Rate-Dependent Plastic Deformation Model of Euler-Bernoulli Beams

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Abstract

This work develops a thermodynamically consistent, geometrically exact, rate-dependent plasticity model for Euler–Bernoulli beams under finite deformation. Beginning with a rigorous formulation of continuum kinematics, the study systematically constructs the mechanical framework by employing the deformation gradient tensor and associated measures of strain, decomposed multiplicatively to separate elastic and plastic contributions. The mechanical balance laws are derived in their weak forms using the principle of virtual power and energy balance, followed by the derivation of the entropy inequality from the second law of thermodynamics, guaranteeing consistency with the Clausius–Duhem inequality.

The constitutive framework is constructed via the Helmholtz free energy approach, incorporating both elastic and plastic energy storage, as well as temperature-dependent contributions. A Perzyna-type overstress flow rule is introduced to model the rate-dependence of plastic flow, complemented by a smooth yield function and isotropic hardening evolution. Dimensional reduction techniques specific to Euler–Bernoulli beam kinematics are applied, leading to beam-specific stress resultants and thermo-mechanical coupling relations. The strong and weak forms of the resulting governing equations are provided, and a locking-free mixed finite element discretization strategy is proposed. The model's time-integration is handled via an exponential map-based update for the plastic deformation gradient, ensuring preservation of plastic incompressibility. Further, the well-posedness of the problem is addressed through functional analytic formulation using Sobolev spaces, and existence and uniqueness of solutions are discussed.

In summary, this work provides a foundational, rate-dependent, large-deformation plasticity model for thermomechanically coupled Euler–Bernoulli beams, suitable for applications in computational mechanics involving thin structures undergoing large, irreversible deformations at finite rates.

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1 Literature Review of Rate-dependent plastic deformation models

Plastic deformation in materials is often sensitive to strain rate, particularly under dynamic loading conditions (e.g., high-speed machining, impact, crash simulations). Classical plasticity theories (e.g., von Mises, Tresca) assume rate independence, but real materials exhibit significant strain-rate effects, necessitating specialized constitutive models.

1.1 Earliest constitutive models

The earliest constitutive models for rate-dependent plasticity were developed to capture the nonlinear relationship between stress, strain, and strain rate in materials subjected to dynamic loading. One of the foundational frameworks was introduced by Perzyna (1966) [1], who formulated an **overstress model** to incorporate viscous effects into plasticity. The key assumption is that plastic strain rate $\dot{\epsilon}^p$ depends on the excess stress beyond the static yield condition. The constitutive equation is expressed as:

$$\dot{\epsilon}^p = \gamma \left\langle \frac{\sigma - \sigma_y}{K} \right\rangle^n \tag{1}$$

where γ is the fluidity parameter governing the rate sensitivity, σ_y is the static yield stress, K is a hardening modulus, and n is a rate-sensitivity exponent in Equation (1). The Macaulay brackets $\langle\cdot\rangle$ in Equation (1) enforce the condition that plastic flow occurs only when $\sigma > \sigma_y$. The model introduces a viscoplastic multiplier $\dot{\lambda}$ such that:

$$\dot{\epsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma} \tag{2}$$

where $f = \sigma - \sigma_y$ is the yield function in Equation (2). The plastic strain rate is thus governed by the overstress $(\sigma - \sigma_y)$ and the material's rate-dependent response.

The Bodner-Partom model (1975) [2] extended this framework by decoupling strain hardening and rate effects into two independent evolutionary equations. The model avoids an explicit yield condition and instead uses an internal state variable Z representing **resistance to plastic flow**, evolving as:

$$\dot{Z} = m(Z_1 - Z)\dot{W}_p \tag{3}$$

where m and Z_1 are material constants, and $\dot{W}_p = \sigma\dot{\epsilon}^p$ is the plastic work rate in Equation (3). The plastic strain rate is given by:

$$\dot{\epsilon}^p = D_0 \exp\left[-\frac{1}{2}\left(\frac{Z^2}{3J_2}\right)^n\right]\frac{\partial J_2}{\partial \sigma} \tag{4}$$

where D_0 is a limiting strain rate, $J_2 = \frac{1}{2}s_{ij}s_{ij}$ is the second invariant of the deviatoric stress s_{ij}, and n controls rate sensitivity in Equation (4). This formulation allows for smooth transition between elastic and plastic regimes without a sharp yield point.

A widely adopted empirical model is the Johnson-Cook (1983) [3] relation, which multiplicatively decomposes strain hardening, strain-rate hardening, and thermal softening effects. The flow stress σ is given by:

$$\sigma = (A + B\epsilon^n)\left(1 + C\ln\frac{\dot{\epsilon}}{\dot{\epsilon}_0}\right)(1 - T^{*m}) \tag{5}$$

where A is the initial yield stress, B and n describe isotropic hardening, C quantifies strain-rate sensitivity, and $\dot{\epsilon}_0$ is a reference strain rate in Equation (5). The thermal term T^* is defined as:

$$T^* = \frac{T - T_{\text{room}}}{T_{\text{melt}} - T_{\text{room}}} \tag{6}$$

where T_{melt} is the melting temperature and m governs thermal softening in Equation (6). The logarithmic strain-rate term $\ln(\dot{\epsilon}/\dot{\epsilon}_0)$ in Equation (5) captures the nonlinear increase in flow stress at higher strain rates, while the power-law hardening term $(A + B\epsilon^n)$ in Equation (5) represents quasi-static plasticity.

These models form the basis for more advanced formulations, incorporating microstructural effects, nonlocal gradients, and multiscale couplings. The Perzyna model introduces a viscoplastic potential, the Bodner-Partom model unifies hardening and rate effects without a yield surface, and the Johnson-Cook model provides a phenomenological but computationally efficient representation for high-rate applications. Each framework has been extended to include additional physics, such as pressure sensitivity in geomaterials or dislocation density evolution in crystal plasticity.

1.2 Physically-Based Models

Physically-based models of rate-dependent plasticity seek to establish a direct connection between microscale dislocation mechanisms and macroscale mechanical response. The **Mechanical Threshold Stress (MTS) model**, developed by Follansbee and Kocks

(1988) [4], is a foundational framework that explicitly accounts for thermal activation barriers governing dislocation motion. The model defines a thermally activated flow stress σ_a and an athermal component σ_G, with the total yield stress σ_y expressed as:

$$\sigma_y = \sigma_a + \sigma_G \tag{7}$$

where σ_G represents long-range barriers (e.g., grain boundaries) that are strain-rate independent, while σ_a captures short-range obstacles (e.g., solute atoms, forest dislocations) that are overcome with thermal assistance in Equation (7). The strain-rate dependence arises from the Arrhenius-type relation for dislocation glide:

$$\dot{\epsilon}^p = \dot{\epsilon}_0 \exp\left[-\frac{\Delta G(\sigma_a)}{k_B T}\right] \tag{8}$$

where $\dot{\epsilon}_0$ is a reference strain rate, $\Delta G(\sigma_a)$ is the activation energy as a function of σ_a, k_B is the Boltzmann constant, and T is absolute temperature in Equation (8). The activation energy is often modeled using the Kocks form:

$$\Delta G(\sigma_a) = \Delta F \left[1 - \left(\frac{\sigma_a}{\hat{\sigma}}\right)^p\right]^q \tag{9}$$

where ΔF is the total free energy required to overcome the barrier without stress, $\hat{\sigma}$ is the threshold stress at 0 K, and p, q are fitting parameters reflecting obstacle profiles in Equation (9). The strain-rate sensitivity is then derived by inverting the Arrhenius relation:

$$\sigma_a = \hat{\sigma}\left[1 - \left(\frac{k_B T}{\Delta F}\ln\frac{\dot{\epsilon}_0}{\dot{\epsilon}^p}\right)^{1/q}\right]^{1/p} \tag{10}$$

The MTS model further incorporates strain hardening through the evolution of $\hat{\sigma}$ with plastic strain ϵ^p:

$$\hat{\sigma} = \hat{\sigma}_0 + \alpha\mu b\sqrt{\rho} \tag{11}$$

where $\hat{\sigma}_0$ is the initial threshold, α is a dimensionless constant, μ is the shear modulus, b is the Burgers vector, and ρ is the dislocation density in Equation (11). The dislocation density evolves according to:

$$\frac{d\rho}{d\epsilon^p} = k_1\sqrt{\rho} - k_2\rho \tag{12}$$

where k_1 represents dislocation generation (e.g., via Frank-Read sources), and k_2 accounts for dynamic recovery in Equation (12).

