
Quantum-TreeSoft Set and Quantum-ForestSoft Set

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Abstract

Fuzzy Sets [1, 2], Neutrosophic Sets [3], and Plithogenic Sets [4] address uncertainty and have diverse applications. The Soft Set framework associates each parameter with subsets of a universe, allowing flexible approximations [5]. Its extensions include Fuzzy Soft Sets [6], Neutrosophic Soft Sets [7], HyperSoft and SuperHyperSoft Sets [8, 9], TreeSoft Sets [10], ForestSoft Sets [11], and GraphicSoft Sets [12]. A TreeSoft Set organizes parameters hierarchically, while a ForestSoft Set unites multiple TreeSoft Set mappings. A Quantum-Soft Set assigns each attribute to a normalized quantum superposition [13].

In this paper, we introduce Quantum TreeSoft Sets and Quantum ForestSoft Sets, extending the Quantum-Soft Set through hierarchical and forest structures. These constructions aim to advance research at the intersection of Quantum Theory and Soft Set Theory.

Keywords: Soft Set, Treesoft set, Quantum Treesoft Set, Quantum Soft Set, Forestsoft set, Quantum Forestsoft Set

1 Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, all sets and structures are assumed to be finite. Unless otherwise stated, the symbol n denotes a non-negative integer.

1.1 Soft Set and TreeSoft Set

A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U. This provides a flexible framework for approximating objects within U [5, 14, 15]. A *TreeSoft Set* is a mapping from subsets of a hierarchical, tree-like parameter structure $\text{Tree}(A)$ to subsets of a universal set U. This structure supports multi-level attributes for more refined and detailed analyses [16–18]. Related concepts include the *Hypersoft Set* [8, 19–22] and the *SuperHypersoft Set* [9, 12, 23, 24]. The definitions of Soft Set and TreeSoft Set are provided below.

Definition 1.1 (Soft Set)**.** [15] Let U be a universal set and E a set of parameters. A *soft set* over U is defined as an ordered pair (F, E), where F is a mapping from E to the power set $\mathcal{P}(U)$:

$$F : E \to \mathcal{P}(U).$$

For each parameter $e \in E$, $F(e) \subseteq U$ represents the set of e-approximate elements in U, with (F, E) forming a parameterized family of subsets of U.

Definition 1.2 (Treesoft Set)**.** [25] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \dots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \dots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \dots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \dots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\dots,i_m}$, where each i_k (for $k = 1, \dots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as $\mathrm{Tree}(A)$, with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\mathrm{Tree}(A)) \to P(H),$$

where $\mathrm{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\mathrm{Tree}(A))$ denotes its power set.

Example 1.3 (Thermoelectric Materials Classification)**.** We model a hierarchical classification of thermoelectric materials using a *TreeSoft Set* $F\colon P(\mathrm{Tree}(A)) \to P(H)$.

- **Universe** U **and Subset** H**:**

  $$U = \{\mathrm{Mat}_1, \dots, \mathrm{Mat}_N\}, \quad H = \{\mathrm{Mat}_i \in U \mid \mathrm{Mat}_i \text{ has both electrical and thermal data}\}.$$

- **Attribute Tree** $\mathrm{Tree}(A)$**:** First-level attributes:

  $$A = \{\, A_1,\, A_2\}, \quad A_1 = \{\text{Resistivity, CarrierConc}\}, \quad A_2 = \{\text{ThermalCond, Seebeck}\}.$$

  Each second-level node is refined into three performance categories:

  $$\begin{aligned} \text{Resistivity} &: \ \{\text{Low, Med, High}\}, \\ \text{CarrierConc} &: \ \{\text{Low, Med, High}\}, \\ \text{ThermalCond} &: \ \{\text{Low, Med, High}\}, \\ \text{Seebeck} &: \ \{\text{Low, Med, High}\}. \end{aligned}$$

  Hence $\mathrm{Tree}(A)$ has depth 3, containing 2 first-level, 4 second-level, and 12 third-level nodes.

- **TreeSoft Mapping** F**:** For any $X \subseteq \mathrm{Tree}(A)$, define

  $$F(X) = \{\mathrm{Mat}_i \in H \mid \text{for each node in } X,\ \mathrm{Mat}_i \text{ satisfies that attribute}\}.$$

  Concretely:

  - If $X = \{\text{Resistivity Low}\}$, then

    $$F(\{\text{Resistivity Low}\}) = \{\mathrm{Mat}_i \in H \mid \rho(\mathrm{Mat}_i) \text{ is low}\}.$$

  - If $X = \{\text{Resistivity Low, Seebeck High}\}$, then

    $$F\big(\{\text{Resistivity Low, Seebeck High}\}\big) = \{\mathrm{Mat}_i \in H \mid \rho(\mathrm{Mat}_i) \text{ is low and } S(\mathrm{Mat}_i) \text{ is high}\}.$$

  - If $X = \{A_1,\, A_2\}$, then

    $$F(\{A_1, A_2\}) = \{\mathrm{Mat}_i \in H \mid \mathrm{Mat}_i \text{ has any electrical data and any thermal data}\}.$$

  - In general,

    $$F\Big(\bigcup_k X_k\Big) = \bigcap_k F(X_k),$$

    since a material must satisfy all chosen nodes.

1.2 ForestSoft Set

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets (cf. [11,26–29]).

Definition 1.4 (ForestSoft Set)**.** [12] Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\big\{\, F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H) \big\}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree $\mathrm{Tree}(A^{(t)})$ of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\mathrm{Forest}\big(\{A^{(t)}\}_{t \in T}\big) \;=\; \bigsqcup_{t \in T} \mathrm{Tree}\big(A^{(t)}\big).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P\big(\mathrm{Forest}(\{A^{(t)}\})\big)$, we set

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t \in T \\ X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

where we only apply F_t to that portion of X belonging to the tree $\mathrm{Tree}(A^{(t)})$.

Example 1.5 (Multifunctional Materials Classification)**.** We classify materials by *thermoelectric* and *mechanical* attributes using a *ForestSoft Set*.

- **Universe U and Subset H:**

  $$U = \{\mathrm{Mat}_1, \ldots, \mathrm{Mat}_N\}, \quad H = \{\mathrm{Mat}_i \in U \mid \text{both thermoelectric and mechanical data exist}\}.$$

- **Two Attribute Trees:**

  (1) $\mathrm{Tree}(A^{(1)})$ (thermoelectric):

  $$A^{(1)} = \{\, E,\ S \,\}, \quad E = \{\text{Resistivity, CarrierConc}\}, \quad S = \{\text{ThermalCond, Seebeck}\}.$$

  (2) $\mathrm{Tree}(A^{(2)})$ (mechanical):

  $$A^{(2)} = \{\, M,\ T \,\}, \quad M = \{\text{YoungMod, Poisson}\}, \quad T = \{\text{Hardness, Toughness}\}.$$

- **Forest $\mathcal{F}$:**

  $$\mathcal{F} = \mathrm{Tree}(A^{(1)}) \;\sqcup\; \mathrm{Tree}(A^{(2)}).$$

  Any $X \subseteq \mathcal{F}$ splits as $X = X_1 \cup X_2$, with $X_1 \subseteq \mathrm{Tree}(A^{(1)})$, $X_2 \subseteq \mathrm{Tree}(A^{(2)})$.

