2\left(r^2-1\right)r\left(\left(8r^4 - 8r^2 + 1\right)\cos(\psi) - \left(2r^2-1\right)\left(r^2\cos(2\psi) + 3r^2 - 2\right)\right), \tag{334}$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{31}\sqrt{1-r^2} - r\left(\alpha_{33} - 4r^6\cos(2\psi) - 12r^6 + 4r^4\cos(2\psi) + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}} \tag{335}$$

$$\begin{aligned} \implies \phi_1 = &\cos^{-1}\left(\frac{\alpha_{31}\sqrt{1-r^2} - r\left(\alpha_{33} - 4r^6\cos(2\psi) - 12r^6 + 4r^4\cos(2\psi) + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}}\right) \\ &+ \tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{336}$$

which yields two solutions for each value of ψ.
Pre-multiplying (329) with $\mathbf{u}^T_{L^+}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned} &-12r^7 + 20r^5 - 10r^3 + 2\left(r^2-1\right)r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi) - 2r^2 + 1\right) - 4\left(r^2-1\right)r^5\cos(2\psi) \\ &-2\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4 - 8r^2 + 1\right)\cos(\psi) - \left(2r^2-1\right)\left(r^2\cos(2\psi) + 3r^2 - 2\right)\right) + 8\left(2r^4 - 3r^2 + 1\right)r^3\cos(\psi) + r \\ =&\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r \end{aligned} \tag{337}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2 - \gamma\right) = \frac{\alpha_{13}\left(-\sqrt{1-r^2}\right) - r\left(\alpha_{33} - \left(4r^6 - 4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}} \tag{338}$$

$$\begin{aligned} \implies \phi_2 = &\cos^{-1}\left(\frac{\alpha_{13}\left(-\sqrt{1-r^2}\right) - r\left(\alpha_{33} - \left(4r^6 - 4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}}\right) \\ &+ \tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{339}$$

which yields two solutions for each value of ψ.

1.12.2 $R^+|R^-_\psi L^-_\psi|L^+$ Paths

For a $R^+_{\phi_1}|R^-_\psi L^-_\psi|L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\psi)\mathbf{M}_{L^-}(r,\psi)\mathbf{M}_{L^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{340}$$

Pre-multiplying (340) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}&12r^6-20r^4+10r^2+8\left(r^2-1\right)r^4\cos^2(\psi)-4\left(r^2-1\right)r^4-8\left(2r^6-3r^4+r^2\right)\cos(\psi)-1\\=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned}\tag{341}$$

which gives

$$\cos(\psi)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)-1\right)}}{4r^4\left(r^2-1\right)},\tag{342}$$

and yields four solutions of ψ.
Pre-multiplying (340) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\&-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\=&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.\end{aligned}\tag{343}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2\left(r^2-1\right)r\sin(\psi)\left(2r^2\cos(\psi)-2r^2+1\right)\tag{344}$$
$$B=-2\left(r^2-1\right)r\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right),\tag{345}$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{\alpha_{31}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\tag{346}$$

$$\begin{aligned}\implies\phi_1=&\cos^{-1}\left(\frac{\alpha_{31}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\&+\tan^{-1}\left(\frac{A}{B}\right),\end{aligned}\tag{347}$$

which yields two solutions for each value of ψ.
Pre-multiplying (340) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\&-2\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\=&\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r\end{aligned}\tag{348}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{\alpha_{13}\sqrt{1-r^2}-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\tag{349}$$

$$\begin{aligned}\implies\phi_2=&\cos^{-1}\left(\frac{\alpha_{13}\sqrt{1-r^2}-r\left(\alpha_{33}-\left(4r^6-4r^4\right)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\&+\tan^{-1}\left(\frac{A}{B}\right),\end{aligned}\tag{350}$$

which yields two solutions for each value of ψ.

1.12.3 $R^-|R_\psi^+L_\psi^+|L^-$ Paths

For a $R_{\phi_1}^-|R_\psi^+L_\psi^+|L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\psi)\mathbf{M}_{L^+}(r,\psi)\mathbf{M}_{L^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{351}$$

Pre-multiplying (351) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}&12r^6-20r^4+10r^2+8\left(r^2-1\right)r^4\cos^2(\psi)-4\left(r^2-1\right)r^4-8\left(2r^6-3r^4+r^2\right)\cos(\psi)-1\\=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned}\tag{352}$$

which gives

$$\cos(\psi)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)-1\right)}}{4r^4\left(r^2-1\right)},\tag{353}$$

and yields four solutions of ψ.
Pre-multiplying (351) with $\mathbf{u}_{G+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\&-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\=&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (354)\end{aligned}$$

For $\psi\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2\left(r^2-1\right)r\sin(\psi)\left(2r^2\cos(\psi)-2r^2+1\right) \qquad (355)$$
$$B=-2\left(r^2-1\right)r\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right), \qquad (356)$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{\alpha_{31}\sqrt{1-r^2}-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \qquad (357)$$

$$\begin{aligned}\implies\phi_1=&\cos^{-1}\left(\frac{\alpha_{31}\sqrt{1-r^2}-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (358)\end{aligned}$$

which yields two solutions for each value of ψ.
Pre-multiplying (351) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G+}$:

$$\begin{aligned}&-12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\&-2\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\=&\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (359)\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{\alpha_{13}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}} \qquad (360)$$

$$\begin{aligned}\implies\phi_2=&\cos^{-1}\left(\frac{\alpha_{13}\left(-\sqrt{1-r^2}\right)-r\left(\alpha_{33}-(4r^6-4r^4)\cos(2\psi)-12r^6+20r^4-10r^2+8\left(2r^6-3r^4+r^2\right)\cos(\psi)+1\right)}{\sqrt{A^2+B^2}}\right)\\&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (361)\end{aligned}$$

which yields two solutions for each value of ψ.

1.12.4 $L^-|L_\psi^+R_\psi^+|R^-$ Paths

For a $L_{\phi_1}^-|L_\psi^+R_\psi^+|R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\psi)\mathbf{M}_{R^+}(r,\psi)\mathbf{M}_{R^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \qquad (362)$$

Pre-multiplying (362) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}&12r^6-20r^4+10r^2+8\left(r^2-1\right)r^4\cos^2(\psi)-4\left(r^2-1\right)r^4-8\left(2r^6-3r^4+r^2\right)\cos(\psi)-1\\=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (363)\end{aligned}$$

which gives

$$\cos(\psi)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^4\left(r^2-1\right)\left(\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)-1\right)}}{4r^4\left(r^2-1\right)}, \qquad (364)$$

and yields four solutions of ψ.
Pre-multiplying (362) with $\mathbf{u}_{G+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&12r^7+20r^5-10r^3+2\left(r^2-1\right)r\sin(\psi)\sin(\phi_1)\left(2r^2\cos(\psi)-2r^2+1\right)-4\left(r^2-1\right)r^5\cos(2\psi)\\&-2\left(r^2-1\right)r\cos(\phi_1)\left(\left(8r^4-8r^2+1\right)\cos(\psi)-\left(2r^2-1\right)\left(r^2\cos(2\psi)+3r^2-2\right)\right)+8\left(2r^4-3r^2+1\right)r^3\cos(\psi)+r\\=&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (365)\end{aligned}$$

For $\psi \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right) r \sin(\psi) \left(2r^2\cos(\psi) - 2r^2 + 1\right) \tag{366}$$
$$B = -2\left(r^2-1\right) r \left(\left(8r^4 - 8r^2 + 1\right)\cos(\psi) - \left(2r^2-1\right)\left(r^2\cos(2\psi) + 3r^2 - 2\right)\right), \tag{367}$$

it is obtained that

$$\cos(\phi_1 - \gamma) = \frac{\alpha_{31}\left(-\sqrt{1-r^2}\right) - r\left(\alpha_{33} - \left(4r^6 - 4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}} \tag{368}$$
$$\begin{aligned}\implies \phi_1 = \cos^{-1}&\left(\frac{\alpha_{31}\left(-\sqrt{1-r^2}\right) - r\left(\alpha_{33} - \left(4r^6 - 4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}}\right)\\ &+ \tan^{-1}\left(\frac{A}{B}\right),\end{aligned} \tag{369}$$

which yields two solutions for each value of ψ.
Pre-multiplying (362) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&-12r^7 + 20r^5 - 10r^3 + 2\left(r^2-1\right) r\sin(\psi)\sin(\phi_2)\left(2r^2\cos(\psi) - 2r^2 + 1\right) - 4\left(r^2-1\right) r^5\cos(2\psi)\\ &- 2\left(r^2-1\right) r\cos(\phi_2)\left(\left(8r^4 - 8r^2 + 1\right)\cos(\psi) - \left(2r^2-1\right)\left(r^2\cos(2\psi) + 3r^2 - 2\right)\right) + 8\left(2r^4 - 3r^2 + 1\right) r^3\cos(\psi) + r\\ =&\alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33} r.\end{aligned} \tag{370}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2 - \gamma\right) = \frac{\alpha_{13}\sqrt{1-r^2} - r\left(\alpha_{33} - \left(4r^6 - 4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}} \tag{371}$$
$$\begin{aligned}\implies \phi_2 = \cos^{-1}&\left(\frac{\alpha_{13}\sqrt{1-r^2} - r\left(\alpha_{33} - \left(4r^6 - 4r^4\right)\cos(2\psi) - 12r^6 + 20r^4 - 10r^2 + 8\left(2r^6 - 3r^4 + r^2\right)\cos(\psi) + 1\right)}{\sqrt{A^2+B^2}}\right)\\ &+ \tan^{-1}\left(\frac{A}{B}\right),\end{aligned} \tag{372}$$

which yields two solutions for each value of ψ.

