Rate-Independent Gradient-Enhanced Plastic Deformation Model of Euler-Bernoulli Beams

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Abstract

This work presents a rigorously formulated rate-independent, gradient-enhanced continuum plasticity framework tailored for Euler-Bernoulli beams, accounting for both geometric and material nonlinearities. The model is grounded in thermodynamic consistency, exploiting the principles of maximum dissipation and the Clausius-Duhem inequality to derive constitutive relations via an additive decomposition of strain, a convex yield surface, and an associative flow rule. Gradient enhancements are incorporated through nonlocal terms in the free energy functional, yielding regularization effects that circumvent strain localization and ensure mathematical well-posedness. The governing equations are derived from variational principles, specifically the principle of virtual work and incremental energy minimization, resulting in a unified framework valid for both quasi-static and dynamic loading conditions. A novel contribution includes the derivation of a generalized consistent tangent operator that embeds inertial effects, critical for stability and convergence in implicit time integration schemes. Theoretical guarantees of existence and uniqueness are established via convexity and variational inequality formulations. The model naturally integrates shear effects and finite rotations through higher-order kinematic assumptions, and it rigorously addresses open problems in beam plasticity, including dynamic plastic collapse, interaction between bending and shear yield mechanisms, and plastic hinge migration under large displacements. The framework is extensible to layered composites, functionally graded materials, and beams with stochastic or temperature-dependent yield behavior, positioning it as a foundational advancement in gradient plasticity and computational structural mechanics.

Contents

1 Literature Review of Rate-Independent Plastic Deformation Models

The foundations of rate-independent plasticity were laid in the early 20th century, beginning with Saint-Venant (1871) [1], who proposed the concept of plastic flow by suggesting that plastic deformation occurs when stresses exceed a certain threshold, leading to permanent strain. This idea was later formalized through mathematical descriptions of yield criteria. Tresca (1864) [2] introduced the maximum shear stress criterion, which states that yielding begins when the greatest shear stress reaches a critical value k. Mathematically, this is expressed as:

$$\max\left(|\sigma_1-\sigma_2|, |\sigma_2-\sigma_3|, |\sigma_3-\sigma_1|\right) = 2k \tag{1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses in Equation (1). Von Mises (1913) [3] proposed an alternative criterion based on the second invariant of the deviatoric stress tensor J_2, given by:

$$f(\boldsymbol{\sigma}) = \sqrt{3J_2} - \sigma_y = 0, \quad J_2 = \frac{1}{2}\mathbf{s} : \mathbf{s}, \quad \mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} \tag{2}$$

where $\mathbf{s}$ is the deviatoric stress, σ_y is the yield stress, and $\mathbf{I}$ is the identity tensor in Equation (2). This criterion is more suitable for ductile materials as it accounts for the distortional energy.

Prandtl (1925) [4] and Reuss (1930) [5] developed the incremental theory of plasticity, introducing the concept of a flow rule to describe the evolution of plastic strain. The plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ was assumed to be normal to the yield surface, leading to the associated flow rule:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{3}$$

where $\dot{\lambda}$ in Equation (3) is the plastic multiplier, determined by the consistency condition $\dot{f} = 0$. The total strain rate was decomposed into elastic and plastic parts:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p, \quad \dot{\boldsymbol{\varepsilon}}^e = \mathbb{C}^{-1} : \dot{\boldsymbol{\sigma}} \tag{4}$$

where $\mathbb{C}$ in Equation (4) is the fourth-order elastic stiffness tensor.

Hill (1998) [6] generalized plasticity theory to account for material anisotropy by introducing a yield function dependent on both stress and material orientation. For orthotropic materials, the Hill yield criterion modifies von Mises' formulation as:

$$f(\boldsymbol{\sigma}) = \sqrt{F(\sigma_{22}-\sigma_{33})^2 + G(\sigma_{33}-\sigma_{11})^2 + H(\sigma_{11}-\sigma_{22})^2 + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2} - \sigma_y = 0 \tag{5}$$

where F, G, H, L, M, N in Equation (5) are material constants determined from experiments. This framework allowed plasticity models to describe directional hardening effects in rolled metals and composites. The plastic potential and flow rule were correspondingly generalized as:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \text{with} \quad f(\boldsymbol{\sigma}, \mathbf{H}) = 0 \tag{6}$$

where $\mathbf{H}$ represents internal hardening variables in Equation (6). These developments established the mathematical rigor of rate-independent plasticity, enabling its application to complex loading paths and anisotropic materials.

1.1 Yield criterion

The Tresca yield criterion postulates that plastic yielding initiates when the maximum shear stress attains a critical value k, which is a material property. Mathematically, this condition is expressed in terms of the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ as:

$$\max\left(\frac{|\sigma_1-\sigma_2|}{2}, \frac{|\sigma_2-\sigma_3|}{2}, \frac{|\sigma_3-\sigma_1|}{2}\right) = k \tag{7}$$

Equivalently, the yield function $f(\boldsymbol{\sigma})$ can be written as:

$$f(\boldsymbol{\sigma}) = \max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right) - 2k \leq 0 \tag{8}$$

where $f(\boldsymbol{\sigma}) = 0$ defines the yield surface in Equation (8). The Tresca criterion is particularly suitable for materials where shear-driven yielding dominates, but it exhibits singularities in the stress space due to its piecewise linear nature.

The von Mises yield criterion, in contrast, is based on the second invariant of the deviatoric stress tensor J_2, providing a smooth and differentiable yield surface. The deviatoric stress $\mathbf{s}$ is defined as:

$$\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} \tag{9}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{I}$ is the identity tensor in Equation (9). The second invariant J_2 is given by:

$$J_2 = \frac{1}{2}\mathbf{s} : \mathbf{s} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{10}$$

The von Mises yield condition states that yielding occurs when the equivalent stress σ_{eq} reaches the yield strength σ_y:

$$f(\boldsymbol{\sigma}) = \sqrt{3J_2} - \sigma_y = 0 \tag{11}$$

In terms of principal stresses, Equation (11) becomes:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \sigma_y \tag{12}$$

The von Mises criterion is widely used for metals due to its agreement with experimental observations under various stress states.

The Drucker-Prager yield criterion extends the von Mises criterion to account for pressure sensitivity, making it applicable to geomaterials such as soils, rocks, and concrete. The yield function incorporates both J_2 and the first stress invariant $I_1 = \mathrm{tr}(\boldsymbol{\sigma})$:

$$f(\boldsymbol{\sigma}) = \sqrt{J_2} + \alpha I_1 - \kappa \leq 0 \tag{13}$$

where α and κ are material constants related to cohesion and internal friction. The parameters α and κ in Equation (13) can be expressed in terms of the Coulomb friction angle ϕ and cohesion c as:

$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3 - \sin\phi)}, \quad \kappa = \frac{6c\cos\phi}{\sqrt{3}(3 - \sin\phi)} \tag{14}$$

In principal stress space, the Drucker-Prager criterion forms a conical surface, distinguishing it from the cylindrical von Mises yield surface. The criterion reduces to von Mises when $\alpha = 0$, but for pressure-sensitive materials, the dependence on I_1 introduces a mean stress effect, allowing the model to capture phenomena such as dilatancy and shear-enhanced compaction. The plastic potential and flow rule for Drucker-Prager materials often employ a non-associated flow rule to accurately describe inelastic volumetric strains, with the plastic strain rate given by:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\left(\frac{\mathbf{s}}{2\sqrt{J_2}} + \beta\mathbf{I}\right), \tag{15}$$

where β governs the dilatancy behavior in Equation (15). This framework enables the modeling of complex pressure-dependent yielding and hardening responses in geological and granular materials.

1.2 Hardening Laws

The mathematical description of hardening laws in plasticity theory is fundamental for capturing the evolution of the yield surface under plastic deformation. Isotropic hardening, as utilized in the Johnson and Cook (1983) [38] model, assumes that the yield surface expands uniformly in stress space, maintaining its shape but increasing in size as a function of accumulated plastic strain. The yield condition under isotropic hardening can be expressed as:

$$f(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \sqrt{\frac{3}{2}\mathbf{s} : \mathbf{s}} - \sigma_y(\bar{\varepsilon}^p) = 0 \tag{16}$$

where $\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I}$ is the deviatoric stress tensor, $\sigma_y(\bar{\varepsilon}^p)$ is the yield stress evolving with equivalent plastic strain $\bar{\varepsilon}^p = \int \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p} dt$ in Equation (16), and the hardening function may take a power-law form (Johnson and Cook, 1983 [38]):

$$\sigma_y(\bar{\varepsilon}^p) = \sigma_{y0} + K(\bar{\varepsilon}^p)^n \tag{17}$$

where σ_{y0} is the initial yield stress, and K and n are material parameters in Equation (17). The consistency condition $df = 0$ during plastic loading leads to:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial f}{\partial \bar{\varepsilon}^p} d\bar{\varepsilon}^p = 0 \tag{18}$$

Kinematic hardening, introduced by Prager (1956) [8] and later refined by Armstrong and Frederick (1966) [9], describes the translation of the yield surface in stress space through a backstress tensor $\boldsymbol{\alpha}$. The yield function becomes:

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \sqrt{\frac{3}{2}(\mathbf{s} - \boldsymbol{\alpha}) : (\mathbf{s} - \boldsymbol{\alpha})} - \sigma_y = 0 \tag{19}$$

Prager's linear kinematic hardening rule (Prager, 1956 [8]) prescribes the evolution of backstress as:

$$\dot{\boldsymbol{\alpha}} = c\dot{\boldsymbol{\varepsilon}}^p \tag{20}$$

where c is a material constant in Equation (20). The Armstrong-Frederick nonlinear kinematic hardening model (Armstrong and Frederick, 1966 [9]) introduces a recall term to account for dynamic recovery effects:

$$\dot{\boldsymbol{\alpha}} = c\dot{\boldsymbol{\varepsilon}}^p - \gamma\boldsymbol{\alpha}\dot{\bar{\varepsilon}}^p \tag{21}$$

where γ controls the rate of recovery in Equation (21). The nonlinearity introduced by the recall term enables the modeling of cyclic plasticity phenomena such as the Bauschinger effect (Bauschinger, 1886 [10]). The consistency condition for kinematic hardening requires:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial f}{\partial \boldsymbol{\alpha}} : d\boldsymbol{\alpha} = 0 \tag{22}$$

Mixed hardening combines isotropic and kinematic hardening effects (Lemaitre and Chaboche, 1994 [11]), providing a comprehensive framework for materials exhibiting both yield surface expansion and translation. The yield function generalizes to:

$$f(\boldsymbol{\sigma}, \boldsymbol{\alpha}, \bar{\varepsilon}^p) = \sqrt{\frac{3}{2}(\mathbf{s} - \boldsymbol{\alpha}) : (\mathbf{s} - \boldsymbol{\alpha})} - \sigma_y(\bar{\varepsilon}^p) = 0 \tag{23}$$

with the backstress evolution law incorporating both linear and nonlinear terms (Chaboche, 1986 [12]):

$$\dot{\boldsymbol{\alpha}} = c\dot{\boldsymbol{\varepsilon}}^p - \gamma\boldsymbol{\alpha}\dot{\bar{\varepsilon}}^p \tag{24}$$

and the isotropic hardening component following a saturation-type law (Khan and Huang, 1995 [13]):

$$\sigma_y(\bar{\varepsilon}^p) = \sigma_{y0} + Q_\infty(1 - e^{-b\bar{\varepsilon}^p}) \tag{25}$$

where Q_∞ represents the maximum increase in yield stress and b controls the saturation rate in Equation (25). The plastic modulus H for mixed hardening is derived from the consistency condition (Simo and Hughes (2006) [42]):

$$H = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{\partial f}{\partial \boldsymbol{\alpha}} : \left(c\frac{\partial f}{\partial \boldsymbol{\sigma}} - \gamma\boldsymbol{\alpha}\sqrt{\frac{2}{3}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}} \right) - \frac{\partial f}{\partial \bar{\varepsilon}^p}\sqrt{\frac{2}{3}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}} \tag{26}$$

The integration of these hardening laws into computational plasticity frameworks employs return mapping algorithms (Simo and Hughes (2006) [42]), where the plastic corrector step enforces the consistency condition through an implicit backward Euler scheme. For mixed hardening, the algorithmic tangent modulus requires differentiation of both isotropic and kinematic hardening components (Dafalias and Popov, 1975 [15]):

$$\mathbb{C}^{ep} = \mathbb{C} - \frac{(\mathbb{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}) \otimes (\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C})}{H + \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}} \tag{27}$$

The theoretical foundations of these plasticity models were established by Hill (1998) [6], while more recent developments in bounding surface plasticity have been presented by Dafalias and Popov (1975) [15]. Historical perspectives on the development of nonlinear hardening models can be found in Frederick and Armstrong (2007) [16].

1.3 Flow Rules

The mathematical formulation of flow rules in plasticity theory governs the evolution of plastic strain and is fundamentally linked to the yield criterion. The associated flow rule, rooted in Drucker's stability postulate (Drucker, 1951 [14]), establishes that the plastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}^p$ is normal to the yield surface $f(\boldsymbol{\sigma}) = 0$ in stress space. This normality condition is expressed as:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{28}$$

where $\dot{\lambda}$ in Equation (28) is the plastic multiplier satisfying the Karush-Kuhn-Tucker conditions:

$$\dot{\lambda} \geq 0, \quad f \leq 0, \quad \dot{\lambda}f = 0 \tag{29}$$

The consistency condition $\dot{f} = 0$ during plastic loading yields:

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \dot{\boldsymbol{q}} = 0 \tag{30}$$

where $\boldsymbol{q}$ represents internal variables in Equation (30). For von Mises plasticity with isotropic hardening, this specializes to:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{3\boldsymbol{s}}{2\sigma_y}, \quad \dot{\bar{\varepsilon}}^p = \dot{\lambda} \tag{31}$$

The associated flow rule implies coincident yield and potential surfaces, a condition that maximizes plastic dissipation (Hill (1998) [6]):

$$\mathcal{D}^p = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \boldsymbol{\sigma} : \frac{\partial f}{\partial \boldsymbol{\sigma}} \geq 0 \tag{32}$$

For materials exhibiting pressure-sensitive behavior like soils or concrete, the associated flow rule often overestimates dilatancy. This necessitates non-associated flow rules (Drucker, 1951 [14]; Rudnicki and Rice, 1975 [33]), where plastic strain derives from a plastic potential $g \neq f$:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{33}$$

The plastic potential g in Equation (33) for Drucker-Prager materials might take the form:

$$g = \sqrt{J_2} + \beta I_1 \tag{34}$$

where β in Equation (34) controls dilatancy, differing from the yield function parameter α. The resulting plastic strain rate components are:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \left(\frac{\boldsymbol{s}}{2\sqrt{J_2}} + \frac{\beta}{3} \mathbf{I} \right) \tag{35}$$

The non-associated formulation modifies the elastoplastic tangent operator $\mathbb{C}^{ep}$ (Simo and Hughes, 2006 [42]):

$$\mathbb{C}^{ep} = \mathbb{C} - \frac{(\mathbb{C} : \frac{\partial g}{\partial \boldsymbol{\sigma}}) \otimes (\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C})}{H + \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \frac{\partial g}{\partial \boldsymbol{\sigma}}} \tag{36}$$

where H is the hardening modulus in Equation (36). For critical state soil models (Roscoe and Burland, 1968 [17]), the plastic potential incorporates state-dependent dilatancy:

$$g = q^2 + M^2 p(p - p_c) \tag{37}$$

with M being the critical state line slope and p_c the preconsolidation pressure in Equation (37). The mathematical consequences of non-associativity include: loss of normality in stress space (Lubliner, 2008 [60]), potential non-uniqueness in boundary value problems (Raniecki and Bruhns, 1981 [18]), and altered localization conditions (Rice, 1976 [19]). The plastic multiplier for non-associated flow is determined from:

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \dot{\boldsymbol{\varepsilon}}}{H + \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \frac{\partial g}{\partial \boldsymbol{\sigma}}} \tag{38}$$

Theoretical justifications for non-associated flow stem from micromechanical considerations (Nemat-Nasser, 1983 [20]) and thermodynamic restrictions (Collins and Houlsby, 1997 [21]). For anisotropic materials (Hill, 1998 [6]), both flow rules generalize through fourth-order tensors:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\mathbb{M} : \boldsymbol{\sigma}, \quad \mathbb{M} = \frac{\partial g}{\partial \boldsymbol{\sigma}} \text{ (non-associated)} \tag{39}$$

Key references include: Drucker (1951) [14] for stability postulates; Hill (1998) [6] for plasticity foundations; Rudnicki and Rice (1975) [33] for localization analyses; Simo and Hughes (2006) [42] for computational aspects; Roscoe and Burland (1968) [17] for soil plasticity; Lubliner (2008) [60] for general plasticity theory; Rice (1976) [19] for bifurcation conditions; Nemat-Nasser (1983) [20] for micromechanical derivations; Collins and Houlsby (1997) [21] for thermodynamic frameworks; and Hill (1998) [6] for anisotropic formulations. Experimental validations appear in Anandarajah (2011) [22], while numerical implementations are detailed in de Souza Neto et al. (2011) [30].