- **TreeSoft Maps F_1, F_2:**

  $$F_1 : P\big(\mathrm{Tree}(A^{(1)})\big) \to P(H), \quad F_2 : P\big(\mathrm{Tree}(A^{(2)})\big) \to P(H).$$

  For example:

  $$F_1(\{\text{Resistivity Low, Seebeck High}\}) = \{\mathrm{Mat}_i \in H \mid \rho(\mathrm{Mat}_i) \text{ is low, } S(\mathrm{Mat}_i) \text{ is high}\},$$

  $$F_2(\{\text{YoungMod Medium, Hardness High}\}) = \{\mathrm{Mat}_i \in H \mid E(\mathrm{Mat}_i) \text{ is medium, } H_V(\mathrm{Mat}_i) \text{ is high}\}.$$

- **ForestSoft Map F:**

  $$\mathbf{F} : P(\mathcal{F}) \;\to\; P(H), \quad \mathbf{F}(X) = F_1(X_1) \;\cup\; F_2(X_2).$$

  Thus, if

  $$X = \{\text{Resistivity Low, Seebeck Med}\} \;\cup\; \{\text{YoungMod High, Toughness Med}\},$$

  then

  $$\mathbf{F}(X) = F_1(\{\text{Resistivity Low, Seebeck Med}\}) \;\cup\; F_2(\{\text{YoungMod High, Toughness Med}\}).$$

1.3 Quantum-Soft Set

A Hilbert space is a complete inner product vector space over real or complex numbers, enabling geometric analysis of functions [30–32]. Hilbert spaces are used in various contexts such as quantum theory. A Quantum-Soft Set maps each classical attribute to a normalized quantum superposition over elements [13]. The definition is stated as follows.

Definition 1.6 (Real Hilbert Space)**.** (cf. [30–32]) A *real Hilbert space* is a pair $(H, \langle\cdot,\cdot\rangle)$ satisfying the following conditions:

1. H is a real vector space.
2. $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ is a function (called an *inner product*) fulfilling:

$$\begin{array}{llll} \text{(Positive-definiteness)} & \langle x, x\rangle \; > \; 0 \quad \text{for all } x \in H \setminus \{0\}, & & \langle x, x\rangle = 0 \iff x = 0, \\ \text{(Symmetry)} & \langle x, y\rangle \; = \; \langle y, x\rangle & & \text{for all } x, y \in H, \\ \text{(Linearity in the first argument)} & \langle \alpha\, x + \beta\, y,\; z\rangle \; = \; \alpha\, \langle x, z\rangle \; + \; \beta\, \langle y, z\rangle & & \text{for all } \alpha, \beta \in \mathbb{R},\; x, y, z \in H. \end{array}$$

3. The norm $\|\cdot\|$ induced by the inner product, defined by

$$\|x\| \; = \; \sqrt{\langle x, x\rangle} \quad \text{for } x \in H,$$

makes $(H, \|\cdot\|)$ into a complete metric space; that is, every Cauchy sequence in $(H, \|\cdot\|)$ converges to some limit in H.

Definition 1.7 (Complex Hilbert Space)**.** (cf. [33]) A *complex Hilbert space* is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ satisfying the following conditions:

1. $\mathcal{H}$ is a vector space over $\mathbb{C}$.
2. $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a function (called a *Hermitian inner product*) fulfilling:

$$\begin{array}{lll} \text{(Positive-definiteness)} & \langle x, x\rangle \; > \; 0 \quad \text{for all } x \in \mathcal{H} \setminus \{0\}, & \langle x, x\rangle = 0 \iff x = 0, \\ \text{(Conjugate symmetry)} & \langle x, y\rangle \; = \; \overline{\langle y, x\rangle} & \text{for all } x, y \in \mathcal{H}, \\ \text{(Sesquilinearity)} & \langle \alpha\, x + \beta\, y,\; z\rangle \; = \; \alpha\, \langle x, z\rangle \; + \; \beta\, \langle y, z\rangle & \text{for all } \alpha, \beta \in \mathbb{C},\; x, y, z \in \mathcal{H}, \\ & \langle x,\; \alpha\, y + \beta\, z\rangle \; = \; \overline{\alpha}\, \langle x, y\rangle \; + \; \overline{\beta}\, \langle x, z\rangle & \text{for all } \alpha, \beta \in \mathbb{C},\; x, y, z \in \mathcal{H}. \end{array}$$

3. The norm $\|\cdot\|$ induced by the inner product, defined by

$$\|x\| \; = \; \sqrt{\langle x, x\rangle} \quad \text{for } x \in \mathcal{H},$$

makes $(\mathcal{H}, \|\cdot\|)$ into a complete metric space; that is, every Cauchy sequence in $(\mathcal{H}, \|\cdot\|)$ converges to some limit in $\mathcal{H}$.

Definition 1.8 (Quantum-Soft Set)**.** [13] Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universe of discourse, and let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of parameters (attributes). Denote by $\mathcal{H}(U)$ the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A *Quantum-Soft Set* over (U, A) is a mapping

$$F : A \longrightarrow \mathcal{H}(U), \quad a_j \;\mapsto\; |\psi_j\rangle = \sum_{i=1}^{n} \alpha_{ij}\; |u_i\rangle\,,$$

subject to the normalization condition

$$\sum_{i=1}^{n} \left|\alpha_{ij}\right|^2 \; = \; 1, \quad j = 1, 2, \ldots, m.$$

Here, $\alpha_{ij} \in \mathbb{C}$ is the amplitude of element u_i under attribute a_j. Measuring the quantum state $|\psi_j\rangle$ in the basis $\{|u_i\rangle\}$ yields $|u_i\rangle$ with probability

$$P(u_i \mid a_j) \; = \; \left|\alpha_{ij}\right|^2.$$

Example 1.9 (Photon Polarization under Different Filter Angles)**.** In this example, we illustrate a *Quantum-Soft Set* that models how a single-photon's polarization state depends on the orientation of a polarizing filter.

- **Universe of Discourse** U**:**
$$U = \{|\mathsf{H}\rangle, |\mathsf{V}\rangle\},$$
where $|\mathsf{H}\rangle$ denotes the horizontal-polarization basis state and $|\mathsf{V}\rangle$ denotes the vertical-polarization basis state. We assume these form an orthonormal basis of a two-dimensional complex Hilbert space $\mathcal{H}(U)$.