More recent **dislocation-dynamics-inspired models**, such as those by Austin and McDowell (2011) [5], extend this framework by explicitly coupling continuum plasticity with dislocation density evolution. These models often employ a tensorial representation of dislocation densities, distinguishing between statistically stored dislocations (SSDs) ρ_S and geometrically necessary dislocations (GNDs) ρ_G. The flow stress is then generalized as:

$$\sigma_y = \sigma_0 + \alpha\mu b\sqrt{\rho_S + \rho_G} \tag{13}$$

where σ_0 is a friction stress in Equation (13). The evolution of ρ_S in Equation (13) follows a modified Kocks-Mecking law:

$$\dot{\rho}_S = \frac{k_1}{b}\sqrt{\rho_S + \rho_G}\,\dot{\epsilon}^p - k_2\rho_S\dot{\epsilon}^p - k_3\rho_S^{3/2} \tag{14}$$

with k_3 introducing a stage IV hardening term in Equation (14). The GND density ρ_G in Equation (14) is derived from the gradient of plastic strain η^p via Nye's tensor Λ:

$$\rho_G = \frac{1}{b}\|\nabla \eta^p\| \tag{15}$$

where $\nabla \eta^p$ in Equation (15) reflects lattice curvature due to GNDs. The plastic strain rate $\dot{\epsilon}^p$ in Equation (15) is linked to dislocation velocity v_d through Orowan's equation:

$$\dot{\epsilon}^p = \rho_m b v_d \tag{16}$$

where ρ_m is the mobile dislocation density. The velocity v_d in Equation (16) is modeled as stress- and temperature-dependent:

$$v_d = v_0 \left(\frac{\tau}{\tau_0}\right)^m \exp\left(-\frac{Q}{k_B T}\right) \tag{17}$$

with v_0 a reference velocity, τ the resolved shear stress, τ_0 a reference stress, m a velocity exponent, and Q an activation energy in Equation (17). These models also incorporate back stresses σ_b due to dislocation pile-ups:

$$\sigma_b = \beta \mu b \sqrt{\rho_G} \tag{18}$$

where β is a scaling factor in Equation (18). The total stress σ is then:

$$\sigma = \sigma_y + \sigma_b + \sigma_{\text{visc}}(\dot{\epsilon}^p) \tag{19}$$

where σ_{visc} is a viscous drag term, often modeled as $\sigma_{\text{visc}} = B(\dot{\epsilon}^p)^m$ in Equation (19). Such formulations enable predictions of size effects, strain localization, and rate-dependent hardening across scales.

1.3 Advanced and Multiscale Models

Advanced and multiscale models of rate-dependent plasticity incorporate higher-order kinematics, microstructure evolution, and data-driven techniques to bridge length and time scales. The **Viscoplastic Self-Consistent (VPSC) model** (Lebensohn and Tomé (1993) [6]) is a mean-field approach that couples crystal plasticity with polycrystalline texture evolution. Each grain is treated as an ellipsoidal inclusion embedded in a homogeneous effective medium (HEM) with properties derived from the collective response of all grains. The viscoplastic strain rate $\dot{\epsilon}_{ij}$ in a grain is governed by the resolved shear stress $\tau^{(s)}$ on slip system s:

$$\dot{\epsilon}_{ij} = \sum_s \gamma_0^{(s)} \left(\frac{|\tau^{(s)}|}{g^{(s)}}\right)^{n^{(s)}} \text{sgn}(\tau^{(s)}) m_{ij}^{(s)} \tag{20}$$

where $\gamma_0^{(s)}$ is a reference shear rate, $g^{(s)}$ the critical resolved shear stress (CRSS), $n^{(s)}$ the rate-sensitivity exponent, and $m_{ij}^{(s)} = \frac{1}{2}(n_i^{(s)} b_j^{(s)} + n_j^{(s)} b_i^{(s)})$ the Schmid tensor for slip system s with normal $n_i^{(s)}$ and Burgers vector $b_i^{(s)}$ in Equation (20). The HEM's macroscopic strain rate $\dot{E}_{ij}$ is obtained by volume-averaging over all grains:

$$\dot{E}_{ij} = \langle \dot{\epsilon}_{ij} \rangle = \sum_{k=1}^{N} w_k \dot{\epsilon}_{ij}^{(k)} \tag{21}$$

where w_k is the volume fraction of grain k in Equation (21). The self-consistent solution enforces stress equilibrium between each grain and the HEM through Eshelby's inclusion formalism:

$$\sigma_{ij}^{(k)} - \Sigma_{ij} = -C_{ijkl}(I_{klmn} - S_{klmn})(\dot{\epsilon}_{mn}^{(k)} - \dot{E}_{mn}) \tag{22}$$

where Σ_{ij} is the macroscopic stress, C_{ijkl} the stiffness tensor, I_{klmn} the identity tensor, and S_{klmn} the Eshelby tensor for the grain's ellipsoidal shape in Equation (22). Texture evolution is tracked via the reorientation of crystallographic axes due to plastic spin:

$$\dot{a}_i^{(k)} = W_{ij}^{(k)} a_j^{(k)} \tag{23}$$

where $a_i^{(k)}$ are the crystal axes and $W_{ij}^{(k)} = \dot{R}_{ik} R_{jk}^T$ the lattice spin, with R_{ij} the rotation matrix linking crystal and sample frames in Equation (23).

Nonlocal and gradient plasticity models (Aifantis (1984) [7]; Abu Al-Rub and Voyiadjis (2006) [8], Gudmundson (2004) [9]) regularize strain localization by introducing higher-order spatial derivatives of strain or internal variables. The flow stress σ_y is augmented with a gradient term:

$$\sigma_y = \sigma_0 + H\epsilon^p + \ell^2 \nabla^2 \epsilon^p \tag{24}$$

where σ_0 is the initial yield stress, H the hardening modulus, ℓ an intrinsic length scale, and $\nabla^2 \epsilon^p$ the Laplacian of plastic strain in Equation (24). The plastic strain rate $\dot{\epsilon}^p$ in Equation (24) is derived from a nonlocal yield condition:

$$\dot{\epsilon}^p = \left\langle \frac{\sigma - (\sigma_0 + H\epsilon^p - \ell^2 \nabla^2 \epsilon^p)}{K} \right\rangle^n \tag{25}$$

The governing partial differential equation (PDE) for equilibrium becomes:

$$\frac{\partial \sigma}{\partial x} + \ell^2 \frac{\partial^3 \epsilon^p}{\partial x^3} = 0 \tag{26}$$

where $\sigma = E(\epsilon - \epsilon^p)$ for elastic modulus E and total strain ϵ in Equation (26). In higher dimensions, the Cauchy stress σ_{ij} couples with the microstress ξ_{ijk}:

$$\sigma_{ij} = C_{ijkl}(\epsilon_{kl} - \epsilon_{kl}^p), \quad \xi_{ijk} = \ell^2 C_{ijmn} \frac{\partial \epsilon_{mn}^p}{\partial x_k} \tag{27}$$

leading to the extended equilibrium condition:

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\partial \xi_{ijk}}{\partial x_j \partial x_k} = 0 \tag{28}$$

Machine learning-augmented models ([10], [11]) replace or supplement constitutive equations with neural networks trained on multiscale data. The flow stress σ is modeled as a function of strain ϵ, strain rate $\dot{\epsilon}$, temperature T, and internal variables q_i:

$$\sigma = \mathcal{N}_\theta(\epsilon, \dot{\epsilon}, T q_i) \tag{29}$$

where $\mathcal{N}_\theta$ is a deep neural network in Equation (29) with weights θ optimized via loss functions:

$$\mathcal{L}(\theta) = \sum_{k=1}^{N} \|\sigma^{(k)} - \mathcal{N}_\theta(\epsilon^{(k)}, \dot{\epsilon}^{(k)}, T^{(k)}, q_i^{(k)})\|^2 + \lambda \|\nabla_\theta \mathcal{N}_\theta\|^2 \tag{30}$$

Physics-informed constraints can be embedded, such as enforcing the second law of thermodynamics:

$$\sigma\dot{\epsilon}^p - \dot{\psi} \geq 0 \tag{31}$$

where ψ is the free energy density in Equation (31), parameterized as:

$$\psi = \mathcal{N}_\theta^\psi(\epsilon^e, q_i) \tag{32}$$

Recurrent neural networks (RNNs) track internal state evolution:

$$q_i^{(t+1)} = \text{RNN}_\phi(q_i^{(t)}, \Delta\epsilon^p, \Delta t) \tag{33}$$

with ϕ denoting RNN weights in Equation (33). Bayesian neural networks quantify uncertainty:

$$p(\sigma|\epsilon, \dot{\epsilon}, T) = \int \delta(\sigma - \mathcal{N}_\theta(\epsilon, \dot{\epsilon}, T))p(\theta|\mathcal{D})d\theta \tag{34}$$

where $p(\theta|\mathcal{D})$ in Equation (34) is the posterior over weights given data $\mathcal{D}$. These models enable adaptive refinement through active learning:

$$\theta^* = \arg\min_\theta \mathbb{E}_{(\epsilon,\dot{\epsilon},T)\sim p_{\text{new}}}[\mathcal{L}(\theta)] \tag{35}$$

where p_{new} in Equation (35) is updated based on model uncertainty.

1.4 Applications and Material-Specific Considerations

The application of rate-dependent plasticity models varies significantly across material classes due to distinct micromechanical mechanisms governing their deformation. For **metals**, particularly FCC (e.g., aluminum, copper) and BCC (e.g., iron, tantalum) crystals, the dominant rate sensitivity arises from thermally activated dislocation glide and phonon drag. The flow stress σ_y in FCC metals is often expressed via a modified Arrhenius law:

$$\dot{\epsilon}^p = \dot{\epsilon}_0 \exp\left[-\frac{\Delta G_0}{k_B T}\left(1 - \left(\frac{\sigma_y - \sigma_a}{\hat{\sigma}}\right)^p\right)^q\right] \tag{36}$$

where σ_a is the athermal stress component, $\hat{\sigma}$ is the mechanical threshold stress, and ΔG_0 is the activation energy at zero stress in Equation (36). For BCC metals, the Peierls barrier introduces stronger temperature and rate dependence, captured by:

$$\sigma_y = \sigma_a + \frac{\Delta G_0}{v^*}\ln\left(\frac{\dot{\epsilon}_0}{\dot{\epsilon}^p}\right) \tag{37}$$

where v^* is the activation volume in Equation (37). Dislocation-density-based hardening is incorporated through:

$$\sigma_a = \alpha\mu b\sqrt{\rho}, \quad \dot{\rho} = k_1\sqrt{\rho}\dot{\epsilon}^p - k_2\rho\dot{\epsilon}^p \tag{38}$$

where α is a geometric factor, μ the shear modulus, and k_1, k_2 represent dislocation multiplication and annihilation rates in Equation (38).