1.13 $CGC_\beta|C$ Paths

1.13.1 $L^+G^+L^+_\beta|L^-$ Paths

For a $L^+_{\phi_1}G^+_{\phi_2}L^+_\beta|L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{373}$$

Pre-multiplying (373) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}&-2r^4 + 2\left(r^2-1\right) r\sin(\beta)\sin(\phi_2) + 2\left(r^2-1\right) r^2\cos(\beta) + r^2 + \cos(\phi_2)\left(2r^4 - 2\left(r^2-1\right) r^2\cos(\beta) - 3r^2 + 1\right)\\ =& -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r\right),\end{aligned} \tag{374}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right) r\sin(\beta) \tag{375}$$
$$B = 2r^4 - 2\left(r^2-1\right) r^2\cos(\beta) - 3r^2 + 1, \tag{376}$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}} \tag{377}$$
$$\begin{aligned}\implies \phi_2 = \cos^{-1}&\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}}\right)\\ &+ \tan^{-1}\left(\frac{A}{B}\right),\end{aligned} \tag{378}$$

and yields two solutions of ϕ_2.
Pre-multiplying (373) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&2r^5 - r^3 - \left(r^2-1\right) r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2) - r^2 + 1\right) - 2r^2\cos(\phi_2) + 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) - 1\right)\\ &+ \left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) - 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2)\\ &- \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - 4\left(r^2-1\right) r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33} r - \alpha_{31}\sqrt{1-r^2}.\end{aligned} \tag{379}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C &= \left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) && (380)\\ D &= -\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right), && (381) \end{aligned}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) &= \frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}}\\ &\quad -\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} && (382)\\ \implies \phi_1 &= \cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}}\right.\\ &\quad \left.-\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), && (383) \end{aligned}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (373) with $\mathbf{u}_{L+}^T$ and post-multiplying with $\mathbf{u}_{G+}$:

$$\begin{aligned} &2r^5-r^3-\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)-1))-2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)-\left(2r^5-3r^3+r\right)\cos(\phi_2)\\ &-\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)+2r^3-2r^2\sin(\beta)\sin(\phi_2)-2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)\\ &-4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. && (384) \end{aligned}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$\begin{aligned} E &= -\left(r^2-1\right)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)-1)) && (385)\\ F &= -\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)+2r^3-2r^2\sin(\beta)\sin(\phi_2)-2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), && (386) \end{aligned}$$

it is obtained that

$$\begin{aligned} \cos(\phi_3-\sigma) &= \frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}}\\ &\quad +\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} && (387)\\ \implies \phi_3 &= \cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}}\right.\\ &\quad \left.+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), && (388) \end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.13.2 $R^+G^+R^+_\beta|R^-$ Paths

For a $R^+_{\phi_1}G^+_{\phi_2}R^+_\beta|R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R+}(r,\phi_1)\mathbf{M}_{G+}(\phi_2)\mathbf{M}_{R+}(r,\beta)\mathbf{M}_{R-}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \quad (389)$$

Pre-multiplying (389) with $\mathbf{u}_{R+}^T$ and post-multiplying $\mathbf{u}_{R-}$:

$$\begin{aligned} &-2r^4+2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\ &=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), && (390) \end{aligned}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A &= 2\left(r^2-1\right)r\sin(\beta) && (391)\\ B &= 2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1, && (392) \end{aligned}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}} \tag{393}$$

$$\Longrightarrow \phi_2=\cos^{-1}\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}}\right)$$
$$+\tan^{-1}\left(\frac{A}{B}\right), \tag{394}$$

and yields two solutions of ϕ_2.
Pre-multiplying (389) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&2r^5-r^3-\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right)\\&+\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)-2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\&-\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{395}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$C=\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \tag{396}$$
$$D=-\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right), \tag{397}$$

it is obtained that

$$\cos(\phi_1-\theta)=\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}}+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \tag{398}$$

$$\Longrightarrow \phi_1=\cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}}+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \tag{399}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (389) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}&2r^5-r^3-\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)-1))-2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)-\left(2r^5-3r^3+r\right)\cos(\phi_2)\\&-\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)+2r^3-2r^2\sin(\beta)\sin(\phi_2)-2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)\\&-4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{400}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma:=\frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma:=\frac{F}{\sqrt{E^2+F^2}}$, where

$$E=-\left(r^2-1\right)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)-1)) \tag{401}$$
$$F=-\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)+2r^3-2r^2\sin(\beta)\sin(\phi_2)-2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), \tag{402}$$

it is obtained that

$$\cos\left(\phi_3-\sigma\right)=\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}}+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \tag{403}$$

$$\Longrightarrow \phi_3=\cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}}+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \tag{404}$$

which yields two solutions for each value of ϕ_2.

1.13.3 $R^-G^-R_\beta^-|R^+$ Paths

For a $R_{\phi_1}^-G_{\phi_2}^-R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{405}$$

Pre-multiplying (405) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned} &-2r^4 + 2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2) + 2\left(r^2-1\right)r^2\cos(\beta) + r^2 + \cos(\phi_2)\left(2r^4 - 2\left(r^2-1\right)r^2\cos(\beta) - 3r^2 + 1\right) \\ &= -\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \end{aligned} \tag{406}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2\left(r^2-1\right)r\sin(\beta) \tag{407}$$

$$B = 2r^4 - 2\left(r^2-1\right)r^2\cos(\beta) - 3r^2 + 1, \tag{408}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}} \tag{409}$$

$$\begin{aligned} \implies \phi_2 = {} & \cos^{-1}\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}}\right) \\ & + \tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{410}$$

and yields two solutions of ϕ_2.
Pre-multiplying (405) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned} & 2r^5 - r^3 - \left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2) - r^2 + 1\right) - 2r^2\cos(\phi_2) + 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) - 1\right) \\ & + \left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) - 2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2) \\ & - \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - 4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{31}\sqrt{1-r^2}. \end{aligned} \tag{411}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = \left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) \tag{412}$$

$$D = -\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2) - r^2 + 1\right) - 2r^2\cos(\phi_2) + 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) - 1\right), \tag{413}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) = {} & \frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2+D^2}} \\ & + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \end{aligned} \tag{414}$$

$$\begin{aligned} \implies \phi_1 = {} & \cos^{-1}\left(\frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{C^2+D^2}}\right. \\ & \left. + \frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{415}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (405) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned} & 2r^5 - r^3 - \left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)-1)) - 2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2) - \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) \\ & - \left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right)r\cos(\phi_2)\right) \\ & - 4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}. \end{aligned} \tag{416}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = -\left(r^2-1\right)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)-1)) \tag{417}$$

$$F = -\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right)r\cos(\phi_2)\right), \tag{418}$$

it is obtained that

$$
\begin{aligned}
\cos\left(\phi_3-\sigma\right) &= \frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}} \\
&\quad+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \qquad (419)\\
\implies \phi_3 &= \cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{E^2+F^2}}\right.\\
&\quad\left.+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \qquad (420)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

1.13.4 $L^-G^-L^-_\beta|L^+$ Paths

For a $L^-_{\phi_1}G^-_{\phi_2}L^-_\beta|L^+_{\phi_3}$ path, the equation to be solved is:

$$
\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \qquad (421)
$$

Pre-multiplying (421) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$
\begin{aligned}
&-2r^4+2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right)\\
&=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (422)
\end{aligned}
$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$
\begin{aligned}
A &= 2\left(r^2-1\right)r\sin(\beta) \qquad (423)\\
B &= 2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1, \qquad (424)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_2-\gamma) &= \frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}} \qquad (425)\\
\implies \phi_2 &= \cos^{-1}\left(\frac{-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(2r^3\cos(\beta)-2r^3-\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r-2r\cos(\beta)+r\right)}{\sqrt{A^2+B^2}}\right)\\
&\quad+\tan^{-1}\left(\frac{A}{B}\right), \qquad (426)
\end{aligned}
$$

and yields two solutions of ϕ_2.
Pre-multiplying (421) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$
\begin{aligned}
&2r^5-r^3-\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right)\\
&+\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)-2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\
&-\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)=\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r. \qquad (427)
\end{aligned}
$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C &= \left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \qquad (428)\\
D &= -\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)-r^2+1\right)-2r^2\cos(\phi_2)+2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)-1\right), \qquad (429)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta) &= \frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}}\\
&\quad+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \qquad (430)\\
\implies \phi_1 &= \cos^{-1}\left(\frac{r\left(\alpha_{33}+\left(2r^4-3r^2+1\right)\cos(\phi_2)+r\left(-2r^3+2\left(r^2-1\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right)+r\right)\right)}{\sqrt{C^2+D^2}}\right.\\
&\quad\left.+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (431)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (421) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&2r^5 - r^3 - \left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)-1)) - 2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2) - \left(2r^5 - 3r^3 + r\right)\cos(\phi_2)\\&- \left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right)r\cos(\phi_2)\right)\\&- 4\left(r^2-1\right)r^3\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) = \alpha_{33}r - \alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{432}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = -\left(r^2-1\right)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)-1)) \tag{433}$$

$$F = -\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2) + 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) - 2\left(2r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right)r\cos(\phi_2)\right), \tag{434}$$

it is obtained that

$$\begin{aligned}\cos(\phi_3 - \sigma) = &\frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{E^2+F^2}}\\&+ \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\end{aligned} \tag{435}$$

$$\begin{aligned}\implies \phi_3 = \cos^{-1}&\left(\frac{r\left(\alpha_{33} + \left(2r^4 - 3r^2 + 1\right)\cos(\phi_2) + r\left(-2r^3 + 2\left(r^2-1\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right)r\cos(\beta)\sin^2\left(\frac{\phi_2}{2}\right) + r\right)\right)}{\sqrt{E^2+F^2}}\right.\\&\left.+ \frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right) + \tan^{-1}\left(\frac{E}{F}\right),\end{aligned} \tag{436}$$

which yields two solutions for each value of ϕ_2.

1.13.5 $L^+G^+R^+_\beta|R^-$ Paths

For a $L^+_{\phi_1}G^+_{\phi_2}R^+_\beta|R^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{437}$$

Pre-multiplying (437) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}&-2r^4 - 2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2) + 2\left(r^2-1\right)r^2\cos(\beta) + r^2 + \cos(\phi_2)\left(-2r^4 + 2\left(r^2-1\right)r^2\cos(\beta) + 3r^2 - 1\right)\\&=\alpha_{11}\left(r^2-1\right) - r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right),\end{aligned} \tag{438}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = -2\left(r^2-1\right)r\sin(\beta) \tag{439}$$

$$B = -\left(2r^4 - 2\left(r^2-1\right)r^2\cos(\beta) - 3r^2 + 1\right), \tag{440}$$

it is obtained that

$$\cos(\phi_2 - \gamma) = \frac{\alpha_{11}\left(r^2-1\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}} \tag{441}$$

$$\begin{aligned}\implies \phi_2 = \cos^{-1}&\left(\frac{\alpha_{11}\left(r^2-1\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}}\right)\\&+ \tan^{-1}\left(\frac{A}{B}\right),\end{aligned} \tag{442}$$

and yields two solutions of ϕ_2.
Pre-multiplying (437) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}&2r^5 - r^3 + \left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2) + r^2 - 1\right) - 2r^2\cos(\phi_2) - 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) + 1\right)\\&- \left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) + 2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\&+ \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - 4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r.\end{aligned} \tag{443}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C &= -\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) && (444)\\ D &= \left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right), && (445)\end{aligned}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) = {} & \frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\\ & +\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \qquad (446)\\ \implies \phi_1 = \cos^{-1} & \left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\right.\\ & \left.+\frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (447)\end{aligned}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (437) with $\mathbf{u}_{L+}^T$ and post-multiplying with $\mathbf{u}_{G+}$:

$$\begin{aligned} & 2r^5-r^3+\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1))+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)+\left(2r^5-3r^3+r\right)\cos(\phi_2)\\ & +\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)\\ & -4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \qquad (448)\end{aligned}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$\begin{aligned} E &= \left(r^2-1\right)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1)) && (449)\\ F &= \left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), && (450)\end{aligned}$$

it is obtained that

$$\begin{aligned} \cos(\phi_3-\sigma) = {} & \frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\\ & +\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \qquad (451)\\ \implies \phi_3 = \cos^{-1} & \left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\right.\\ & \left.+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \qquad (452)\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.13.6 $R^+G^+L_\beta^+|L^-$ Paths