- **Set of Attributes** A**:** Let
$$A = \{a_1, a_2, a_3\},$$
where each attribute a_j corresponds to a polarizer oriented at a specific angle θ_j. Concretely, we take:
$$a_1 : \theta_1 = 0°, \quad a_2 : \theta_2 = 45°, \quad a_3 : \theta_3 = 30°.$$

- **Mapping to Quantum States:** We define the Quantum-Soft Set
$$F : A \longrightarrow \mathcal{H}(U), \quad a_j \mapsto |\psi_j\rangle,$$
by assigning to each polarizer orientation θ_j the corresponding photon polarization state $|\psi_j\rangle$. In the $\{|\mathsf{H}\rangle, |\mathsf{V}\rangle\}$ basis:
$$|\psi_j\rangle = \cos(\theta_j)\, |\mathsf{H}\rangle + \sin(\theta_j)\, |\mathsf{V}\rangle.$$
Thus:
$$|\psi_1\rangle = \cos(0°)\, |\mathsf{H}\rangle + \sin(0°)\, |\mathsf{V}\rangle = 1\, |\mathsf{H}\rangle + 0\, |\mathsf{V}\rangle,$$
$$|\psi_2\rangle = \cos(45°)\, |\mathsf{H}\rangle + \sin(45°)\, |\mathsf{V}\rangle = \tfrac{1}{\sqrt{2}}\, |\mathsf{H}\rangle + \tfrac{1}{\sqrt{2}}\, |\mathsf{V}\rangle,$$
$$|\psi_3\rangle = \cos(30°)\, |\mathsf{H}\rangle + \sin(30°)\, |\mathsf{V}\rangle = \tfrac{\sqrt{3}}{2}\, |\mathsf{H}\rangle + \tfrac{1}{2}\, |\mathsf{V}\rangle.$$
One checks that each $|\psi_j\rangle$ is normalized:
$$\langle \psi_j \mid \psi_j \rangle = \cos^2(\theta_j) + \sin^2(\theta_j) = 1, \quad j = 1, 2, 3.$$

- **Interpretation of Measurements:** Measuring $|\psi_j\rangle$ in the $\{|\mathsf{H}\rangle, |\mathsf{V}\rangle\}$ basis yields the outcome $|\mathsf{H}\rangle$ with probability $\cos^2(\theta_j)$ and $|\mathsf{V}\rangle$ with probability $\sin^2(\theta_j)$. For example:
$$P(\mathsf{H} \mid a_2) = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}, \quad P(\mathsf{V} \mid a_2) = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}.$$

- **Summary:** The mapping
$$F(a_j) = |\psi_j\rangle = \cos(\theta_j)\, |\mathsf{H}\rangle + \sin(\theta_j)\, |\mathsf{V}\rangle \quad (j = 1, 2, 3)$$
constitutes a *Quantum-Soft Set* over (U, A). Physically, F encodes how a single-photon's polarization state changes as one rotates a linear polarizer through angles 0°, 45°, and 30°. Upon measurement, each $|\psi_j\rangle$ collapses to $|\mathsf{H}\rangle$ or $|\mathsf{V}\rangle$ with the appropriate probabilities $\cos^2(\theta_j)$ and $\sin^2(\theta_j)$.

2 Result: Quantum-Treesoft Set

We now introduce the notion of a *Quantum-TreeSoft Set*, which simultaneously generalizes the concepts of a Quantum-Soft Set and a TreeSoft Set.

Definition 2.1 (Quantum-TreeSoft Set)**.** Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universe, and let
$$A = \{A_1, A_2, \ldots, A_m\}$$
be a finite set of first-level attributes. As in Definition 1.2, let $\mathrm{Tree}(A)$ denote the finite rooted tree of attributes obtained by iterating sub-attribute expansions
$$A_i = \{A_{i,1}, A_{i,2}, \ldots\}, \quad A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\}, \quad \ldots$$

up to depth m. Denote by $P\big(\mathrm{Tree}(A)\big)$ the power set of all nodes (including leaves) in the attribute-tree.

Let $\mathcal{H}(U)$ be the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A *Quantum-TreeSoft Set* over $(U,\ \mathrm{Tree}(A))$ is a mapping

$$F\ :\ P\big(\mathrm{Tree}(A)\big)\ \longrightarrow\ \mathcal{H}(U),$$

subject to the following normalization condition:

$$\text{For each } X\ \in\ P\big(\mathrm{Tree}(A)\big),\quad F(X)\ =\ |\psi_X\rangle\ =\ \sum_{i=1}^{n}\alpha_i(X)\ |u_i\rangle\,,\quad \sum_{i=1}^{n}\big|\alpha_i(X)\big|^2\ =\ 1.$$

Here, $\alpha_i(X)\in\mathbb{C}$ is the amplitude associated with element u_i under the tree-node subset X. Measuring the state $|\psi_X\rangle$ in the basis $\{|u_i\rangle\}$ yields outcome $|u_i\rangle$ with probability

$$P\big(u_i\ \big|\ X\big)=\big|\alpha_i(X)\big|^2.$$

Remark 2.2. (a) If $\mathrm{Tree}(A)$ has depth 1, so that $\mathrm{Tree}(A)\cong A$ (i.e., no sub-attributes beyond level 1), then

$$P\big(\mathrm{Tree}(A)\big)\ =\ P(A),$$

and a Quantum-TreeSoft Set F restricts to a mapping $\widetilde{F}\,:\,P(A)\to\mathcal{H}(U)$. In particular, if one only evaluates F on singleton sets $\{\,A_j\}\subseteq A$, one recovers a Quantum-Soft Set as in Definition 1.8 by setting $|\psi_{A_j}\rangle=F(\{\,A_j\})$.

(b) Conversely, if for every $X\subseteq\mathrm{Tree}(A)$ the amplitude vector $(\alpha_1(X),\ldots,\alpha_n(X))$ has support exactly equal to a classical subset $H(X)\subseteq U$ (i.e., $\alpha_i(X)\neq 0$ if and only if $u_i\in H(X)$, and we choose $\alpha_i(X)=1/\sqrt{|H(X)|}$ for each $u_i\in H(X)$), then measuring $|\psi_X\rangle$ yields uniformly distributed membership in $H(X)$. In that case, one can identify $\widetilde{H}\,:\,P\big(\mathrm{Tree}(A)\big)\to P(U)$ via

$$\widetilde{H}(X)\ =\ \Big\{u_i\in U\ \Big|\ \alpha_i(X)\neq 0\Big\}.$$

This $\widetilde{H}$ is exactly a TreeSoft Set as in Definition 1.2, and thus a Quantum-TreeSoft Set whose amplitudes are restricted to such uniform-support states recovers a classical TreeSoft Set.

Example 2.3 (Polarization and Spatial-Mode Superpositions of a Single Photon)**.** In this example, we model a single photon that can occupy one of two orthogonal polarization states and one of two orthogonal spatial-mode states. We organize these physical properties into a hierarchical (tree-like) set of attributes and then define a *Quantum-TreeSoft Set* that assigns to each subset of those attributes a normalized superposition state in the photon's four-dimensional Hilbert space.