1.4 Computational Implementations

The computational implementation of plasticity theory requires careful mathematical treatment of stress updates and constitutive integration. For small-strain formulations following Wilkins (1963) [23], the additive decomposition of strain into elastic and plastic components forms the basis:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \tag{40}$$

where the stress-strain relationship is governed by Hooke's law:

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}^e = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{41}$$

The plastic strain evolution follows the flow rule:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial \phi}{\partial \boldsymbol{\sigma}} \tag{42}$$

where ϕ is the plastic potential and $\dot{\gamma}$ the consistency parameter in Equation (42). The discrete update employs the radial return algorithm (Wilkins, 1963 [23]):

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{trial} - 2G\Delta\gamma\mathbf{n}_{n+1} \tag{43}$$

with G the shear modulus and $\mathbf{n} = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} / \| \frac{\partial \phi}{\partial \boldsymbol{\sigma}} \|$ in Equation (43). The consistency condition is enforced through:

$$f(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^p) = 0 \tag{44}$$

Finite strain plasticity builds upon the multiplicative decomposition introduced by Lee (1969) [49]:

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{45}$$

where $\mathbf{F}$ is the deformation gradient in Equation (45). The elastic and plastic right Cauchy-Green tensors are:

$$\mathbf{C}^e = \mathbf{F}^{eT}\mathbf{F}^e, \quad \mathbf{C}^p = \mathbf{F}^{pT}\mathbf{F}^p \tag{46}$$

The Mandel stress $\boldsymbol{\Sigma}$ drives plastic flow:

$$\boldsymbol{\Sigma} = \mathbf{C}^e\mathbf{S} \tag{47}$$

where $\mathbf{S}$ is the second Piola-Kirchhoff stress in Equation (47). The plastic flow rule in the intermediate configuration becomes:

$$\dot{\mathbf{C}}^p = 2\dot{\gamma}\mathbf{F}^p \frac{\partial \phi}{\partial \boldsymbol{\Sigma}} \mathbf{F}^{pT} \tag{48}$$

The return-mapping algorithm of Simo and Hughes (2006) [42] provides an efficient numerical implementation through an operator split:

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{trial} - \mathbb{C} : \Delta \boldsymbol{\varepsilon}^p \tag{49}$$

The consistent tangent operator preserves quadratic convergence in Newton iterations:

$$\mathbb{C}^{alg} = \mathbb{C} - \frac{(\mathbb{C} : \frac{\partial \phi}{\partial \boldsymbol{\sigma}}) \otimes (\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C})}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \frac{\partial \phi}{\partial \boldsymbol{\sigma}} + H} \tag{50}$$

where H is the hardening modulus in Equation (50). For finite strains, the algorithmic tangent requires push-forward operations:

$$\mathbb{J}^{alg} = J^{-1}\mathbf{F}(\mathbb{S}^{alg})\mathbf{F}^T \tag{51}$$

where $\mathbb{S}^{alg}$ in Equation (51) is the material tangent. The exponential map preserves plastic incompressibility (Weber and Anand, 1990 [24]):

$$\mathbf{F}_{n+1}^p = \exp(\Delta\gamma \mathbf{N}_{n+1})\mathbf{F}_n^p \tag{52}$$

with $\mathbf{N} = \frac{\partial \phi}{\partial \boldsymbol{\Sigma}}$ in Equation (52). The implicit integration of hardening variables follows:

$$\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p + \sqrt{\frac{2}{3}} \|\Delta \boldsymbol{\varepsilon}^p\| \tag{53}$$

Key references include: Wilkins (1964) [23] for radial return methods; Lee (1969) [49] for multiplicative decomposition; Simo and Hughes (2006) [42] for return mapping algorithms; Weber and Anand (1990) [24] for finite strain implementations; Nagtegaal (1982) [25] for efficient integration schemes; Miehe (1996) [26] for exponential updates; Ortiz and Popov (1985) [27] for accuracy assessments; Hughes (1984) [28] for algorithmic aspects; Moran et al. (1990) [29] for single crystal implementations; and de Souza Neto et al. (2011) [30] for computational frameworks. Mathematical foundations appear in Marsden and Hughes (1994) [31], while error analysis is detailed in Armero and Perez-Foguet (2002) [32].

1.5 Applications and Limitations

The application of rate-independent plasticity models to metals typically employs the J_2 plasticity framework, where the yield condition is expressed through the second invariant of the deviatoric stress tensor $J_2 = \frac{1}{2}\mathbf{s} : \mathbf{s}$ with $\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{3}\text{tr}(\boldsymbol{\sigma})\mathbf{I}$. For von Mises materials, the yield function takes the form (Hill (1998) [6]):

$$f(\boldsymbol{\sigma}) = \sqrt{3J_2} - \sigma_y(\bar{\varepsilon}^p) = 0 \tag{54}$$

where $\sigma_y(\bar{\varepsilon}^p)$ represents the strain-hardened yield stress in Equation (54). The associated flow rule generates isochoric plastic strain rates:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{3\mathbf{s}}{2\sigma_y} \tag{55}$$

This formulation (Equation (55)) accurately captures the plastic response of polycrystalline metals like steel and aluminum under monotonic loading (Johnson and Cook, 1983 [38]), with hardening laws often specified as:

$$\sigma_y(\bar{\varepsilon}^p) = \sigma_{y0} + h(\bar{\varepsilon}^p)^n \tag{56}$$

For geomaterials exhibiting pressure-sensitivity and dilatancy, the Drucker-Prager criterion modifies the J_2 framework through (Drucker and Prager, 1952 [7]):

$$f(\boldsymbol{\sigma}) = \sqrt{J_2} + \alpha I_1 - \kappa = 0 \tag{57}$$

In Equation (57) $I_1 = \mathrm{tr}(\boldsymbol{\sigma})$, and α, κ depend on friction angle ϕ and cohesion c:

$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3-\sin\phi)}, \quad \kappa = \frac{6c\cos\phi}{\sqrt{3}(3-\sin\phi)} \tag{58}$$

Non-associated flow becomes essential for realistic volumetric predictions (Rudnicki and Rice, 1975 [33]):

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\left(\frac{\mathbf{s}}{2\sqrt{J_2}} + \frac{\beta}{3}\mathbf{I}\right) \tag{59}$$

with $\beta \neq \alpha$ controlling dilatancy in Equation (59). The Mohr-Coulomb criterion provides superior accuracy for frictional materials (Nova, 1982 [34]):

$$f(\boldsymbol{\sigma}) = \frac{\sigma_1 - \sigma_3}{2} - \frac{\sigma_1 + \sigma_3}{2}\sin\phi - c\cos\phi = 0 \tag{60}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are principal stresses in Equation (60). The plastic potential g introduces a dilatancy angle ψ:

$$g(\boldsymbol{\sigma}) = \frac{\sigma_1 - \sigma_3}{2} - \frac{\sigma_1 + \sigma_3}{2}\sin\psi \tag{61}$$

These models exhibit several fundamental limitations. The assumption of rate-independence renders them incapable of capturing creep strains governed by Norton's law (Norton, 1929 [35]):

$$\dot{\varepsilon}^c = A\sigma^n e^{-Q/RT} \tag{62}$$

or stress relaxation phenomena described by Maxwell-type models (Christensen, 1982 [36]):

$$\dot{\boldsymbol{\sigma}} + \frac{\mathbb{C}}{\eta} : \boldsymbol{\sigma} = \mathbb{C} : \dot{\boldsymbol{\varepsilon}} \tag{63}$$

where η is viscosity in Equation (63). High-strain-rate effects observable in Taylor impact tests (Taylor, 1948) require modifications like the Johnson-Cook rate term (Johnson and Cook, 1983 [38]):

$$\sigma_y = (A + B(\bar{\varepsilon}^p)^n)(1 + C\ln\dot{\varepsilon}^*) \tag{64}$$

with $\dot{\varepsilon}^* = \dot{\bar{\varepsilon}}^p/\dot{\varepsilon}_0$ in Equation (64). The absence of microscopic length scales in classical models also fails to predict size effects captured by gradient plasticity (Fleck and Hutchinson, 2001 [39]):

$$\sigma_y = \sigma_\infty + \ell\|\nabla\bar{\varepsilon}^p\| \tag{65}$$

where ℓ is the material length scale in Equation (65). Thermodynamic restrictions (Noll et. al. (1974) [40]) further constrain admissible hardening laws when considering thermal coupling:

$$\dot{s} = \frac{r}{\theta} - \frac{1}{\rho\theta}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \frac{1}{\rho\theta}\sum_{\alpha} A_\alpha \dot{\xi}_\alpha \geq 0 \tag{66}$$

where s is entropy, θ temperature, and A_α thermodynamic forces in Equation (66).

Key references include: Hill (1998) [6] for J_2 theory; Drucker and Prager (1952) [7] for geomaterials; Rudnicki and Rice (1975) [33] for non-associativity; Nova (1982) [34] for Mohr-Coulomb; Norton (1929) [35] for creep; Christensen (2013) [36] for viscoelasticity; Taylor (1948) [37] for dynamic yielding; Johnson and Cook (1983) [38] for rate-dependence; Fleck and Hutchinson (2001) [39] for gradient effects; Noll et. al. (1974) [40] for thermodynamics. Comparative studies appear in Chen and Han (1988) [41], while computational implementations are detailed in Simo and Hughes (2006) [42].

1.6 Recent Advances

Recent advances in rate-independent plasticity theory have significantly expanded the modeling capabilities for complex material behaviors. Crystal plasticity frameworks, pioneered by Asaro (1983) [43], model single crystal deformation through slip system kinematics, where the plastic velocity gradient $\mathbf{L}^p$ is constructed from crystallographic slip rates $\dot{\gamma}^\alpha$ on slip systems α:

$$\mathbf{L}^p = \sum_{\alpha=1}^{N} \dot{\gamma}^\alpha \mathbf{s}^\alpha \otimes \mathbf{m}^\alpha \tag{67}$$

with $\mathbf{s}^\alpha$ and $\mathbf{m}^\alpha$ denoting the slip direction and normal vectors, respectively in Equation (67). The resolved shear stress τ^α activates slip when:

$$|\tau^\alpha| = |\boldsymbol{\sigma} : (\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha)| \geq \tau_y^\alpha \tag{68}$$

where τ_y^α is the critical resolved shear stress in Equation (68). Hardening is governed by matrix evolution equations (Peirce et al., 1983 [45]):

$$\dot{\tau}_y^\alpha = \sum_{\beta=1}^{N} h_{\alpha\beta}|\dot{\gamma}^\beta|, \quad h_{\alpha\beta} = q_{\alpha\beta} h_0 \text{sech}^2\left(\frac{h_0\gamma}{\tau_s - \tau_0}\right) \tag{69}$$

where $q_{\alpha\beta}$ is the latent hardening matrix, h_0 the initial hardening modulus, and τ_s the saturation stress in Equation (69). The multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ (Lee, 1969 [49]) leads to finite strain formulations where the plastic flow rule becomes:

$$\dot{\mathbf{F}}^p\mathbf{F}^{p-1} = \sum_{\alpha=1}^{N} \dot{\gamma}^\alpha \mathbf{s}^\alpha \otimes \mathbf{m}^\alpha \tag{70}$$

Gradient plasticity theories (Fleck and Hutchinson, 2001 [39]) introduce higher-order stresses $\boldsymbol{\Xi}$ to capture size effects through strain gradient dependencies:

$$\boldsymbol{\Xi} = \frac{\partial\psi}{\partial\nabla\boldsymbol{\varepsilon}^p}, \quad \psi = \psi(\boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, \nabla\boldsymbol{\varepsilon}^p) \tag{71}$$

The extended yield condition incorporates a gradient-enhanced effective stress:

$$f = \sigma_{eq}(\boldsymbol{\sigma}, \Xi) - \sigma_y(\bar{\varepsilon}^p, \ell\|\nabla\bar{\varepsilon}^p\|) \leq 0 \tag{72}$$

where ℓ is the intrinsic material length scale in Equation (72). The governing higher-order equilibrium equations become:

$$\text{div}\boldsymbol{\sigma} + \text{div}(\text{div}\Xi) = 0 \tag{73}$$

Data-driven approaches (Kirchdoerfer and Ortiz, 2016 [44]) reformulate plasticity as a minimization problem in the phase space $\mathcal{Z} = \{\boldsymbol{\varepsilon}, \boldsymbol{\sigma}\}$:

$$\min_{(\boldsymbol{\varepsilon},\boldsymbol{\sigma})\in\mathcal{C}} \min_{(\boldsymbol{\varepsilon}^*,\boldsymbol{\sigma}^*)\in\mathcal{D}} \left[\|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^*\|^2_{\mathbb{C}} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\|^2_{\mathbb{C}^{-1}}\right] \tag{74}$$

where $\mathcal{C}$ is the set of compatible strain-stress pairs and $\mathcal{D}$ the material dataset in Equation (74). Machine learning enhancements employ neural networks to approximate yield functions (Ghaboussi et al., 1991 [46]):

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) \approx \mathcal{NN}(\boldsymbol{\sigma}, \boldsymbol{q}; \mathbf{w}), \quad \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial\mathcal{NN}}{\partial\boldsymbol{\sigma}} \tag{75}$$

with weights $\mathbf{w}$ trained on multiscale data in Equation (75). Thermodynamically consistent formulations (Lefik and Schrefler (2003) [47]) ensure:

$$\mathcal{NN}(\boldsymbol{\sigma}, \boldsymbol{q}) \leq 0, \quad \dot{\boldsymbol{q}} = -\dot{\lambda}\frac{\partial\mathcal{NN}}{\partial\boldsymbol{q}} \tag{76}$$

Micromechanically-informed models (Latypov and Kalidindi (2017) [48]) couple crystal plasticity with data science through:

$$\dot{\gamma}^\alpha = \mathcal{F}\left(\tau^\alpha, \tau_y^\alpha; \boldsymbol{\theta}\right), \quad \boldsymbol{\theta} = \text{argmin}\sum_{i=1}^{N}\|\mathbf{T}_i^{exp} - \mathbf{T}_i^{CPFE}(\boldsymbol{\theta})\|^2 \tag{77}$$

where $\mathcal{F}$ is a machine-learned flow rule and $\mathbf{T}$ represents stress-strain responses in Equation (77).

Key references include: Asaro (1983) [43] for crystal plasticity fundamentals; Fleck and Hutchinson (2001) [39] for strain gradient theory; Kirchdoerfer and Ortiz (2016) [44] for data-driven mechanics; Peirce et al. (1983) [45] for hardening models; Ghaboussi et al. (1991) [46] for early neural network applications; Lefik and Schrefler (2003) [47] for thermodynamic machine learning; Latypov and Kalidindi (2017) [48] for multiscale-data fusion; Lee (1969) [49] for kinematic decomposition; Nye (1953) [50] for dislocation density tensors; and Roters et al. (2010) [51] for computational crystal plasticity implementations. Mathematical foundations appear in Ortiz and Repetto (1999) [52], while Bayesian approaches are detailed in Wang and Sun (2018) [53].

2 Literature Review of Rate-Independent Plastic Deformation Models of Euler-Bernoulli Beams

The Euler-Bernoulli beam theory, which postulates that plane sections stay plane and normal to the neutral axis, has been extensively modified to include plastic deformation for analyzing structures subjected to large loads. Rate-independent plasticity models are very significant to quasi-static loading conditions (e.g., civil engineering and mechanical engineering), where the strain rate has little effect on material behavior.

2.1 Classical Plasticity Theory in Beam Models

Classical plasticity theory for beam models stems from ground-breaking work by Hill (1998) [6] and Prager & Hodge (1951) [54], when the theories of plastic deformation for structural element were first developed. The Euler-Bernoulli beam assumption, where plane sections remain plane and normal to the neutral axis, means that plasticity modeling can be reduced to a uniaxial stress state analysis. The yield criterion for such beams is typically expressed in terms of the bending moment M, with the plastic limit given by

$$|M| \leq M_Y \tag{78}$$

where M_Y is the fully plastic moment capacity of the cross-section. For a rectangular beam, this moment is derived as $M_Y = \sigma_Y \frac{bh^2}{4}$, where σ_Y is the yield stress, and b and h are the width and height of the cross-section, respectively.

The theory employs an associated flow rule to describe the evolution of plastic curvature κ^p. When the yield condition

$$f(M) = |M| - M_Y = 0 \tag{79}$$

is satisfied, the plastic curvature rate is given by

$$\dot{\kappa}^p = \dot{\lambda}\,\text{sgn}(M) \tag{80}$$

where $\dot{\lambda}$ in Equation (80) is the plastic multiplier determined from the consistency condition $\dot{f} = 0$. For perfectly plastic materials, M_Y in Equation (79) remains constant while hardening models introduce dependencies such as isotropic where M_Y evolves as

$$M_Y = M_{Y0} + H\kappa^p \tag{81}$$

with H being the hardening modulus in Equation (81).

Hodge (1959) [55] extended these concepts to limit analysis, determining collapse loads for beams under various boundary conditions. The kinematic approach, based on upper-bound theorems, and the static approach, using lower-bound theorems, were developed to bracket the true collapse load. For instance, a simply supported beam with a central point load P collapses when $P = \frac{4M_Y}{L}$, where L is the span length. Prager's work further incorporated kinematic hardening to account for the Bauschinger effect, where the yield surface translates in stress space, modifying the yield condition to

$$f(M, \alpha) = |M - \alpha| - M_Y \leq 0 \tag{82}$$

with α representing the backstress in Equation (82).

Martin (1975) [56] later formalized incremental plasticity for beams using variational principles, enabling numerical implementations. The governing equations combine equilibrium, compatibility, and constitutive laws, with the equilibrium condition for a beam under distributed load $q(x)$ given by $\frac{d^2M}{dx^2} = -q(x)$. Numerical solutions often employ return-mapping algorithms to enforce plasticity constraints, ensuring that stresses remain on the yield surface. These classical models remain foundational for modern computational plasticity in beam structures, though they neglect shear effects and geometric nonlinearities, which later theories address.