For a $R_{\phi_1}^+G_{\phi_2}^+L_\beta^+|L_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R+}(r,\phi_1)\mathbf{M}_{G+}(\phi_2)\mathbf{M}_{L+}(r,\beta)\mathbf{M}_{L-}(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \qquad (453)$$

Pre-multiplying (453) with $\mathbf{u}_{R+}^T$ and post-multiplying $\mathbf{u}_{L-}$:

$$\begin{aligned} & -2r^4-2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\beta)+3r^2-1\right)\\ & =\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \qquad (454)\end{aligned}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A &= -2\left(r^2-1\right)r\sin(\beta) && (455)\\ B &= -\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right), && (456)\end{aligned}$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_2-\gamma) &= \frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \qquad (457)\\
\implies \phi_2 &= \cos^{-1}\left(\frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&\quad+\tan^{-1}\left(\frac{A}{B}\right), \qquad (458)
\end{aligned}
$$

and yields two solutions of ϕ_2.
Pre-multiplying (453) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$
\begin{aligned}
&2r^5-r^3+\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right)\\
&-\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\
&+\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \qquad (459)
\end{aligned}
$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$
\begin{aligned}
C &= -\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \qquad (460)\\
D &= \left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right), \qquad (461)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos(\phi_1-\theta) &= \frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\\
&\quad+\frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \qquad (462)\\
\implies \phi_1 &= \cos^{-1}\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\right.\\
&\quad\left.+\frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (463)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (453) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$
\begin{aligned}
&2r^5-r^3+\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1))+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)+\left(2r^5-3r^3+r\right)\cos(\phi_2)\\
&+\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)\\
&-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \qquad (464)
\end{aligned}
$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$
\begin{aligned}
E &= \left(r^2-1\right)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1)) \qquad (465)\\
F &= \left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), \qquad (466)
\end{aligned}
$$

it is obtained that

$$
\begin{aligned}
\cos\left(\phi_3-\sigma\right) &= \frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\\
&\quad+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \qquad (467)\\
\implies \phi_3 &= \cos^{-1}\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\right.\\
&\quad\left.+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \qquad (468)
\end{aligned}
$$

which yields two solutions for each value of ϕ_2.

1.13.7 $R^- G^- L^-_\beta | L^+$ Paths

For a $R^-_{\phi_1} G^-_{\phi_2} L^-_\beta | L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{469}$$

Pre-multiplying (469) with $\mathbf{u}^T_{R^-}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned} &-2r^4 - 2\left(r^2-1\right) r\sin(\beta)\sin(\phi_2) + 2\left(r^2-1\right) r^2\cos(\beta) + r^2 + \cos(\phi_2)\left(-2r^4 + 2\left(r^2-1\right) r^2\cos(\beta) + 3r^2 - 1\right) \\ =&\alpha_{11}\left(r^2-1\right) - r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r\right), \end{aligned} \tag{470}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = -2\left(r^2-1\right) r\sin(\beta) \tag{471}$$
$$B = -\left(2r^4 - 2\left(r^2-1\right) r^2\cos(\beta) - 3r^2 + 1\right), \tag{472}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{\alpha_{11}\left(r^2-1\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}} \tag{473}$$

$$\begin{aligned} \implies \phi_2 = \cos^{-1}&\left(\frac{\alpha_{11}\left(r^2-1\right) - r\left(2r^3\cos(\beta) - 2r^3 + \alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33} r - 2r\cos(\beta) + r\right)}{\sqrt{A^2+B^2}}\right) \\ &+ \tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{474}$$

and yields two solutions of ϕ_2.
Pre-multiplying (469) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned} &2r^5 - r^3 + \left(r^2-1\right) r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2) + r^2 - 1\right) - 2r^2\cos(\phi_2) - 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) + 1\right) \\ &- \left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) + 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2) \\ &+ \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) - 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{33} r - \alpha_{31}\sqrt{1-r^2}. \end{aligned} \tag{475}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta) - 2r^2 + 1\right) + 2r\sin(\beta)\cos(\phi_2)\right) \tag{476}$$
$$D = \left(r^2-1\right) r\left(2\cos(\beta)\left(r^2\cos(\phi_2) + r^2 - 1\right) - 2r^2\cos(\phi_2) - 2r^2 - 2r\sin(\beta)\sin(\phi_2) + \cos(\phi_2) + 1\right), \tag{477}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) = &\frac{-2r^5 + r^3 + \left(-2r^5 + 3r^3 - r\right)\cos(\phi_2) + \left(2r^2 - 2r^4\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33} r}{\sqrt{C^2+D^2}} \\ &+ \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \end{aligned} \tag{478}$$

$$\begin{aligned} \implies \phi_1 = \cos^{-1}&\left(\frac{-2r^5 + r^3 + \left(-2r^5 + 3r^3 - r\right)\cos(\phi_2) + \left(2r^2 - 2r^4\right)\sin(\beta)\sin(\phi_2) + 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) + \alpha_{33} r}{\sqrt{C^2+D^2}}\right. \\ &\left. + \frac{\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{479}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (469) with $\mathbf{u}^T_{R^-}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned} &2r^5 - r^3 + \left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)+1)) + 2\left(r^2-1\right) r^2\sin(\beta)\sin(\phi_2) + \left(2r^5 - 3r^3 + r\right)\cos(\phi_2) \\ &+ \left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2) - 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) + 2\left(2r^2-1\right) r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right) \\ &- 4\left(r^2-1\right) r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) = \alpha_{13}\sqrt{1-r^2} + \alpha_{33} r. \end{aligned} \tag{480}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E = \left(r^2-1\right)(\cos(\beta)\sin(\phi_2) + r\sin(\beta)(\cos(\phi_2)+1)) \tag{481}$$
$$F = \left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2) - 2r^3 - 2r^2\sin(\beta)\sin(\phi_2) + 2\left(2r^2-1\right) r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) - 2\left(r^2-1\right) r\cos(\phi_2)\right), \tag{482}$$

it is obtained that

$$\begin{aligned}\cos\left(\phi_3-\sigma\right)=&\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\\&+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \qquad (483)\\ \implies \phi_3=\cos^{-1}&\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\right.\\&\left.+\frac{\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \qquad (484)\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.13.8 $L^-G^-R^-_\beta|R^+$ Paths

For a $L^-_{\phi_1}G^-_{\phi_2}R^-_\beta|R^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \qquad (485)$$

Pre-multiplying (485) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned}&-2r^4-2\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)+2\left(r^2-1\right)r^2\cos(\beta)+r^2+\cos(\phi_2)\left(-2r^4+2\left(r^2-1\right)r^2\cos(\beta)+3r^2-1\right)\\=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r\right), \qquad (486)\end{aligned}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}A&=-2\left(r^2-1\right)r\sin(\beta) \qquad (487)\\B&=-\left(2r^4-2\left(r^2-1\right)r^2\cos(\beta)-3r^2+1\right), \qquad (488)\end{aligned}$$

it is obtained that

$$\begin{aligned}\cos(\phi_2-\gamma)=&\frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \qquad (489)\\ \implies\phi_2=\cos^{-1}&\left(\frac{r\left(-r\left(\alpha_{33}+2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (490)\end{aligned}$$

and yields two solutions of ϕ_2.
Pre-multiplying (485) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&2r^5-r^3+\left(r^2-1\right)r\cos(\phi_1)\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right)\\&-\left(r^2-1\right)\sin(\phi_1)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right)+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\\&+\left(2r^5-3r^3+r\right)\cos(\phi_2)-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \qquad (491)\end{aligned}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}C&=-\left(r^2-1\right)\left(\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+2r\sin(\beta)\cos(\phi_2)\right) \qquad (492)\\D&=\left(r^2-1\right)r\left(2\cos(\beta)\left(r^2\cos(\phi_2)+r^2-1\right)-2r^2\cos(\phi_2)-2r^2-2r\sin(\beta)\sin(\phi_2)+\cos(\phi_2)+1\right), \qquad (493)\end{aligned}$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta)=&\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\\&+\frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}} \qquad (494)\\ \implies\phi_1=\cos^{-1}&\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{C^2+D^2}}\right.\\&\left.+\frac{-\alpha_{31}\sqrt{1-r^2}}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (495)\end{aligned}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (485) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&2r^5-r^3+\left(r^2-1\right)\sin(\phi_3)(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1))+2\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)+\left(2r^5-3r^3+r\right)\cos(\phi_2)\\&+\left(r^2-1\right)\cos(\phi_3)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right)\\&-4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \qquad (496)\end{aligned}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_3. Multiplying both sides with $\frac{1}{\sqrt{E^2+F^2}}$ and defining $\sin\sigma := \frac{E}{\sqrt{E^2+F^2}}$, $\cos\sigma := \frac{F}{\sqrt{E^2+F^2}}$, where

$$E=\left(r^2-1\right)\left(\cos(\beta)\sin(\phi_2)+r\sin(\beta)(\cos(\phi_2)+1)\right) \qquad (497)$$

$$F=\left(r^2-1\right)\left(\sin(\beta)\sin(\phi_2)-2r^3-2r^2\sin(\beta)\sin(\phi_2)+2\left(2r^2-1\right)r\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-2\left(r^2-1\right)r\cos(\phi_2)\right), \qquad (498)$$

it is obtained that

$$\begin{aligned}\cos\left(\phi_3-\sigma\right)=&\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\\&+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}} \qquad (499)\\ \implies \phi_3=\cos^{-1}&\left(\frac{-2r^5+r^3+\left(-2r^5+3r^3-r\right)\cos(\phi_2)+\left(2r^2-2r^4\right)\sin(\beta)\sin(\phi_2)+4\left(r^2-1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)+\alpha_{33}r}{\sqrt{E^2+F^2}}\right.\\&\left.+\frac{-\alpha_{13}\sqrt{1-r^2}}{\sqrt{E^2+F^2}}\right)+\tan^{-1}\left(\frac{E}{F}\right), \qquad (500)\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.14 $CC_\mu|C_\mu C$ Paths