- **Universe of Discourse U.** Let

$$U\ =\ \big\{\,|\mathsf{H},0\rangle\,,\ |\mathsf{H},1\rangle\,,\ |\mathsf{V},0\rangle\,,\ |\mathsf{V},1\rangle\big\},$$

  where each basis vector $|\mathsf{P},m\rangle$ denotes a Fock-state for a single photon with polarization $\mathsf{P}\in\{\mathsf{H},\mathsf{V}\}$ (horizontal or vertical) and spatial mode index $m\in\{0,1\}$ (e.g., two orthogonal transverse modes). These four vectors form an orthonormal basis of the complex Hilbert space $\mathcal{H}(U)$ of dimension $n=4$.

- **Attribute Tree** $\mathrm{Tree}(A)$**.** We introduce a two-level hierarchy of attributes:

$$A=\{\,A_1,\ A_2\},$$

  where

$$A_1\ =\ \{\ A_{1,1},\ A_{1,2}\ \},\quad A_2\ =\ \{\ A_{2,1},\ A_{2,2}\ \}.$$

  Concretely:

  - A_1 (first-level) = "Polarization." Its children (second-level) are

$$A_{1,1}=\text{“Horizontal (H)”},\quad A_{1,2}=\text{“Vertical (V)”}.$$

- A_2 (first-level) = "Spatial Mode." Its children (second-level) are

$$A_{2,1} = \text{“Mode 0”}, \quad A_{2,2} = \text{“Mode 1”}.$$

Thus

$$\text{Tree}(A) \;=\; \{A_1,\, A_{1,1},\, A_{1,2},\, A_2,\, A_{2,1},\, A_{2,2}\}.$$

Its power set $P(\text{Tree}(A))$ consists of all subsets of these six attribute-nodes (including the empty set).

- **Hilbert Space $\mathcal{H}(U)$.** We denote by $\{|u_i\rangle\}_{i=1}^4$ the orthonormal basis

$$|u_1\rangle = |\mathsf{H}, 0\rangle, \quad |u_2\rangle = |\mathsf{H}, 1\rangle, \quad |u_3\rangle = |\mathsf{V}, 0\rangle, \quad |u_4\rangle = |\mathsf{V}, 1\rangle.$$

Any normalized state $|\psi\rangle \in \mathcal{H}(U)$ can be written as

$$|\psi\rangle = \alpha_1 |\mathsf{H}, 0\rangle \;+\; \alpha_2 |\mathsf{H}, 1\rangle \;+\; \alpha_3 |\mathsf{V}, 0\rangle \;+\; \alpha_4 |\mathsf{V}, 1\rangle, \quad \sum_{i=1}^{4} |\alpha_i|^2 = 1.$$

- **Definition of the Quantum-TreeSoft Set F.** We define

$$F : P(\text{Tree}(A)) \;\longrightarrow\; \mathcal{H}(U), \quad X \;\mapsto\; |\psi_X\rangle,$$

by prescribing $|\psi_X\rangle$ according to which attribute-nodes appear in X. In each case below, we ensure $\|\, |\psi_X\rangle \,\| = 1$. Concretely:

(a) $X = \{A_{1,1}, A_{2,2}\}$ (Photon is "Horizontal" polarization and "Mode 1"). Then we set

$$|\psi_{\{A_{1,1},\, A_{2,2}\}}\rangle = |\mathsf{H}, 1\rangle.$$

In the basis $\{|u_i\rangle\}$, this corresponds to amplitudes $\alpha_1 = \alpha_3 = 0,\ \alpha_2 = 1,\ \alpha_4 = 0$.

(b) $X = \{A_{1,1}\}$ (Photon is "Horizontal" polarization, spatial mode unspecified). We define

$$|\psi_{\{A_{1,1}\}}\rangle \;=\; \frac{1}{\sqrt{2}}\,\big(|\mathsf{H}, 0\rangle \;+\; |\mathsf{H}, 1\rangle\big).$$

Equivalently, $\alpha_1 = \alpha_2 = 1/\sqrt{2}$, $\alpha_3 = \alpha_4 = 0$. Measuring $|\psi_{\{A_{1,1}\}}\rangle$ in the basis yields "$|\mathsf{H}, 0\rangle$" or "$|\mathsf{H}, 1\rangle$" each with probability $1/2$.

(c) $X = \{A_{2,1}\}$ (Photon is in "Spatial Mode 0," polarization unspecified). We define

$$|\psi_{\{A_{2,1}\}}\rangle \;=\; \frac{1}{\sqrt{2}}\,\big(|\mathsf{H}, 0\rangle \;+\; |\mathsf{V}, 0\rangle\big).$$

Here $\alpha_1 = \alpha_3 = 1/\sqrt{2}$, $\alpha_2 = \alpha_4 = 0$. A measurement yields "$|\mathsf{H}, 0\rangle$" or "$|\mathsf{V}, 0\rangle$" with equal probability.

(d) $X = \{A_1\}$ (Photon's polarization type is known but unspecified among "H" or "V"; spatial mode unspecified). We define

$$|\psi_{\{A_1\}}\rangle \;=\; \frac{1}{2}\,\big(|\mathsf{H}, 0\rangle \;+\; |\mathsf{H}, 1\rangle \;+\; |\mathsf{V}, 0\rangle \;+\; |\mathsf{V}, 1\rangle\big).$$

In amplitude form: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$. A measurement then yields any of the four basis states with probability $1/4$.

(e) $X = \{A_2\}$ (Photon's spatial-mode type is known but unspecified between "0" or "1"; polarization unspecified). We define

$$|\psi_{\{A_2\}}\rangle \;=\; \frac{1}{2}\,\big(|\mathsf{H}, 0\rangle \;+\; |\mathsf{V}, 0\rangle \;+\; |\mathsf{H}, 1\rangle \;+\; |\mathsf{V}, 1\rangle\big),$$

again with each amplitude $= 1/2$. A measurement yields any basis state with probability $1/4$.

(f) $X = \{A_{1,2}, A_2\}$ (Photon is "Vertical" polarization but spatial mode unspecified between "0" or "1"). We define

$$|\psi_{\{A_{1,2}, A_2\}}\rangle = \frac{1}{\sqrt{2}} \big(|\mathsf{V}, 0\rangle + |\mathsf{V}, 1\rangle\big),$$

so that $\alpha_3 = \alpha_4 = 1/\sqrt{2}$, $\alpha_1 = \alpha_2 = 0$. In effect, the presence of both $A_{1,2}$ ("V") and A_2 ("spatial mode unspecified") collapses the photon into the uniform superposition over $|\mathsf{V}, 0\rangle$ and $|\mathsf{V}, 1\rangle$.

