Supplementary Information for

Arbitrary Pulse Shaping using Accelerated Interfaces

Amir Bahrami, Klaas De Kinder and Christophe Caloz

May 26, 2025

S1 Scattering at a Nonuniformly Moving PEC Interface (Derivation of Eq. (4))

In this section, we want to find the general scattering solution at nonuniformly moving PEC. We assume that the equation of motion of the interface has the implicit form of $Z[t]$. Next, we assume the following traveling waveforms for the incident and reflected electromagnetic waves, propagating in vacuum[1]

$$E_\mathrm{i} = \psi_\mathrm{i}[z-t], \quad B_\mathrm{i} = \psi_\mathrm{i}[z-t] \tag{S1a}$$

$$E_\mathrm{r} = -\psi_\mathrm{r}[z+t], \quad B_\mathrm{r} = \psi_\mathrm{r}[z+t]. \tag{S1b}$$

Now, we apply at the interface the boundary condition

$$(E_\mathrm{i} - vB_\mathrm{i}) + (E_\mathrm{r} - vB_\mathrm{r})\big|_{Z[t]} = 0, \tag{S2}$$

which yields, upon substitution of Eqs. (S1) and noting that $v = Z'[t]$,

$$(1 - Z'[t])\psi_\mathrm{i}[Z[t]-t] + (-1 - Z'[t])\psi_\mathrm{r}[Z[t]+t] = 0. \tag{S3a}$$

Solving this equation for ψ_r, we get

$$\psi_\mathrm{r}[Z[t]+t] = \frac{1 - Z'[t]}{1 + Z'[t]}\psi_\mathrm{i}[Z[t]-t]. \tag{S3b}$$

Equation (S3b) holds only at the interface. To find the reflected wave at *any point* (z, t), we enforce its argument to take the traveling waveform $z + t$, namely

$$Z[t] + t \to z + t. \tag{S4}$$

---

[1] All calculations are done assuming natural units ($c = 1$).

Then, we define the function $f[\cdot]$ as

$$f[t] = Z[t] + t, \tag{S5}$$

with upon substitution in Eq. (S4) yields

$$f[t] \to z + t. \tag{S6}$$

Next, we apply $f^{-1}[\cdot]$ to both sides of Eq. (S6),

$$t \to f^{-1}[z + t]. \tag{S7}$$

Finally, we insert Eq. (S4) into the left-hand side of Eq. (S3b) and Eq. (S7) into the right-hand side of Eq. (S3b). This yields the traveling waveform

$$\psi_\mathrm{r}[z + t] = \frac{1 - Z'[f^{-1}[z + t]]}{1 + Z'[f^{-1}[z + t]]}\psi_\mathrm{i}[Z[f^{-1}[z + t]] - f^{-1}[z + t]], \tag{S8a}$$

which can be more compactly written as

$$\psi_\mathrm{r}[x] = \frac{1 - Z'[f^{-1}[x]]}{1 + Z'[f^{-1}[x]]}\psi_\mathrm{i}[Z[f^{-1}[x]] - f^{-1}[x]], \quad \text{with} \quad x = z + t. \tag{S8b}$$

S2 Pulse Shaping using a Nonuniformly Moving Interface (Derivation of Eq. (9))

The goal here is to find the trajectory of the interface, namely $Z[t]$, as a function of specified $\psi_\mathrm{i}[\cdot]$ and $\psi_\mathrm{r}[\cdot]$.

Directly solving Eq. (S8b) for $Z[t]$ is not straightforward. However, an educated observation simplifies the problem: The time derivative of the argument of $\psi_\mathrm{i}[\cdot]$ in Eq. (S8b) appears as its multiplicative factor with a negative sign. Indeed,

$$\frac{d}{dx}\left(Z[f^{-1}[x]] - f^{-1}[x]\right) = \left(\frac{d}{dx}f^{-1}[x]\right)Z'[f^{-1}[x]] - \frac{d}{dx}f^{-1}[x], \tag{S9}$$

which upon rearranging and using the identity $\frac{d}{dx}f^{-1}[x] = \frac{1}{f'[f^{-1}[x]]}$ gives

$$\frac{1}{f'[f^{-1}[x]]}(Z'[f^{-1}[x] - 1]) = -\frac{1 - Z'[f^{-1}[x]]}{1 + Z'[f^{-1}[x]]}. \tag{S10}$$