For **polymers and soft materials**, the Bergström-Boyce hyperelastic-viscoplastic model decomposes the deformation gradient $\mathbf{F}$ into elastic $\mathbf{F}^e$ and plastic $\mathbf{F}^p$ parts:

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \quad \dot{\mathbf{F}}^p = \mathbf{D}^p\mathbf{F}^p \tag{39}$$

where $\mathbf{D}^p$ is the plastic rate tensor in Equation (39). The stress response derives from a free energy potential Ψ:

$$\mathbf{T} = \frac{\partial \Psi}{\partial \mathbf{E}^e}, \quad \Psi = \Psi_{\text{vol}}(J^e) + \Psi_{\text{iso}}(\bar{\mathbf{E}}^e) \tag{40}$$

with $J^e = \det \mathbf{F}^e$ and $\bar{\mathbf{E}}^e$ the isochoric elastic Green strain in Equation (40). The plastic flow rule is:

$$\mathbf{D}^p = \dot{\gamma}^p \frac{\mathbf{T}'}{\|\mathbf{T}'\|}, \quad \dot{\gamma}^p = \dot{\gamma}_0 \left(\frac{\tau}{\tau_{\text{base}}} \right)^m \left(\frac{\tau}{\tau_{\text{cut}}} + 1 \right)^{n-m} \tag{41}$$

where $\tau = \sqrt{\frac{1}{2}\mathbf{T}' : \mathbf{T}'}$ is the equivalent shear stress, τ_{base} and τ_{cut} are material parameters, and m, n control rate sensitivity in Equation (41). The viscoelastic relaxation is modeled via a Prony series:

$$\mathbf{T}(t) = \mathbf{T}_\infty + \sum_{i=1}^{N} \mathbf{T}_i e^{-t/\lambda_i} \tag{42}$$

where λ_i are relaxation times and $\mathbf{T}_\infty$ the equilibrium stress in Equation (42).

Geomaterials under dynamic loads require pressure-sensitive yield criteria. The rate-dependent Drucker-Prager model modifies the yield function F as:

$$F = \sqrt{J_2} + \alpha_\phi p - \kappa(\dot{\epsilon}^p), \quad p = -\frac{1}{3}\text{tr}(\boldsymbol{\sigma}) \tag{43}$$

In Equation (43) $J_2 = \frac{1}{2}\mathbf{s} : \mathbf{s}$ is the second deviatoric invariant, α_ϕ the friction coefficient, and $\kappa(\dot{\epsilon}^p)$ the rate-hardened cohesion:

$$\kappa(\dot{\epsilon}^p) = \kappa_0 \left(1 + \beta \ln \frac{\dot{\epsilon}^p}{\dot{\epsilon}_0} \right) \tag{44}$$

The cap model introduces a compaction-dependent yield surface:

$$F_c = \sqrt{(J_2 - \chi p)^2} + \alpha_c (p - p_a) - \kappa_c \tag{45}$$

where χ governs the transition from shear to compaction, p_a the cap pressure, and κ_c the cap hardening in Equation (45). The evolution of p_a is:

$$\dot{p}_a = H_c \dot{\epsilon}_v^p, \quad \epsilon_v^p = \text{tr}(\boldsymbol{\epsilon}^p) \tag{46}$$

with H_c in Equation (46) the cap hardening modulus. The viscoplastic flow rule follows:

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\lambda} \frac{\partial Q}{\partial \boldsymbol{\sigma}}, \quad \dot{\lambda} = \frac{1}{\eta} \langle F \rangle \tag{47}$$

where Q is the plastic potential and η the viscosity in Equation (47). Strain-rate effects in rocks are often described by a power-law creep:

$$\dot{\epsilon}^p = A \sigma^n e^{-Q/RT} \tag{48}$$

where A is a pre-exponential factor and Q the activation energy in Equation (48). These material-specific formulations ensure accurate prediction of rate-dependent phenomena across diverse loading regimes.

1.5 Challenges and Open Questions

1.5.1 Thermo-Mechanical Coupling

The interplay between plastic dissipation and thermal softening introduces significant challenges in rate-dependent plasticity modeling. Under adiabatic conditions (common at strain rates $\dot{\epsilon}^p > 10^3\,\mathrm{s}^{-1}$), the temperature rise ΔT is governed by the heat equation coupled with plastic work:

$$\rho c_p \dot{T} = \chi \sigma : \dot{\epsilon}^p + \nabla \cdot (k \nabla T) \tag{49}$$

where ρ is density, c_p specific heat, χ the Taylor-Quinney coefficient (typically 0.9), and k thermal conductivity in Equation (49). The flow stress σ_y becomes temperature-dependent:

$$\sigma_y(\epsilon^p, \dot{\epsilon}^p, T) = \underbrace{(A + B(\epsilon^p)^n)}_{\text{Isotropic hardening}} \underbrace{\left(1 + C \ln \frac{\dot{\epsilon}^p}{\dot{\epsilon}_0}\right)}_{\text{Rate hardening}} \underbrace{\left(1 - \left(\frac{T - T_0}{T_m - T_0}\right)^m\right)}_{\text{Thermal softening}} \tag{50}$$

where T_0 is reference temperature and T_m the melting point in Equation (50). The competition between hardening and softening leads to non-monotonic stress-strain curves, requiring incremental updates in computational algorithms:

$$\sigma_{n+1} = \sigma_n + \mathbb{C} : (\Delta\epsilon - \Delta\epsilon^p) - \beta \Delta T \mathbf{I} \tag{51}$$

where $\mathbb{C}$ is the elastic stiffness tensor and β the thermal expansion coefficient in Equation (51). Numerical instabilities arise when thermal softening dominates, necessitating regularization techniques like gradient plasticity [12].

1.5.2 Dynamic Recrystallization

At high strain rates ($\dot{\epsilon}^p > 10^2\,\mathrm{s}^{-1}$), dynamic recrystallization (DRX) alters microstructure via nucleation of strain-free grains. The recrystallized volume fraction X is modeled empirically [13]:

$$X = 1 - \exp\left(-k_d \left(\frac{\epsilon^p - \epsilon_c}{\epsilon_0}\right)^{n_d}\right) \tag{52}$$

where ϵ_c is critical strain for DRX initiation, ϵ_0 a reference strain, and k_d, n_d material parameters in Equation (52). The grain size evolution follows:

$$\frac{dD}{d\epsilon^p} = -k_r (D - D_{\min}) + \frac{\Gamma}{\sigma_y} \dot{\epsilon}^p \tag{53}$$

where D is grain size, $D_{\min}$ a saturation size, k_r a kinetic coefficient, and Γ a grain boundary energy term in Equation (53). The Zener-Hollomon parameter Z couples temperature and strain rate:

$$Z = \dot{\epsilon}^p \exp\left(\frac{Q_d}{RT}\right) \tag{54}$$

where Q_d is activation energy for DRX in Equation (54). The resulting flow stress must account for both dislocation density ρ and grain size D via the Hall-Petch relation:

$$\sigma_y = \sigma_0 + \alpha \mu b \sqrt{\rho} + k_y D^{-1/2} \tag{55}$$

Phase-field methods [14] introduce an order parameter ϕ to track recrystallized regions:

$$\dot{\phi} = -M\frac{\delta \mathcal{F}}{\delta \phi}, \quad \mathcal{F} = \int \left[f(\phi) + \frac{\kappa}{2}|\nabla \phi|^2 \right] dV \tag{56}$$

where M is mobility, $\mathcal{F}$ free energy, and $f(\phi)$ a double-well potential in Equation (56).

1.5.3 Experimental Validation at Extreme Rates

Split-Hopkinson pressure bar (SHPB) tests infer stress σ and strain ϵ from elastic wave propagation:

$$\sigma(t) = \frac{A_e}{A_s} E \epsilon_r(t), \quad \epsilon(t) = -\frac{2c_0}{L_s}\int_0^t \epsilon_r(\tau) d\tau \tag{57}$$

where A_e, A_s are bar/sample areas, E Young's modulus, c_0 wave speed, L_s sample length, and $\epsilon_r(t)$ reflected strain in Equation (57). At rates $> 10^4\,\mathrm{s}^{-1}$, wave dispersion and inertia complicate analysis, requiring viscoelastic bar corrections [15]:

$$\sigma(t) = \frac{A_e}{A_s}\int_0^t E(t-\tau)\dot{\epsilon}_r(\tau) d\tau \tag{58}$$

Nanoindentation strain rates $\dot{\epsilon} \sim h^{-1}\dot{h}$ (for depth h) in Equation (58) are limited by dislocation nucleation statistics [16]:

$$P(\dot{\epsilon}) \propto \exp\left(-\frac{\Delta G(\sigma)}{k_B T}\right), \quad \Delta G(\sigma) = \Delta G_0 \left(1 - \left(\frac{\sigma}{\sigma_0}\right)^p\right)^q \tag{59}$$

Discrepancies between experiments and models persist due to:

1. **Shear banding instabilities**: Resolved by nonlocal models [17]:

$$\sigma_y = \sigma_0 + \ell^2 \nabla^2 \epsilon^p \tag{60}$$

2. **Phase transformations under loading**: E.g., $\alpha \to \epsilon$ martensite in steels, modeled via:

$$\dot{f}_\epsilon = f_\alpha(1 - f_\epsilon)\left[A \exp\left(-\frac{Q}{RT}\right)(\sigma - \sigma_c)^n\right] \tag{61}$$

   where f_ϵ is martensite fraction and σ_c critical transformation stress in Equation (61).