(g) $X = \varnothing$ (empty subset) (No attributes specified). We must still assign a normalized state. One natural choice is the equal superposition over all four basis states:

$$|\psi_{\varnothing}\rangle = \frac{1}{2} \big(|\mathsf{H}, 0\rangle + |\mathsf{H}, 1\rangle + |\mathsf{V}, 0\rangle + |\mathsf{V}, 1\rangle\big),$$

identical to $|\psi_{\{A_1\}}\rangle$ or $|\psi_{\{A_2\}}\rangle$. One could equally choose any other fixed normalized state if one wishes a different convention for the empty attribute set.

In each case, the state $|\psi_X\rangle = F(X)$ is normalized and the amplitudes $\alpha_i(X)$ satisfy $\sum_{i=1}^{4} |\alpha_i(X)|^2 = 1$.

- **Measurement Interpretation.** For any chosen subset $X \subseteq \mathrm{Tree}(A)$, the Quantum-TreeSoft Set F yields the state $|\psi_X\rangle$. A measurement in the computational basis $\{|\mathsf{P}, m\rangle\}$ produces outcome $|\mathsf{P}, m\rangle$ with probability $|\alpha_i(X)|^2$. For example:

$$P\big(|\mathsf{H}, 1\rangle \;\big|\; X = \{A_{1,1}, A_{2,2}\}\big) = 1, \quad P\big(|\mathsf{H}, 1\rangle \;\big|\; X = \{A_1\}\big) = \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4}.$$

  Thus the TreeSoft attributes determine which superposition or basis state the photon occupies.

- **Summary.** The mapping

$$F : P\big(\mathrm{Tree}(A)\big) \longrightarrow \mathcal{H}(U), \quad X \mapsto |\psi_X\rangle,$$

  with $|\psi_X\rangle$ defined as above for each possible subset X, constitutes a valid *Quantum-TreeSoft Set* over the attribute-tree $\mathrm{Tree}(A)$. Physically, one can think of preparing a single photon in a specific superposition of polarization and spatial-mode basis states according to which nodes of the tree one "selects." This model can represent, for instance, configurable single-photon sources in quantum optical experiments, where one chooses polarization filters (horizontal vs. vertical) and spatial filters (mode 0 vs. mode 1) hierarchically, and the resulting photon state is the appropriate normalized superposition of the corresponding basis states.

Theorem 2.4 (Generalization Property)**.** *Let $F : P\big(\mathrm{Tree}(A)\big) \to \mathcal{H}(U)$ be a Quantum-TreeSoft Set as in Definition 2.1. Then:*

(i) ***(Reduction to Quantum-Soft Set).*** *If* $\mathrm{Tree}(A)$ *has no proper sub-attributes (i.e.,* $\mathrm{Tree}(A) = A$ *at depth 1), then the restriction*

$$F_{\text{flat}} : P(A) \longrightarrow \mathcal{H}(U), \quad X \mapsto F(X)$$

satisfies that, upon further restricting to singleton subsets $\{a_j\} \subseteq A$, the mapping $\{a_j\} \mapsto F(\{a_j\})$ is exactly the definition of a Quantum-Soft Set over (U, A).

(ii) ***(Recovery of TreeSoft Set).*** *Suppose that for each $X \subseteq \mathrm{Tree}(A)$, the state $|\psi_X\rangle = F(X)$ satisfies*

$$\alpha_i(X) \neq 0 \iff u_i \in H(X), \quad \textit{and} \quad \alpha_i(X) = \frac{1}{\sqrt{|H(X)|}} \textit{ for } u_i \in H(X),$$

where $H(X) \subseteq U$ is some classical subset of U. Define $\widetilde{H} : P\big(\mathrm{Tree}(A)\big) \to P(U)$ by $\widetilde{H}(X) = H(X)$. Then $\widetilde{H}$ is a TreeSoft Set in the sense of Definition 1.2.

Proof. (i) If $\mathrm{Tree}(A)$ has depth 1, then $\mathrm{Tree}(A) = A$ and $P\big(\mathrm{Tree}(A)\big) = P(A)$. By Definition 2.1, F is a map $P(A) \to \mathcal{H}(U)$. In particular, for each singleton $\{A_j\} \subseteq A$,

$$F\big(\{A_j\}\big) = |\psi_{\{A_j\}}\rangle = \sum_{i=1}^{n} \alpha_i\big(\{A_j\}\big)\, |u_i\rangle, \quad \sum_{i=1}^{n} \big|\alpha_i(\{A_j\})\big|^2 = 1.$$

Setting $|\psi_j\rangle = F(\{A_j\})$ for $j = 1, 2, \ldots, m$ recovers exactly a Quantum-Soft Set $\widetilde{F} : A \to \mathcal{H}(U)$, $\widetilde{F}(A_j) = |\psi_j\rangle$, as in Definition 1.8. Hence F restricted to singletons is a Quantum-Soft Set.

(ii) Suppose for each $X \subseteq \mathrm{Tree}(A)$ we have

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle\,, \quad \sum_{i=1}^{n} \bigl|\alpha_i(X)\bigr|^2 = 1,$$

and the nonzero amplitudes $\alpha_i(X)$ occur precisely on indices i such that u_i lies in some classical subset $H(X) \subseteq U$. Further assume $\alpha_i(X) = 1/\sqrt{|H(X)|}$ whenever $u_i \in H(X)$. Define $\widetilde{H} : P\bigl(\mathrm{Tree}(A)\bigr) \to P(U)$ by $\widetilde{H}(X) = H(X)$.

We must check that $\widetilde{H}$ satisfies the axioms of a TreeSoft Set (Definition 1.2). Namely:

(a) $\widetilde{H}$ is well-defined: for each $X \subseteq \mathrm{Tree}(A)$, $H(X)$ is determined uniquely as the support of $|\psi_X\rangle$.

(b) For each X, $\widetilde{H}(X) \subseteq U$ is some classical subset. Thus $\widetilde{H}$ indeed takes values in $P(U)$.

(c) If $X \cap \mathrm{Tree}(A^{(j)}) \neq \emptyset$ for finitely many sub-trees $\mathrm{Tree}(A^{(j)})$, then measuring $|\psi_X\rangle$ uniformly yields elements in the corresponding union of the classical images. Concretely, if X decomposes as $X = \bigcup_j X_j$ with $X_j \subseteq \mathrm{Tree}(A^{(j)})$ across disjoint sub-trees, one checks

$$H(X) \;=\; \bigcup_j H(X_j),$$

by construction of amplitudes: the nonzero amplitudes of $|\psi_X\rangle$ must coincide with those coming from each $|\psi_{X_j}\rangle$. Thus $\widetilde{H}$ behaves exactly as the union of the corresponding TreeSoft Set mappings on each component sub-tree.

(d) Therefore $\widetilde{H}$ satisfies the requirement of being the (point-wise) union of the images under each TreeSoft component, as in Definition 1.4. In particular, if $\mathrm{Tree}(A)$ itself is a single tree, then this reduces exactly to the axioms for a TreeSoft Set.

Hence $\widetilde{H}$ is a TreeSoft Set.