2.2 Numerical implementations of Rate-Independent Plasticity in Beam Models

Numerical implementations of rate-independent plasticity in beam models employ several key computational strategies to ensure accuracy and efficiency. The return-mapping algorithm, first formalized by Simo and Hughes (1998) [57], provides a robust framework for plastic stress integration, decomposing each load step into an elastic predictor and plastic corrector phase. For an Euler-Bernoulli beam, the trial elastic moment is computed as

$$M_{n+1}^{\text{trial}} = M_n + EI\Delta\kappa \tag{83}$$

where $\Delta\kappa$ in Equation (83) is the incremental curvature. When the yield condition

$$f(M_{n+1}^{\text{trial}}) = |M_{n+1}^{\text{trial}}| - M_Y > 0 \tag{84}$$

is violated, the plastic corrector step solves the nonlinear equation

$$M_{n+1} = M_{n+1}^{\text{trial}} - EI\Delta\kappa^p \tag{85}$$

with

$$\Delta\kappa^p = \Delta\lambda\,\text{sgn}(M_{n+1}) \tag{86}$$

where the plastic multiplier $\Delta\lambda$ in Equation (86) is determined through the consistency condition $f(M_{n+1}) = 0$. This algorithm maintains numerical stability even for large load increments through its implicit formulation. For thin beams prone to locking phenomena, mixed formulations have been developed that independently interpolate moments and rotations, as demonstrated by de Borst et al. (2012) [58]. The Hu-Washizu variational principle provides the theoretical foundation for these approaches, introducing separate interpolation fields for moments M, curvatures κ, and rotations θ, with the weak form equations

$$\int_L (\delta M(\kappa - \theta') + \delta\kappa(M - EI\kappa^e) + \delta\theta' M)dx = 0 \tag{87}$$

where κ^e in Equation (87) represents the elastic part of curvature. Analytical solutions remain valuable for verification, particularly for fundamental cases like three-point bending where the collapse load can be derived as $P_{\text{limit}} = 4M_Y/L$ for a simply supported beam of length L. More complex loading scenarios often employ piecewise analytical solutions combined with numerical root-finding to locate plastic hinge formations. Recent advances incorporate meshfree methods and isogeometric analysis to handle large deformations, using NURBS basis functions for smoother curvature fields that improve accuracy in plastic hinge regions. The numerical treatment of hardening effects introduces additional complexity, with Armstrong-Frederick type nonlinear hardening requiring the solution of evolution equations like

$$\dot{\alpha} = H\dot{\kappa}^p - \gamma\alpha|\dot{\kappa}^p| \tag{88}$$

for the backstress α, where H and γ are material parameters in Equation (88). Parallel computing implementations have significantly enhanced the efficiency of these algorithms for large-scale structural analysis, particularly when combined with adaptive mesh refinement strategies near plastic hinge locations. The choice between displacement-controlled and force-controlled solution schemes depends on the specific application, with arc-length methods proving particularly effective for tracing equilibrium paths through limit points in post-yield behavior. These numerical approaches collectively enable the accurate simulation of plastic deformation in beam structures while balancing computational cost and solution fidelity.

2.3 Application in Structural Engineering

The rate-independent plastic deformation models of Euler-Bernoulli beams find extensive application in structural engineering, particularly in seismic analysis and progressive collapse assessment of framed structures. The plastic hinge methodology, based on the fundamental yield condition

$$f(M) = |M| - M_Y \leq 0 \tag{89}$$

has been successfully implemented in performance-based earthquake engineering, where the formation sequence of plastic hinges determines the structural ductility. Izzuddin (2008) [59] demonstrated its effectiveness in progressive collapse analysis through the concept of "sudden column removal" scenarios, where the plastic moment capacity M_Y governs the redistribution of forces. For composite steel-concrete beams, the model has been extended by considering layered sections with different yield criteria for each material, modifying the moment capacity to $M_Y = \sum_{i=1}^{n} \sigma_{Y,i} A_i d_i$ where $\sigma_{Y,i}$, A_i, and d_i represent the yield stress, area, and moment arm of the i-th layer respectively. The approach has been further generalized to account for axial force-moment interaction through yield surfaces of the form

$$f(N, M) = (N/N_Y)^2 + |M/M_Y| - 1 \leq 0 \tag{90}$$

where N_Y is the axial yield capacity in Equation (90). In structural optimization, these models have been coupled with reliability analysis to design minimum-weight beams satisfying probabilistic plastic collapse constraints, formulated as $P[M_Y \geq M_{\max}] \geq P_{\text{target}}$. Recent extensions incorporate stochastic material properties through random field representations of the yield stress

$$\sigma_Y(x) = \bar{\sigma}_Y(1 + \epsilon(x)) \tag{91}$$

where $\epsilon(x)$ is a spatially correlated random process in Equation (91). For fire engineering applications, temperature-dependent yield criteria

$$M_Y(T) = M_{Y0}(1 - T/T_c)^\beta \tag{92}$$

have been developed, with T_c representing a critical temperature and β an empirical exponent in Equation (92). The models have also been adapted for functionally graded materials by considering continuously varying yield stress $\sigma_Y(y)$ across the beam depth, requiring numerical integration of

$$M_Y = \int_{-h/2}^{h/2} y \sigma_Y(y) b(y) dy \tag{93}$$

In the context of structural health monitoring, inverse problems have been formulated to estimate plastic deformation history from measured displacements using the relation

$$\kappa^p(x) = \kappa(x) - M(x)/EI \tag{94}$$

where $\kappa^p(x)$ in Equation (94) represents the elastic part of curvature. These applications demonstrate the versatility of the basic rate-independent plastic beam formulation while highlighting the need for ongoing extensions to address emerging engineering challenges in extreme loading environments and novel material systems.

Lubliner's 2008 [60] formulation rigorously addresses the coupling between rate-independent plasticity and dynamic effects through a variational framework that maintains thermodynamic consistency while properly accounting for inertial forces. The fundamental governing equations are derived from the principle of virtual work extended to dynamic conditions:

$$\int_V (\sigma_{ij}\delta\epsilon_{ij} + \rho\ddot{u}_i\delta u_i)dV = \int_{\partial V} t_i\delta u_i dS + \int_V f_i\delta u_i dV \tag{95}$$

where ρ is the mass density and $\ddot{u}_i$ the acceleration field in Equation (95). The key innovation lies in the decomposition of the stress power:

$$\mathcal{P} = \sigma_{ij}\dot{\epsilon}_{ij} = \sigma_{ij}(\dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}) + \rho\ddot{u}_i\dot{u}_i \tag{96}$$

For rate-independent materials, the plastic flow rule remains governed by the consistency condition:

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha = 0 \tag{97}$$

where q_α in Equation (97) represents internal hardening variables. Lubliner's formulation introduces an inertial modification to the yield function:

$$f(\sigma_{ij}, q_\alpha, \ddot{u}_i) = \bar{f}(\sigma_{ij}, q_\alpha) - \frac{1}{2}\rho\|\ddot{u}\|^2 \leq 0 \tag{98}$$

Equation (98) creates a dynamic yield surface that contracts with increasing acceleration, capturing the interaction between plastic dissipation and kinetic energy. The plastic multiplier $\dot{\lambda}$ satisfies the Kuhn-Tucker conditions:

$$\dot{\lambda} \geq 0, \quad f \leq 0, \quad \dot{\lambda}f = 0 \tag{99}$$

even under dynamic loading. The coupled system (Equation (99)) is closed by the equation of motion:

$$\sigma_{ij,j} + f_i = \rho\ddot{u}_i \tag{100}$$

with the constitutive update:

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\epsilon}_{kl} - \dot{\lambda}\frac{\partial f}{\partial \sigma_{kl}}) \tag{101}$$

Lubliner's formulation preserves the rate-independent character of plasticity while properly accounting for inertia through several key features: (1) The dynamic yield condition maintains the consistency condition $\dot{\lambda}\dot{f} = 0$ even during rapid loading; (2) The plastic dissipation inequality:

$$\mathcal{D}^p = \sigma_{ij}\dot{\epsilon}^p_{ij} - q_\alpha\dot{\xi}_\alpha \geq 0 \tag{102}$$

remains satisfied; (3) The formulation reduces exactly to quasi-static plasticity when $\ddot{u}_i \to 0$. For wave propagation problems, the characteristic equation:

$$\det\left[C_{ijkl}n_j n_l - \rho c^2\delta_{ik}\right] = 0 \tag{103}$$

The characteristic equation (Equation (103)) shows how plastic deformation modifies wave speeds c through the tangent modulus C_{ijkl}. The work also establishes uniqueness criteria for the dynamic plastic boundary value problem through the inequality:

$$\int_V (\Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij} + \rho\Delta\ddot{u}_i\Delta\dot{u}_i)dV \geq 0 \tag{104}$$

where Δ in Equation (104) denotes differences between two solutions. The derivation of Equation (104) is done in **Subsubsection** 2.3.1. Numerical implementation requires careful treatment of the inertial-plastic coupling term in the consistent tangent operator:

$$\frac{\partial \Delta\sigma_{ij}}{\partial \Delta\epsilon_{kl}} = C_{ijkl} - \frac{C_{ijmn}\frac{\partial f}{\partial \sigma_{mn}}\frac{\partial f}{\partial \sigma_{pq}}C_{pqkl}}{\frac{\partial f}{\partial \sigma_{rs}}C_{rstu}\frac{\partial f}{\partial \sigma_{tu}} + H + \rho\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}}\delta_{ij}} \tag{105}$$

The derivation of Equation (105) is done in **Subsubsection** 2.3.2. This framework has been particularly influential in problems involving impact and seismic excitations where the timescale separation between loading and plastic response becomes ambiguous. The theory demonstrates that while the material response remains rate-independent, the structural response exhibits apparent rate effects due to the kinetic energy contribution to the dynamic yield condition.

Lubliner's dynamic plasticity formulation requires careful derivation of the consistent tangent operator to maintain numerical stability in time integration schemes. The complete expression for the tangent modulus in the presence of inertial coupling is derived as follows. Beginning with the rate form of the constitutive equations:

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\epsilon}_{kl} - \dot{\lambda}\frac{\partial f}{\partial \sigma_{kl}}) \tag{106}$$

where the plastic multiplier $\dot{\lambda}$ in Equation (106) is determined from the consistency condition:

$$\dot{\lambda} = \frac{\frac{\partial f}{\partial \sigma_{mn}}C_{mnkl}\dot{\epsilon}_{kl} + \frac{\partial f}{\partial \ddot{u}_i}\ddot{u}_i}{\frac{\partial f}{\partial \sigma_{pq}}C_{pqrs}\frac{\partial f}{\partial \sigma_{rs}} + H} \tag{107}$$

with H in Equation (107) representing the hardening modulus. The inertial coupling term appears in the numerator through $\frac{\partial f}{\partial \ddot{u}_i} = -\rho\ddot{u}_i$. Using Equations (106) and (107), the complete consistent tangent operator shall be:

$$\mathbb{C}^{ep}_{ijkl} = \frac{\partial \Delta\sigma_{ij}}{\partial \Delta\epsilon_{kl}} = C_{ijkl} - \frac{\left(C_{ijmn}\frac{\partial f}{\partial \sigma_{mn}} + \rho\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}}\delta_{ij}\right)\left(C_{pqkl}\frac{\partial f}{\partial \sigma_{pq}} + \rho\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}}\delta_{pq}\right)}{\frac{\partial f}{\partial \sigma_{ab}}C_{abcd}\frac{\partial f}{\partial \sigma_{cd}} + \rho\left\|\frac{\partial \ddot{u}}{\partial \epsilon}\right\|^2 + H} \tag{108}$$

where the acceleration derivative term $\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}}$ in Equation (108) is obtained from the discretized equation of motion:

$$\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}} = M^{-1}_{ij}\frac{\partial F^{int}_j}{\partial \epsilon_{kl}} \tag{109}$$

with M_{ij} being the consistent mass matrix and F^{int}_j the internal force vector in Equation (109). This exact formulation preserves several crucial properties:

1. Quadratic convergence in Newton-Raphson iterations when properly linearized
2. Preservation of the symmetry of the tangent operator when the mass matrix is consistent
3. Thermodynamic consistency through the maintained inequality:

$$\sigma_{ij}\dot{\epsilon}^p_{ij} - q_\alpha\dot{\xi}_\alpha - \frac{1}{2}\rho\frac{d}{dt}(\dot{u}_i\dot{u}_i) \geq 0 \tag{110}$$

The formulation also properly handles the dynamic yield surface evolution:

$$f(\sigma_{ij}, q_\alpha, \ddot{u}_i) = \bar{f}(\sigma_{ij}, q_\alpha) - \frac{1}{2}\rho \ddot{u}_i \ddot{u}_i \Delta t^2 \leq 0 \tag{111}$$

where Δt in Equation (111) appears due to the time discretization. This completes the rigorous description of the inertial coupling in Lubliner's rate-independent plasticity framework.

2.3.1 Derivation of Inequality (Equation (104)) that establishes the Uniqueness criteria for the dynamic plastic boundary value problem

To derive the uniqueness inequality for the dynamic plastic boundary value problem, we begin by considering two distinct solutions to the governing equations. Let the first solution be denoted by $(\sigma_{ij}^{(1)}, \dot{\epsilon}_{ij}^{(1)}, \ddot{u}_i^{(1)}, \dot{u}_i^{(1)})$ and the second solution by $(\sigma_{ij}^{(2)}, \dot{\epsilon}_{ij}^{(2)}, \ddot{u}_i^{(2)}, \dot{u}_i^{(2)})$. The differences between these solutions are defined as $\Delta\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}$, $\Delta\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{(1)} - \dot{\epsilon}_{ij}^{(2)}$, $\Delta\ddot{u}_i = \ddot{u}_i^{(1)} - \ddot{u}_i^{(2)}$, and $\Delta\dot{u}_i = \dot{u}_i^{(1)} - \dot{u}_i^{(2)}$. The momentum balance equation for each solution is given by $\sigma_{ij,j}^{(k)} + f_i = \rho\ddot{u}_i^{(k)}$ for $k = 1, 2$. Subtracting these two equations yields the difference in momentum balance:

$$\Delta\sigma_{ij,j} = \rho\Delta\ddot{u}_i \tag{112}$$

The strain rate decomposition for each solution is $\dot{\epsilon}_{ij}^{(k)} = \dot{\epsilon}_{ij}^{e(k)} + \dot{\epsilon}_{ij}^{p(k)}$, and the elastic constitutive relation is $\sigma_{ij}^{(k)} = C_{ijkl}\dot{\epsilon}_{kl}^{e(k)}$. Taking the difference of Equation (112) between the two solutions gives:

$$\Delta\sigma_{ij} = C_{ijkl}\Delta\dot{\epsilon}_{kl}^{e} \tag{113}$$

The plastic flow rule in Equation (113) for associative plasticity is $\dot{\epsilon}_{ij}^{p(k)} = \dot{\gamma}^{(k)}\frac{\partial f}{\partial\sigma_{ij}}\big|_{(k)}$, leading to the difference in plastic strain rates:

$$\Delta\dot{\epsilon}_{ij}^{p} = \dot{\gamma}^{(1)}\frac{\partial f}{\partial\sigma_{ij}}\Big|_{(1)} - \dot{\gamma}^{(2)}\frac{\partial f}{\partial\sigma_{ij}}\Big|_{(2)} \tag{114}$$

To derive the uniqueness condition, we multiply the difference in momentum balance (Equation (112)) by $\Delta\dot{u}_i$ and integrate over the volume V:

$$\int_V \Delta\sigma_{ij,j}\Delta\dot{u}_i\, dV = \int_V \rho\Delta\ddot{u}_i\Delta\dot{u}_i\, dV \tag{115}$$

Applying integration by parts to the left-hand side of Equation (115) gives:

$$\int_{\partial V} \Delta\sigma_{ij}n_j\Delta\dot{u}_i\, dS - \int_V \Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}\, dV = \int_V \rho\Delta\ddot{u}_i\Delta\dot{u}_i\, dV \tag{116}$$

Assuming homogeneous boundary conditions ($\Delta\dot{u}_i = 0$ on ∂V), the boundary term in LHS of Equation (116) vanishes, and we obtain:

$$-\int_V \Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}\, dV = \int_V \rho\Delta\ddot{u}_i\Delta\dot{u}_i\, dV \tag{117}$$

Rearranging the above Equation (117) yields:

$$\int_V \Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}\, dV + \int_V \rho\Delta\ddot{u}_i\Delta\dot{u}_i\, dV = 0 \tag{118}$$

Using Equation (114), The difference in plastic dissipation is given by:

$$\Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}^{p} = \Delta\sigma_{ij}\left(\dot{\gamma}^{(1)}\frac{\partial f}{\partial\sigma_{ij}}\Big|_{(1)} - \dot{\gamma}^{(2)}\frac{\partial f}{\partial\sigma_{ij}}\Big|_{(2)}\right) \tag{119}$$

Due to the principle of maximum plastic dissipation and the convexity of the yield surface, this term $\Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}^{p}$ in LHS of Equation (119) is non-negative:

$$\Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}^{p} \geq 0 \tag{120}$$

The total strain rate difference can be written as:

$$\Delta\dot{\epsilon}_{ij} = \Delta\dot{\epsilon}_{ij}^{e} + \Delta\dot{\epsilon}_{ij}^{p} \tag{121}$$

Using Equations (113) and (121), we can therefore write:

$$\Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij} = C_{ijkl}\Delta\dot{\epsilon}_{ij}^{e}\Delta\dot{\epsilon}_{kl}^{e} + \Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}^{p} \geq 0 \tag{122}$$

Combining these results with the earlier equation, we have:

$$\int_V \Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}\,dV + \int_V \rho\Delta\ddot{u}_i\Delta\dot{u}_i\,dV \geq 0 \tag{123}$$

This inequality (Equation (123)) ensures that the difference between two solutions does not grow, guaranteeing uniqueness for the dynamic plastic boundary value problem. The non-negativity of $\Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij}$ arises from the convexity of the yield condition and the associative flow rule, while the kinetic energy term $\rho\Delta\ddot{u}_i\Delta\dot{u}_i$ ensures stability in the dynamic response.

The final uniqueness criterion is therefore:

$$\boxed{\int_V \left(\Delta\sigma_{ij}\Delta\dot{\epsilon}_{ij} + \rho\Delta\ddot{u}_i\Delta\dot{u}_i\right) dV \geq 0} \tag{124}$$

This inequality (Equation (124)) holds due to the dissipative nature of plasticity and the energy stability of the dynamic system. It ensures that any two solutions cannot diverge, thereby establishing uniqueness for the problem.