These challenges demand advanced computational frameworks (e.g., MD-FEM coupling [18]) and high-fidelity experiments (e.g., ultrafast XRD [19]) for resolution.

1.6 Recent Trends

1.6.1 Multiscale Modeling: Coupling MD/FEM

Recent advances in multiscale modeling focus on bridging atomistic (Molecular Dynamics, MD) and continuum (Finite Element Method, FEM) scales to capture rate-dependent plasticity across spatial and temporal domains. The key challenge lies in ensuring energy consistency between scales. The bridging is often achieved through the **concurrent**

coupling method, where the displacement field $\mathbf{u}$ is decomposed into coarse ($\mathbf{u}^c$) and fine ($\mathbf{u}^f$) scales:

$$\mathbf{u} = \mathbf{u}^c + \mathbf{u}^f, \quad \mathbf{u}^c = \mathbf{N}\mathbf{d}, \quad \mathbf{u}^f = \mathcal{M}(\mathbf{r}, t) \tag{62}$$

where $\mathbf{N}$ are FEM shape functions, $\mathbf{d}$ nodal displacements, and $\mathcal{M}$ the MD resolution in critical regions in Equation (62). The Hamiltonian $\mathcal{H}$ of the coupled system is:

$$\mathcal{H} = \underbrace{\sum_{i=1}^{N_{\text{MD}}} \left(\frac{1}{2} m_i \|\mathbf{v}_i\|^2 + V(\mathbf{r}_i) \right)}_{\text{MD energy}} + \underbrace{\int_\Omega \left(\frac{1}{2} \rho \|\dot{\mathbf{u}}^c\|^2 + \Psi(\nabla \mathbf{u}^c) \right) d\Omega}_{\text{FEM energy}} \tag{63}$$

where m_i and $\mathbf{v}_i$ are atomic masses/velocities, $V(\mathbf{r}_i)$ the interatomic potential, ρ density, and Ψ the strain energy density in Equation (63). The coupling forces at the interface Γ are enforced via Lagrange multipliers $\boldsymbol{\lambda}$:

$$\mathbf{f}_{\text{couple}} = \int_\Gamma \boldsymbol{\lambda} \cdot (\mathbf{u}^c - \mathbf{u}^f) \, d\Gamma \tag{64}$$

The **bridging scale method** [20] eliminates high-frequency MD modes in the continuum region via a projection matrix $\mathbf{P}$:

$$\mathbf{u}^f = (\mathbf{I} - \mathbf{P})\mathbf{u}^{\text{MD}}, \quad \mathbf{P} = \mathbf{N}(\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T \tag{65}$$

For rate-dependent plasticity, the Cauchy stress $\boldsymbol{\sigma}$ in the FEM region is informed by MD-averaged virial stresses:

$$\boldsymbol{\sigma}^{\text{MD}} = -\frac{1}{V} \sum_{i=1}^{N} \left(m_i \mathbf{v}_i \otimes \mathbf{v}_i + \frac{1}{2} \sum_{j \neq i} \mathbf{r}_{ij} \otimes \mathbf{f}_{ij} \right) \tag{66}$$

where V is the MD volume, $\mathbf{r}_{ij}$ interatomic distances, and $\mathbf{f}_{ij}$ forces in Equation (66). The coupled stress update becomes:

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \mathbb{C} : (\Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}^p) + \underbrace{\beta \left(\boldsymbol{\sigma}^{\text{MD}} - \boldsymbol{\sigma}^{\text{FEM}} \right)}_{\text{Correction term}} \tag{67}$$

where β is a blending parameter in Equation (67).

1.6.2 Data-Driven Approaches: Bayesian Calibration and ML

Parameter identification for rate-dependent models increasingly employs **Bayesian inference** to quantify uncertainty. Given experimental data $\mathcal{D} = \{\boldsymbol{\epsilon}_i, \dot{\boldsymbol{\epsilon}}_i, \boldsymbol{\sigma}_i\}$, the posterior distribution of parameters $\boldsymbol{\theta} = \{A, B, n, C\}$ (e.g., Johnson-Cook) is:

$$p(\boldsymbol{\theta}|\mathcal{D}) = \frac{p(\mathcal{D}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathcal{D})}, \quad p(\mathcal{D}|\boldsymbol{\theta}) = \prod_{i=1}^{N} \mathcal{N}(\boldsymbol{\sigma}_i | \boldsymbol{\sigma}_{\text{model}}(\boldsymbol{\epsilon}_i, \dot{\boldsymbol{\epsilon}}_i; \boldsymbol{\theta}), \Sigma) \tag{68}$$

where $\mathcal{N}$ denotes a normal distribution with covariance Σ in Equation (68). Markov Chain Monte Carlo (MCMC) sampling generates parameter ensembles:

$$\boldsymbol{\theta}^{(k+1)} = \boldsymbol{\theta}^{(k)} + \sqrt{2\Delta t}\mathbf{R}, \quad \mathbf{R} \sim \mathcal{N}(0, \mathbf{I}) \tag{69}$$

with acceptance probability $\alpha = \min\left(1, \frac{p(\boldsymbol{\theta}^{(k+1)}|\mathcal{D})}{p(\boldsymbol{\theta}^{(k)}|\mathcal{D})}\right)$.

Neural networks replace constitutive models entirely or correct their residuals. A physics-informed neural network (PINN) for flow stress predicts:

$$\boldsymbol{\sigma} = \mathrm{NN}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}, \dot{\boldsymbol{\epsilon}}, T) + \lambda_r \mathcal{R}(\boldsymbol{\epsilon}, \dot{\boldsymbol{\epsilon}}, T) \tag{70}$$

where $\mathcal{R}$ in Equation (70) enforces the yield condition $f(\boldsymbol{\sigma}, \boldsymbol{\theta}) \leq 0$:

$$\mathcal{R} = \|\nabla_{\boldsymbol{\sigma}} f \cdot \dot{\boldsymbol{\epsilon}}^p - \dot{\lambda}\|^2 \tag{71}$$

The loss function combines data mismatch and physics constraints:

$$\mathcal{L}(\boldsymbol{\theta}) = \sum_{i=1}^{N} \|\boldsymbol{\sigma}_i - \mathrm{NN}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_i, \dot{\boldsymbol{\epsilon}}_i, T_i)\|^2 + \lambda_p \|\mathcal{R}\|^2 \tag{72}$$

1.6.3 Strain-Rate-Dependent Damage Models

The Johnson-Cook damage criterion [21] extends plasticity to fracture with a rate-dependent damage accumulation D:

$$D = \sum \frac{\Delta \epsilon^p}{\epsilon_f}, \quad \epsilon_f = \left[D_1 + D_2 \exp(D_3 \sigma^*)\right] \left(1 + D_4 \ln \dot{\epsilon}^*\right) \left(1 + D_5 T^*\right) \tag{73}$$

where $\sigma^* = \sigma_m / \sigma_{\mathrm{eq}}$ is stress triaxiality, $\dot{\epsilon}^* = \dot{\epsilon}^p / \dot{\epsilon}_0$, and D_i material constants in Equation (73). The failure condition $D \geq 1$ triggers element deletion in FEM.

Gradient-enhanced damage models [22] regularize mesh dependence by introducing a nonlocal damage field $\bar{D}$:

$$\bar{D} - \ell^2 \nabla^2 \bar{D} = D, \quad \dot{D} = \left(\frac{Y}{S}\right)^s \dot{\epsilon}^p \tag{74}$$

where $Y = \frac{1}{2} \boldsymbol{\epsilon}^e : \mathbb{C} : \boldsymbol{\epsilon}^e$ is the strain energy release rate, S and s damage parameters in Equation (74). The stress-strain relation softens with damage:

$$\boldsymbol{\sigma} = (1 - \bar{D}) \mathbb{C} : \boldsymbol{\epsilon}^e \tag{75}$$

Coupled plasticity-damage models [23] use Helmholtz free energy Ψ:

$$\Psi = (1 - D) \left(\frac{1}{2} \boldsymbol{\epsilon}^e : \mathbb{C} : \boldsymbol{\epsilon}^e + \Psi_p(\alpha)\right) \tag{76}$$

where α is a hardening internal variable in Equation (76). The thermodynamic force Y drives damage evolution:

$$Y = -\frac{\partial \Psi}{\partial D} = \frac{1}{2} \boldsymbol{\epsilon}^e : \mathbb{C} : \boldsymbol{\epsilon}^e + \Psi_p(\alpha) \tag{77}$$

These trends highlight the integration of high-fidelity simulations, stochastic methods, and machine learning to advance predictive modeling of rate-dependent material behavior.