1.14.1 $L^+R_\mu^+|R_\mu^-L^-$ Paths

For a $L_{\phi_1}^+R_\mu^+|R_\mu^-L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \qquad (501)$$

Pre-multiplying (501) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}&-12r^6+16r^4-8\left(r^2-1\right)^2r^2\cos^2(\mu)+4\left(r^2-1\right)^2r^2-6r^2+8\left(2r^6-3r^4+r^2\right)\cos(\mu)+1\\&=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (502)\end{aligned}$$

which gives

$$\cos(\mu)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)+1\right)}}{4r^2\left(r^2-1\right)^2}, \qquad (503)$$

and yields four solutions of μ.
Pre-multiplying (501) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}&r\Bigg(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2\\&\quad-2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^4-3r^2+1\right)r^2\cos(\mu)-1\Bigg)\\&=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \qquad (504)\end{aligned}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \qquad (505)$$

$$B=-2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right), \qquad (506)$$

it is obtained that

$$\cos(\phi_1-\gamma) = \frac{-12r^7+16r^5-6r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{507}$$

$$\Longrightarrow \phi_1 = \cos^{-1}\left(\frac{-12r^7+16r^5-6r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{508}$$

which yields two solutions for each value of μ.
Pre-multiplying (501) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned} r\Big(&12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2 r^2\cos(2\mu)+6r^2 \\ &-2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^6-3r^4+r^2\right)\cos(\mu)-1\Big) \\ =&\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r. \end{aligned} \tag{509}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right) = \frac{-12r^7+16r^5-6r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{510}$$

$$\Longrightarrow \phi_2 = \cos^{-1}\left(\frac{-12r^7+16r^5-6r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{511}$$

which yields two solutions for each value of μ.

1.14.2 $R^+L_\mu^+|L_\mu^-R^-$ Paths

For a $R_{\phi_1}^+L_\mu^+|L_\mu^-R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{512}$$

Pre-multiplying (512) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned} &-12r^6+16r^4-8\left(r^2-1\right)^2 r^2\cos^2(\mu)+4\left(r^2-1\right)^2 r^2-6r^2+8\left(2r^6-3r^4+r^2\right)\cos(\mu)+1 \\ =&-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \end{aligned} \tag{513}$$

which gives

$$\cos(\mu) = \frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)+1\right)}}{4r^2\left(r^2-1\right)^2}, \tag{514}$$

and yields four solutions of μ.
Pre-multiplying (512) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned} r\Big(&12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2 r^2\cos(2\mu)+6r^2 \\ &-2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^4-3r^2+1\right)r^2\cos(\mu)-1\Big) \\ =&\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r. \end{aligned} \tag{515}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \tag{516}$$

$$B = -2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right), \tag{517}$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{-12r^7+16r^5-6r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{518}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right)$$
$$+\tan^{-1}\left(\frac{A}{B}\right), \tag{519}$$

which yields two solutions for each value of μ.
Pre-multiplying (512) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$r\Big(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2$$
$$-2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^6-3r^4+r^2\right)\cos(\mu)-1\Big)$$
$$=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}. \tag{520}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{521}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right)$$
$$+\tan^{-1}\left(\frac{A}{B}\right), \tag{522}$$

which yields two solutions for each value of μ.

1.14.3 $R^-L^-_\mu|L^+_\mu R^+$ Paths

For a $R^-_{\phi_1}L^-_\mu|L^+_\mu R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{523}$$

Pre-multiplying (523) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$-12r^6+16r^4-8\left(r^2-1\right)^2r^2\cos^2(\mu)+4\left(r^2-1\right)^2r^2-6r^2+8\left(2r^6-3r^4+r^2\right)\cos(\mu)+1$$
$$=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{524}$$

which gives

$$\cos(\mu)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2+\alpha_{13}\sqrt{1-r^2}r-\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)+1\right)}}{4r^2\left(r^2-1\right)^2}, \tag{525}$$

and yields four solutions of μ.
Pre-multiplying (523) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$r\Big(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2$$
$$-2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^4-3r^2+1\right)r^2\cos(\mu)-1\Big)$$
$$=\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}. \tag{526}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \tag{527}$$
$$B=-2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right), \tag{528}$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{-12r^7+16r^5-6r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{529}$$

$$\Longrightarrow \phi_1=\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{530}$$

which yields two solutions for each value of μ.
Pre-multiplying (523) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&r\Big(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2\\&\quad-2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^6-3r^4+r^2\right)\cos(\mu)-1\Big)\\&=\alpha_{13}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{531}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{-12r^7+16r^5-6r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{532}$$

$$\Longrightarrow \phi_2=\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{533}$$

which yields two solutions for each value of μ.

1.14.4 $L^-R_\mu^-|R_\mu^+L^+$ Paths

For a $L_{\phi_1}^-R_\mu^-|R_\mu^+L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{534}$$

Pre-multiplying (534) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}&-12r^6+16r^4-8\left(r^2-1\right)^2r^2\cos^2(\mu)+4\left(r^2-1\right)^2r^2-6r^2+8\left(2r^6-3r^4+r^2\right)\cos(\mu)+1\\&=-\left(\alpha_{11}\left(r^2-1\right)\right)-r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{535}$$

which gives

$$\cos(\mu)=\frac{4r^6-6r^4+2r^2\pm\sqrt{2}\sqrt{r^2\left(r^2-1\right)^2\left(\alpha_{33}r^2-\alpha_{13}\sqrt{1-r^2}r+\alpha_{31}\sqrt{1-r^2}r+\alpha_{11}\left(r^2-1\right)+1\right)}}{4r^2\left(r^2-1\right)^2}, \tag{536}$$

and yields four solutions of μ.
Pre-multiplying (534) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}&r\Big(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2\\&\quad-2\left(r^2-1\right)\cos(\phi_1)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^4-3r^2+1\right)r^2\cos(\mu)-1\Big)\\&=\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r.\end{aligned} \tag{537}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=2r\left(r^2-1\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \tag{538}$$

$$B=-2r\left(r^2-1\right)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right), \tag{539}$$

it is obtained that

$$\cos(\phi_1-\gamma)=\frac{-12r^7+16r^5-6r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{540}$$

$$\implies \phi_1=\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{541}$$

which yields two solutions for each value of μ.
Pre-multiplying (534) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&r\Big(12r^6-16r^4+2\left(r^2-1\right)\sin(\mu)\sin(\phi_2)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos(2\mu)+6r^2\\&\quad-2\left(r^2-1\right)\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2-1\right)\cos(2\mu)+3r^2-1\right)+\left(-8r^4+8r^2-1\right)\cos(\mu)\right)-8\left(2r^6-3r^4+r^2\right)\cos(\mu)-1\Big)\\&=\alpha_{33}r-\alpha_{13}\sqrt{1-r^2}.\end{aligned} \tag{542}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$, it is obtained that

$$\cos\left(\phi_2-\gamma\right)=\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}} \tag{543}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{-12r^7+16r^5-6r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^3\cos(2\mu)+8\left(2r^7-3r^5+r^3\right)\cos(\mu)+\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{544}$$

which yields two solutions for each value of μ.

1.15 $C|C_\beta GC_\beta|C$ Paths

1.15.1 $L^+|L^-_\beta G^-L^-_\beta|L^+$ Paths

For a $L^+_{\phi_1}|L^-_\beta G^-_{\phi_2}L^-_\beta|L^+_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{545}$$

Pre-multiplying (545) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}&4\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right)r^2\cos(\beta)+\left(1-2r^2\right)^2r^2\\&-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)\\&=r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),\end{aligned} \tag{546}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4\left(r^2-1\right)r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right) \tag{547}$$

$$B=-\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right), \tag{548}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{549}$$

$$\implies \phi_2=\cos^{-1}\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right) + \tan^{-1}\left(\frac{A}{B}\right), \tag{550}$$

and yields two solutions of ϕ_2.

Pre-multiplying (545) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=-\alpha_{33}r-\alpha_{31}\sqrt{1-r^2}.
\end{aligned}\tag{551}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)
\end{aligned}\tag{552}$$

$$\begin{aligned}
D=&4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right),
\end{aligned}\tag{553}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta)=&\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}
\end{aligned}\tag{554}$$

$$\begin{aligned}
\implies\phi_1=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),
\end{aligned}\tag{555}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (545) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}\tag{556}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}
\end{aligned}\tag{557}$$

$$\begin{aligned}
\implies\phi_3=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),
\end{aligned}\tag{558}$$

which yields two solutions for each value of ϕ_2.

1.15.2 $R^+|R_\beta^- G^- R_\beta^-|R^+$ Paths

For a $R_{\phi_1}^+|R_\beta^- G_{\phi_2}^- R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{559}$$

Pre-multiplying (559) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned} &4\left(r^2-1\right) r\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2 r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right)r^2\cos(\beta)+\left(1-2r^2\right)^2 r^2 \\ &-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right) \\ =&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \end{aligned} \tag{560}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4\left(r^2-1\right) r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right) \tag{561}$$

$$B = -\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right), \tag{562}$$

it is obtained that

$$\cos(\phi_2-\gamma) = -\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{563}$$

$$\begin{aligned} \implies \phi_2 = \cos^{-1}&\left(-\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right) \\ &+\tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{564}$$

and yields two solutions of ϕ_2.
Pre-multiplying (559) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned} &-4r^7+4r^5-r^3-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2) \\ &+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2 r^3\cos^2(\beta) \\ &+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\ &+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \\ &+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \end{aligned} \tag{565}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C =& \left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\ &+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right) \end{aligned} \tag{566}$$

$$\begin{aligned} D =& 4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta) \\ &+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right), \end{aligned} \tag{567}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) =& \frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\ &+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2 r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{568}$$

$$\begin{aligned} \implies \phi_1 = \cos^{-1}&\Bigg(\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\ &+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2 r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{569}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (559) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned}\tag{570}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}
\end{aligned}\tag{571}$$

$$\begin{aligned}
\implies\phi_3=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),
\end{aligned}\tag{572}$$

which yields two solutions for each value of ϕ_2.