2.3.2 Derivation of the Consistent Tangent Operator (Equation (105))

The derivation of the Consistent Tangent Operator (Equation (105)) begins with the constitutive update for the stress increment in elastoplasticity, expressed as

$$\Delta\sigma_{ij} = C_{ijkl}(\Delta\epsilon_{kl} - \Delta\epsilon_{kl}^{p}) \tag{125}$$

where C_{ijkl} is the elastic stiffness tensor, $\Delta\epsilon_{kl}$ is the total strain increment, and $\Delta\epsilon_{kl}^{p}$ is the plastic strain increment in Equation (125). For associative plasticity, the plastic strain increment follows the flow rule

$$\Delta\epsilon_{ij}^{p} = \Delta\gamma\frac{\partial f}{\partial\sigma_{ij}} \tag{126}$$

with $\Delta\gamma$ being the plastic multiplier increment and $f(\sigma_{ij}, q_\alpha)$ the yield function in Equation (126). The consistency condition during plastic loading requires that the updated stress state remains on the yield surface, leading to

$$f(\sigma_{ij} + \Delta\sigma_{ij}, q_\alpha + \Delta q_\alpha) = 0 \tag{127}$$

Linearizing the above condition (Equation (127)) gives

$$\frac{\partial f}{\partial \sigma_{ij}}\Delta\sigma_{ij} + \frac{\partial f}{\partial q_\alpha}\Delta q_\alpha = 0 \tag{128}$$

Substituting the stress increment and assuming isotropic hardening ($\Delta q_\alpha = -H\Delta\gamma$) in Equation (128), the consistency condition becomes

$$\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl}\Delta\epsilon_{kl} - \Delta\gamma\left(\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl}\frac{\partial f}{\partial \sigma_{kl}} + H\right) = 0 \tag{129}$$

Solving for $\Delta\gamma$ in Equation (129) yields

$$\Delta\gamma = \frac{\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl}\Delta\epsilon_{kl}}{\frac{\partial f}{\partial \sigma_{rs}} C_{rstu}\frac{\partial f}{\partial \sigma_{tu}} + H} \tag{130}$$

The inertial effects are incorporated through the dynamic yield condition

$$f(\sigma_{ij}, q_\alpha, \ddot{u}_i) = \bar{f}(\sigma_{ij}, q_\alpha) - \frac{1}{2}\rho\|\ddot{u}\|^2 \leq 0 \tag{131}$$

where ρ is the mass density and $\ddot{u}_i$ the acceleration in Equation (131). The acceleration is related to the strain via the equation of motion $\rho\ddot{u}_i = \sigma_{ij,j} + f_i$. For small increments, the linearized dependence of acceleration on strain is approximated as $\rho\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}} \approx \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}}\delta_{ij}$. The total stress increment is then expressed as

$$\Delta\sigma_{ij} = C_{ijkl}\Delta\epsilon_{kl} - \Delta\gamma C_{ijkl}\frac{\partial f}{\partial \sigma_{kl}} \tag{132}$$

Substituting $\Delta\gamma$ from the consistency condition in to Equation (132), we get

$$\Delta\sigma_{ij} = C_{ijkl}\Delta\epsilon_{kl} - \frac{C_{ijmn}\frac{\partial f}{\partial \sigma_{mn}}\frac{\partial f}{\partial \sigma_{pq}} C_{pqkl}\Delta\epsilon_{kl}}{\frac{\partial f}{\partial \sigma_{rs}} C_{rstu}\frac{\partial f}{\partial \sigma_{tu}} + H + \rho\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}}\delta_{ij}} \tag{133}$$

The consistent tangent operator is obtained by differentiating $\Delta\sigma_{ij}$ with respect to $\Delta\epsilon_{kl}$. The hardening modulus H in Equation (133) is generalized to include gradient effects or other internal variables, but for simplicity, it is retained as a scalar. The inertial coupling term $\rho\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}}\delta_{ij}$ in Equation (133) modifies the denominator, ensuring the tangent operator accounts for dynamic effects. This leads to the final form of the consistent tangent operator:

$$\boxed{\frac{\partial \Delta\sigma_{ij}}{\partial \Delta\epsilon_{kl}} = C_{ijkl} - \frac{C_{ijmn}\frac{\partial f}{\partial \sigma_{mn}}\frac{\partial f}{\partial \sigma_{pq}} C_{pqkl}}{\frac{\partial f}{\partial \sigma_{rs}} C_{rstu}\frac{\partial f}{\partial \sigma_{tu}} + H + \rho\frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}}\delta_{ij}}} \tag{134}$$

The first term C_{ijkl} in Equation (134) represents the elastic stiffness, while the second term accounts for plastic softening, scaled by the hardening modulus and inertial coupling. In Equation (134), the denominator includes the plastic modulus $\frac{\partial f}{\partial \sigma_{rs}} C_{rstu} \frac{\partial f}{\partial \sigma_{tu}}$, the hardening term H, and the inertial coupling term $\rho \frac{\partial \ddot{u}_i}{\partial \epsilon_{kl}} \delta_{ij}$, which introduces rate-dependence. This formulation ensures numerical stability in dynamic problems where plastic deformation and inertial effects are strongly coupled, such as impact or seismic loading. The derivation highlights the interplay between the consistency condition in plasticity, the linearization of the dynamic yield condition, and the coupling between stress and acceleration through the equation of motion. The final form of the tangent operator is critical for implicit dynamic simulations, where accurate stress updates are essential for convergence and stability.

2.4 Challenges and Open Questions

Significant challenges persist in extending rate-independent plastic beam models to address complex real-world behaviors. The neglect of shear effects in Euler-Bernoulli formulations becomes particularly problematic for thick beams ($L/h < 5$) where shear deformation contributes substantially to plastic work dissipation. While modified theories propose yield criteria of the form:

$$f(M, V) = (M/M_Y)^2 + (V/V_Y)^2 - 1 \leq 0 \tag{135}$$

the interaction parameter α in $V_Y = \alpha \sigma_Y A/\sqrt{3}$ in Equation (135) remains poorly quantified experimentally for various materials. Geometric nonlinearity introduces additional complexity through the P-Δ coupling term in the equilibrium equations:

$$\frac{d^2}{dx^2}(EI\frac{d^2w}{dx^2}) + P\frac{d^2w}{dx^2} = q(x) - \frac{d^2M_p}{dx^2} \tag{136}$$

where large deflections alter both the stiffness matrix:

$$\mathbf{K}_G = \int_0^L P(\frac{d\mathbf{N}}{dx})^T(\frac{d\mathbf{N}}{dx})dx \tag{137}$$

and plastic hinge formation sequence. Experimental validation poses another major challenge, as evidenced by the 15-20% discrepancy between predicted and observed collapse loads in slender steel beams under combined loading, where:

$$\lambda_{exp}/\lambda_{pred} = 0.82 \pm 0.07 \tag{138}$$

across 32 tests. The calibration of length scale parameters in nonlocal formulations:

$$\bar{\sigma}(x) = \int_0^L \phi(|x - \xi|, \ell)\sigma(\xi)d\xi \tag{139}$$

remains ambiguous for different material classes. For composite beams, the interface plasticity condition:

$$[\tau] = \tau_Y(1 - \exp(-\beta[\![u]\!])) \tag{140}$$

lacks universal acceptance due to conflicting experimental data. Open questions persist regarding: (1) The proper weighting of shear vs. bending plasticity contributions in the flow rule:

$$\dot{\epsilon}^p = \dot{\lambda}((1-\alpha)\frac{\partial f}{\partial \sigma} + \alpha\frac{\partial f}{\partial \tau}) \tag{141}$$

(2) The validity of quadratic interaction terms:

$$f = (\frac{M}{M_Y})^n + (\frac{V}{V_Y})^m - 1 \leq 0 \tag{142}$$

where n, m in Equation (142) may differ from 2; (3) The treatment of unloading/reloading paths in large rotation plasticity where $\theta_p > 0.2$ rad; (4) The development of unified experimental protocols for plastic hinge characterization under combined loading; and (5) The extension to stochastic material properties where:

$$\sigma_Y(x) = \bar{\sigma}_Y(1 + \epsilon(x)) \tag{143}$$

with $\epsilon(x)$ in Equation (143) satisfying the following equation:

$$\langle \epsilon(x)\epsilon(y) \rangle = \sigma^2 \exp(-|x - y|/\ell_c) \tag{144}$$

These unresolved issues highlight the need for coordinated computational, theoretical, and experimental efforts to advance plastic beam modeling.

2.4.1 Shear Effects

Reddy's higher-order beam theories [61] [62] address the critical limitation of Euler-Bernoulli theory by incorporating shear deformation effects through a parabolic variation of transverse shear strain across the beam depth. The displacement field in Reddy's beam theory is expressed as:

$$u(x, z) = u_0(x) - z\frac{dw_0}{dx} + f(z)\phi_x(x), \quad f(z) = z\left(1 - \frac{4z^2}{3h^2}\right) \tag{145}$$

where u_0 and w_0 are midplane displacements, and ϕ_x represents the rotation due to shear in Equation (145). The shear yield condition is incorporated through a modified von Mises criterion that accounts for through-thickness stress variations:

$$f(\sigma, \tau) = \left(\frac{\sigma_x}{\sigma_Y}\right)^2 + \left(\frac{\tau_{xz}}{\tau_Y}\right)^2 - 1 \leq 0, \quad \tau_Y = \frac{\sigma_Y}{\sqrt{3}} \tag{146}$$

The resultant constitutive equations couple bending and shear plasticity through:

$$\begin{bmatrix} M \\ Q \end{bmatrix} = \begin{bmatrix} D_b & D_{bs} \\ D_{bs} & D_s \end{bmatrix} \begin{bmatrix} \kappa^p \\ \gamma^p \end{bmatrix} \tag{147}$$

where $D_b = EI$, $D_s = \frac{5}{6}GA$, and D_{bs} in Equation (147) represents the bending-shear coupling modulus. The plastic flow rule incorporates shear effects through:

$$\dot{\epsilon}_{ij}^p = \dot{\lambda}\left(\frac{\partial f}{\partial \sigma_{ij}} + \beta\frac{\partial f}{\partial \tau_{ij}}\right) \tag{148}$$

with β in Equation (148) quantifies the shear-bending interaction strength. Numerical implementations employ a modified return-mapping algorithm that simultaneously satisfies both bending and shear yield conditions:

$$\Delta\gamma^p = \Delta\lambda\left(\frac{\tau_{xz}}{|\tau_{xz}|} + \beta\frac{\partial f}{\partial \sigma_x}\right) \tag{149}$$

The theory demonstrates superior accuracy for thick beams where shear deformation contributes significantly to plastic hinge formation, particularly in composite laminates and functionally graded materials. Experimental validations show prediction errors below 8 percentage for $L/h < 5$ compared to 25-30 percentage errors from classical plasticity theories. Recent extensions incorporate strain gradient effects through a length-scale dependent yield condition:

$$f = \bar{f}(\sigma, \tau) + \ell^2 \nabla^2 \bar{f} - 1 \leq 0 \tag{150}$$

where ℓ in Equation (150) is the material length scale capturing microstructural effects on shear yield behavior. These developments have proven particularly valuable in analyzing polymer composites and sandwich structures where shear yielding precedes bending failure.

2.4.2 Geometric Nonlinearity

The treatment of geometric nonlinearity in plastic beam analysis represents a critical advancement in structural mechanics, particularly for scenarios involving large deflections and post-buckling behavior. Bazant and Cedolin's (1991) [63] foundational work demonstrates that when beam deflections become comparable to member dimensions, the conventional small-deflection plasticity theory becomes inadequate due to two primary effects: (1) significant changes in internal force distributions caused by P-Δ effects, and (2) altered plastic hinge formation patterns. The nonlinear moment-curvature relationship must be modified to account for finite rotations:

$$M = M_Y \left(1 + \frac{3}{2}\theta_p^2\right) \quad \text{for} \quad \theta_p > 0.1 \text{ rad} \tag{151}$$

where θ_p is the plastic rotation at the hinge in Equation (151). The equilibrium equations in the deformed configuration introduce additional coupling terms:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 w}{dx^2}\right) + P\frac{d^2 w}{dx^2} = q(x) - \frac{d^2 M_p}{dx^2} \tag{152}$$

with P representing the axial force and w the lateral deflection in Equation (152). The plastic hinge formation criterion becomes displacement-dependent:

$$f(M, P, w) = \left|\frac{M}{M_Y(w)}\right| + \left(\frac{P}{P_Y}\right)^2 - 1 \leq 0 \tag{153}$$

where $M_Y(w) = M_{Y0}(1 - \alpha\|w\|/L)$ in Equation (153) accounts for deflection-induced strength reduction. Bazant and Cedolin's stability analysis reveals that large deflections can precipitate premature hinge formation through the amplification factor:

$$\lambda_{cr} = \frac{\pi^2 EI}{L^2}\left(1 - \frac{M_p}{M_Y}\right) \tag{154}$$

Numerical implementation requires incremental-iterative procedures due to the path-dependent nature of the problem. The tangent stiffness matrix incorporates both geometric and material nonlinearities:

$$\mathbf{K}_T = \mathbf{K}_E + \mathbf{K}_G(w) + \mathbf{K}_P(\epsilon^p) \tag{155}$$

where $\mathbf{K}_G$ is the geometric stiffness dependent on current displacements, and $\mathbf{K}_P$ reflects plastic softening in Equation (155). Experimental validations show that neglecting these nonlinearities can overestimate collapse loads by up to 40% in slender members (Bazant & Cedolin, 1991, p. 427). The theory has been extended to address dynamic progressive collapse through the nonlinear dynamic amplification factor:

$$D_{nl} = 1 + \beta \left(\frac{\theta_{max}}{\theta_Y} - 1 \right)^2 \tag{156}$$

where θ_{max} is the maximum achieved rotation and θ_Y the yield rotation in Equation (156). These formulations enable accurate prediction of plastic hinge migration and collapse mechanisms in real-world structures experiencing large displacements.

3 Kinematics: Beam Geometry and Strain Decomposition

The kinematic description of the Euler-Bernoulli beam with gradient plasticity requires a rigorous mathematical framework that precisely captures finite deformations while maintaining consistency with the geometrically exact strain measures. Let $\mathcal{B}_0 \subset \mathbb{R}^3$ denote the reference configuration of the beam with axial coordinate $x \in [0, L]$ and cross-sectional coordinates $(y, z) \in A$, where A is the compact, connected cross-section. The displacement field $\mathbf{u} : \mathcal{B}_0 \to \mathbb{R}^3$ is decomposed as:

$$\mathbf{u}(x, y, z) = u(x)\mathbf{e}_x + w(x)\mathbf{e}_z - z\theta(x)\mathbf{e}_x + \mathbf{u}_{\text{warp}}(x, y, z) \tag{157}$$

where $u(x)$ represents the axial displacement, $w(x)$ the transverse displacement, $\theta(x) = w'(x)$ the rotation under the Euler-Bernoulli constraint, and $\mathbf{u}_{\text{warp}}$ accounts for cross-sectional warping (assumed negligible in standard beam theory) in Equation (157). The total strain tensor $\boldsymbol{\epsilon}$ is derived from the Green-Lagrange strain measure:

$$\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \underbrace{\begin{bmatrix} u' + \frac{1}{2}(w')^2 - zw'' & \text{sym} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{axial strain}} + \underbrace{\begin{bmatrix} 0 & \frac{1}{2}(\gamma_{xy}) & \frac{1}{2}(\gamma_{xz}) \\ \text{sym} & 0 & 0 \\ \text{sym} & 0 & 0 \end{bmatrix}}_{\text{shear terms}} \tag{158}$$

where $\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$ is the deformation gradient in Equation (158). Under the Euler-Bernoulli hypothesis ($\gamma_{xy} = \gamma_{xz} = 0$) in Equation (158), the only non-zero strain component is:

$$\epsilon_{xx}(x, z) = \underbrace{u' + \frac{1}{2}(w')^2}_{\text{membrane strain } \epsilon_0(x)} + \underbrace{(-zw'')}_{\text{bending strain } z\kappa(x)} = \epsilon_0(x) + z\kappa(x) \tag{159}$$

$\kappa(x)$ in Equation (159) is the curvature of the beam at the point x. The plastic strain decomposition follows a multiplicative split $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ of the deformation gradient, which reduces to an additive decomposition of the Green-Lagrange strain for small elastic strains but finite plastic deformations:

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^p + \frac{1}{2}\boldsymbol{\epsilon}^e\boldsymbol{\epsilon}^p + \frac{1}{2}\boldsymbol{\epsilon}^p\boldsymbol{\epsilon}^e + \mathcal{O}((\boldsymbol{\epsilon}^e)^2, (\boldsymbol{\epsilon}^p)^2) \tag{160}$$

For the beam formulation, Equation (160) simplifies to the additive decomposition of the axial strain component:

$$\epsilon_{xx} = \epsilon_{xx}^e + \epsilon_{xx}^p + \frac{1}{2}(\epsilon_{xx}^e \epsilon_{xx}^p + \epsilon_{xx}^p \epsilon_{xx}^e) \approx \epsilon_{xx}^e + \epsilon_{xx}^p \tag{161}$$

when $\|\boldsymbol{\epsilon}^e\|, \|\boldsymbol{\epsilon}^p\| \ll 1$. The plastic strain ϵ_{xx}^p in Equation (161) is assumed constant through the thickness, while the elastic strain ϵ_{xx}^e in Equation (161) varies linearly:

$$\epsilon_{xx}^e(x, z) = \epsilon_0^e(x) + z\kappa^e(x), \quad \epsilon_{xx}^p(x, z) = \epsilon_0^p(x) + z\kappa^p(x) \tag{162}$$

The curvature measures are derived from the geometrically exact expression:

$$\kappa(x) = \frac{w''(x)}{(1 + (w'(x))^2)^{3/2}} \approx w''(x) \quad \text{(for moderate rotations)} \tag{163}$$

Note that the $\kappa(x)$ in Equation (163) is related to the elastic curvature $\kappa^e(x)$ and the plastic curvature $\kappa^e(x)$ in Equation (162) by the following equation

$$\kappa(x) = \kappa^e(x) + \kappa^p(x) \tag{164}$$

The kinematic relations are closed by introducing the internal variable $\alpha(x)$ for isotropic hardening and its gradient $\nabla\alpha$ for gradient effects, which modify the yield condition but do not directly enter the strain decomposition. The complete kinematic description maintains frame indifference under rigid body motions and satisfies the compatibility conditions:

$$\frac{\partial^2 \epsilon_{xx}}{\partial z^2} = 0, \quad \frac{\partial^2 \epsilon_{xx}}{\partial x \partial z} = -\frac{\partial \kappa}{\partial x} \tag{165}$$

which are automatically satisfied by the Euler-Bernoulli ansatz. The displacement field resides in the Sobolev space:

$$(u, w) \in H^1(0, L) \times H^2(0, L) \tag{166}$$

ensuring finite energy for both membrane and bending deformations, while the plastic strains in Equation (162) satisfy:

$$\epsilon_{xx}^p \in BV(0, L), \quad \kappa^p \in BV(0, L) \tag{167}$$

where BV denotes functions of bounded variation, accounting for possible plastic strain localization. This kinematic framework provides the foundation for deriving the equilibrium equations and constitutive relations while maintaining mathematical consistency with both finite deformation theory and the infinitesimal strain limit.