1.7 Conclusion

The evolution of rate-dependent plasticity models has transitioned from purely empirical formulations to sophisticated microstructure-aware theories, reflecting a paradigm shift in constitutive modeling. Early empirical models, such as the Johnson-Cook relation, provided a phenomenological foundation through multiplicative decompositions of strain hardening, rate sensitivity, and thermal softening:

$$\sigma_y = (A + B(\epsilon^p)^n)\left(1 + C\ln\frac{\dot{\epsilon}^p}{\dot{\epsilon}_0}\right)(1 - T^{*m}) \tag{78}$$

where $T^* = (T - T_{\text{room}})/(T_{\text{melt}} - T_{\text{room}})$ in Equation (78). However, these models lacked physical mechanisms, prompting the development of theories like the Mechanical Threshold Stress (MTS) framework, which explicitly links dislocation dynamics to macroscale response:

$$\dot{\epsilon}^p = \dot{\epsilon}_0 \exp\left[-\frac{\Delta G_0}{k_B T}\left(1 - \left(\frac{\sigma_a}{\hat{\sigma}}\right)^p\right)^q\right] \tag{79}$$

In Equation (79) $\sigma_a = \sigma_y - \sigma_G$ is the thermally activated stress component, and $\hat{\sigma} = \hat{\sigma}_0 + \alpha\mu b\sqrt{\rho}$ incorporates dislocation density ρ evolution:

$$\frac{d\rho}{d\epsilon^p} = k_1\sqrt{\rho} - k_2\rho \tag{80}$$

Modern approaches integrate multiscale methods to bridge atomistic and continuum descriptions. Concurrent coupling techniques enforce compatibility between Molecular Dynamics (MD) and Finite Element Method (FEM) domains through energy-conserving constraints:

$$\mathcal{H} = \sum_{i=1}^{N_{\text{MD}}}\left(\frac{1}{2}m_i\|\mathbf{v}_i\|^2 + V(\mathbf{r}_i)\right) + \int_\Omega\left(\frac{1}{2}\rho\|\dot{\mathbf{u}}^c\|^2 + \Psi(\nabla\mathbf{u}^c)\right)d\Omega \tag{81}$$

where $\mathbf{u}^c = \mathbf{N}\mathbf{d}$ is the coarse-scale displacement field, and $\mathbf{u}^f = \mathcal{M}(\mathbf{r}, t)$ represents fine-scale atomic motion in Equation (81). The Cauchy stress $\boldsymbol{\sigma}$ in the continuum region is informed by MD-averaged virial stresses:

$$\boldsymbol{\sigma}^{\text{MD}} = -\frac{1}{V}\sum_{i=1}^{N}\left(m_i\mathbf{v}_i \otimes \mathbf{v}_i + \frac{1}{2}\sum_{j\neq i}\mathbf{r}_{ij} \otimes \mathbf{f}_{ij}\right) \tag{82}$$

Data-driven methods now augment traditional modeling, with Bayesian inference quantifying parameter uncertainty:

$$p(\boldsymbol{\theta}|\mathcal{D}) \propto \prod_{i=1}^{N}\mathcal{N}(\boldsymbol{\sigma}_i|\boldsymbol{\sigma}_{\text{model}}(\boldsymbol{\epsilon}_i, \dot{\boldsymbol{\epsilon}}_i; \boldsymbol{\theta}), \Sigma)p(\boldsymbol{\theta}) \tag{83}$$

where $\boldsymbol{\theta} = \{A, B, n, C\}$ are material parameters in Equation (83). Physics-informed neural networks (PINNs) encode constitutive laws directly into machine learning architectures:

$$\mathcal{L}(\boldsymbol{\theta}) = \sum_{i=1}^{N}\|\boldsymbol{\sigma}_i - \text{NN}_{\boldsymbol{\theta}}(\boldsymbol{\epsilon}_i, \dot{\boldsymbol{\epsilon}}_i, T_i)\|^2 + \lambda\|\nabla_{\boldsymbol{\sigma}} f \cdot \dot{\boldsymbol{\epsilon}}^p - \dot{\lambda}\|^2 \tag{84}$$

Equation (84) ensures thermodynamic consistency through the yield condition $f(\boldsymbol{\sigma}, \boldsymbol{\theta}) \leq 0$. Strain-rate-dependent damage models further extend these frameworks to fracture, as in the Johnson-Cook failure criterion:

$$D = \sum \frac{\Delta \epsilon^p}{\epsilon_f}, \quad \epsilon_f = \left[D_1 + D_2 \exp(D_3 \sigma^*)\right] \left(1 + D_4 \ln \dot{\epsilon}^*\right) \left(1 + D_5 T^*\right) \tag{85}$$

where $\sigma^* = \sigma_m / \sigma_{\text{eq}}$ is the stress triaxiality in Equation (85). Gradient-enhanced formulations regularize localization:

$$\bar{D} - \ell^2 \nabla^2 \bar{D} = D, \quad \dot{D} = \left(\frac{Y}{S}\right)^s \dot{\epsilon}^p \tag{86}$$

for nonlocal damage field $\bar{D}$ and strain energy release rate $Y = \frac{1}{2} \boldsymbol{\epsilon}^e : \mathbb{C} : \boldsymbol{\epsilon}^e$ in Equation (86). Future advancements will require tighter integration of high-throughput experiments (e.g., in situ XRD [24]) with computational models, leveraging scalable algorithms for coupled MD-FEM simulations [25] and probabilistic machine learning [26] to achieve predictive accuracy across extreme loading regimes.

2 Kinematical Foundations

2.1 Configurations and Deformation Gradient

The kinematic framework for finite strain plasticity begins by defining the motion of a material body through the deformation map $\varphi(\mathbf{X}, t)$, which maps points $\mathbf{X}$ from the reference configuration B_0 to their current positions $\mathbf{x} = \varphi(\mathbf{X}, t)$ in the deformed configuration B_t. The deformation gradient $\mathbf{F}$ is derived as the material gradient of this map:

$$\mathbf{F}(\mathbf{X}, t) = \nabla_X \varphi(\mathbf{X}, t) = \frac{\partial \varphi_i}{\partial X_J} \mathbf{e}_i \otimes \mathbf{E}_J \tag{87}$$

where $\mathbf{e}_i$ and $\mathbf{E}_J$ are basis vectors in the current and reference configurations respectively in Equation (87). The multiplicative decomposition of $\mathbf{F}$ into elastic and plastic components is expressed as:

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p \tag{88}$$

Here, $\mathbf{F}^p$ represents the plastic deformation from B_0 to an intermediate (stress-free) configuration $\bar{B}$, while $\mathbf{F}^e$ maps from $\bar{B}$ to B_t in Equation (88). The plastic incompressibility condition is enforced through:

$$\det \mathbf{F}^p = 1 \tag{89}$$

The elastic Green-Lagrange strain tensor in the intermediate configuration is given by:

$$\mathbf{E}^e = \frac{1}{2} \left((\mathbf{F}^e)^T \mathbf{F}^e - \mathbf{I} \right) \tag{90}$$

The plastic flow is characterized by the plastic velocity gradient $\mathbf{L}^p$ in the intermediate configuration:

$$\mathbf{L}^p = \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} \tag{91}$$

$\mathbf{L}^p$ in Equation (91) is typically decomposed into symmetric and skew-symmetric parts:

$$\mathbf{L}^p = \mathbf{D}^p + \mathbf{W}^p \tag{92}$$

where $\mathbf{D}^p$ is the plastic stretching rate and $\mathbf{W}^p$ is the plastic spin in Equation (92). The evolution of $\mathbf{F}^p$ is governed by the flow rule:

$$\dot{\mathbf{F}}^p = \mathbf{L}^p \mathbf{F}^p \tag{93}$$

The right Cauchy-Green tensor $\mathbf{C}^e$ in the intermediate configuration is:

$$\mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e \tag{94}$$

and the corresponding elastic Finger tensor is:

$$\mathbf{b}^e = \mathbf{F}^e (\mathbf{F}^e)^T \tag{95}$$

These kinematic quantities (Elastic right Cauchy-Green tensor (Equation (94)) and the Elastic Finger tensor (Equation (95))) are fundamental for constructing thermodynamically consistent constitutive equations. The multiplicative decomposition maintains the transformation properties under superposed rigid body motions, ensuring objectivity of the formulation. The plastic deformation preserves volume (as per $\det \mathbf{F}^p = 1$), while the elastic deformation may include both volumetric and distortional components. The intermediate configuration serves as a crucial reference for defining stress measures and evolution laws in finite strain plasticity theory.

2.2 Beam-Specific Kinematics

The Euler-Bernoulli beam kinematics impose a constrained deformation field that captures the characteristic behavior of slender beams undergoing finite deformations. The deformation map $\varphi(x, y, z)$ is expressed in terms of the midline deformation $\varphi_0(x)$ and two orthonormal directors $\mathbf{d}_2(x)$ and $\mathbf{d}_3(x)$, which remain normal to the beam's tangent vector $\mathbf{d}_1(x) = \varphi_0'(x)$ after deformation. This constraint enforces the classical Euler-Bernoulli hypothesis that plane sections remain plane and normal to the deformed midline, eliminating transverse shear deformation. The directors satisfy the orthonormality conditions $\mathbf{d}_i \cdot \mathbf{d}_j = \delta_{ij}$ with $\mathbf{d}_3 = \mathbf{d}_1 \times \mathbf{d}_2$, forming a local coordinate system along the beam.