□

3 Result: Quantum-Forestsoft Set

We now introduce the concept of a *Quantum-ForestSoft Set*, which simultaneously generalizes the notions of a Quantum-Soft Set, a Quantum-TreeSoft Set, and a ForestSoft Set.

Definition 3.1 (Attribute Forest)**.** Let $\{\,\mathrm{Tree}(A^{(t)})\,\}_{t\in T}$ be a (finite or countable) collection of rooted attribute-trees, each constructed as in Definition 1.2. The *forest* of these trees is the disjoint union

$$\mathrm{Forest}\bigl(\{A^{(t)}\}_{t\in T}\bigr) \;=\; \bigsqcup_{t\in T} \mathrm{Tree}\bigl(A^{(t)}\bigr).$$

Denote by $P\bigl(\mathrm{Forest}(\{A^{(t)}\})\bigr)$ the power set of all nodes (including leaves) in this forest.

Definition 3.2 (Quantum-ForestSoft Set)**.** Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universe of discourse. Let

$$\mathcal{F} \;=\; \mathrm{Forest}\bigl(\{A^{(t)}\}_{t\in T}\bigr)$$

be an attribute forest comprising the disjoint union of trees $\mathrm{Tree}(A^{(t)})$ (for each $t \in T$), as in the preceding definition. We write

$$P(\mathcal{F}) \;=\; P\bigl(\mathrm{Forest}(\{A^{(t)}\})\bigr)$$

for its power set.

Let $\mathcal{H}(U)$ be the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A *Quantum-ForestSoft Set* over $(U, \mathcal{F})$ is a mapping

$$F \;:\; P(\mathcal{F}) \;\longrightarrow\; \mathcal{H}(U),$$

satisfying the following normalization condition:

$$\text{For each } X \;\in\; P(\mathcal{F}), \quad F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle\,, \quad \sum_{i=1}^{n} \bigl|\alpha_i(X)\bigr|^2 = 1.$$

Here, $\alpha_i(X) \in \mathbb{C}$ is the amplitude associated with element u_i under the forest-subset X. Measuring $|\psi_X\rangle$ in the basis $\{|u_i\rangle\}$ yields outcome $|u_i\rangle$ with probability

$$P(u_i \mid X) = |\alpha_i(X)|^2.$$

Remark 3.3. (a) If the forest $\mathcal{F}$ consists of a single tree of depth 1, so that $\mathcal{F} = A$ is merely a flat set of first-level attributes, then $P(\mathcal{F}) = P(A)$. Restricting F to singleton subsets $\{ a_j \} \subseteq A$ recovers exactly a *Quantum-Soft Set* (Definition 1.8) via $|\psi_{a_j}\rangle = F(\{ a_j \})$.

(b) If each tree in the forest has arbitrary finite depth but we choose each vector $F(X)$ to have support equal to a classical subset $H_t(X) \subseteq U$ whenever $X \subseteq \mathrm{Tree}(A^{(t)})$, with uniform amplitudes $\frac{1}{\sqrt{|H_t(X)|}}$ on that support, then F restricted to each individual tree recovers a *TreeSoft Set* (Definition 1.2). In particular, if $X \in P(\mathcal{F})$ decomposes into disjoint pieces $X = \bigsqcup_{j\in J} X_j$ with $X_j \subseteq \mathrm{Tree}(A^{(t_j)})$, then the support of $|\psi_X\rangle$ is the union $\bigcup_{j\in J} H_{t_j}(X_j) \subseteq U$, recovering the union property of a *ForestSoft Set* (Definition 1.4).

Example 3.4 (Single-Photon Polarization and Path Encoding)**.** In this example, we model a single photon that can occupy one of two orthogonal polarization states and one of two distinct spatial paths (e.g., in a Mach–Zehnder interferometer). We organize these two independent degrees of freedom into a *forest* of two attribute-trees—one for polarization and one for path—and then define a *Quantum-ForestSoft Set* that assigns to each subset of those forest nodes a normalized superposition in the four-dimensional Hilbert space of the photon.

Universe of Discourse U. Let

$$U = \{ |\mathsf{H}, A\rangle, |\mathsf{H}, B\rangle, |\mathsf{V}, A\rangle, |\mathsf{V}, B\rangle \},$$

where:

- $|\mathsf{P}, Pth\rangle$ denotes a single-photon Fock state with polarization $\mathsf{P} \in \{\mathsf{H}, \mathsf{V}\}$ (horizontal or vertical) and spatial path $Pth \in \{A, B\}$ (path A or path B).

These four basis vectors form an orthonormal basis of the complex Hilbert space $\mathcal{H}(U)$ of dimension $n = 4$.

Attribute Forest $\mathcal{F}$. We use two rooted attribute-trees indexed by $t = 1, 2$:

$$\mathrm{Tree}(A^{(1)}) \quad \text{and} \quad \mathrm{Tree}(A^{(2)}),$$

where:

(1) **Polarization Tree** $\mathrm{Tree}(A^{(1)})$**.**

$$A^{(1)} = \{ \mathsf{Pol} \}, \quad \mathsf{Pol} = \{ \mathsf{H}, \mathsf{V} \}.$$

Thus $\mathrm{Tree}(A^{(1)}) = \{ \mathsf{Pol}, \mathsf{H}, \mathsf{V} \}$.

(2) **Path Tree** $\mathrm{Tree}(A^{(2)})$**.**

$$A^{(2)} = \{ \mathsf{Path} \}, \quad \mathsf{Path} = \{ A, B \}.$$

Thus $\mathrm{Tree}(A^{(2)}) = \{ \mathsf{Path}, A, B \}$.

The attribute *forest* is the disjoint union:

$$\mathcal{F} = \mathrm{Tree}(A^{(1)}) \bigsqcup \mathrm{Tree}(A^{(2)}) = \{ \mathsf{Pol}, \mathsf{H}, \mathsf{V}, \mathsf{Path}, A, B \}.$$

We write $P(\mathcal{F})$ for the power set of these six nodes. Any element

$$X \in P(\mathcal{F})$$

can be decomposed uniquely as

$$X = X_1 \cup X_2, \quad X_1 \subseteq \{\mathsf{Pol}, \mathsf{H}, \mathsf{V}\}, \quad X_2 \subseteq \{\mathsf{Path}, A, B\}.$$

Hilbert Space $\mathcal{H}(U)$. Label the four orthonormal basis vectors as:

$$|u_1\rangle = |\mathsf{H}, A\rangle, \quad |u_2\rangle = |\mathsf{H}, B\rangle, \quad |u_3\rangle = |\mathsf{V}, A\rangle, \quad |u_4\rangle = |\mathsf{V}, B\rangle.$$

A general normalized state $|\psi\rangle \in \mathcal{H}(U)$ can be written as

$$|\psi\rangle = \alpha_1\ |\mathsf{H}, A\rangle\ +\ \alpha_2\ |\mathsf{H}, B\rangle\ +\ \alpha_3\ |\mathsf{V}, A\rangle\ +\ \alpha_4\ |\mathsf{V}, B\rangle, \quad |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 + |\alpha_4|^2 = 1.$$

Definition of the Quantum-ForestSoft Set F. We define

$$F : P(\mathcal{F}) \longrightarrow \mathcal{H}(U), \quad X \mapsto |\psi_X\rangle,$$

by prescribing $|\psi_X\rangle$ according to which nodes appear in X. Below we list representative cases; each $|\psi_X\rangle$ is normalized and determines amplitudes $\alpha_i(X)$ for $i = 1, \ldots, 4$.