The mathematical formulation of rate-independent plasticity in Euler-Bernoulli beams builds upon the fundamental kinematic assumption that plane sections remain plane and normal to the neutral axis, reducing the three-dimensional plasticity problem to a uniaxial stress state. The constitutive framework begins with the additive decomposition of the curvature

$$\kappa = \kappa^e + \kappa^p \tag{168}$$

where κ^e represents the elastic curvature and κ^p the plastic curvature in Equation (168). The elastic response follows Hooke's law $M = EI\kappa^e$, with M being the bending moment,

E Young's modulus, and I the second moment of area. The yield criterion is typically expressed as

$$f(M) = |M| - M_Y \leq 0 \tag{169}$$

where M_Y denotes the yield moment capacity, calculated for a rectangular cross-section as $M_Y = \sigma_Y(bh^2/4)$ with σ_Y being the yield stress in Equation (169). When the yield condition is satisfied ($f = 0$), plastic flow occurs according to the associated flow rule

$$\dot{\kappa}^p = \dot{\lambda}(\partial f/\partial M) = \dot{\lambda}\,\mathrm{sgn}(M) \tag{170}$$

where $\dot{\lambda}$ is the plastic multiplier subject to the Kuhn-Tucker conditions $\dot{\lambda} \geq 0$, $f \leq 0$, and $\dot{\lambda}f = 0$ in Equation (170). The consistency condition $\dot{\lambda}\dot{f} = 0$ ensures the stress state remains on the yield surface during plastic loading. For materials exhibiting hardening behavior, the yield surface evolves according to either isotropic hardening

$$M_Y = M_{Y0} + H\kappa^p \tag{171}$$

or kinematic hardening through a backstress variable α in Equation (171) modifying the yield function to

$$f(M, \alpha) = |M - \alpha| - M_{Y0} \leq 0 \tag{172}$$

with the evolution law $\dot{\alpha} = H\dot{\kappa}^p$ in Equation (172). The numerical implementation typically employs a return-mapping algorithm, where an elastic predictor step

$$M_{n+1}^{\text{trial}} = M_n + EI\Delta\kappa \tag{173}$$

is followed by a plastic corrector step that projects the trial stress back onto the yield surface when $f(M_{n+1}^{\text{trial}}) > 0$, solving the nonlinear equation

$$M_{n+1} = M_{n+1}^{\text{trial}} - EI\Delta\kappa^p \tag{174}$$

with $\Delta\kappa^p = \Delta\lambda\,\mathrm{sgn}(M_{n+1})$ in Equation (174). This formulation maintains objectivity under large deformations when coupled with proper geometric nonlinearity treatment, though the classical theory neglects shear deformation effects. The equilibrium equations complete the system, with the strong form given by

$$d^2M/dx^2 + q(x) = 0 \tag{175}$$

for distributed loading $q(x)$ in Equation (175), requiring appropriate boundary conditions for well-posed solutions. These mathematical constructs form the basis for both analytical solutions of simple plastic hinge problems and finite element implementations for complex structural analyses.

4 Balance Laws: Mechanics and Thermodynamics

4.1 Linear and Angular Momentum Balance

For any arbitrary subdomain $\omega \subset [0, L]$, the rate of change of linear momentum equals the sum of external forces:

$$\frac{d}{dt}\int_\omega \rho A\dot{u}\,dx = \underbrace{[N]_{\partial\omega}}_{\text{internal}} + \underbrace{\int_\omega b_x\,dx}_{\text{external}} \tag{176}$$

where ρ is the mass density per unit volume, A is the cross-sectional area and b_x is the axial body force in Equation (176). Applying the divergence theorem and localizing yields:

$$\rho A\ddot{u} = \frac{\partial N}{\partial x} + b_x \tag{177}$$

For quasi-static problems ($\ddot{u} \approx 0$) in Equation (177) with no body forces:

$$\boxed{\frac{\partial N}{\partial x} = 0 \quad \text{(Strong form of linear momentum balance)}} \tag{178}$$

Similarly, The moment equilibrium about the z-axis requires:

$$\frac{d}{dt}\int_\omega \rho I \ddot{w}'\,dx = \underbrace{[M]_{\partial\omega} - \int_\omega V\,dx}_{\text{internal}} + \underbrace{\int_\omega (q + zb_z)\,dx}_{\text{external}} \tag{179}$$

where $V = \frac{\partial M}{\partial x}$ is the shear force (reactive quantity), I is the second moment of area and b_z is the transverse body force in Equation (179). Localizing and neglecting rotational inertia ($\ddot{w}' \approx 0$) in Equation (179) and body forces:

$$\frac{\partial M}{\partial x} - V = 0 \tag{180}$$

$$\frac{\partial V}{\partial x} + q = 0 \tag{181}$$

Combining the above 2 Equations (180) and (181) gives:

$$\boxed{\frac{\partial^2 M}{\partial x^2} + q = 0 \quad \text{(Strong form of angular momentum balance)}} \tag{182}$$

The stress resultants (axial force N and bending moment M) therefore satisfy:

$$N' = 0, \quad M'' + q = 0 \tag{183}$$

where $q(x)$ is the transverse load in Equation (183). The stress resultants are defined through the thickness integration:

$$N(x,t) = \int_{-h/2}^{h/2} \sigma(x,z,t)\,dz, \quad M(x,t) = -\int_{-h/2}^{h/2} z\sigma(x,z,t)\,dz \tag{184}$$

where σ is the axial Cauchy stress in Equation (184). The linear momentum balance in strong form is derived by considering forces on an infinitesimal element:

$$\rho A\frac{\partial^2 u}{\partial t^2} = \frac{\partial N}{\partial x} + b_x \tag{185}$$

$$\rho A\frac{\partial^2 w}{\partial t^2} = \frac{\partial V}{\partial x} + q + b_z \tag{186}$$

where V is the shear force, q the distributed transverse load, and b_x, b_z body forces in Equations (185) and (186). For quasi-static conditions, these 2 Equations (185) and (186) reduce to:

$$\frac{\partial N}{\partial x} = 0 \tag{187}$$

$$\frac{\partial V}{\partial x} + q = 0 \tag{188}$$

The angular momentum balance about the neutral axis gives:

$$\frac{\partial M}{\partial x} - V = 0 \tag{189}$$

Combining Equation (189) with the transverse equilibrium Equation (188) yields the fundamental equation:

$$\frac{\partial^2 M}{\partial x^2} + q = 0 \tag{190}$$

4.2 Thermodynamic Framework

4.2.1 Helmholtz Free Energy Density

The Helmholtz free energy density ψ for the rate-independent plasticity model of an Euler-Bernoulli beam must rigorously satisfy the principles of continuum thermodynamics and account for both elastic and plastic contributions. We construct it as a function of the elastic strain measures (ϵ^e, κ^e) and internal hardening variables α:

$$\psi(\epsilon^e, \kappa^e, \alpha) = \underbrace{\frac{1}{2}EA(\epsilon^e)^2 + \frac{1}{2}EI(\kappa^e)^2}_{\text{elastic storage}} + \underbrace{\frac{1}{2}H\alpha^2}_{\text{hardening potential}} \tag{191}$$

Here EA and EI represent the axial and bending stiffness, while H is the hardening modulus in Equation (191). The quadratic form ensures convexity, guaranteeing thermodynamic stability. The elastic terms are derived from the Saint-Venant-Kirchhoff constitutive assumption, linearizing the stored elastic energy for small elastic strains. The hardening term models isotropic hardening through the internal variable α in Equation (191), which evolves with plastic deformation. The thermodynamic conjugates are obtained via exact differentiation:

$$N = \frac{\partial \psi}{\partial \epsilon^e} = EA\epsilon^e, \quad M = \frac{\partial \psi}{\partial \kappa^e} = EI\kappa^e, \quad Q = -\frac{\partial \psi}{\partial \alpha} = -H\alpha \tag{192}$$

where N is the axial force, M the bending moment, and Q the thermodynamic force conjugate to α in Equation (192). The negative sign in Q in Equation (192) reflects that hardening opposes plastic flow. The free energy must satisfy the Coleman-Noll inequality for dissipation:

$$\mathcal{D} = N\dot{\epsilon}^p + M\dot{\kappa}^p + Q\dot{\alpha} \geq 0 \tag{193}$$

For rate-independent plasticity, this reduces to the constraint that plastic work must be non-negative. The hardening law $\dot{\alpha} = \dot{\gamma}$ with $\dot{\gamma} \geq 0$ ensures $\mathcal{D} = (N\partial_N f + M\partial_M f + Q)\dot{\gamma} \geq 0$, where f is the yield function in Equation (193). The free energy's convexity in all arguments ($\partial^2_{\epsilon^e}\psi = EA > 0$, $\partial^2_{\kappa^e}\psi = EI > 0$, $\partial^2_{\alpha}\psi = H > 0$) ensures:

1. Uniqueness of the elastic response
2. Positive dissipation for arbitrary loading paths
3. Ellipticity of the incremental equilibrium equations

For gradient-enhanced models, we augment ψ with non-local terms:

$$\psi_{\text{grad}} = \psi(\epsilon^e, \kappa^e, \alpha) + \frac{1}{2}H_g\|\nabla\alpha\|^2 \tag{194}$$

where H_g is the gradient hardening modulus in Equation (194). This introduces a microstress $\sigma = H_g \nabla\alpha$ in Equation (194) and modifies the yield condition to include higher-order stresses. The free energy density thus provides the complete thermodynamic foundation for both local and non-local plasticity formulations.

4.2.2 Clausius-Duhem Inequality (2nd Law) applied to the rate-independent plasticity model without gradient enhancing

The Clausius-Duhem inequality for the rate-independent plasticity model emerges as the fundamental thermodynamic restriction governing irreversible processes. For the Euler-Bernoulli beam formulation, we consider the local dissipation inequality in its reduced form, derived from the second law of thermodynamics under isothermal conditions:

$$\mathcal{D} = \sigma : \dot{\epsilon} - \dot{\psi} \geq 0 \tag{195}$$

where σ represents the Cauchy stress tensor, $\dot{\epsilon}$ the strain rate tensor, and ψ the Helmholtz free energy density in Equation (195). For the beam model, this translates to stress resultants working on their conjugate strain rates:

$$\mathcal{D} = N\dot{\epsilon} + M\dot{\kappa} - \dot{\psi}(\epsilon^e, \kappa^e, \alpha) \geq 0 \tag{196}$$

Substituting the additive strain decomposition $\epsilon = \epsilon^e + \epsilon^p$ and $\kappa = \kappa^e + \kappa^p$, along with the time derivative of the free energy $\psi = \frac{1}{2}EA(\epsilon^e)^2 + \frac{1}{2}EI(\kappa^e)^2 + \frac{1}{2}H\alpha^2$ in Equation (196), we obtain:

$$\mathcal{D} = N(\dot{\epsilon}^e + \dot{\epsilon}^p) + M(\dot{\kappa}^e + \dot{\kappa}^p) - (EA\epsilon^e\dot{\epsilon}^e + EI\kappa^e\dot{\kappa}^e + H\alpha\dot{\alpha}) \geq 0 \tag{197}$$

The reversible (elastic) terms in Equation (197) cancel exactly through the constitutive relations $N = EA\epsilon^e$ and $M = EI\kappa^e$, leaving only the plastic dissipation:

$$\mathcal{D} = N\dot{\epsilon}^p + M\dot{\kappa}^p - Q\dot{\alpha} \geq 0 \tag{198}$$

where $Q = -H\alpha$ in Equation (198) is the thermodynamic force conjugate to the hardening variable. For associative plasticity with yield function $f(N, M, Q)$, the plastic flow rules are derived from the principle of maximum dissipation:

$$\dot{\epsilon}^p = \dot{\gamma}\frac{\partial f}{\partial N}, \quad \dot{\kappa}^p = \dot{\gamma}\frac{\partial f}{\partial M}, \quad \dot{\alpha} = -\dot{\gamma}\frac{\partial f}{\partial Q} \tag{199}$$

Substituting Equation (199) into the dissipation inequality yields:

$$\mathcal{D} = \dot{\gamma}\left(N\frac{\partial f}{\partial N} + M\frac{\partial f}{\partial M} + Q\frac{\partial f}{\partial Q}\right) \geq 0 \tag{200}$$

For the classical von Mises-type yield condition $f(N, M, Q) = \sqrt{N^2 + (M/c)^2} - (N_Y + Q) \leq 0$, Equation (200) becomes:

$$\mathcal{D} = \dot{\gamma}\left(\frac{N^2 + (M/c)^2}{\sqrt{N^2 + (M/c)^2}} + Q\right) = \dot{\gamma}(N_Y + Q) \geq 0 \tag{201}$$

The Kuhn-Tucker conditions $\dot{\gamma} \geq 0$, $f \leq 0$, and $\dot{\gamma}f = 0$ ensure that plastic loading only occurs when $f = 0$, making the dissipation exactly zero during elastic processes and strictly positive during plastic flow. The hardening modulus $H > 0$ in Equation (201) guarantees $Q = -H\alpha \leq 0$ for $\alpha \geq 0$, ensuring $\mathcal{D} \geq 0$ is always satisfied. This framework rigorously maintains thermodynamic consistency by enforcing that all admissible plastic processes must satisfy the inequality pointwise in both space and time. The dissipation measure is furthermore convex and lower semicontinuous in its arguments, ensuring existence of solutions to the incremental variational problem.

4.2.3 Clausius-Duhem Inequality (2nd Law) applied to the rate-independent plasticity model with gradient enhancing

The Clausius-Duhem inequality constitutes the fundamental restriction imposed by the second law of thermodynamics on constitutive equations. For the gradient-enhanced plasticity model with Helmholtz free energy $\psi(\epsilon^e, \kappa^e, \alpha, \nabla\alpha)$, the local dissipation inequality in spatial form reads:

$$\mathcal{D} = \sigma : \dot{\epsilon} + \boldsymbol{\sigma} \cdot \dot{\nabla\alpha} - \dot{\psi} \geq 0 \tag{202}$$

where $\boldsymbol{\sigma}$ represents the microstress conjugate to the gradient of internal variable $\nabla\alpha$ in Equation (202). For the Euler-Bernoulli beam formulation, this translates to stress resultants and higher-order stresses working on their conjugate rates:

$$\mathcal{D} = N\dot{\epsilon} + M\dot{\kappa} + \boldsymbol{\sigma} \cdot \dot{\nabla\alpha} - \dot{\psi}(\epsilon^e, \kappa^e, \alpha, \nabla\alpha) \geq 0 \tag{203}$$

The extended free energy density including gradient effects is given by:

$$\psi = \underbrace{\frac{1}{2}EA(\epsilon^e)^2 + \frac{1}{2}EI(\kappa^e)^2}_{\text{elastic}} + \underbrace{\frac{1}{2}H\alpha^2}_{\text{local hardening}} + \underbrace{\frac{1}{2}H_g\|\nabla\alpha\|^2}_{\text{gradient hardening}} \tag{204}$$

Taking the material time derivative and substituting the strain decomposition $\epsilon = \epsilon^e + \epsilon^p$, $\kappa = \kappa^e + \kappa^p$ in Equation (204) yields:

$$\dot{\psi} = EA\epsilon^e\dot{\epsilon}^e + EI\kappa^e\dot{\kappa}^e + H\alpha\dot{\alpha} + H_g\nabla\alpha \cdot \nabla\dot{\alpha} \tag{205}$$

The dissipation inequality (Equation (203)) thus becomes:

$$\mathcal{D} = N(\dot{\epsilon}^e + \dot{\epsilon}^p) + M(\dot{\kappa}^e + \dot{\kappa}^p) + \boldsymbol{\sigma} \cdot \dot{\nabla\alpha} - (EA\epsilon^e\dot{\epsilon}^e + EI\kappa^e\dot{\kappa}^e + H\alpha\dot{\alpha} + H_g\nabla\alpha \cdot \nabla\dot{\alpha}) \geq 0 \tag{206}$$

Using the constitutive relations $N = EA\epsilon^e$, $M = EI\kappa^e$, $Q = -H\alpha$, and $\boldsymbol{\sigma} = H_g\nabla\alpha$, the elastic terms cancel exactly, leaving the plastic dissipation (Equation (206)) as:

$$\mathcal{D} = N\dot{\epsilon}^p + M\dot{\kappa}^p - Q\dot{\alpha} + \boldsymbol{\sigma} \cdot \dot{\nabla\alpha} - H_g\nabla\alpha \cdot \nabla\dot{\alpha} \geq 0 \tag{207}$$

For associative plasticity with yield function $f(N, M, Q, \operatorname{div}\boldsymbol{\sigma})$, the flow rules follow from maximum dissipation:

$$\dot{\epsilon}^p = \dot{\gamma}\frac{\partial f}{\partial N}, \dot{\kappa}^p = \dot{\gamma}\frac{\partial f}{\partial M}, \dot{\alpha} = -\dot{\gamma}\frac{\partial f}{\partial Q}, \dot{\nabla\alpha} = -\dot{\gamma}\nabla\left(\frac{\partial f}{\partial Q}\right) \tag{208}$$

Substituting Equation (208) into the dissipation inequality (Equation (207)) and applying integration by parts to the gradient terms shows the pointwise inequality:

$$\mathcal{D} = \dot{\gamma}\left[N\frac{\partial f}{\partial N} + M\frac{\partial f}{\partial M} + Q\frac{\partial f}{\partial Q} + \operatorname{div}\left(\boldsymbol{\sigma}\frac{\partial f}{\partial Q}\right)\right] \geq 0 \tag{209}$$

The extended yield condition incorporating gradient effects:

$$f(N, M, Q, \operatorname{div} \boldsymbol{\sigma}) = \sqrt{N^2 + (M/c)^2} - (N_Y + Q - \operatorname{div} \boldsymbol{\sigma}) \leq 0 \tag{210}$$

Equation (210) ensures non-negative dissipation (Equation (209)) through the relation:

$$\mathcal{D} = \dot{\gamma}(N_Y + Q - \operatorname{div} \boldsymbol{\sigma}) \geq 0 \tag{211}$$

The Kuhn-Tucker conditions $\dot{\gamma} \geq 0$, $f \leq 0$, $\dot{\gamma} f = 0$ and the positive hardening moduli $H > 0$, $H_g > 0$ guarantee thermodynamic consistency. The dissipation measure remains convex and lower semicontinuous, preserving the existence of solutions to the incremental variational problem while accounting for non-local gradient effects in the hardening response. This formulation rigorously extends classical plasticity to include gradient hardening while maintaining exact thermodynamic admissibility.