The deformation gradient $\mathbf{F}$ derived from this kinematic assumption takes the form:

$$\mathbf{F} = \nabla\varphi = \varphi_0' \otimes \mathbf{e}_1 + \mathbf{d}_2 \otimes \mathbf{e}_2 + \mathbf{d}_3 \otimes \mathbf{e}_3 + y\mathbf{d}_2' \otimes \mathbf{e}_1 + z\mathbf{d}_3' \otimes \mathbf{e}_1 \tag{96}$$

where primes denote derivatives with respect to x in Equation (96). The Green-Lagrange strain tensor $\mathbf{E}$ measures the finite strain relative to the reference configuration:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}\begin{bmatrix} \varphi_0' \cdot \varphi_0' - 1 + y^2\mathbf{d}_2' \cdot \mathbf{d}_2' + z^2\mathbf{d}_3' \cdot \mathbf{d}_3' & \text{sym} & \\ \varphi_0' \cdot \mathbf{d}_2 + y\mathbf{d}_2' \cdot \mathbf{d}_2 & \mathbf{d}_2 \cdot \mathbf{d}_2 - 1 & \\ \varphi_0' \cdot \mathbf{d}_3 + z\mathbf{d}_3' \cdot \mathbf{d}_3 & \mathbf{d}_2 \cdot \mathbf{d}_3 & \mathbf{d}_3 \cdot \mathbf{d}_3 - 1 \end{bmatrix}. \tag{97}$$

For small strains but potentially large rotations, the axial strain component of $\mathbf{E}$ in Equation (97) simplifies to:

$$E_{11} \approx \varepsilon(x) + y\kappa_2(x) + z\kappa_3(x) \tag{98}$$

where $\varepsilon(x) = \|\varphi_0'\| - 1$ is the midline extensional strain, and $\kappa_2(x)$, $\kappa_3(x)$ are curvature components related to director derivatives in Equation (98). The transverse normal

strains E_{22} and E_{33} vanish due to the Euler-Bernoulli constraint, while the shear components E_{12} and E_{13} are zero by the orthogonality of directors to the tangent vector. This kinematic framework enables the derivation of beam stress resultants through thickness integration of the second Piola-Kirchhoff stress $\mathbf{S}$:

$$N = \int_A S_{11}\, dA, \quad M_2 = -\int_A z S_{11}\, dA, \quad M_3 = \int_A y S_{11}\, dA \tag{99}$$

where N is the axial force and M_2, M_3 are bending moments in Equation (99). The directors' evolution equations complete the kinematic description by specifying how the cross-section orientation changes along the beam.

3 Balance Laws in Weak Form

3.1 Virtual Power Principle

The Virtual Power Principle (VPP) constitutes a fundamental weak formulation of the momentum balance laws in continuum mechanics, providing a rigorous framework for constructing numerical discretizations while maintaining consistency with the underlying physics. In the context of finite-strain thermo-elasto-viscoplasticity, the Virtual Power Principle (VPP) is expressed as:

$$\int_{B_0} \mathbf{P} : \delta\mathbf{F}\, dV = \int_{B_0} \mathbf{f} \cdot \delta\varphi\, dV + \int_{\partial B_0} \mathbf{t} \cdot \delta\varphi\, dA \tag{100}$$

In Equation (100) $\mathbf{P}$ denotes the first Piola-Kirchhoff stress tensor, $\delta\mathbf{F} = \nabla(\delta\varphi)$ represents the virtual deformation gradient induced by admissible variations $\delta\varphi$ of the deformation map φ, $\mathbf{f}$ signifies the body force density per unit reference volume, and $\mathbf{t}$ corresponds to surface tractions acting on the boundary ∂B_0. The left-hand side constitutes the internal virtual power, while the right-hand side embodies the external virtual power. The mathematical rigor of this formulation stems from its derivation through a weighted residual approach applied to the strong form of the local momentum balance

$$\mathrm{Div}(\mathbf{P}) + \mathbf{f} = \rho_0 \ddot{\varphi} \tag{101}$$

Multiplication by test functions $\delta\varphi$ (vanishing on essential boundaries) and integration by parts of Equation (101) yields the weak form. The deformation gradient $\mathbf{F} = \nabla\varphi$ appears naturally through this process, maintaining geometric exactness in finite deformations.

The virtual power statement must satisfy precise functional-analytic conditions to ensure well-posedness. The test functions $\delta\varphi$ reside in the Sobolev space $W_0^{1,p}(B_0)^3$ for $p \geq 2$, guaranteeing that both $\delta\varphi$ and $\delta\mathbf{F}$ possess sufficient regularity. The stress $\mathbf{P}$ belongs to the dual space $L^q(B_0)^{3\times 3}$ with $1/p+1/q = 1$, ensuring the internal power integral is finite via Hölder's inequality. Thermodynamic consistency is embedded through the constitutive relation $\mathbf{P} = \rho_0 \partial\psi/\partial\mathbf{F}$, where ψ is the Helmholtz free energy density. This connection enforces that the virtual power principle remains compatible with the second law of thermodynamics, as evidenced by the Clausius-Duhem inequality:

$$\mathcal{D} = \mathbf{P} : \dot{\mathbf{F}} - \rho_0(\dot{\psi} + \eta\dot{\theta}) - \frac{\mathbf{q} \cdot \nabla\theta}{\theta} \geq 0 \tag{102}$$

The Virtual Power Principle (VPP) naturally extends to beam theories through dimensional reduction. For Euler-Bernoulli beams, the virtual deformation gradient specializes to:

$$\delta\mathbf{F} = \delta u'\mathbf{e}_1 \otimes \mathbf{e}_1 - y\delta w''\mathbf{e}_1 \otimes \mathbf{e}_1 \tag{103}$$

where u and w are axial and transverse displacements in Equation (103). This leads to the beam-specific virtual power statement:

$$\int_0^L (N\delta u' - M\delta w'')\, dx = \text{External Virtual Work} \tag{104}$$

with $N = \int_A P_{11} dA$ and $M = -\int_A yP_{11} dA$ in Equation (104) being stress resultants. The rigorous foundation of the Virtual Power Principle (VPP) thus provides a consistent pathway from three-dimensional continuum mechanics to structural theories while preserving all nonlinear and thermodynamic constraints.

3.1.1 Proof of the Equation (103)

The virtual deformation gradient for Euler-Bernoulli beams emerges from a rigorous dimensional reduction of the three-dimensional continuum formulation, constrained by the kinematic assumptions of the theory. The beam's displacement field $\mathbf{u}(x, y) = u(x)\mathbf{e}_1 + [w(x) - yw'(x)]\mathbf{e}_2$ incorporates both axial displacement $u(x)$ and transverse displacement $w(x)$, with the term $-yw'(x)$ enforcing the Euler-Bernoulli hypothesis that cross-sections remain plane and normal to the deformed midline. The corresponding virtual displacement field follows as

$$\delta\mathbf{u}(x, y) = \delta u(x)\mathbf{e}_1 + [\delta w(x) - y\delta w'(x)]\mathbf{e}_2 \tag{105}$$

where δu and δw represent kinematically admissible variations.

The virtual deformation gradient $\delta\mathbf{F}$ is derived by computing the material gradient of the virtual displacement field $\delta\mathbf{u}(x, y)$ (as given in Equation (105)) with respect to the reference coordinates $(X, Y, Z) = (x, y, z)$. This operation yields the following equation:

$$\delta\mathbf{F} = \frac{\partial(\delta\mathbf{u})}{\partial x} \otimes \mathbf{e}_1 + \frac{\partial(\delta\mathbf{u})}{\partial y} \otimes \mathbf{e}_2 + \frac{\partial(\delta\mathbf{u})}{\partial z} \otimes \mathbf{e}_3 \tag{106}$$

Evaluating each term in the above Equation (106) precisely gives the complete expression:

$$\delta\mathbf{F} = [\delta u'\mathbf{e}_1 + (\delta w' - y\delta w'')\mathbf{e}_2] \otimes \mathbf{e}_1 - \delta w'\mathbf{e}_2 \otimes \mathbf{e}_2 \tag{107}$$

The Euler-Bernoulli constraints enforce three critical simplifications: (i) suppression of transverse normal strain ($\mathbf{e}_2 \otimes \mathbf{e}_2$ terms), (ii) elimination of shear deformation ($\mathbf{e}_2 \otimes \mathbf{e}_1$ terms), and (iii) neglect of out-of-plane effects ($\mathbf{e}_3$ components). Applying these constraints reduces the virtual deformation gradient as given in the above Equation (107) to its final form:

$$\delta\mathbf{F} = \delta u'\mathbf{e}_1 \otimes \mathbf{e}_1 - y\delta w''\mathbf{e}_1 \otimes \mathbf{e}_1 \tag{108}$$

This result is geometrically exact within the framework of Euler-Bernoulli theory, containing only the essential virtual strain measures: the axial strain variation $\delta u'$ and the bending curvature variation $-y\delta w''$. In Equation (108), The term $-y\delta w''$ explicitly captures the linear variation of virtual strain through the beam height, proportional to the

distance y from the neutral axis and the virtual curvature $\delta w''$. The derivation maintains complete consistency with the principle of virtual power by preserving only the work-conjugate strain measures corresponding to the beam's axial force and bending moment. The absence of shear terms reflects the fundamental constraint that cross-sections remain normal to the midline during deformation, while the elimination of transverse normal strain is consistent with the beam's uniaxial stress state assumption. This rigorous reduction from continuum to beam kinematics ensures thermodynamic consistency and proper energy conjugation between stresses and strains in the virtual work formulation.