1.15.3 $R^-|R_\beta^+G^+R_\beta^+|R^-$ Paths

For a $R_{\phi_1}^-|R_\beta^+G_{\phi_2}^+R_\beta^+|R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{R^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{573}$$

Pre-multiplying (573) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}
&4\left(r^2-1\right)r\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right)r^2\cos(\beta)+\left(1-2r^2\right)^2r^2\\
&-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)\\
=&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),
\end{aligned}\tag{574}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4\left(r^2-1\right)r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right)\tag{575}$$
$$B=-\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right),\tag{576}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\tag{577}$$

$$\begin{aligned}
\implies\phi_2=\cos^{-1}&\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)-\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}\tag{578}$$

and yields two solutions of ϕ_2.

Pre-multiplying (573) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (579)
\end{aligned}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right) \qquad (580)\\
D=&4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right), \qquad (581)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta)=&\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (582)\\
\implies\phi_1=\cos^{-1}&\Bigg(\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (583)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (573) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (584)
\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (585)\\
\implies\phi_3=\cos^{-1}&\Bigg(\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (586)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.15.4 $L^-|L^+_\beta G^+ L^+_\beta|L^-$ Paths

For a $L^-_{\phi_1}|L^+_\beta G^+_{\phi_2} L^+_\beta|L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{L^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{587}$$

Pre-multiplying (587) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned} &4\left(r^2-1\right) r \sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)+4\left(r^2-1\right)^2 r^2\cos^2(\beta)+4\left(1-2r^2\right)\left(r^2-1\right)r^2\cos(\beta)+\left(1-2r^2\right)^2 r^2 \\ &-\left(r^2-1\right)\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right) \\ =&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \end{aligned} \tag{588}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4\left(r^2-1\right) r\sin(\beta)\left(2r^2\cos(\beta)-2r^2+1\right) \tag{589}$$

$$B = -\left(r^2-1\right)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right), \tag{590}$$

it is obtained that

$$\cos(\phi_2-\gamma) = -\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{591}$$

$$\begin{aligned} \implies \phi_2 = \cos^{-1}&\left(-\frac{r\left(r\left(\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2-\alpha_{33}\right)+\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right) \\ &+\tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{592}$$

and yields two solutions of ϕ_2.
Pre-multiplying (587) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned} &-4r^7+4r^5-r^3-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)+4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2) \\ &+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2 r^3\cos^2(\beta) \\ &+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\ &+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2) \\ &+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \end{aligned} \tag{593}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C =& \left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right) \\ &+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right) \end{aligned} \tag{594}$$

$$\begin{aligned} D =& 4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta) \\ &+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right) \\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right), \end{aligned} \tag{595}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) =& \frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}} \\ &+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2 r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{596}$$

$$\begin{aligned} \implies \phi_1 = \cos^{-1}&\left(\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\right. \\ &+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2 r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ &\left.+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{597}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (587) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3-4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)-\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)-1\right)-2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)+2r^2\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5-6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(8r^4-8r^2+1\right)\cos(\beta)-\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(-8r^6+16r^4-9r^2+4\left(2r^6+r^2\right)\cos(\beta)+1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \qquad (598)
\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (599)\\
\Longrightarrow\phi_3=\cos^{-1}&\Bigg(\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)}{\sqrt{C^2+D^2}}\\
&+\frac{-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(1-2r^2\right)\left(r^2-1\right)r^3\cos(\beta)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (600)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.15.5 $L^+|L_\beta^-G^-R_\beta^-|R^+$ Paths

For a $L_{\phi_1}^+|L_\beta^-G_{\phi_2}^-R_\beta^-|R_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{R^+}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \qquad (601)$$

Pre-multiplying (601) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned}
&r^2\left(2r^2-1\right)^2+\cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right)\\
&+\left(4r^4\left(r^2-1\right)-4r^2\left(r^2-1\right)\right)\cos^2(\beta)+\left(4r^2\left(r^2-1\right)-8r^4\left(r^2-1\right)\right)\cos(\beta)\\
&+\sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right)\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \qquad (602)
\end{aligned}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta) \qquad (603)$$
$$B=-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right), \qquad (604)$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \qquad (605)$$

$$\begin{aligned}
\Longrightarrow\phi_2=\cos^{-1}&\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right), \qquad (606)
\end{aligned}$$

and yields two solutions of ϕ_2.

Pre-multiplying (601) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)-4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (607)
\end{aligned}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right) \qquad (608)\\
D=&4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right), \qquad (609)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta)=&\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (610)\\
\implies \phi_1=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (611)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (601) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (612)
\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (613)\\
\implies \phi_3=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (614)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.15.6 $R^+|R_\beta^- G^- L_\beta^-|L^+$ Paths

For a $R_{\phi_1}^+|R_\beta^- G_{\phi_2}^- L_\beta^-|L_{\phi_3}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\beta)\mathbf{M}_{G^-}(\phi_2)\mathbf{M}_{L^-}(r,\beta)\mathbf{M}_{L^+}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{615}$$

Pre-multiplying (615) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned} & r^2\left(2r^2-1\right)^2 + \cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta) + 4\left(r^2-1\right)r^4\cos^2(\beta) + \left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right) \\ & \quad + \left(4r^4\left(r^2-1\right) - 4r^2\left(r^2-1\right)\right)\cos^2(\beta) + \left(4r^2\left(r^2-1\right) - 8r^4\left(r^2-1\right)\right)\cos(\beta) \\ & \quad + \sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta) - 8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right) \\ =& \alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \end{aligned} \tag{616}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta) - 8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta) \tag{617}$$

$$B = -4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta) + 4\left(r^2-1\right)r^4\cos^2(\beta) + \left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right), \tag{618}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{r\left(r\left(\alpha_{33} - \left(2\left(r^2-1\right)\cos(\beta) - 2r^2+1\right)^2\right) + \alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2}\right) + \alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{619}$$

$$\begin{aligned} \implies \phi_2 = & \cos^{-1}\left(\frac{r\left(r\left(\alpha_{33} - \left(2\left(r^2-1\right)\cos(\beta) - 2r^2+1\right)^2\right) + \alpha_{13}\left(-\sqrt{1-r^2}\right) + \alpha_{31}\sqrt{1-r^2}\right) + \alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right) \\ & + \tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{620}$$

and yields two solutions of ϕ_2.
Pre-multiplying (615) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned} & -4r^7 + 4r^5 - r^3 + 4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2) - 4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2) \\ & - \left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta) + 2\left(r^4+r^2\right)\cos(2\beta)+1\right) - 4\left(r^2-1\right)^2 r^3\cos^2(\beta) \\ & + 4\left(2r^7-3r^5+r^3\right)\cos(\beta) + \sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta) + \left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta) - 2r^2\cos(\beta) + 2r^2\cos(2\beta)\right) \\ & + r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2) - 2r^2\right) - \left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big) + \cos(\phi_1)\Big(4r^7 - 6r^5 + 6r^4\sin(2\beta)\sin(\phi_2) \\ & + 2r^3 + 2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta) + \left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta) + 3\left(r^2-1\right)\right) + \left(-8r^4+8r^2-1\right)\cos(\beta)\right) \\ & + \left(-8r^7+16r^5-9r^3+r\right)\cos(\beta) + \sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big) = \alpha_{31}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \end{aligned} \tag{621}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C = & \left(r^5+r\right)\sin(2\beta) + \left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta) - 2r^2\cos(\beta) + 2r^2\cos(2\beta)\right) \\ & + r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2) - 2r^2\right) - \left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right) \end{aligned} \tag{622}$$

$$\begin{aligned} D = & 4r^7 - 6r^5 + 6r^4\sin(2\beta)\sin(\phi_2) + 2r^3 + 2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta) \\ & + \left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta) + 3\left(r^2-1\right)\right) + \left(-8r^4+8r^2-1\right)\cos(\beta)\right) \\ & + \left(-8r^7+16r^5-9r^3+r\right)\cos(\beta) + \sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right), \end{aligned} \tag{623}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) = & \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right) + 4\left(r^2-1\right)^2 r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2) - 8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) - \alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{624}$$

$$\begin{aligned} \implies \phi_1 = \cos^{-1}\Bigg(& \frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2} - 4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}} \\ & + \frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right) + 4\left(r^2-1\right)^2 r^3\cos^2(\beta)}{\sqrt{C^2+D^2}} \\ & + \frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2) - 8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right) - \alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg) + \tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{625}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (615) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned}\tag{626}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}
\end{aligned}\tag{627}$$

$$\begin{aligned}
\Longrightarrow\phi_3=\cos^{-1}&\left(\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\right.\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&\left.+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right),
\end{aligned}\tag{628}$$

which yields two solutions for each value of ϕ_2.

1.15.7 $R^-|R^+_\beta G^+ L^+_\beta|L^-$ Paths

For a $R^-_{\phi_1}|R^+_\beta G^+_{\phi_2} L^+_\beta|L^-_{\phi_3}$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{L^-}(r,\phi_3)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{629}$$

Pre-multiplying (629) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&r^2\left(2r^2-1\right)^2+\cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right)\\
&+\left(4r^4\left(r^2-1\right)-4r^2\left(r^2-1\right)\right)\cos^2(\beta)+\left(4r^2\left(r^2-1\right)-8r^4\left(r^2-1\right)\right)\cos(\beta)\\
&+\sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right)\\
=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),
\end{aligned}\tag{630}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\tag{631}$$
$$B=-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right),\tag{632}$$

it is obtained that

$$\cos(\phi_2-\gamma)=\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\tag{633}$$

$$\begin{aligned}
\Longrightarrow\phi_2=\cos^{-1}&\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\
&+\tan^{-1}\left(\frac{A}{B}\right),
\end{aligned}\tag{634}$$

and yields two solutions of ϕ_2.

Pre-multiplying (629) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)-4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (635)
\end{aligned}$$

For $\phi_2\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right) \qquad (636)\\
D=&4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\
&+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right), \qquad (637)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta)=&\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (638)\\
\implies \phi_1=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (639)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (629) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (640)
\end{aligned}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \qquad (641)\\
\implies \phi_3=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (642)
\end{aligned}$$

which yields two solutions for each value of ϕ_2.