5 Constitutive Theory: Plasticity

5.1 Yield Criterion

The yield criterion for the rate-independent plasticity model must rigorously satisfy three fundamental requirements: (1) convexity to ensure uniqueness of solutions, (2) invariance under rigid body transformations, and (3) thermodynamic consistency with the dissipation inequality. For the Euler-Bernoulli beam with combined axial-bending plasticity and gradient hardening, we construct the yield function f as a mapping from the stress resultant space to $\mathbb{R}$:

$$f(N, M, Q, \operatorname{div} \boldsymbol{\sigma}) = \sqrt{N^2 + \left(\frac{M}{c}\right)^2} - (N_Y + Q - \operatorname{div} \boldsymbol{\sigma}) \leq 0 \tag{212}$$

where $c = h/2$ is the half-depth of the beam cross-section, N_Y the initial axial yield strength, $Q = -H\alpha$ the isotropic hardening stress, and $\boldsymbol{\sigma} = H_g \nabla \alpha$ the higher-order microstress in Equation (212). The square root term represents the equivalent stress resultant combining axial force and bending moment, derived from the J_2-plasticity analogy in stress-resultant space. The term $\operatorname{div} \boldsymbol{\sigma}$ in Equation (212) introduces non-local gradient effects through the microstress divergence, modifying the yield surface based on spatial variations of plastic strain. The convexity of f follows from:

1. The L^2-norm $\sqrt{N^2 + (M/c)^2}$ being strictly convex in (N, M)
2. Linear hardening terms Q and $\operatorname{div} \boldsymbol{\sigma}$ preserving convexity
3. The positive definiteness of Hessian matrix $\nabla^2 f$:

$$\nabla^2 f = \begin{bmatrix} \frac{(M/c)^2}{(N^2+(M/c)^2)^{3/2}} & -\frac{NM/c^2}{(N^2+(M/c)^2)^{3/2}} & 0 \\ -\frac{NM/c^2}{(N^2+(M/c)^2)^{3/2}} & \frac{N^2/c^2}{(N^2+(M/c)^2)^{3/2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0 \tag{213}$$

The associative flow rule derives from the principle of maximum plastic dissipation. For the yield function f as defined in Equation (211) we have

$$\dot{\epsilon}^p = \dot{\gamma}\frac{\partial f}{\partial N} = \dot{\gamma}\frac{N}{\sqrt{N^2 + (M/c)^2}} \tag{214}$$

$$\dot{\kappa}^p = \dot{\gamma}\frac{\partial f}{\partial M} = \dot{\gamma}\frac{M/c^2}{\sqrt{N^2 + (M/c)^2}} \tag{215}$$

$$\dot{\alpha} = -\dot{\gamma}\frac{\partial f}{\partial Q} = \dot{\gamma} \tag{216}$$

$$\nabla\dot{\alpha} = -\dot{\gamma}\nabla\left(\frac{\partial f}{\partial Q}\right) = \dot{\gamma}\nabla(1) = \mathbf{0} \tag{217}$$

The Kuhn-Tucker complementary conditions enforce consistency between yielding and loading:

$$\dot{\gamma} \geq 0, \quad f \leq 0, \quad \dot{\gamma} f = 0 \tag{218}$$

During plastic loading ($f = 0$), the consistency condition $\dot{f} = 0$ governs the evolution:

$$\frac{N\dot{N} + (M/c^2)\dot{M}}{\sqrt{N^2 + (M/c)^2}} - \dot{Q} + \operatorname{div}\dot{\boldsymbol{\sigma}} = 0 \tag{219}$$

Substituting the constitutive relations $\dot{N} = EA\dot{\epsilon}^e$, $\dot{M} = EI\dot{\kappa}^e$, $\dot{Q} = -H\dot{\alpha}$, and $\dot{\boldsymbol{\sigma}} = H_g\nabla\dot{\alpha}$ in Equation (219) yields the plastic multiplier:

$$\dot{\gamma} = \frac{\frac{EAN}{\sqrt{N^2+(M/c)^2}}\dot{\epsilon} + \frac{EIM/c^2}{\sqrt{N^2+(M/c)^2}}\dot{\kappa}}{H + H_g\|\nabla\|^2 + \frac{EAN^2}{N^2+(M/c)^2} + \frac{EIM^2/c^4}{N^2+(M/c)^2}} \tag{220}$$

Equation (220) demonstrates the regularizing effect of gradient terms ($H_g\|\nabla\|^2$) on the plastic multiplier, preventing localization pathologies. The yield criterion thus provides a complete description of the elastic domain boundary, flow direction, and hardening evolution while maintaining exact thermodynamic consistency with the Clausius-Duhem inequality.

5.2 Flow Rule (Associative Plasticity)

The associative flow rule for the rate-independent plasticity model emerges from the principle of maximum plastic dissipation, which constitutes a fundamental postulate in the thermodynamic formulation of plasticity theory. For the Euler-Bernoulli beam with combined axial-force/bending-moment yielding and gradient hardening effects, the plastic flow direction derives rigorously from the subdifferential of the yield function $f(N, M, Q, \operatorname{div}\boldsymbol{\sigma})$ in generalized stress space. The flow rule takes the form of a normality condition:

$$(\dot{\epsilon}^p, \dot{\kappa}^p, \dot{\alpha}, \nabla\dot{\alpha}) = \dot{\gamma}\partial f(N, M, Q, \operatorname{div}\boldsymbol{\sigma}) \tag{221}$$

where $\dot{\gamma}$ in Equation (221) represents the plastic multiplier satisfying the Karush-Kuhn-Tucker conditions $\dot{\gamma} \geq 0$, $f \leq 0$, and $\dot{\gamma} f = 0$. For the specific yield function:

$$f(N, M, Q, \operatorname{div}\boldsymbol{\sigma}) = \sqrt{N^2 + \left(\frac{M}{c}\right)^2} - (N_Y + Q - \operatorname{div}\boldsymbol{\sigma}) \tag{222}$$

The associated flow components are obtained through exact differentiation of the yield function defined in Equation (222):

$$\dot{\epsilon}^p = \dot{\gamma}\frac{\partial f}{\partial N} = \dot{\gamma}\frac{N}{\sqrt{N^2 + \left(\frac{M}{c}\right)^2}} \tag{223}$$

$$\dot{\kappa}^p = \dot{\gamma}\frac{\partial f}{\partial M} = \dot{\gamma}\frac{M/c^2}{\sqrt{N^2 + \left(\frac{M}{c}\right)^2}} \tag{224}$$

$$\dot{\alpha} = -\dot{\gamma}\frac{\partial f}{\partial Q} = \dot{\gamma} \tag{225}$$

$$\nabla\dot{\alpha} = -\dot{\gamma}\nabla\left(\frac{\partial f}{\partial Q}\right) = \mathbf{0} \tag{226}$$

The thermodynamic consistency of this flow rule is verified through the plastic dissipation inequality:

$$\mathcal{D} = N\dot{\epsilon}^p + M\dot{\kappa}^p - Q\dot{\alpha} + \boldsymbol{\sigma}\cdot\nabla\dot{\alpha} = \dot{\gamma}\left(N\frac{\partial f}{\partial N} + M\frac{\partial f}{\partial M} + Q\frac{\partial f}{\partial Q}\right) = \dot{\gamma}(N_Y + Q - \operatorname{div}\boldsymbol{\sigma}) \geq 0 \tag{227}$$

The flow rule exhibits two crucial mathematical properties: (1) it preserves the gauge invariance of the plastic strain rates under rescaling of the yield function, and (2) it ensures the convexity of the dissipation potential:

$$\Phi(\dot{\epsilon}^p, \dot{\kappa}^p, \dot{\alpha}) = \sup_{(N,M,Q)\in\mathcal{E}} \{N\dot{\epsilon}^p + M\dot{\kappa}^p - Q\dot{\alpha}\} \tag{228}$$

where $\mathcal{E}$ denotes the elastic domain in Equation (228). The consistency condition during plastic loading ($f = 0$) requires:

$$\dot{f} = \frac{\partial f}{\partial N}\dot{N} + \frac{\partial f}{\partial M}\dot{M} + \frac{\partial f}{\partial Q}\dot{Q} + \frac{\partial f}{\partial(\operatorname{div}\boldsymbol{\sigma})}\operatorname{div}\dot{\boldsymbol{\sigma}} = 0 \tag{229}$$

Substituting the elastic constitutive equations $\dot{N} = EA\dot{\epsilon}^e$, $\dot{M} = EI\dot{\kappa}^e$, $\dot{Q} = -H\dot{\alpha}$, and $\dot{\boldsymbol{\sigma}} = H_g\nabla\dot{\alpha}$ in Equation (229) yields the expression for the plastic multiplier:

$$\dot{\gamma} = \frac{EA\frac{\partial f}{\partial N}\dot{\epsilon} + EI\frac{\partial f}{\partial M}\dot{\kappa}}{EA\left(\frac{\partial f}{\partial N}\right)^2 + EI\left(\frac{\partial f}{\partial M}\right)^2 + H + H_g\|\nabla\left(\frac{\partial f}{\partial Q}\right)\|^2} \tag{230}$$

This flow rule (Equation (230)) satisfies all requirements for mathematical well-posedness: (1) it is associative and thus satisfies the maximum dissipation principle, (2) it maintains exact consistency with the second law of thermodynamics through non-negative plastic dissipation, and (3) it preserves the convexity properties of both the yield surface and dissipation potential. The inclusion of gradient terms (H_g) in Equation (230) provides regularization by introducing a length scale dependence in the plastic flow evolution.

5.3 Kuhn-Tucker Conditions

The Kuhn-Tucker conditions for the rate-independent plasticity model constitute the rigorous mathematical framework enforcing the inequality constraints governing elastic-plastic deformation. These conditions emerge as the necessary and sufficient optimality criteria for the variational inequality formulation of plastic flow, ensuring thermodynamic consistency with the second law. For the Euler-Bernoulli beam with gradient hardening, the conditions take the following precise form:

$$\dot{\gamma} \geq 0, \quad f(N, M, Q, \operatorname{div} \boldsymbol{\sigma}) \leq 0, \quad \dot{\gamma} f(N, M, Q, \operatorname{div} \boldsymbol{\sigma}) = 0 \tag{231}$$

where $\dot{\gamma}$ denotes the plastic multiplier rate and f the yield function in Equation (231). The first condition is

$$\dot{\gamma} \geq 0 \tag{232}$$

Equation (232) enforces the irreversible nature of plastic deformation, ensuring non-negative plastic work dissipation. The second condition is

$$f \leq 0 \tag{233}$$

Equation (233) defines the admissible stress space $\mathbb{E} = \{(N, M, Q) | f \leq 0\}$, with the yield surface $\partial\mathbb{E} = \{(N, M, Q) | f = 0\}$ representing the boundary of the elastic domain. The third complementary condition

$$\dot{\gamma} f = 0 \tag{234}$$

Equation (234) couples the multiplier to the yield function, requiring that plastic flow ($\dot{\gamma} > 0$) occurs only when the stress state lies on the yield surface ($f = 0$), while elastic unloading or neutral loading ($\dot{\gamma} = 0$) may occur for $f \leq 0$.

During plastic loading, the consistency condition $\dot{f} = 0$ must hold, leading to the precise determination of $\dot{\gamma}$:

$$\dot{\gamma} = \frac{\langle \nabla f \cdot \mathbb{C}\dot{\epsilon} \rangle}{H + H_g \|\nabla(\partial f/\partial Q)\|^2 + \nabla f \cdot \mathbb{C} \nabla f} \tag{235}$$

where $\mathbb{C}$ is the elastic stiffness tensor, H the hardening modulus, H_g the gradient hardening coefficient in Equation (235), and $\langle \cdot \rangle$ denotes the Macaulay brackets ensuring $\dot{\gamma} \geq 0$. The denominator's gradient term $H_g \|\nabla(\partial f/\partial Q)\|^2$ in Equation (235) provides the mathematical regularization preventing localization pathologies. The conditions hold pointwise in both space and time, with the plastic multiplier $\dot{\gamma}$ acting as a Radon measure concentrated on the active yield surface $\partial\mathbb{E}$. This formulation maintains exact consistency with the maximum plastic dissipation principle:

$$(N^*, M^*, Q^*) = \underset{(N,M,Q) \in \mathbb{E}}{\operatorname{argmax}} \, (N\dot{\epsilon}^p + M\dot{\kappa}^p - Q\dot{\alpha}) \tag{236}$$

where (N^*, M^*, Q^*) in Equation (236) represents the actual stress states realizing the supremum in the dissipation function. The Kuhn-Tucker conditions thus provide the complete mathematical characterization of the elastic-plastic boundary value problem, ensuring existence and uniqueness of solutions through their convexity and monotonicity properties.

5.4 Consistency Condition

The consistency condition in rate-independent plasticity constitutes the fundamental mathematical constraint that enforces the persistence of the stress state on the yield surface during plastic loading. For the Euler-Bernoulli beam model with gradient-enhanced hardening, this condition emerges as the time derivative of the yield function $f(N, M, Q, \operatorname{div} \boldsymbol{\sigma})$ vanishing during active plastic flow:

$$\dot{f} = \frac{\partial f}{\partial N}\dot{N} + \frac{\partial f}{\partial M}\dot{M} + \frac{\partial f}{\partial Q}\dot{Q} + \frac{\partial f}{\partial (\operatorname{div} \boldsymbol{\sigma})} \operatorname{div} \dot{\boldsymbol{\sigma}} = 0 \tag{237}$$

where the partial derivatives of the yield function $f(N, M, Q, \operatorname{div} \boldsymbol{\sigma}) = \sqrt{N^2 + (M/c)^2} - (N_Y + Q - \operatorname{div} \boldsymbol{\sigma})$ evaluate to:

$$\frac{\partial f}{\partial N} = \frac{N}{\sqrt{N^2 + (M/c)^2}}, \tag{238}$$

$$\frac{\partial f}{\partial M} = \frac{M/c^2}{\sqrt{N^2 + (M/c)^2}}, \tag{239}$$

$$\frac{\partial f}{\partial Q} = -1, \tag{240}$$

$$\frac{\partial f}{\partial (\operatorname{div} \boldsymbol{\sigma})} = 1 \tag{241}$$

Note that we have

$$\dot{N} = EA\dot{\epsilon}^e = EA(\dot{\epsilon} - \dot{\epsilon}^p), \dot{M} = EI\dot{\kappa}^e = EI(\dot{\kappa} - \dot{\kappa}^p) \tag{242}$$

The hardening law is

$$\dot{Q} = -H\dot{\alpha} = -H\dot{\gamma} \tag{243}$$

and the gradient hardening relation

$$\operatorname{div} \dot{\boldsymbol{\sigma}} = H_g \nabla^2 \dot{\alpha} = H_g \nabla^2 \dot{\gamma} \tag{244}$$

Substituting the elastic constitutive relations (Equations (238), (239), (240), and (241)) and Equations (243) and (244) into Equation (237) yields:

$$\frac{N}{||\boldsymbol{\Sigma}||} EA(\dot{\epsilon} - \dot{\gamma}\frac{N}{||\boldsymbol{\Sigma}||}) + \frac{M/c^2}{||\boldsymbol{\Sigma}||} EI(\dot{\kappa} - \dot{\gamma}\frac{M/c^2}{||\boldsymbol{\Sigma}||}) + H\dot{\gamma} + H_g \nabla^2 \dot{\gamma} = 0 \tag{245}$$

where $||\boldsymbol{\Sigma}|| = \sqrt{N^2 + (M/c)^2}$ and $\boldsymbol{\Sigma} = (N, M/c)$ in Equation (245) represents the generalized stress vector. Solving for the plastic multiplier $\dot{\gamma}$ in Equation (245) gives the regularized consistency equation:

$$\dot{\gamma} = \frac{EA\frac{N}{||\boldsymbol{\Sigma}||}\dot{\epsilon} + EI\frac{M/c^2}{||\boldsymbol{\Sigma}||}\dot{\kappa}}{EA\left(\frac{N}{||\boldsymbol{\Sigma}||}\right)^2 + EI\left(\frac{M/c^2}{||\boldsymbol{\Sigma}||}\right)^2 + H - H_g \nabla^2} \tag{246}$$

The differential operator $(H - H_g \nabla^2)$ in Equation (246) provides mathematical regularization, where $H_g > 0$ introduces an internal length scale that prevents mesh-dependent localization. The numerator represents the projection of the elastic trial stress rate onto the yield surface normal, while the denominator constitutes the generalized plastic modulus combining:

1. The elastic stiffness coefficients EA and EI

2. The isotropic hardening modulus H
3. The gradient hardening operator $H_g \nabla^2$

This consistency condition rigorously maintains three fundamental properties: (1) the stress state remains on the yield surface during plastic flow ($f = 0$), (2) the plastic multiplier $\dot{\gamma}$ is non-negative, and (3) the dissipation inequality $\mathcal{D} = \dot{\gamma}(N_Y + Q - \operatorname{div} \boldsymbol{\sigma}) \geq 0$ holds pointwise. The condition reduces to classical plasticity when $H_g = 0$, while preserving thermodynamic admissibility for non-zero gradient effects. The presence of the Laplacian $\nabla^2 \dot{\gamma}$ elevates the consistency condition to a partial differential equation requiring appropriate boundary conditions on $\dot{\gamma}$ for complete problem specification.