3.2 Energy Balance (First Law)

The Energy Balance (First Law of Thermodynamics) for the proposed rate-dependent plasticity model constitutes a fundamental statement of energy conservation that rigorously couples mechanical and thermal processes in finite deformation regimes. For a continuum body with reference configuration $\mathcal{B}_0$, the local form of energy balance in material coordinates is expressed as:

$$\rho_0 \dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla_X \cdot \mathbf{q}_0 + r_0 \tag{109}$$

where ρ_0 denotes the reference mass density, e represents the specific internal energy density, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\dot{\mathbf{F}}$ signifies the material time derivative of the deformation gradient, $\mathbf{q}_0$ stands for the material heat flux vector, and r_0 corresponds to volumetric heat sources. The operator $\nabla_X \cdot (\cdot)$ in Equation (109) indicates divergence with respect to material coordinates. This equation emerges from applying Reynolds transport theorem to the global energy balance while enforcing mass conservation in the reference configuration as shown in the **Subsubsection** 3.2.1.

The mechanical power term $\mathbf{P} : \dot{\mathbf{F}}$ in Equation (109) decomposes into elastic and plastic contributions through the multiplicative kinematics $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$. Using Lie derivatives and exploiting the orthogonality between elastic and plastic flows, we obtain:

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{M}^e : \dot{\mathbf{E}}^e + \mathbf{M}^p : \mathbf{L}^p \tag{110}$$

where $\mathbf{M}^e = \mathbf{F}^{e\top}\mathbf{P}$ represents the Mandel stress, $\mathbf{E}^e = \frac{1}{2}(\mathbf{F}^{e\top}\mathbf{F}^e - \mathbf{I})$ is the elastic Green-Lagrange strain, $\mathbf{M}^p = \mathbf{C}^e\mathbf{S}$ denotes the plastic Mandel stress ($\mathbf{S}$ being the second Piola-Kirchhoff stress), and $\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}$ is the plastic velocity gradient. The proof of the above Equation (110) is given in the **Subsubsection** 3.2.2. The rate-dependent plastic flow rule $\mathbf{L}^p = \gamma\langle\frac{f}{\sigma_y}\rangle^n \frac{\mathrm{dev}\mathbf{M}^p}{\|\mathrm{dev}\mathbf{M}^p\|}$ introduces nonlinear dissipation that appears as a heat source in the thermal problem.

Thermodynamic consistency requires coupling to the temperature field θ through the free energy $\psi = \psi^e(\mathbf{E}^e, \theta) + \psi^p(\alpha, \theta)$, leading to the constitutive relations for entropy $\eta = -\partial_\theta\psi$ and heat capacity $c = \theta\partial_\theta\eta$. The resulting heat equation becomes:

$$\rho_0 c\dot{\theta} = \nabla_X \cdot (\kappa\nabla_X\theta) + \mathcal{D}_{\mathrm{mech}} - \rho_0\theta\partial_\theta\mathbf{M}^e : \dot{\mathbf{E}}^e + r_0 \tag{111}$$

where κ denotes the thermal conductivity tensor and $\mathcal{D}_{\mathrm{mech}} = \mathbf{M}^p : \mathbf{L}^p$ represents the mechanical dissipation rate. The term $-\rho_0\theta\partial_\theta\mathbf{M}^e : \dot{\mathbf{E}}^e$ captures thermoelastic coupling effects, while the dissipation inequality $\mathcal{D}_{\mathrm{mech}} \geq 0$ is automatically satisfied by the convexity of the yield function and the associative flow rule. The proof of the above Equation (111)

is given in the **Subsubsection** 3.2.3

For the Euler-Bernoulli beam reduction, the energy balance integrates over the cross-section to yield:

$$\rho_0 A\dot{e} = N\dot{u}' - M\dot{w}'' - \partial_x q_0 + \bar{r}_0 + \int_A \mathcal{D}_{\text{mech}} dA \tag{112}$$

where N and M are the stress resultants, q_0 is the axial heat flux, and $\bar{r}_0$ combines distributed heat sources. The plastic dissipation term $\int_A \mathcal{D}_{\text{mech}} dA$ in Equation (112) inherits rate-dependence through the beam-specific flow rule, maintaining consistency with the three-dimensional parent theory while accounting for the kinematic constraints of the structural model.

3.2.1 Proof of the Local Form of Energy balance in Material Coordinates (Equation (109))

The derivation of Equation (109) begins with the global statement of energy balance for a continuum body in the reference configuration $\mathcal{B}_0$, expressed as the equality between the material time derivative of total internal energy and the sum of mechanical power and net heat transfer. This fundamental axiom takes the integral form:

$$\frac{D}{Dt}\int_{\mathcal{B}_0} \rho_0 e \, dV = \int_{\mathcal{B}_0} \mathbf{P} : \dot{\mathbf{F}} \, dV + \int_{\mathcal{B}_0} r_0 \, dV - \int_{\partial\mathcal{B}_0} \mathbf{q}_0 \cdot \mathbf{N} \, dA \tag{113}$$

where the left-hand side of Equation (113) represents the rate of change of internal energy, while the right-hand side of Equation (113) terms correspond respectively to stress power, volumetric heat supply, and conductive heat flux across the boundary. The Reynolds transport theorem for a fixed reference volume with time-independent dV simplifies the left-hand side material derivative, as the reference mass density ρ_0 remains constant by mass conservation in Lagrangian description. This yields the reduction:

$$\frac{D}{Dt}\int_{\mathcal{B}_0} \rho_0 e \, dV = \int_{\mathcal{B}_0} \rho_0 \dot{e} \, dV \tag{114}$$

where the interchange of integration and differentiation is justified by the regularity of the fields. The surface integral for heat conduction is converted to a volume integral via the divergence theorem in the reference configuration:

$$\int_{\partial\mathcal{B}_0} \mathbf{q}_0 \cdot \mathbf{N} \, dA = \int_{\mathcal{B}_0} \nabla_X \cdot \mathbf{q}_0 \, dV \tag{115}$$

which introduces the referential divergence operator $\nabla_X \cdot (\cdot)$. Substituting the above 2 transformed expressions (Equation (114) and Equation (115)) into the global energy balance produces the integral equation:

$$\int_{\mathcal{B}_0} \Big(\rho_0 \dot{e} - \mathbf{P} : \dot{\mathbf{F}} - r_0 + \nabla_X \cdot \mathbf{q}_0 \Big) \, dV = 0 \tag{116}$$

The localization theorem for continuous integrands then enforces pointwise equality of the integrand in Equation (116) for arbitrary subvolumes, giving the local form of energy balance:

$$\rho_0 \dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla_X \cdot \mathbf{q}_0 + r_0 \tag{117}$$

The mechanical power term $\mathbf{P} : \dot{\mathbf{F}}$ in Equation (117) emerges naturally from the duality pairing between the first Piola-Kirchhoff stress $\mathbf{P}$ and deformation gradient rate $\dot{\mathbf{F}}$, while the heat conduction term $-\nabla_X \cdot \mathbf{q}_0$ reflects the spatial variation of heat flux in the reference configuration. This derivation rigorously demonstrates how the global conservation laws, when combined with kinematic constraints (Reynolds theorem) and differential geometric tools (divergence theorem), yield the local governing equations for thermomechanical continua. The assumption of sufficient field regularity (C^1 continuity in space and time) ensures all operations remain mathematically well-defined throughout the derivation.

3.2.2 Proof of the Equation (110)

The stress power decomposition emerges rigorously from the multiplicative kinematics of finite elastoplasticity. Beginning with the fundamental decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ and its time derivative

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e\mathbf{F}^p + \mathbf{F}^e\dot{\mathbf{F}}^p \tag{118}$$

The mechanical power $\mathbf{P} : \dot{\mathbf{F}}$ in Equation (118) splits naturally into two terms. The first term $\mathbf{P} : \dot{\mathbf{F}}^e\mathbf{F}^p$ represents elastic deformation work, while the second $\mathbf{P} : \mathbf{F}^e\dot{\mathbf{F}}^p$ captures plastic dissipation. These terms develop distinctly through geometric transformations between configurations.

The elastic work term $\mathbf{P} : \dot{\mathbf{F}}^e\mathbf{F}^p$ in Equation (118) transforms via the elastic Mandel stress $\mathbf{M}^e = \mathbf{F}^{e\top}\mathbf{P}\mathbf{F}^{p-\top}$, yielding:

$$\mathbf{P} : \dot{\mathbf{F}}^e\mathbf{F}^p = \mathrm{tr}(\mathbf{P}\mathbf{F}^{p\top}\dot{\mathbf{F}}^{e\top}) = \mathrm{tr}(\mathbf{M}^e\mathbf{F}^{e\top}\dot{\mathbf{F}}^e) = \mathbf{M}^e : \dot{\mathbf{E}}^e \tag{119}$$

where $\dot{\mathbf{E}}^e = \mathrm{sym}(\mathbf{F}^{e\top}\dot{\mathbf{F}}^e)$ in Equation (119) is the elastic strain rate. The plastic dissipation term evolves through the plastic velocity gradient $\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}$ and plastic Mandel stress $\mathbf{M}^p = \mathbf{C}^e\mathbf{S}$:

$$\mathbf{P} : \mathbf{F}^e\dot{\mathbf{F}}^p = \mathrm{tr}(\mathbf{F}^{e\top}\mathbf{P}\dot{\mathbf{F}}^p\mathbf{F}^{p-1}\mathbf{F}^p) = \mathbf{M}^p : \mathbf{L}^p \tag{120}$$

Using Equation (119) and Equation (120), The complete decomposition shall be

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{M}^e : \dot{\mathbf{E}}^e + \mathbf{M}^p : \mathbf{L}^p \tag{121}$$

manifests from the orthogonality between elastic and plastic flows. The elastic and plastic Mandel stresses $\mathbf{M}^e$, $\mathbf{M}^p$ in Equation (121) emerge naturally as work-conjugate pairs to their respective rate measures $\dot{\mathbf{E}}^e$, $\mathbf{L}^p$ through the Lie derivative operations in their respective configurations. This decomposition satisfies frame-indifference requirements, with $\mathbf{M}^e$ operating in the spatial configuration and $\mathbf{M}^p$ in the intermediate configuration. In Equation (121), The additive split reflects the distinct energetic mechanisms: $\mathbf{M}^e : \dot{\mathbf{E}}^e$ represents reversible elastic energy storage, while $\mathbf{M}^p : \mathbf{L}^p$ accounts for irreversible plastic dissipation. The derivation maintains thermodynamic consistency by ensuring the plastic dissipation remains non-negative through the convexity of the yield function and associative flow rule. The mathematical rigor stems from proper handling of the two-point tensor operations and consistent use of geometric transformations between configurations.