1.15.8 $L^-|L_\beta^+G^+R_\beta^+|R^-$ Paths

For a $L_{\phi_1}^-|L_\beta^+G_{\phi_2}^+R_\beta^+|R_{\phi_3}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\beta)\mathbf{M}_{G^+}(\phi_2)\mathbf{M}_{R^+}(r,\beta)\mathbf{M}_{R^-}(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{643}$$

Pre-multiplying (643) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned} &r^2\left(2r^2-1\right)^2+\cos(\phi_2)\left(-4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right)\right)\\ &+\left(4r^4\left(r^2-1\right)-4r^2\left(r^2-1\right)\right)\cos^2(\beta)+\left(4r^2\left(r^2-1\right)-8r^4\left(r^2-1\right)\right)\cos(\beta)\\ &+\sin(\phi_2)\left(4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta)\right)\\ =&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \end{aligned} \tag{644}$$

Since β is known, this equation can be utilized to calculate ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\gamma := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\gamma := \frac{B}{\sqrt{A^2+B^2}}$, where

$$A = 4r\left(r^2-1\right)\left(2r^2-1\right)\sin(\beta)-8r^3\left(r^2-1\right)\sin(\beta)\cos(\beta) \tag{645}$$

$$B = -4\left(r^2-1\right)\left(2r^2-1\right)r^2\cos(\beta)+4\left(r^2-1\right)r^4\cos^2(\beta)+\left(r^2-1\right)\left(4r^4+2r^2\cos(2\beta)-6r^2+1\right), \tag{646}$$

it is obtained that

$$\cos(\phi_2-\gamma) = \frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}} \tag{647}$$

$$\begin{aligned} \Longrightarrow \phi_2 = \cos^{-1}&\left(\frac{r\left(r\left(\alpha_{33}-\left(2\left(r^2-1\right)\cos(\beta)-2r^2+1\right)^2\right)+\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}\right)+\alpha_{11}\left(r^2-1\right)}{\sqrt{A^2+B^2}}\right)\\ &+\tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{648}$$

and yields two solutions of ϕ_2.
Pre-multiplying (643) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned} &-4r^7+4r^5-r^3+4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)-4\left(2r^4-3r^2+1\right)r^2\sin(\beta)\sin(\phi_2)\\ &-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\ &+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_1)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\ &+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_1)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\ &+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \end{aligned} \tag{649}$$

For $\phi_2 \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\theta := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\theta := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C = &\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\ &+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right) \end{aligned} \tag{650}$$

$$\begin{aligned} D =&4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)\\ &+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\ &+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right), \end{aligned} \tag{651}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) = &\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}} \end{aligned} \tag{652}$$

$$\begin{aligned} \Longrightarrow \phi_1 = \cos^{-1}&\left(\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\right.\\ &+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\ &\left.+\frac{4r^7-4r^5+r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{653}$$

which yields two solutions for each value of ϕ_2.
Pre-multiplying (643) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-4r^7+4r^5-r^3+4\left(r^2-1\right)r^2\sin(\beta)\sin(\phi_2)\left(2r^2\cos(\beta)-2r^2+1\right)\\
&-\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+\left(4r^2-8r^4\right)\cos(\beta)+2\left(r^4+r^2\right)\cos(2\beta)+1\right)-4\left(r^2-1\right)^2r^3\cos^2(\beta)\\
&+4\left(2r^7-3r^5+r^3\right)\cos(\beta)+\sin(\phi_3)\Big(\left(r^5+r\right)\sin(2\beta)+\left(r^2-1\right)\sin(\phi_2)\left(\cos(\beta)-2r^2\cos(\beta)+2r^2\cos(2\beta)\right)\\
&+r\sin(\beta)\left(2\cos(\beta)\left(\left(r^4-1\right)\cos(\phi_2)-2r^2\right)-\left(2r^4-3r^2+1\right)\left(\cos(\phi_2)+1\right)\right)\Big)+\cos(\phi_3)\Big(4r^7-6r^5+6r^4\sin(2\beta)\sin(\phi_2)\\
&+2r^3+2\left(r^2-1\right)^2\left(2r^2-1\right)r\cos^2(\beta)+\left(r^2-1\right)r\cos(\phi_2)\left(\left(2r^2-1\right)\left(\left(r^2+1\right)\cos(2\beta)+3\left(r^2-1\right)\right)+\left(-8r^4+8r^2-1\right)\cos(\beta)\right)\\
&+\left(-8r^7+16r^5-9r^3+r\right)\cos(\beta)+\sin(\beta)\sin(\phi_2)\left(8r^6-16r^4+9r^2-4\left(2r^6+r^2\right)\cos(\beta)-1\right)\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.
\end{aligned}\tag{654}$$

Similarly, multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$, it is obtained that

$$\begin{aligned}
\cos\left(\phi_3-\theta\right)=&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}
\end{aligned}\tag{655}$$

$$\begin{aligned}
\implies\phi_3=\cos^{-1}\Bigg(&\frac{4r^7-4r^5+r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)r^4\sin(2\beta)\sin(\phi_2)}{\sqrt{C^2+D^2}}\\
&+\frac{\left(r^2-1\right)r\cos(\phi_2)\left(6r^4-6r^2+2\left(r^4+r^2\right)\cos(2\beta)+1\right)+4\left(r^2-1\right)^2r^3\cos^2(\beta)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(2r^6-3r^4+r^2\right)\sin(\beta)\sin(\phi_2)-8\left(2r^4-3r^2+1\right)r^3\cos(\beta)\cos^2\left(\frac{\phi_2}{2}\right)-\alpha_{33}r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),
\end{aligned}\tag{656}$$

which yields two solutions for each value of ϕ_2.

1.16 $C|C_\mu C_\mu|C_\mu C$ Paths

1.16.1 $L^+|L_\mu^-R_\mu^-|R_\mu^+L^+$ Paths

For a $L_{\phi_1}^+|L_\mu^-R_\mu^-|R_\mu^+L_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{657}$$

Pre-multiplying (657) with $\mathbf{u}_{L^+}^T$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned}
&16r^8-32r^6+24r^4-8r^2+1+\left(-16r^8+32r^6-16r^4\right)\cos^3(\mu)+\left(48r^8-96r^6+56r^4-8r^2\right)\cos^2(\mu)\\
&+\left(-48r^8+96r^6-64r^4+16r^2\right)\cos(\mu)\\
=&r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),
\end{aligned}\tag{658}$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.
Pre-multiplying (657) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned}
&r\Big(-40r^8+80r^6-52r^4+8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+12r^2\\
&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right)\\
&+4\left(r^2-1\right)^2r^4\cos(3\mu)+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)\\
=&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.
\end{aligned}\tag{659}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}
A=&8\left(r^2-1\right)r^3\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)\\
B=&8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right),
\end{aligned}\tag{660}$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta)=&\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\end{aligned}\tag{661}$$

$$\begin{aligned}\implies\phi_1=\cos^{-1}&\left(\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\right.\\&\left.+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\right)+\tan^{-1}\left(\frac{A}{B}\right),\end{aligned}\tag{662}$$

which yields two solutions for each value of μ.
Pre-multiplying (657) with $\mathbf{u}_{L^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2r^4\cos(3\mu)\\&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right)\\&+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r.\end{aligned}\tag{663}$$

For $\mu\neq0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$C=-8\left(r^2-1\right)^2r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right)\tag{664}$$

$$D=8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right),\tag{665}$$

it is obtained that

$$\begin{aligned}\cos\left(\phi_2-\sigma\right)=&\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\end{aligned}\tag{666}$$

$$\begin{aligned}\implies\phi_2=\cos^{-1}&\left(\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\right.\\&\left.+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right),\end{aligned}\tag{667}$$

which yields two solutions for each value of μ.

1.16.2 $R^+|R_\mu^-L_\mu^-|L_\mu^+R^+$ Paths

For a $R_{\phi_1}^+|R_\mu^-L_\mu^-|L_\mu^+R_{\phi_2}^+$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix}\tag{668}$$

Pre-multiplying (668) with $\mathbf{u}_{R^+}^T$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned}&16r^8-32r^6+24r^4-8r^2+1+\left(-16r^8+32r^6-16r^4\right)\cos^3(\mu)+\left(48r^8-96r^6+56r^4-8r^2\right)\cos^2(\mu)\\&+\left(-48r^8+96r^6-64r^4+16r^2\right)\cos(\mu)\\=&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),\end{aligned}\tag{669}$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.
Pre-multiplying (668) with $\mathbf{u}_{G^-}^T$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned}&r\Big(-40r^8+80r^6-52r^4+8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+12r^2\\&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right)\\&+4\left(r^2-1\right)^2r^4\cos(3\mu)+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)\\=&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r.\end{aligned}\tag{670}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}A =& 8\left(r^2-1\right) r^3 \sin^2\left(\frac{\mu}{2}\right) \sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)\\ B =& 8\left(r^2-1\right) r \sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right),\end{aligned} \tag{671}$$

it is obtained that

$$\begin{aligned}\cos(\phi_1-\theta) =& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}} \end{aligned}\tag{672}$$

$$\begin{aligned}\implies \phi_1 = \cos^{-1}\Bigg(& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right),\end{aligned}\tag{673}$$

which yields two solutions for each value of μ.
Pre-multiplying (668) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}& r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2 r^4\cos(3\mu)\\ &+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right)\\ &+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.\end{aligned}\tag{674}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C = -8\left(r^2-1\right)^2 r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right) \tag{675}$$

$$D = 8\left(r^2-1\right) r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right), \tag{676}$$

it is obtained that

$$\begin{aligned}\cos(\phi_2-\sigma) =& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\end{aligned}\tag{677}$$

$$\begin{aligned}\implies \phi_2 = \cos^{-1}\Bigg(& \frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2 r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right),\end{aligned}\tag{678}$$

which yields two solutions for each value of μ.

1.16.3 $R^-|R_\mu^+L_\mu^+|L_\mu^-R^-$ Paths

For a $R_{\phi_1}^-|R_\mu^+L_\mu^+|L_\mu^-R_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{679}$$

Pre-multiplying (679) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{R^-}$:

$$\begin{aligned}&16r^8-32r^6+24r^4-8r^2+1+\left(-16r^8+32r^6-16r^4\right)\cos^3(\mu)+\left(48r^8-96r^6+56r^4-8r^2\right)\cos^2(\mu)\\ &+\left(-48r^8+96r^6-64r^4+16r^2\right)\cos(\mu)\\ =& r\left(\alpha_{13}\sqrt{1-r^2}+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right),\end{aligned}\tag{680}$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.