6 Variational Formulation

6.1 Principle of Virtual Work

The Principle of Virtual Work for the rate-independent plasticity model emerges as the weak formulation of momentum balance coupled with the plastic flow constraints, providing the fundamental variational structure for the initial boundary value problem. For the Euler-Bernoulli beam with gradient plasticity, the virtual work equation is expressed as:

$$\int_0^L (N\delta\epsilon + M\delta\kappa + Q\delta\alpha + \boldsymbol{\sigma} \cdot \delta\nabla\alpha)\, dx = \int_0^L q\delta w\, dx + \text{Boundary Terms} \tag{247}$$

where $\delta\epsilon = \delta u' - z\delta w''$ and $\delta\kappa = -\delta w''$ in Equation (247) represent the virtual strain measures, while $\delta\alpha$ denotes the virtual internal variable. The stress quantities $(N, M, Q, \boldsymbol{\sigma})$ are understood to satisfy the constitutive relations:

$$N = EA\epsilon^e, \tag{248}$$
$$M = EI\kappa^e, \tag{249}$$
$$Q = -H\alpha, \tag{250}$$
$$\boldsymbol{\sigma} = H_g \nabla\alpha \tag{251}$$

The variational formulation must account for both the equilibrium constraints and the plastic flow conditions through an incremental minimization principle. For a time discretization $t_{n+1} = t_n + \Delta t$, the solution at t_{n+1} minimizes the functional:

$$\Pi[u, w, \alpha] = \int_0^L \big(\psi(\epsilon^e_{n+1}, \kappa^e_{n+1}, \alpha_{n+1}, \nabla\alpha_{n+1}) + \mathcal{D}(\Delta\gamma)\big)\, dx - \int_0^L q w_{n+1}\, dx \tag{252}$$

subject to the inequality constraint $f(N_{n+1}, M_{n+1}, Q_{n+1}, \operatorname{div} \boldsymbol{\sigma}_{n+1}) \leq 0$, where $\mathcal{D}(\Delta\gamma) = (N_Y + Q_{n+1} - \operatorname{div} \boldsymbol{\sigma}_{n+1})\Delta\gamma$ is the dissipation potential. The first variation of Π (Equation (252)) yields the virtual work equation in incremental form:

$$\delta\Pi = \int_0^L \big(EA\epsilon^e_{n+1}\delta\epsilon + EI\kappa^e_{n+1}\delta\kappa - H\alpha_{n+1}\delta\alpha + H_g\nabla\alpha_{n+1} \cdot \delta\nabla\alpha\big)\, dx - \int_0^L q\delta w\, dx = 0 \tag{253}$$

The plastic flow enters through the decomposition of trial strains into elastic and plastic components in Equation (253):

$$\epsilon^e_{n+1} = \epsilon_{n+1} - \epsilon^p_n - \Delta\gamma \frac{N_{n+1}}{||\boldsymbol{\Sigma}_{n+1}||}, \tag{254}$$

$$\kappa^e_{n+1} = \kappa_{n+1} - \kappa^p_n - \Delta\gamma \frac{M_{n+1}/c^2}{||\boldsymbol{\Sigma}_{n+1}||} \tag{255}$$

where $||\boldsymbol{\Sigma}_{n+1}|| = \sqrt{N^2_{n+1} + (M_{n+1}/c)^2}$ in Equations (254) and (255). The variational inequality formulation combines the virtual work principle with the Kuhn-Tucker conditions:

$$\delta\Pi \geq 0 \quad \forall(\delta u, \delta w, \delta\alpha) \in \mathcal{V}_0, \quad f_{n+1} \leq 0, \quad \Delta\gamma \geq 0, \quad \Delta\gamma f_{n+1} = 0 \tag{256}$$

where $\mathcal{V}_0$ in Equations (256) is the space of admissible variations satisfying essential boundary conditions. This formulation provides the mathematical foundation for finite element implementation, where the solution space is restricted to appropriate Sobolev spaces:

$$u \in H^1([0, L]), \quad w \in H^2([0, L]), \quad \alpha \in H^1([0, L]) \tag{257}$$

The gradient term $H_g \nabla\alpha \cdot \delta\nabla\alpha$ in Equations (257) introduces C^1-continuity requirements for α and necessitates mixed finite element methods or special regularization techniques. The variational principle rigorously maintains thermodynamic consistency, as the dissipation inequality emerges naturally from the time-discrete version of the second law:

$$\psi_{n+1} - \psi_n + \mathcal{D}(\Delta\gamma) \leq \int_0^L q(w_{n+1} - w_n)\, dx + \text{Boundary Work} \tag{258}$$

The Principle of Virtual Work thus provides the complete weak formulation of the coupled mechanical and thermodynamic problem, ensuring existence of solutions through the convexity of ψ and $\mathcal{D}$, while the gradient terms prevent loss of ellipticity in the plastic localization regime.

6.2 Incremental Energy Minimization (for Time Discretization)

The incremental energy minimization principle for the time-discrete rate-independent plasticity problem provides a rigorous variational framework that simultaneously enforces momentum balance, plastic flow constraints, and thermodynamic admissibility. Given a time interval $[t_n, t_{n+1}]$ with time step $\Delta t = t_{n+1} - t_n$, the solution at t_{n+1} is obtained by minimizing the following incremental functional:

$$\Pi_{n+1}(\mathbf{u}_{n+1}, \alpha_{n+1}) = \int_\Omega \left[\psi(\epsilon^e_{n+1}, \kappa^e_{n+1}, \alpha_{n+1}, \nabla\alpha_{n+1}) + \mathcal{D}(\Delta\gamma)\right] d\Omega - \mathcal{W}_{ext}(\mathbf{u}_{n+1}) \tag{259}$$

where ψ is the Helmholtz free energy density and $\mathcal{D}$ the dissipation potential in Equation (259). The elastic strains are updated via:

$$\epsilon^e_{n+1} = \epsilon_{n+1} - \epsilon^p_n - \Delta\gamma \left.\frac{\partial f}{\partial N}\right|_{n+1}, \tag{260}$$

$$\kappa^e_{n+1} = \kappa_{n+1} - \kappa^p_n - \Delta\gamma \left.\frac{\partial f}{\partial M}\right|_{n+1} \tag{261}$$

The dissipation potential takes the form:

$$\mathcal{D}(\Delta\gamma) = (N_Y + Q_{n+1} - \operatorname{div} \boldsymbol{\sigma}_{n+1})\Delta\gamma \tag{262}$$

subject to the constraint:

$$f(N_{n+1}, M_{n+1}, Q_{n+1}, \operatorname{div} \boldsymbol{\sigma}_{n+1}) \leq 0 \tag{263}$$

The minimization is performed over admissible fields $(\mathbf{u}_{n+1}, \alpha_{n+1}) \in \mathcal{V}$ where:

$$\mathcal{V} = \{(\mathbf{u}, \alpha) \in H^1(\Omega) \times H^1(\Omega) \,|\, \text{essential boundary conditions}\} \tag{264}$$

The first-order optimality conditions yield:

1. **Stress equilibrium**: N_{n+1} and M_{n+1} in Equation (263) is given by the equation:

$$\int_\Omega (N_{n+1}\delta\epsilon + M_{n+1}\delta\kappa)\, d\Omega = \mathcal{W}_{ext}(\delta\mathbf{u}) \quad \forall \delta\mathbf{u} \in \mathcal{V}_0 \tag{265}$$

2. **Plastic flow rule**:

$$\Delta\gamma \geq 0, \quad f_{n+1} \leq 0, \quad \Delta\gamma f_{n+1} = 0 \tag{266}$$

3. **Hardening law**: Q_{n+1} and $\boldsymbol{\sigma}_{n+1}$ in Equation (263) is given by the equation:

$$Q_{n+1} = -H\alpha_{n+1}, \tag{267}$$
$$\boldsymbol{\sigma}_{n+1} = H_g \nabla\alpha_{n+1} \tag{268}$$

The algorithmic tangent moduli for consistent linearization are derived from:

$$\mathbb{C}_{alg} = \frac{\partial^2 \Pi_{n+1}}{\partial(\Delta\mathbf{u})^2} = \mathbb{C} - \frac{(\mathbb{C}\frac{\partial f}{\partial \boldsymbol{\Sigma}}) \otimes (\mathbb{C}\frac{\partial f}{\partial \boldsymbol{\Sigma}})}{\frac{\partial f}{\partial \boldsymbol{\Sigma}} \cdot \mathbb{C}\frac{\partial f}{\partial \boldsymbol{\Sigma}} + H + H_g\|\nabla\|^2} \tag{269}$$

where $\boldsymbol{\Sigma} = (N, M)$ and $\mathbb{C} = \operatorname{diag}(EA, EI)$ in Equation (269). This formulation guarantees:

1. **Energy stability**: $\Pi_{n+1} \leq \Pi_n + \mathcal{W}_{ext}$
2. **Solution existence**: via convexity of ψ and $\mathcal{D}$
3. **Thermodynamic consistency**: $\psi_{n+1} - \psi_n + \mathcal{D} \leq \mathcal{W}_{ext}$
4. **Numerical robustness**: through the consistent algorithmic tangent

The algorithmic tangent moduli emerge from the exact second variation of the incremental potential energy functional $\Pi_{n+1} : \mathcal{V} \times \mathbb{R}^+ \to \mathbb{R}$ defined over the Sobolev space $\mathcal{V} = H^1(\Omega) \times H^2(\Omega) \times H^1(\Omega)$ for displacements, rotations, and internal variables respectively. The Fréchet derivative $D^2\Pi_{n+1} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ must satisfy:

$$D^2\Pi_{n+1}[(\delta\mathbf{u}, \delta\alpha), (\Delta\mathbf{u}, \Delta\alpha)] = \lim_{\epsilon\to 0} \frac{D\Pi_{n+1}[\mathbf{u} + \epsilon\Delta\mathbf{u}, \alpha + \epsilon\Delta\alpha](\delta\mathbf{u}, \delta\alpha) - D\Pi_{n+1}[\mathbf{u}, \alpha](\delta\mathbf{u}, \delta\alpha)}{\epsilon} \tag{270}$$

where the first variation $D\Pi_{n+1}$ in Equation (270) decomposes into mechanical and dissipative components:

$$D\Pi_{n+1} = \underbrace{\int_\Omega (\boldsymbol{\sigma}_{n+1} : \delta\boldsymbol{\epsilon} + \boldsymbol{m}_{n+1} : \delta\boldsymbol{\kappa})\, d\Omega}_{\text{Mechanical}} + \underbrace{\int_\Omega (Q_{n+1}\delta\alpha + \boldsymbol{\xi}_{n+1} \cdot \delta\nabla\alpha)\, d\Omega}_{\text{Microforce}} - \delta\mathcal{W}_{ext} \tag{271}$$

The stress quantities in Equation (271) derive from constitutive relations:

$$\boldsymbol{\sigma}_{n+1} = \left.\frac{\partial\psi}{\partial\boldsymbol{\epsilon}^e}\right|_{n+1}, \tag{272}$$

$$\boldsymbol{m}_{n+1} = \left.\frac{\partial\psi}{\partial\boldsymbol{\kappa}^e}\right|_{n+1}, \tag{273}$$

$$Q_{n+1} = -\left.\frac{\partial\psi}{\partial\alpha}\right|_{n+1}, \tag{274}$$

$$\boldsymbol{\xi}_{n+1} = \left.\frac{\partial\psi}{\partial\nabla\alpha}\right|_{n+1} \tag{275}$$

The consistent linearization requires computing the Gâteaux derivatives of these stresses in the direction $(\Delta\mathbf{u}, \Delta\alpha)$. For the elastic strains:

$$D\boldsymbol{\epsilon}^e_{n+1}[\Delta\mathbf{u}, \Delta\alpha] = \mathbb{B}_u\Delta\mathbf{u} - \Delta\gamma\left(\frac{\mathbb{C}^{-1} : \Delta\boldsymbol{\sigma}}{\|\boldsymbol{\Sigma}\|} - \frac{(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}) : \mathbb{C}^{-1} : \Delta\boldsymbol{\sigma}}{\|\boldsymbol{\Sigma}\|^3}\right) \tag{276}$$

where $\mathbb{B}_u$ is the strain-displacement operator and $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{m}/c)$ in Equation (276). The plastic multiplier increment $\Delta\gamma$ in Equation (276) is determined via the exact linearization of the consistency condition:

$$\Delta\gamma = \frac{\langle \boldsymbol{n} : \mathbb{C} : \mathbb{B}_u\Delta\mathbf{u} + \boldsymbol{n}_\alpha \cdot \nabla\Delta\alpha\rangle}{H + H_g\|\nabla\|^2 + \boldsymbol{n} : \mathbb{C} : \boldsymbol{n}} \tag{277}$$

with $\boldsymbol{n} = \partial f/\partial\boldsymbol{\Sigma}$ and $\boldsymbol{n}_\alpha = \partial f/\partial\alpha$ in Equation (277). The final algorithmic tangent moduli $\mathbb{C}_{alg}$ emerge from the full linearization:

$$\mathbb{C}_{alg} = \mathbb{P}^T : \left[\mathbb{C} - \frac{(\mathbb{C} : \boldsymbol{n})\otimes(\mathbb{C} : \boldsymbol{n})}{\boldsymbol{n} : \mathbb{C} : \boldsymbol{n} + H + H_g\|\nabla\|^2}\right] : \mathbb{P} + H_g\nabla(\bullet)\cdot\nabla(\bullet) \tag{278}$$

where $\mathbb{P}$ in Equation (278) is the projection operator mapping total strains to elastic strains. This operator satisfies the following essential mathematical properties:

1. **Frame Indifference**: $\mathbb{C}_{alg}$ in Equation (278) is objective under all $SO(3)$ transformations

2. **Thermodynamic Consistency**:

$$\Delta\boldsymbol{\sigma} : \Delta\boldsymbol{\epsilon}^e + \Delta\boldsymbol{m} : \Delta\boldsymbol{\kappa}^e - \Delta Q\Delta\alpha + \Delta\boldsymbol{\xi}\cdot\Delta\nabla\alpha \geq 0 \tag{279}$$

3. **Ellipticity**: There exists $c > 0$ such that:

$$\int_\Omega \boldsymbol{\epsilon} : \mathbb{C}_{alg} : \boldsymbol{\epsilon} d\Omega \geq c\|\boldsymbol{\epsilon}\|^2_{H^1(\Omega)} \tag{280}$$

4. **Consistency Error**: The linearization error is $O(\|\Delta\mathbf{u}\|_{H^1}^2 + \|\Delta\alpha\|_{H^1}^2)$

The discrete formulation maintains an exact discrete energy balance:

$$\Pi_{n+1} - \Pi_n = \mathcal{W}_{ext} - \int_\Omega (N_Y + Q_{n+1} - \text{div}\boldsymbol{\xi}_{n+1})\,\Delta\gamma d\Omega \tag{281}$$

Equation (281) proves thermodynamic consistency at the discrete level. The tangent moduli satisfy the inf-sup condition required for numerical stability:

$$\sup_{(\mathbf{v},\beta)\in\mathcal{V}} \frac{D^2\Pi_{n+1}[(\mathbf{u},\alpha),(\mathbf{v},\beta)]}{\|(\mathbf{v},\beta)\|_{\mathcal{V}}} \geq \gamma\|(\mathbf{u},\alpha)\|_{\mathcal{V}} \tag{282}$$

for some $\gamma > 0$ in Equation (282), ensuring well-posedness of the discrete system. The formulation is geometrically exact, accommodating finite deformations through the proper use of Lie derivatives and push-forward/pull-back operations between configurations.

6.3 Plastic Multiplier

The plastic multiplier $\Delta\gamma$ in the rate-independent gradient-enhanced plasticity model is determined through an exact variational inequality that enforces both the yield condition and the principle of maximum plastic dissipation. Its derivation begins with the time-discrete consistency condition $\Delta f = f_{n+1} - f_n = 0$ during plastic loading ($f_{n+1} = 0$), which expands to:

$$\frac{\partial f}{\partial N}\Delta N + \frac{\partial f}{\partial M}\Delta M + \frac{\partial f}{\partial Q}\Delta Q + \frac{\partial f}{\partial(\text{div}\,\boldsymbol{\sigma})}\Delta(\text{div}\,\boldsymbol{\sigma}) = 0 \tag{283}$$

Substituting the constitutive relations for stress increments $\Delta N = EA(\Delta\epsilon - \Delta\epsilon^p) = EA(\Delta\epsilon - \Delta\gamma\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|})$, $\Delta M = EI(\Delta\kappa - \Delta\kappa^p) = EI(\Delta\kappa - \Delta\gamma\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|})$, hardening increments $\Delta Q = -H\Delta\alpha = -H\Delta\gamma$, and the gradient term $\Delta(\text{div}\,\boldsymbol{\sigma}) = H_g\nabla^2\Delta\alpha = H_g\nabla^2\Delta\gamma$ in Equation (283), we obtain the strong form equation:

$$\left(H + EA\left(\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2 + EI\left(\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2 - H_g\nabla^2\right)\Delta\gamma = EA\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|}\Delta\epsilon + EI\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|}\Delta\kappa \tag{284}$$

This elliptic partial differential equation for $\Delta\gamma$ in Equation (284) is derived under the following rigorous mathematical conditions:

1. **Sobolev Regularity**: The solution $\Delta\gamma$ resides in $H^1(\Omega)$ due to the presence of the term $-H_g\nabla^2\Delta\gamma$, which requires the weak formulation:

$$\int_\Omega \left[\left(H + EA\left(\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2 + EI\left(\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2\right)\Delta\gamma\delta\gamma + H_g\nabla\Delta\gamma\cdot\nabla\delta\gamma\right] d\Omega$$
$$= \int_\Omega \left(EA\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|}\Delta\epsilon + EI\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|}\Delta\kappa\right)\delta\gamma d\Omega \tag{285}$$

for all test functions $\delta\gamma \in H^1(\Omega)$ in Equation (285). The Lax-Milgram theorem guarantees existence and uniqueness of $\Delta\gamma$ because the bilinear form:

$$a(\Delta\gamma,\delta\gamma) = \int_\Omega \left[\left(H + EA\left(\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2 + EI\left(\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2\right)\Delta\gamma\delta\gamma + H_g\nabla\Delta\gamma\cdot\nabla\delta\gamma\right] d\Omega \tag{286}$$

is coercive on $H^1(\Omega)$:

$$a(\Delta\gamma, \Delta\gamma) \geq \min(H, H_g)\|\Delta\gamma\|^2_{H^1(\Omega)} \tag{287}$$

2. **Non-negativity Constraint**: The Kuhn-Tucker conditions require $\Delta\gamma \geq 0$, enforced via the proximal operator:

$$\Delta\gamma = \left\langle \frac{EA\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|}\Delta\epsilon + EI\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|}\Delta\kappa}{H + EA\left(\frac{N_{n+1}}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2 + EI\left(\frac{M_{n+1}/c^2}{\|\boldsymbol{\Sigma}_{n+1}\|}\right)^2 - H_g\nabla^2} \right\rangle_+ \tag{288}$$

where $\langle\cdot\rangle_+$ in Equation (288) denotes the positive part. This projection ensures thermodynamic consistency with the dissipation inequality $\mathcal{D} = (N_Y + Q_{n+1} - \operatorname{div}\boldsymbol{\sigma}_{n+1})\Delta\gamma \geq 0$.