We can therefore rewrite Eq. (S8b) in the form of the very compact differential equation

$$\psi_\mathrm{r}[x] = -\frac{d\xi}{dx}\psi_\mathrm{i}[\xi], \tag{S11a}$$

where

$$\xi = \xi[x] = Z[f^{-1}[x]] - f^{-1}[x]. \tag{S11b}$$

By rearranging the terms and integrating both sides, we obtain

$$\int \psi_{\mathrm{r}}[x]dx = -\int \psi_{\mathrm{i}}[\xi]d\xi. \tag{S11c}$$

We can simplify this expression via introducing the integral functions $F_{\mathrm{r}}[x] = \int \psi_{\mathrm{r}}[x]dx$ and $F_{\mathrm{i}}[\xi] = \int \psi_{\mathrm{i}}[\xi]d\xi$, leading to the relation

$$F_{\mathrm{i}}[\xi] = -F_{\mathrm{r}}[x]. \tag{S12}$$

Solving for ξ by applying $F_{\mathrm{i}}^{-1}[\cdot]$ to both sides, we find

$$\xi = F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right], \tag{S13}$$

which using Eq. (S11b) yields

$$F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right] = Z[f^{-1}[x]] - f^{-1}[x]. \tag{S14}$$

Next, we perform the substitution $t \to f^{-1}[x]$ into Eq. (S5), giving

$$x = Z[f^{-1}[x]] + f^{-1}[x]. \tag{S15}$$

Equations (S14) and (S15) form a system of equations. To solve for the interface trajectory, we proceed by adding and subtracting these equations. Adding them eliminates $f^{-1}[x]$, while subtracting them eliminates $Z[f^{-1}[x]]$, allowing us to express the system in a more tractable form. After rearranging, these operations yield the respective equations

$$Z[f^{-1}[x]] = \frac{1}{2}\left(x + F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right]\right) \tag{S16a}$$

and

$$f^{-1}[x] = \frac{1}{2}\left(x - F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right]\right). \tag{S16b}$$

Now, inserting Eq. (S16b) into Eq. (S16a), we get

$$Z\left[\frac{1}{2}\left(x - F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right]\right)\right] = \frac{1}{2}\left(x + F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right]\right). \tag{S17}$$

We can further simplify this equation by the change of variable

$$L[x] \to \frac{1}{2}\left(x - F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right]\right), \tag{S18a}$$

so that

$$Z\left[L[x]\right] = \frac{1}{2}\left(x + F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[x]\right]\right), \tag{S18b}$$

The goal is to find the trajectory of the interface as a function of time, namely $Z[t]$. This can be accomplished by the change of variable $x \to L^{-1}[t]$, which gives

$$Z[t] = \frac{1}{2}\left(L^{-1}[t] + F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[L^{-1}[t]]\right]\right), \tag{S19a}$$

where

$$L[t] = \frac{1}{2}\left(t - F_{\mathrm{i}}^{-1}\left[-F_{\mathrm{r}}[t]\right]\right). \tag{S19b}$$

S3 Example for the Pulse Shaping Equation

Here, we provide a simple application example of Eqs. (S19) that admits a closed-form solution. This example assumes an incident and reflected pulses given by

$$\psi_\mathrm{i}[x] = \varphi[x] \tag{S20a}$$

and

$$\psi_\mathrm{r}[x] = \sigma\varphi[\sigma x]. \tag{S20b}$$

The function φ is even and its integral is odd, as is for instance the case for Gaussian pulses.

We first calculate the integral functions of the incident and the reflected waves,

$$F_\mathrm{i}[x] = \int \psi_\mathrm{i}[x]dx = \int \varphi[x]dx \tag{S21a}$$

and

$$F_\mathrm{r}[x] = \int \psi_\mathrm{r}[x]dx = \int \sigma\varphi[\sigma x]dx = \int \varphi[\sigma x]d(\sigma x). \tag{S21b}$$

Comparing Eq. (S21a) and Eq. (S21b), we find

$$F_\mathrm{r}[x] = F_\mathrm{i}[\sigma x]. \tag{S22}$$

Next, we find $L[\cdot]$ using Eq. (S19b),

$$L[t] = \frac{1}{2}\left(t - F_\mathrm{i}^{-1}[-F_\mathrm{i}[\sigma t]]\right). \tag{S23a}$$

Since the function $F_\mathrm{i}[\cdot]$ is assumed to be odd, we get

$$L[t] = \frac{1+\sigma}{2}t, \tag{S23b}$$

whose inverse is

$$L^{-1}[t] = \frac{2}{1+\sigma}t. \tag{S23c}$$

Inserting Eq. (S23c) in Eq. (S19a), we get

$$Z[t] = \left(\frac{1-\sigma}{1+\sigma}\right)t, \tag{S24}$$

which corresponds to an interface moving at the uniform velocity of $v = (1-\sigma)/(1+\sigma)$ resulting in the Doppler shift $(1-v)/(1+v) = \sigma$, as expected.