(a) $X = \{\mathsf{H},\ A\}$. (Photon is "Horizontal" polarization and in path A.)

$$|\psi_{\{\mathsf{H},A\}}\rangle\ =\ |\mathsf{H}, A\rangle,$$

so that $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$.

(b) $X = \{\mathsf{H}\}$. (Photon is "Horizontal" polarization; path unspecified.)

$$|\psi_{\{\mathsf{H}\}}\rangle\ =\ \frac{1}{\sqrt{2}}\left(|\mathsf{H}, A\rangle\ +\ |\mathsf{H}, B\rangle\right),$$

i.e. $\alpha_1 = \alpha_2 = 1/\sqrt{2},\ \alpha_3 = \alpha_4 = 0$.

(c) $X = \{A\}$. (Photon is in path A; polarization unspecified.)

$$|\psi_{\{A\}}\rangle\ =\ \frac{1}{\sqrt{2}}\left(|\mathsf{H}, A\rangle\ +\ |\mathsf{V}, A\rangle\right),$$

i.e. $\alpha_1 = \alpha_3 = 1/\sqrt{2},\ \alpha_2 = \alpha_4 = 0$.

(d) $X = \{\mathsf{Pol}\}$. (Polarization known but unspecified between H or V; path unspecified.)

$$|\psi_{\{\mathsf{Pol}\}}\rangle\ =\ \frac{1}{2}\left(|\mathsf{H}, A\rangle\ +\ |\mathsf{H}, B\rangle\ +\ |\mathsf{V}, A\rangle\ +\ |\mathsf{V}, B\rangle\right),$$

so that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$.

(e) $X = \{\mathsf{Path}\}$. (Path known but unspecified between A or B; polarization unspecified.)

$$|\psi_{\{\mathsf{Path}\}}\rangle\ =\ \frac{1}{2}\left(|\mathsf{H}, A\rangle\ +\ |\mathsf{V}, A\rangle\ +\ |\mathsf{H}, B\rangle\ +\ |\mathsf{V}, B\rangle\right),$$

again with each amplitude $= 1/2$.

(f) $X = \{\mathsf{H},\ \mathsf{Path}\}$. (Photon is "Horizontal" polarization; path unspecified between A or B.)

$$|\psi_{\{\mathsf{H},\,\mathsf{Path}\}}\rangle\ =\ \frac{1}{\sqrt{2}}\left(|\mathsf{H}, A\rangle\ +\ |\mathsf{H}, B\rangle\right),$$

identical to case (b).

(g) $X = \{\mathsf{V},\ A\}$. (Photon is "Vertical" polarization and in path A.)

$$|\psi_{\{\mathsf{V},A\}}\rangle\ =\ |\mathsf{V}, A\rangle,$$

so that $\alpha_3 = 1,\ \alpha_1 = \alpha_2 = \alpha_4 = 0$.

(h) $X = \{\mathsf{H},\ B\}$. (Photon is "Horizontal" polarization and in path B.)

$$|\psi_{\{\mathsf{H},B\}}\rangle\ =\ |\mathsf{H}, B\rangle,$$

i.e. $\alpha_2 = 1,\ \alpha_1 = \alpha_3 = \alpha_4 = 0$.

(i) $X = \varnothing$. (No attributes specified.) We choose

$$|\psi_\varnothing\rangle\ =\ \frac{1}{2}\left(|\mathsf{H}, A\rangle\ +\ |\mathsf{H}, B\rangle\ +\ |\mathsf{V}, A\rangle\ +\ |\mathsf{V}, B\rangle\right),$$

so that each basis amplitude $= 1/2$. One could also select any other fixed normalized state if desired.

Measurement Interpretation. For any subset $X \subseteq \mathcal{F}$, the Quantum-ForestSoft Set F yields the normalized state $|\psi_X\rangle$. Measuring in the computational basis $\{|\mathsf{H}, A\rangle, |\mathsf{H}, B\rangle, |\mathsf{V}, A\rangle, |\mathsf{V}, B\rangle\}$ produces outcome $|\mathsf{P}, Pth\rangle$ with probability $|\alpha_i(X)|^2$. For example:

$$P\big(|\mathsf{H}, A\rangle \;\big|\; X = \{\mathsf{H}\}\big) \;=\; \left|\tfrac{1}{\sqrt{2}}\right|^2 \;=\; \tfrac{1}{2}, \quad P\big(|\mathsf{V}, A\rangle \;\big|\; X = \{\mathsf{H}\}\big) \;=\; 0.$$

Summary. The mapping

$$F : P(\mathcal{F}) \;\longrightarrow\; \mathcal{H}(U), \quad X \;\mapsto\; |\psi_X\rangle,$$

with $|\psi_X\rangle$ defined above for each X, constitutes a valid *Quantum-ForestSoft Set* over the attribute forest $\mathcal{F} = \{\mathsf{Pol}, \mathsf{H}, \mathsf{V}, \mathsf{Path}, A, B\}$. Physically, one can think of preparing a single photon in a superposition of polarization and path states according to which nodes in the forest are "selected," thereby modeling a configurable photon source in an interferometric quantum optics experiment.

Theorem 3.5 (Generalization of Quantum-Soft, Quantum-TreeSoft, and ForestSoft)**.** *Let $F : P(\mathcal{F}) \to \mathcal{H}(U)$ be a Quantum-ForestSoft Set as in Definition 3.2. Then:*

(i) ***(Reduction to Quantum-Soft Set).*** *If every tree in $\mathcal{F}$ has depth 1, so that $\mathcal{F} = A$ is a single flat set of attributes, then $P(\mathcal{F}) = P(A)$, and restricting F to singleton subsets $\{a_j\} \subseteq A$ yields a Quantum-Soft Set*

$$\widetilde{F} : A \;\longrightarrow\; \mathcal{H}(U), \quad a_j \;\mapsto\; F(\{a_j\}).$$

(ii) ***(Recovery of Quantum-TreeSoft Set).*** *Suppose $\mathcal{F}$ consists of exactly one rooted tree* Tree(A) *(so $\mathcal{F} =$* Tree(A)*), and assume that for each $X \subseteq$* Tree(A)*, the state $|\psi_X\rangle = F(X)$ has support equal to a classical tree-soft set image $H(X) \subseteq U$ with uniform amplitudes $\frac{1}{\sqrt{|H(X)|}}$. Then the restricted mapping*

$$\widetilde{H} : P\big(\mathrm{Tree}(A)\big) \;\longrightarrow\; P(U), \quad X \;\mapsto\; H(X),$$

is a TreeSoft Set *(Definition 1.2), and F recovers the corresponding Quantum-TreeSoft Set of Definition 2.1.*