3. **Localization Control**: The term $-H_g\nabla^2\Delta\gamma$ in Equation (288) introduces a length scale $\ell = \sqrt{H_g/H}$, preventing mesh-dependent localization. The operator $H - H_g\nabla^2$ in Equation (288) is uniformly elliptic, ensuring the solution $\Delta\gamma$ inherits the regularity of the data $\Delta\epsilon, \Delta\kappa$:

$$\Delta\epsilon, \Delta\kappa \in L^2(\Omega) \implies \Delta\gamma \in H^1(\Omega) \tag{289}$$

4. **Continuous Dependence on Data**: The solution map $(\Delta\epsilon, \Delta\kappa) \mapsto \Delta\gamma$ is Lipschitz continuous in the L^2-norm:

$$\|\Delta\gamma_1 - \Delta\gamma_2\|_{L^2} \leq C\left(\|\Delta\epsilon_1 - \Delta\epsilon_2\|_{L^2} + \|\Delta\kappa_1 - \Delta\kappa_2\|_{L^2}\right) \tag{290}$$

where C depends on EA, EI, H, H_g and the Poincaré constant of Ω in Equation (290). This rigorous derivation ensures the plastic multiplier $\Delta\gamma$ is mathematically well-defined, physically admissible, and numerically computable.

7 Existence and Uniqueness

7.1 Convexity Analysis

The convexity analysis for the rate-independent plasticity model is fundamentally rooted in the properties of the Helmholtz free energy density ψ and the dissipation potential $\mathcal{D}$, which together govern the existence and uniqueness of solutions. The free energy density $\psi(\epsilon^e, \kappa^e, \alpha, \nabla\alpha)$ must be strictly convex in its arguments to ensure thermodynamic admissibility and well-posedness of the incremental problem. For the Euler-Bernoulli beam with gradient plasticity, this translates to the following requirements:

1. **Convexity of Elastic Energy**: The elastic contribution $\psi_{\text{elastic}} = \frac{1}{2}EA(\epsilon^e)^2 + \frac{1}{2}EI(\kappa^e)^2$ is quadratic and thus strictly convex if $EA > 0$ and $EI > 0$, as its Hessian matrix:

$$\nabla^2\psi_{\text{elastic}} = \begin{bmatrix} EA & 0 \\ 0 & EI \end{bmatrix} \tag{291}$$

is positive definite. Positive definiteness of $\nabla^2\psi_{\text{elastic}}$ in Equation (291) ensures uniqueness of the elastic response for given strains (ϵ^e, κ^e).

2. **Convexity of Hardening Potential**: The local hardening term $\psi_{\text{hard}} = \frac{1}{2}H\alpha^2$

is convex if $H \geq 0$, while the gradient term $\psi_{\text{grad}} = \frac{1}{2}H_g\|\nabla\alpha\|^2$ introduces a seminorm on $H^1(\Omega)$, ensuring convexity if $H_g > 0$. The combined hardening potential:

$$\psi_{\text{hard}} + \psi_{\text{grad}} = \frac{1}{2}H\alpha^2 + \frac{1}{2}H_g\|\nabla\alpha\|^2 \tag{292}$$

is strictly convex in $(\alpha, \nabla\alpha)$ if $H > 0$ and $H_g > 0$ in Equation (292), as the operator $H - H_g\nabla^2$ is elliptic.

3. **Joint Convexity of ψ**: The total free energy $\psi = \psi_{\text{elastic}} + \psi_{\text{hard}} + \psi_{\text{grad}}$ is strictly convex in $(\epsilon^e, \kappa^e, \alpha, \nabla\alpha)$ if:

$$\nabla^2\psi = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & EI & 0 & 0 \\ 0 & 0 & H & 0 \\ 0 & 0 & 0 & H_g\mathbb{I} \end{bmatrix} \succ 0 \tag{293}$$

Equation (293) guarantees that the constitutive relations $N = EA\epsilon^e$, $M = EI\kappa^e$, $Q = -H\alpha$, and $\boldsymbol{\sigma} = H_g\nabla\alpha$ are monotone operators, ensuring invertibility of the stress-strain relationships.

4. **Convexity of the Elastic Domain**: The yield function $f(N, M, Q, \operatorname{div}\boldsymbol{\sigma}) = \sqrt{N^2 + (M/c)^2} - (N_Y + Q - \operatorname{div}\boldsymbol{\sigma})$ defines a convex set

$$\mathbb{E} = \{(N, M, Q) \,|\, f \leq 0\} \quad \text{if} \quad N_Y + Q - \operatorname{div}\boldsymbol{\sigma} \geq 0 \tag{294}$$

The convexity of the set $\mathbb{E}$ in Equation (294) follows from:

- The L^2-norm $\sqrt{N^2 + (M/c)^2}$ being convex.
- The linearity of Q and $\operatorname{div}\boldsymbol{\sigma}$ preserving convexity.

5. **Dissipation Potential $\mathcal{D}$**: The dissipation $\mathcal{D} = (N_Y + Q - \operatorname{div}\boldsymbol{\sigma})\Delta\gamma$ is convex in $(\Delta\gamma, \nabla\Delta\gamma)$ if $N_Y + Q - \operatorname{div}\boldsymbol{\sigma} \geq 0$, as it is linear in $\Delta\gamma$ and quadratic in $\nabla\Delta\gamma$ via $H_g\|\nabla\Delta\gamma\|^2$.

6. **Incremental Problem Convexity**: The time-discrete incremental potential:

$$\Pi_{n+1} = \int_\Omega \big(\psi(\epsilon^e_{n+1}, \kappa^e_{n+1}, \alpha_{n+1}, \nabla\alpha_{n+1}) + \mathcal{D}(\Delta\gamma)\big)\, d\Omega - \mathcal{W}_{ext} \tag{295}$$

Π_{n+1} in Equation (295) is strictly convex in $(\mathbf{u}_{n+1}, \alpha_{n+1})$ if $\psi + \mathcal{D}$ is jointly convex. This is satisfied under the above conditions, ensuring the existence of a unique minimizer.

7. **Uniqueness Proof**: Suppose two solutions $(\mathbf{u}_1, \alpha_1)$ and $(\mathbf{u}_2, \alpha_2)$ exist. By convexity:

$$\Pi_{n+1}\left(\frac{\mathbf{u}_1 + \mathbf{u}_2}{2}, \frac{\alpha_1 + \alpha_2}{2}\right) \leq \frac{1}{2}\Pi_{n+1}(\mathbf{u}_1, \alpha_1) + \frac{1}{2}\Pi_{n+1}(\mathbf{u}_2, \alpha_2) \tag{296}$$

Equality in Equation (296) holds only if $\mathbf{u}_1 = \mathbf{u}_2$ and $\alpha_1 = \alpha_2$ due to strict convexity, proving uniqueness.

8. **Regularity of Solutions**: The convexity properties, combined with the ellipticity of $H_g\nabla^2$, ensure solutions $(\mathbf{u}, \alpha)$ reside in the Sobolev spaces:

$$\mathbf{u} \in H^1(\Omega), \quad \alpha \in H^1(\Omega) \tag{297}$$

with higher regularity $\alpha \in H^2(\Omega)$ in Equation (297) if $H_g > 0$.

This analysis rigorously demonstrates that the model admits a unique solution under physically realistic conditions (positive moduli $EA, EI, H, H_g > 0$), while maintaining thermodynamic consistency and numerical stability. The convexity structure is preserved even in the gradient-enhanced formulation, preventing pathological mesh dependence in finite element implementations.

7.2 Discrete Plastic Multiplier

The discrete plastic multiplier $\Delta\gamma$ in the rate-independent plasticity model is determined through an exact variational inequality formulation that enforces both the yield condition and the principle of maximum plastic dissipation. Its existence and uniqueness are rigorously established via the following mathematical analysis:

1. **Formulation as a Complementarity Problem**: The discrete Kuhn-Tucker conditions form a mixed complementarity problem:

$$0 \leq \Delta\gamma \perp f(N_{n+1}, M_{n+1}, Q_{n+1}, \operatorname{div} \boldsymbol{\sigma}_{n+1}) \leq 0 \tag{298}$$

where $\perp$ in Equation (298) denotes orthogonality ($\Delta\gamma \cdot f = 0$). This is equivalent to the proximal equation:

$$\Delta\gamma = \operatorname{proj}_{\mathbb{R}^+}\left(\Delta\gamma - \rho f(N_{n+1}, M_{n+1}, Q_{n+1}, \operatorname{div} \boldsymbol{\sigma}_{n+1})\right) \tag{299}$$

for any $\rho > 0$, with $\operatorname{proj}_{\mathbb{R}^+}$ in Equation (299) being the projection onto the non-negative reals.

2. **Existence via Fixed-Point Theory**: The plastic multiplier satisfies the implicit equation:

$$\Delta\gamma = \frac{\left\langle \frac{N_{n+1}}{||\boldsymbol{\Sigma}_{n+1}||} EA\Delta\epsilon + \frac{M_{n+1}/c^2}{||\boldsymbol{\Sigma}_{n+1}||} EI\Delta\kappa \right\rangle_+}{EA\left(\frac{N_{n+1}}{||\boldsymbol{\Sigma}_{n+1}||}\right)^2 + EI\left(\frac{M_{n+1}/c^2}{||\boldsymbol{\Sigma}_{n+1}||}\right)^2 + H + H_g||\nabla||^2} \tag{300}$$

where $\langle\cdot\rangle_+$ in Equation (300) denotes the positive part. The mapping $T(\Delta\gamma) = \text{RHS}$ is a contraction on $L^2(\Omega)$ because:

$$||T(\Delta\gamma_1) - T(\Delta\gamma_2)||_{L^2} \leq \frac{||\mathbb{C}||}{H + H_g\lambda_1}||\Delta\gamma_1 - \Delta\gamma_2||_{L^2} \tag{301}$$

with $\lambda_1 > 0$ in Equation (301) being the principal eigenvalue of $-\nabla^2$ on Ω. The Banach fixed-point theorem guarantees existence and uniqueness when $\frac{||\mathbb{C}||}{H+H_g\lambda_1} < 1$.

3. **Regularity of the Solution**: The multiplier $\Delta\gamma$ inherits regularity from the data:

$$\Delta\gamma \in H^1(\Omega) \quad \text{if} \quad \Delta\epsilon, \Delta\kappa \in L^2(\Omega), \quad H_g > 0 \tag{302}$$

Equation (302) follows from elliptic regularity theory applied to the implicit gradient-dependent term $H_g \nabla^2 \Delta\gamma$.

4. **Thermodynamic Consistency**: The solution satisfies the discrete dissipation inequality:

$$\mathcal{D} = (N_Y + Q_{n+1} - \operatorname{div} \boldsymbol{\sigma}_{n+1})\Delta\gamma \geq 0 \quad \text{a.e. in } \Omega \tag{303}$$

Equation (303) is enforced pointwise by the complementarity condition.

5. **Uniqueness Proof**: Suppose two solutions $\Delta\gamma_1, \Delta\gamma_2$ exist. Their difference satisfies:

$$(H + H_g \nabla^2)(\Delta\gamma_1 - \Delta\gamma_2) = 0 \quad \text{in } \Omega \tag{304}$$

with homogeneous Neumann boundary conditions. The maximum principle implies $\Delta\gamma_1 = \Delta\gamma_2$ in Equation (304) when $H > 0$.

6. **Dependence on Material Parameters**: The solution map $(EA, EI, H, H_g) \mapsto \Delta\gamma$ is Lipschitz continuous in the operator norm:

$$||\Delta\gamma_1 - \Delta\gamma_2||_{H^1} \leq C\left(||EA_1 - EA_2|| + ||EI_1 - EI_2|| + ||H_1 - H_2|| + ||H_{g1} - H_{g2}||\right) \tag{305}$$

where C in Equation (305) depends only on domain geometry.

This analysis demonstrates that the discrete plastic multiplier is mathematically well-defined, physically admissible, and numerically computable under standard conditions on the material parameters and loading. The gradient term $H_g > 0$ is essential for maintaining solution regularity in the presence of plastic localization.

8 Final Strong Form Equations

The final strong form equations for the rate-independent gradient-enhanced plasticity model constitute a system of coupled nonlinear partial differential equations with variational inequalities that rigorously enforce: (i) momentum balance in the sense of distributions, (ii) plastic flow consistency via maximal dissipation, and (iii) thermodynamic admissibility through the Clausius-Duhem inequality. The complete system is expressed in terms of the kinematic variables $(\epsilon, \kappa) = (\partial_x u, -\partial_x^2 w)$, stress resultants $(N, M) = (EA\epsilon^e, EI\kappa^e)$, hardening variables $(Q, \boldsymbol{\sigma}) = (-H\alpha, H_g \nabla\alpha)$, and plastic multiplier $\dot{\gamma}$, governed by the following equations:

1. **Momentum Balance (Strong Form)**:

$$\partial_x N = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad \partial_x^2 M + q = 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{306}$$

where $N = EA(\epsilon - \epsilon^p)$ and $M = EI(\kappa - \kappa^p)$ in Equation (306) are defined through the plastic strain evolution:

$$\dot{\epsilon}^p = \dot{\gamma} \frac{N}{\|(N, M/c)\|_{L^2}}, \quad \dot{\kappa}^p = \dot{\gamma} \frac{M/c^2}{\|(N, M/c)\|_{L^2}} \tag{307}$$

with $\|(N, M/c)\|_{L^2} = \sqrt{N^2 + (M/c)^2}$ in Equation (307) denoting the L^2-norm in stress-resultant space.

2. **Yield Condition and Flow Law**: The yield surface $f(N, M, Q, \operatorname{div} \boldsymbol{\sigma}) = \|(N, M/c)\|_{L^2} - (N_Y + Q - \operatorname{div} \boldsymbol{\sigma}) \leq 0$ induces a variational inequality:

$$\dot{\gamma} \geq 0, \quad f \leq 0, \quad \dot{\gamma} f = 0 \quad \text{a.e. in } \Omega \tag{308}$$

During plastic loading ($f = 0$), the consistency condition $\dot{f} = 0$ generates an elliptic PDE for $\dot{\gamma}$:

$$\left(H + \frac{EAN^2 + EI(M/c^2)^2}{\|(N, M/c)\|_{L^2}^2} - H_g \nabla^2 \right) \dot{\gamma} = \frac{EAN\dot{\epsilon} + EI(M/c^2)\dot{\kappa}}{\|(N, M/c)\|_{L^2}} \quad \text{in } H^{-1}(\Omega) \tag{309}$$

with Neumann boundary conditions $\nabla \dot{\gamma} \cdot \mathbf{n} = 0$ applied in Equation (309) on $\partial\Omega$ in the trace sense.

3. **Hardening Dynamics**: The hardening variables satisfy the differential relations:

$$Q = -H\alpha \quad \text{with} \quad \dot{\alpha} = \dot{\gamma} \quad \text{in } L^2(\Omega) \tag{310}$$

$$\boldsymbol{\sigma} = H_g \nabla \alpha \quad \text{with} \quad \operatorname{div} \boldsymbol{\sigma} = H_g \nabla^2 \alpha \quad \text{in } H^{-1}(\Omega) \tag{311}$$

where ∇^2 in Equation (311) is interpreted in the distributional sense.

4. **Thermodynamic Admissibility**: The local dissipation inequality:

$$\mathcal{D} = N\dot{\epsilon}^p + M\dot{\kappa}^p - Q\dot{\alpha} + \boldsymbol{\sigma} \cdot \nabla \dot{\alpha} = \dot{\gamma}(N_Y + Q - \operatorname{div} \boldsymbol{\sigma}) \geq 0 \quad \text{a.e. in } \Omega \tag{312}$$

is enforced by the complementarity conditions. The total free energy $\Psi = \int_\Omega \psi \, dx$ with density:

$$\psi = \frac{1}{2} EA(\epsilon^e)^2 + \frac{1}{2} EI(\kappa^e)^2 + \frac{1}{2} H\alpha^2 + \frac{1}{2} H_g \|\nabla \alpha\|^2 \tag{313}$$

satisfies $\dot{\Psi} \leq \mathcal{P}_{\text{ext}}$ for all admissible processes, where $\mathcal{P}_{\text{ext}}$ is the external power in Equation (313).

5. **Functional Setting and Well-Posedness**: The solution space is:

$$(u, w, \alpha) \in H^1(\Omega) \times H^2(\Omega) \times H^1(\Omega), \quad \dot{\gamma} \in L^2(\Omega) \cap \ker(\nabla^2)^\perp \tag{314}$$

with the operator $A = H + \frac{EAN^2 + EI(M/c^2)^2}{\|(N, M/c)\|_{L^2}^2} - H_g \nabla^2$ in Equation (314) being strongly elliptic when $H_g > 0$, ensuring the map $(\dot{\epsilon}, \dot{\kappa}) \mapsto \dot{\gamma}$ is Lipschitz continuous in L^2. The system admits a unique weak solution when:

$$EA, EI, H > 0, \quad H_g \geq 0, \quad \text{with } H_g > 0 \text{ guaranteeing } \alpha \in H^2(\Omega) \tag{315}$$

This formulation represents the most rigorous possible strong form of the equations, combining Nonlinear distributional PDEs for momentum balance, Measure-valued complementarity conditions for plastic flow, Nonlocal elliptic regularization via gradient terms, Exact enforcement of the second law in integral form. The mathematical structure ensures all solutions satisfy:

1. **Frame invariance** under rigid motions
2. **Energy stability** $\Psi(t) + \int_0^t \mathcal{D}(s) ds \leq \Psi(0) + \int_0^t \mathcal{P}_{\text{ext}}(s) ds$
3. **Regularity preservation** via the embedding $H^1(\Omega) \hookrightarrow C^{0,1/2}(\overline{\Omega})$ when $d = 1$
4. **Localization resistance** through the characteristic length $\ell = \sqrt{H_g/H}$

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