Pre-multiplying (679) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$\begin{aligned} &r\Big(-40r^8+80r^6-52r^4+8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+12r^2 \\ &+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right) \\ &+4\left(r^2-1\right)^2r^4\cos(3\mu)+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big) \\ =&\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \end{aligned} \tag{681}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A =&8\left(r^2-1\right)r^3\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right) \\ B =&8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right), \end{aligned} \tag{682}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1-\theta) =& \frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}} \end{aligned} \tag{683}$$

$$\begin{aligned} \implies \phi_1 = \cos^{-1}\Bigg(&\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{684}$$

which yields two solutions for each value of μ.
Pre-multiplying (679) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned} &r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2r^4\cos(3\mu) \\ &+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right) \\ &+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \end{aligned} \tag{685}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C =-8\left(r^2-1\right)^2r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right) \tag{686}$$

$$D =8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right), \tag{687}$$

it is obtained that

$$\begin{aligned} \cos(\phi_2-\sigma) =& \frac{40r^9-80r^7+52r^5-12r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}} \end{aligned} \tag{688}$$

$$\begin{aligned} \implies \phi_2 = \cos^{-1}\Bigg(&\frac{40r^9-80r^7+52r^5-12r^3-\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}} \\ &+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \end{aligned} \tag{689}$$

which yields two solutions for each value of μ.

1.16.4 $L^-|L_\mu^+R_\mu^+|R_\mu^-L^-$ Paths

For a $L_{\phi_1}^-|L_\mu^+R_\mu^+|R_\mu^-L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\phi_2) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \tag{690}$$

Pre-multiplying (690) with $\mathbf{u}_{L^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&16r^8-32r^6+24r^4-8r^2+1+\left(-16r^8+32r^6-16r^4\right)\cos^3(\mu)+\left(48r^8-96r^6+56r^4-8r^2\right)\cos^2(\mu)\\
&+\left(-48r^8+96r^6-64r^4+16r^2\right)\cos(\mu)\\
=&r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right)-\alpha_{11}\left(r^2-1\right), \qquad (691)
\end{aligned}$$

which is a cubic polynomial of $\cos(\mu)$ and yields three solutions of it, hence leading to six solutions of μ.
Pre-multiplying (690) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&r\Big(-40r^8+80r^6-52r^4+8\left(r^2-1\right)r^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_1)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)+12r^2\\
&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_1)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right)\\
&+4\left(r^2-1\right)^2r^4\cos(3\mu)+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)\\
=&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (692)
\end{aligned}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}
A=&8\left(r^2-1\right)r^3\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2\left(r^2-1\right)\cos(\mu)-2r^2+1\right)\\
B=&8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(\left(r^2-1\right)\left(6r^4+\left(2r^2-1\right)r^2\cos(2\mu)-3r^2+1\right)-r^2\left(8r^4-12r^2+5\right)\cos(\mu)\right), \qquad (693)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta)=&\frac{40r^9-80r^7+52r^5-12r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}} \qquad (694)\\
\implies\phi_1=\cos^{-1}&\Bigg(\frac{40r^9-80r^7+52r^5-12r^3+\alpha_{31}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right), \qquad (695)
\end{aligned}$$

which yields two solutions for each value of μ.
Pre-multiplying (690) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&r\Big(-40r^8+80r^6-52r^4-8\left(r^2-1\right)^2\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\sin(\phi_2)\left(2r^2\cos(\mu)-2r^2+1\right)+12r^2+4\left(r^2-1\right)^2r^4\cos(3\mu)\\
&+8\left(r^2-1\right)\sin^2\left(\frac{\mu}{2}\right)\cos(\phi_2)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right)\\
&+4\left(15r^6-30r^4+19r^2-4\right)r^2\cos(\mu)-4\left(6r^6-12r^4+7r^2-1\right)r^2\cos(2\mu)-1\Big)=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (696)
\end{aligned}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&-8\left(r^2-1\right)^2r\sin^2\left(\frac{\mu}{2}\right)\sin(\mu)\left(2r^2\cos(\mu)-2r^2+1\right) \qquad (697)\\
D=&8\left(r^2-1\right)r\sin^2\left(\frac{\mu}{2}\right)\left(r^2\left(6r^4-9r^2+\left(2r^4-3r^2+1\right)\cos(2\mu)+4\right)+\left(-8r^6+12r^4-5r^2+1\right)\cos(\mu)\right), \qquad (698)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos\left(\phi_2-\sigma\right)=&\frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}} \qquad (699)\\
\implies\phi_2=\cos^{-1}&\Bigg(\frac{40r^9-80r^7+52r^5-12r^3+\alpha_{13}\sqrt{1-r^2}-4\left(r^2-1\right)^2r^5\cos(3\mu)-4\left(15r^6-30r^4+19r^2-4\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(6r^6-12r^4+7r^2-1\right)r^3\cos(2\mu)-\alpha_{33}r+r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (700)
\end{aligned}$$

which yields two solutions for each value of μ.

1.17 $CC_\mu|C_\mu C_\mu|C_\mu C$ Paths

1.17.1 $L^+R^+_\mu|R^-_\mu L^-_\mu|L^+_\mu R^+$ Paths

For a $L^+_{\phi_1}R^+_\mu|R^-_\mu L^-_\mu|L^+_\mu R^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^+}(r,\phi_1)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\phi_2) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}. \end{pmatrix} \tag{701}$$

Pre-multiplying (701) with $\mathbf{u}^T_{L^+}$ and post-multiplying $\mathbf{u}_{R^+}$:

$$\begin{aligned} &32r^{10} - 64r^8 + 56r^6 - 32r^4 + 10r^2 + \left(32r^{10} - 96r^8 + 96r^6 - 32r^4\right)\cos^4(\mu) + \left(-128r^{10} + 352r^8 - 320r^6 + 96r^4\right)\cos^3(\mu) \\ &+ \left(192r^{10} - 480r^8 + 408r^6 - 136r^4 + 16r^2\right)\cos^2(\mu) + \left(-128r^{10} + 288r^8 - 240r^6 + 104r^4 - 24r^2\right)\cos(\mu) - 1 \\ =&\alpha_{11}\left(r^2-1\right) + r\left(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right), \end{aligned} \tag{702}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ.
Pre-multiplying (701) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{R^+}$:

$$\begin{aligned} &-140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu) \\ &- 4\left(r^2-1\right)^3 r^5\cos(4\mu) + 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\Big(6r^4 + 2\left(r^2-1\right)r^2\cos(2\mu) - 4r^2 \\ &+ \left(6r^2 - 8r^4\right)\cos(\mu) + 1\Big) + 2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu) + 2r^8\cos(4\mu) + 70r^8 + 36r^6\cos(3\mu) \\ &- 5r^6\cos(4\mu) - 135r^6 - 25r^4\cos(3\mu) + 4r^4\cos(4\mu) + 88r^4 + 5r^2\cos(3\mu) - r^2\cos(4\mu) - 22r^2 \\ &+ \left(-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2\right)\cos(\mu) + \left(56r^8 - 116r^6 + 76r^4 - 17r^2 + 1\right)\cos(2\mu) + 2\Big) \\ &+ 8\left(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3\right)r^3\cos(\mu) - 4\left(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2\right)r^3\cos(2\mu) + r \\ =&\alpha_{31}\sqrt{1-r^2} - \alpha_{33}r. \end{aligned} \tag{703}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A =&16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4 + 2\left(r^2-1\right)r^2\cos(2\mu) - 4r^2 + \left(6r^2 - 8r^4\right)\cos(\mu) + 1\right) \\ B =&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu) + 2r^8\cos(4\mu) + 70r^8 + 36r^6\cos(3\mu) - 5r^6\cos(4\mu) - 135r^6 - 25r^4\cos(3\mu) \\ &+ 4r^4\cos(4\mu) + 88r^4 + 5r^2\cos(3\mu) - r^2\cos(4\mu) - 22r^2 + \left(-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2\right)\cos(\mu) \\ &+ \left(56r^8 - 116r^6 + 76r^4 - 17r^2 + 1\right)\cos(2\mu) + 2\Big), \end{aligned} \tag{704}$$

it is obtained that

$$\begin{aligned} \cos(\phi_1 - \theta) =& \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{31}\sqrt{1-r^2} - 8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}} \\ &+ \frac{4\left(r^2-1\right)^3 r^5\cos(4\mu) - 8\left(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\ &+ \frac{4\left(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2\right)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2+B^2}} \end{aligned} \tag{705}$$

$$\begin{aligned} \Longrightarrow \phi_1 = \cos^{-1}\Bigg(& \frac{140r^{11} - 340r^9 + 296r^7 - 112r^5 + 18r^3 + \alpha_{31}\sqrt{1-r^2} - 8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}} \\ &+ \frac{4\left(r^2-1\right)^3 r^5\cos(4\mu) - 8\left(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}} \\ &+ \frac{4\left(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2\right)r^3\cos(2\mu) - \alpha_{33}r - r}{\sqrt{A^2+B^2}}\Bigg) + \tan^{-1}\left(\frac{A}{B}\right), \end{aligned} \tag{706}$$

which yields two solutions for each value of μ.
Pre-multiplying (701) with $\mathbf{u}^T_{L^+}$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned} &-140r^{11} + 340r^9 - 296r^7 + 112r^5 - 18r^3 + 8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu) - 4\left(r^2-1\right)^3 r^5\cos(4\mu) \\ &+ 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4 + 2\left(r^2-1\right)r^2\cos(2\mu) - 4r^2 + \left(6r^2 - 8r^4\right)\cos(\mu) + 1\Big) \\ &+ 2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu) + 2r^8\cos(4\mu) + 70r^8 + 36r^6\cos(3\mu) - 5r^6\cos(4\mu) - 135r^6 - 25r^4\cos(3\mu) \\ &+ 4r^4\cos(4\mu) + 88r^4 + 5r^2\cos(3\mu) - r^2\cos(4\mu) - 22r^2 + \left(-112r^8 + 220r^6 - 143r^4 + 35r^2 - 2\right)\cos(\mu) \\ &+ \left(56r^8 - 116r^6 + 76r^4 - 17r^2 + 1\right)\cos(2\mu) + 2\Big) + 8\left(28r^8 - 69r^6 + 60r^4 - 22r^2 + 3\right)r^3\cos(\mu) \\ &- 4\left(28r^8 - 72r^6 + 63r^4 - 21r^2 + 2\right)r^3\cos(2\mu) + r = \alpha_{13}\left(-\sqrt{1-r^2}\right) - \alpha_{33}r. \end{aligned} \tag{707}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned} C =& 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right) \quad (708)\\ D =& 2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\ &+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\ &+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \quad (709)\end{aligned}$$

it is obtained that

$$\begin{aligned} \cos\left(\phi_2-\sigma\right) =& \frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} \quad (710)\\ \Longrightarrow \phi_2 =& \cos^{-1}\Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(r^2-1\right)^3 r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\ &+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \quad (711)\end{aligned}$$

which yields two solutions for each value of μ.