(iii) ***(Recovery of ForestSoft Set).*** *Suppose each tree $t \in T$ has an associated classical TreeSoft Set*

$$H_t : P\big(\mathrm{Tree}(A^{(t)})\big) \;\longrightarrow\; P(U),$$

and that for every $X \in P(\mathcal{F})$ (where $\mathcal{F} = \bigsqcup_{t\in T}$ Tree$(A^{(t)})$*) we have*

$$F(X) = |\psi_X\rangle \;=\; \sum_{i=1}^{n} \alpha_i(X)\; |u_i\rangle, \quad \sum_{i=1}^{n} \big|\alpha_i(X)\big|^2 = 1,$$

with the property that

$$\alpha_i(X) \neq 0 \quad\Longleftrightarrow\quad u_i \in \bigsqcup_{\substack{t\in T \\ X\cap\mathrm{Tree}(A^{(t)})\neq\varnothing}} H_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

and whenever u_i belongs to that union, $\alpha_i(X) = 1/\sqrt{\big|\bigcup_t H_t(X \cap \mathrm{Tree}(A^{(t)}))\big|}$. Then defining

$$\widetilde{H} : P(\mathcal{F}) \;\longrightarrow\; P(U), \quad X \;\mapsto\; \bigcup_{\substack{t\in T \\ X\cap\mathrm{Tree}(A^{(t)})\neq\varnothing}} H_t\big(X \cap \mathrm{Tree}(A^{(t)})\big)$$

produces a ForestSoft Set *(Definition 1.4). In particular, F recovers the classical ForestSoft Set under this uniform-amplitude support assumption.*

Proof. (i) If each tree in $\mathcal{F}$ has depth 1, then $\mathcal{F}$ is a single flat set A of first-level attributes. Hence $P(\mathcal{F}) = P(A)$. By Definition 3.2, F is a map $P(A) \to \mathcal{H}(U)$. For each singleton subset $\{a_j\} \subseteq A$, we have

$$F\big(\{a_j\}\big) \;=\; |\psi_{\{a_j\}}\rangle \;=\; \sum_{i=1}^{n} \alpha_i\big(\{a_j\}\big)\; |u_i\rangle, \quad \sum_{i=1}^{n} \big|\alpha_i(\{a_j\})\big|^2 = 1.$$

Defining $\widetilde{F} : A \to \mathcal{H}(U)$ by $\widetilde{F}(a_j) = F(\{a_j\})$ yields exactly a *Quantum-Soft Set* as in Definition 1.8.

(*ii*) Suppose $\mathcal{F} = \mathrm{Tree}(A)$ is a single rooted tree. For each $X \subseteq \mathrm{Tree}(A)$, let

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle\,, \quad \sum_{i=1}^{n} \bigl|\alpha_i(X)\bigr|^2 = 1,$$

and assume that the nonzero amplitudes $\alpha_i(X)$ occur if and only if $u_i \in H(X) \subseteq U$, where H is a classical TreeSoft Set on $\mathrm{Tree}(A)$. Furthermore, assume $\alpha_i(X) = 1/\sqrt{|H(X)|}$ for $u_i \in H(X)$. Then define

$$\widetilde{H} : P\bigl(\mathrm{Tree}(A)\bigr) \longrightarrow P(U), \quad X \mapsto H(X).$$

We verify that $\widetilde{H}$ satisfies the axioms of a TreeSoft Set (Definition 1.2):

- For each $X \subseteq \mathrm{Tree}(A)$, $\widetilde{H}(X) = H(X) \subseteq U$ is well-defined, since $H(X)$ is exactly the support of $|\psi_X\rangle$.
- The mapping $\widetilde{H}$ takes values in $P(U)$ by hypothesis.
- If $X = \bigcup_{j=1}^{k} X_j$ is a disjoint union of subsets each contained in some sub-tree of $\mathrm{Tree}(A)$, then the support of $|\psi_X\rangle$ is the union $\bigcup_{j=1}^{k} H(X_j)$. This matches the union property required of TreeSoft Sets.

Hence $\widetilde{H}$ is a TreeSoft Set, and F restricted to uniform-support states recovers the *Quantum-TreeSoft Set* of Definition 2.1.

(*iii*) Now let $\mathcal{F} = \bigsqcup_{t\in T} \mathrm{Tree}(A^{(t)})$ be a disjoint union of several trees. For each tree $t \in T$, suppose there is a classical TreeSoft Set

$$H_t : P\bigl(\mathrm{Tree}(A^{(t)})\bigr) \longrightarrow P(U).$$

For any $X \in P(\mathcal{F})$, write

$$X = \bigsqcup_{t\in T_0} \bigl(X \cap \mathrm{Tree}(A^{(t)})\bigr),$$

where $T_0 = \{\, t \in T \mid X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing \,\}$. By hypothesis,

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle\,, \quad \sum_{i=1}^{n} \bigl|\alpha_i(X)\bigr|^2 = 1,$$

with

$$\alpha_i(X) \neq 0 \quad \Longleftrightarrow \quad u_i \in \bigcup_{t\in T_0} H_t\bigl(X \cap \mathrm{Tree}(A^{(t)})\bigr),$$

and whenever u_i lies in that union, $\alpha_i(X) = \bigl|\bigcup_{t\in T_0} H_t(X \cap \mathrm{Tree}(A^{(t)}))\bigr|^{-1/2}$. Define

$$\widetilde{H} : P(\mathcal{F}) \longrightarrow P(U), \quad X \mapsto \bigcup_{t\in T_0} H_t\bigl(X \cap \mathrm{Tree}(A^{(t)})\bigr).$$

We must check that $\widetilde{H}$ satisfies the axioms of a *ForestSoft Set* (Definition 1.4):

- For each X, $\widetilde{H}(X) \subseteq U$ is well-defined, since it is the union of the supports of the quantum states $|\psi_{X\cap \mathrm{Tree}(A^{(t)})}\rangle$ over $t \in T_0$.
- If $X \cap \mathrm{Tree}(A^{(t)}) = \emptyset$, then $H_t(X \cap \mathrm{Tree}(A^{(t)})) = \emptyset$ by the TreeSoft Set property, so such t does not contribute to the union.
- If $X = \bigcup_{j=1}^{k} X_j$ with each $X_j \subseteq \mathrm{Tree}(A^{(t_j)})$, then

  $$\widetilde{H}(X) = \bigcup_{j=1}^{k} H_{t_j}(X_j),$$

  matching exactly the union rule in Definition 1.4.
- Thus $\widetilde{H}$ is a ForestSoft Set, and F recovers it under the uniform-amplitude support assumption.

Hence all three reductions are verified.

□

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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