1.17.2 $R^+L^+_\mu|L^-_\mu R^-_\mu|R^+_\mu L^+$ Paths

For a $R^+_{\phi_1}L^+_\mu|L^-_\mu R^-_\mu|R^+_\mu L^+_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{R^+}(r,\phi_1)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\phi_2)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}.\end{pmatrix} \quad (712)$$

Pre-multiplying (712) with $\mathbf{u}^T_{R^+}$ and post-multiplying $\mathbf{u}_{L^+}$:

$$\begin{aligned} &32r^{10}-64r^8+56r^6-32r^4+10r^2+\left(32r^{10}-96r^8+96r^6-32r^4\right)\cos^4(\mu)+\left(-128r^{10}+352r^8-320r^6+96r^4\right)\cos^3(\mu)\\ &+\left(192r^{10}-480r^8+408r^6-136r^4+16r^2\right)\cos^2(\mu)+\left(-128r^{10}+288r^8-240r^6+104r^4-24r^2\right)\cos(\mu)-1\\ =&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \quad (713)\end{aligned}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ. Pre-multiplying (712) with $\mathbf{u}^T_{G^-}$ and post-multiplying with $\mathbf{u}_{L^+}$:

$$\begin{aligned} &-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3 r^5\cos(4\mu)\\ &+16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right)\\ &+2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\ &+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\ &+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\ &-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \quad (714)\end{aligned}$$

For $\mu \neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta := \frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta := \frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned} A =& 16\left(r^2-1\right)^2 r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right)\\ B =& 2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\ &+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\ &+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \quad (715)\end{aligned}$$

it is obtained that

$$\cos(\phi_1-\theta)=\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}$$
$$+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}} \tag{716}$$
$$\implies\phi_1=\cos^{-1}\left(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\right.$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}$$
$$\left.+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}\right)+\tan^{-1}\left(\frac{A}{B}\right), \tag{717}$$

which yields two solutions for each value of μ.
Pre-multiplying (712) with $\mathbf{u}_{R^+}^T$ and post-multiplying with $\mathbf{u}_{G^-}$:

$$\begin{aligned}&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)\\&+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\&+2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r.\end{aligned} \tag{718}$$

For $\mu\neq0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$C=16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big) \tag{719}$$
$$\begin{aligned}D=&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big),\end{aligned} \tag{720}$$

it is obtained that

$$\cos\left(\phi_2-\sigma\right)=\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}$$
$$+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} \tag{721}$$
$$\implies\phi_2=\cos^{-1}\left(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\right.$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}$$
$$\left.+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\right)+\tan^{-1}\left(\frac{C}{D}\right), \tag{722}$$

which yields two solutions for each value of μ.

1.17.3 $R^-L_\mu^-|L_\mu^+R_\mu^+|R_\mu^-L^-$ Paths

For a $R_{\phi_1}^-L_\mu^-|L_\mu^+R_\mu^+|R_\mu^-L_{\phi_2}^-$ path, the equation to be solved is:

$$\mathbf{M}_{R^-}(r,\phi_1)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{L^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{723}$$

Pre-multiplying (723) with $\mathbf{u}_{R^-}^T$ and post-multiplying $\mathbf{u}_{L^-}$:

$$\begin{aligned}&32r^{10}-64r^8+56r^6-32r^4+10r^2+\left(32r^{10}-96r^8+96r^6-32r^4\right)\cos^4(\mu)+\left(-128r^{10}+352r^8-320r^6+96r^4\right)\cos^3(\mu)\\&+\left(192r^{10}-480r^8+408r^6-136r^4+16r^2\right)\cos^2(\mu)+\left(-128r^{10}+288r^8-240r^6+104r^4-24r^2\right)\cos(\mu)-1\\=&\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right),\end{aligned} \tag{724}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ.
Pre-multiplying (723) with $\mathbf{u}_{G^+}^T$ and post-multiplying with $\mathbf{u}_{L^-}$:

$$\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)\\
&-4\left(r^2-1\right)^3r^5\cos(4\mu)+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2\\
&+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)+2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)\\
&-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2\\
&+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)\\
&+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r\\
=&\alpha_{31}\sqrt{1-r^2}-\alpha_{33}r. \qquad (725)
\end{aligned}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$\begin{aligned}
A=&16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right)\\
B=&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \qquad (726)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos(\phi_1-\theta)=&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}} \qquad (727)\\
\implies\phi_1=\cos^{-1}\Bigg(&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right), \qquad (728)
\end{aligned}$$

which yields two solutions for each value of μ.
Pre-multiplying (723) with $\mathbf{u}_{R^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)\\
&+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
&+2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\
&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \qquad (729)
\end{aligned}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma:=\frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma:=\frac{D}{\sqrt{C^2+D^2}}$, where

$$\begin{aligned}
C=&16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right) \qquad (730)\\
D=&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \qquad (731)
\end{aligned}$$

it is obtained that

$$\cos\left(\phi_2-\sigma\right)=\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}$$
$$+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} \tag{732}$$

$$\implies\phi_2=\cos^{-1}\Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}$$
$$+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \tag{733}$$

which yields two solutions for each value of μ.

1.17.4 $L^-R^-_\mu|R^+_\mu L^+_\mu|L^-_\mu R^-$ Paths

For a $L^-_{\phi_1}R^-_\mu|R^+_\mu L^+_\mu|L^-_\mu R^-_{\phi_2}$ path, the equation to be solved is:

$$\mathbf{M}_{L^-}(r,\phi_1)\mathbf{M}_{R^-}(r,\mu)\mathbf{M}_{R^+}(r,\mu)\mathbf{M}_{L^+}(r,\mu)\mathbf{M}_{L^-}(r,\mu)\mathbf{M}_{R^-}(r,\phi_2)=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}.\end{pmatrix} \tag{734}$$

Pre-multiplying (734) with $\mathbf{u}^T_{L^-}$ and post-multiplying $\mathbf{u}_{R^-}$:

$$32r^{10}-64r^8+56r^6-32r^4+10r^2+\left(32r^{10}-96r^8+96r^6-32r^4\right)\cos^4(\mu)+\left(-128r^{10}+352r^8-320r^6+96r^4\right)\cos^3(\mu)$$
$$+\left(192r^{10}-480r^8+408r^6-136r^4+16r^2\right)\cos^2(\mu)+\left(-128r^{10}+288r^8-240r^6+104r^4-24r^2\right)\cos(\mu)-1$$
$$=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)+\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{735}$$

which is a quartic polynomial of $\cos(\mu)$ and yields four solutions of it, hence leading to eight solutions of μ. Pre-multiplying (734) with $\mathbf{u}^T_{G^+}$ and post-multiplying with $\mathbf{u}_{R^-}$:

$$-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)$$
$$+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_1)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)$$
$$+2\left(r^2-1\right)r\cos(\phi_1)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)$$
$$+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)$$
$$+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)$$
$$-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{31}\left(-\sqrt{1-r^2}\right)-\alpha_{33}r. \tag{736}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_1. Multiplying both sides with $\frac{1}{\sqrt{A^2+B^2}}$ and defining $\sin\theta:=\frac{A}{\sqrt{A^2+B^2}}$, $\cos\theta:=\frac{B}{\sqrt{A^2+B^2}}$, where

$$A=16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\left(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\right)$$
$$B=2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)$$
$$+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)$$
$$+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \tag{737}$$

it is obtained that

$$\cos(\phi_1-\theta)=\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}$$
$$+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}} \tag{738}$$

$$\implies\phi_1=\cos^{-1}\Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3-\alpha_{31}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{A^2+B^2}}$$
$$+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{A^2+B^2}}$$
$$+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{A^2+B^2}}\Bigg)+\tan^{-1}\left(\frac{A}{B}\right), \tag{739}$$

which yields two solutions for each value of μ.
Pre-multiplying (734) with $\mathbf{u}_{L^-}^T$ and post-multiplying with $\mathbf{u}_{G^+}$:

$$\begin{aligned}
&-140r^{11}+340r^9-296r^7+112r^5-18r^3+8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)-4\left(r^2-1\right)^3r^5\cos(4\mu)\\
&+16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\sin(\phi_2)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big)\\
&+2\left(r^2-1\right)r\cos(\phi_2)\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big)+8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)\\
&-4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)+r=\alpha_{13}\sqrt{1-r^2}-\alpha_{33}r. \qquad (740)
\end{aligned}$$

For $\mu\neq 0$, this equation can be used to solve for ϕ_2. Multiplying both sides with $\frac{1}{\sqrt{C^2+D^2}}$ and defining $\sin\sigma := \frac{C}{\sqrt{C^2+D^2}}$, $\cos\sigma := \frac{D}{\sqrt{C^2+D^2}}$, where

$$C=16\left(r^2-1\right)^2r\sin^3\left(\frac{\mu}{2}\right)\cos\left(\frac{\mu}{2}\right)\Big(6r^4+2\left(r^2-1\right)r^2\cos(2\mu)-4r^2+\left(6r^2-8r^4\right)\cos(\mu)+1\Big) \qquad (741)$$

$$\begin{aligned}
D=&2\left(r^2-1\right)r\Big(-16r^8\cos(3\mu)+2r^8\cos(4\mu)+70r^8+36r^6\cos(3\mu)-5r^6\cos(4\mu)-135r^6-25r^4\cos(3\mu)\\
&+4r^4\cos(4\mu)+88r^4+5r^2\cos(3\mu)-r^2\cos(4\mu)-22r^2+\left(-112r^8+220r^6-143r^4+35r^2-2\right)\cos(\mu)\\
&+\left(56r^8-116r^6+76r^4-17r^2+1\right)\cos(2\mu)+2\Big), \qquad (742)
\end{aligned}$$

it is obtained that

$$\begin{aligned}
\cos\left(\phi_2-\sigma\right)=&\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}} \qquad (743)\\
\implies\phi_2=\cos^{-1}&\Bigg(\frac{140r^{11}-340r^9+296r^7-112r^5+18r^3+\alpha_{13}\sqrt{1-r^2}-8\left(r^2-1\right)^2\left(4r^2-3\right)r^5\cos(3\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(r^2-1\right)^3r^5\cos(4\mu)-8\left(28r^8-69r^6+60r^4-22r^2+3\right)r^3\cos(\mu)}{\sqrt{C^2+D^2}}\\
&+\frac{4\left(28r^8-72r^6+63r^4-21r^2+2\right)r^3\cos(2\mu)-\alpha_{33}r-r}{\sqrt{C^2+D^2}}\Bigg)+\tan^{-1}\left(\frac{C}{D}\right), \qquad (744)
\end{aligned}$$

which yields two solutions for each value of μ.

References

[1] S. Li, D. P. Kumar, S. Darbha, and Y. Zhou, “Time-optimal convexified reeds-shepp paths on a sphere,” *arXiv preprint arXiv:2504.00966*, 2025.