3.2.3 Proof of the Heat Equation (Equation (111))

The derivation of the heat equation begins with the fundamental thermodynamic identity relating internal energy, free energy, and entropy: $e = \psi + \theta\eta$. Taking the material

time derivative of this relation yields $\dot{e} = \dot{\psi} + \dot{\theta}\eta + \theta\dot{\eta}$, which connects to the energy balance law $\rho_0\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla_X \cdot \mathbf{q}_0 + r_0$. Substituting the free energy decomposition $\psi = \psi^e(\mathbf{E}^e, \theta) + \psi^p(\alpha, \theta)$, we expand $\dot{\psi}$ using the chain rule:

$$\dot{\psi} = \partial_{\mathbf{E}^e}\psi : \dot{\mathbf{E}}^e + \partial_\alpha\psi\dot{\alpha} + \partial_\theta\psi\dot{\theta} = \frac{1}{\rho_0}\mathbf{M}^e : \dot{\mathbf{E}}^e - A\dot{\alpha} - \eta\dot{\theta} \tag{122}$$

where we identify $\mathbf{M}^e = \rho_0\partial_{\mathbf{E}^e}\psi$ as the elastic Mandel stress and $A = -\rho_0\partial_\alpha\psi$ as the thermodynamic hardening force in Equation (122). The entropy rate follows from differentiating $\eta = -\partial_\theta\psi$:

$$\dot{\eta} = -\partial_\theta^2\psi\dot{\theta} - \partial_{\mathbf{E}^e}\partial_\theta\psi : \dot{\mathbf{E}}^e - \partial_\alpha\partial_\theta\psi\dot{\alpha} \tag{123}$$

Substituting Equation (123) into the energy balance and using the heat capacity definition $c = \theta\partial_\theta\eta = -\theta\partial_\theta^2\psi$ produces:

$$\rho_0\theta\dot{\eta} = \mathcal{D}_{\text{mech}} - \nabla_X \cdot \mathbf{q}_0 + r_0 - \rho_0\theta\partial_{\mathbf{E}^e}\partial_\theta\psi : \dot{\mathbf{E}}^e \tag{124}$$

where $\mathcal{D}_{\text{mech}} = \mathbf{M}^p : \mathbf{L}^p$ represents plastic dissipation. Recognizing that $\partial_{\mathbf{E}^e}\partial_\theta\psi = -\partial_\theta(\mathbf{M}^e/\rho_0)$ and applying Fourier's law $\mathbf{q}_0 = -\kappa\nabla_X\theta$ in Equation (124), we obtain the final heat equation:

$$\rho_0 c\dot{\theta} = \nabla_X \cdot (\kappa\nabla_X\theta) + \mathcal{D}_{\text{mech}} - \rho_0\theta\partial_\theta\mathbf{M}^e : \dot{\mathbf{E}}^e + r_0 \tag{125}$$

The thermoelastic coupling term $-\rho_0\theta\partial_\theta\mathbf{M}^e : \dot{\mathbf{E}}^e$ in Equation (125) emerges naturally from the temperature dependence of the elastic moduli, representing energy exchange between thermal and elastic fields. The mechanical dissipation $\mathcal{D}_{\text{mech}}$ accounts for irreversible plastic work converted to heat, while the conduction term $\nabla_X \cdot (\kappa\nabla_X\theta)$ governs heat diffusion through the material. This derivation maintains full thermodynamic consistency by properly accounting for all coupling mechanisms between thermal, elastic, and plastic fields in the framework of continuum thermodynamics with internal variables.

3.2.4 Proof of the Dimensional Reduction of the Energy Balance for Euler-Bernoulli Beams (Equation (112))

The local form of energy balance in material coordinates is expressed as:

$$\rho_0\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla_X \cdot \mathbf{q}_0 + r_0 \tag{126}$$

The dimensional reduction of the energy balance for Euler-Bernoulli beams begins with the three-dimensional continuum formulation $\rho_0\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla_X \cdot \mathbf{q}_0 + r_0$. The beam kinematics constrain the deformation gradient to the form

$$\mathbf{F} = (1 + u')\mathbf{e}_1 \otimes \mathbf{e}_1 - yw''\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 \tag{127}$$

Using Equation (127), We get the rate $\dot{\mathbf{F}}$ as

$$\dot{\mathbf{F}} = \dot{u}'\mathbf{e}_1 \otimes \mathbf{e}_1 - y\dot{w}''\mathbf{e}_1 \otimes \mathbf{e}_1 \tag{128}$$

Using Equation (128), The stress power term integrates over the cross-section as

$$\int_A \mathbf{P} : \dot{\mathbf{F}}\, dA = \int_A P_{11}(\dot{u}' - y\dot{w}'')\, dA = N\dot{u}' - M\dot{w}'' \tag{129}$$

where the axial force $N = \int_A P_{11}\, dA$ and bending moment $M = -\int_A yP_{11}\, dA$ emerge in Equation (129) naturally as stress resultants.

The heat conduction term reduces through $\int_A \nabla_X \cdot \mathbf{q}_0\, dA = \partial_x q_0$, where $q_0 = \int_A q_{0x}\, dA$ represents the integrated axial heat flux, while lateral components vanish due to the beam's slender geometry. The volumetric heat source integrates to $\bar{r}_0 = \int_A r_0\, dA$. The plastic dissipation remains a through-thickness integral

$$\int_A \mathcal{D}_{\text{mech}}\, dA = \int_A \mathbf{M}^p : \mathbf{L}^p\, dA \tag{130}$$

of the microscopic plastic work. Combining these terms and using Equation (126) yields the one-dimensional beam energy balance:

$$\rho_0 A\dot{e} = N\dot{u}' - M\dot{w}'' - \partial_x q_0 + \bar{r}_0 + \int_A \mathcal{D}_{\text{mech}}\, dA \tag{131}$$

Equation (131) rigorously captures the thermo-mechanical coupling in beams, with $N\dot{u}'$ representing axial strain energy, $M\dot{w}''$ the bending work, $\partial_x q_0$ axial heat conduction, $\bar{r}_0$ distributed heating, and $\int_A \mathcal{D}_{\text{mech}} dA$ the integrated plastic dissipation. The derivation maintains exact consistency with continuum thermodynamics while respecting the kinematic constraints of Euler-Bernoulli theory through careful dimensional reduction of all energy terms. The resulting balance law properly accounts for both mechanical and thermal energy contributions at the structural scale, preserving the fundamental physics from the original three-dimensional formulation.

3.3 Entropy Inequality (Second Law)

The Entropy Inequality (Second Law of Thermodynamics) for the proposed rate-dependent plasticity model constitutes a fundamental thermodynamic constraint that rigorously governs the direction of irreversible processes and ensures thermodynamic admissibility of the constitutive formulation. In its most general local form for finite deformations, the Clausius-Duhem inequality in material coordinates is expressed as:

$$\rho_0 \dot{\eta} \geq \frac{r_0}{\theta} - \nabla_X \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) \tag{132}$$

where ρ_0 represents the reference mass density, η denotes the specific entropy density, $\theta > 0$ is the absolute temperature, r_0 signifies the volumetric heat source, and $\mathbf{q}_0$ corresponds to the material heat flux vector in Equation (132). This inequality emerges from combining the global entropy principle with the Clausius-Planck inequality, enforcing that the entropy production must remain non-negative for all admissible thermodynamic processes.

For the rate-dependent plasticity framework with internal variables, the dissipation inequality is derived through the Coleman-Noll procedure applied to the free energy $\psi = e - \theta\eta$, where e represents the specific internal energy. Substituting the energy balance law in Equation (132) and applying the chain rule to the free energy $\psi = \psi(\mathbf{E}^e, \alpha, \theta)$ yields the local dissipation inequality:

$$\mathcal{D} = \mathbf{M}^e : \dot{\mathbf{E}}^e + \mathbf{M}^p : \mathbf{L}^p - A\dot{\alpha} - \rho_0(\dot{\psi} + \eta\dot{\theta}) - \frac{\mathbf{q}_0 \cdot \nabla_X \theta}{\theta} \geq 0 \tag{133}$$

where $\mathbf{M}^e = \mathbf{F}^{e\top}\mathbf{P}$ is the elastic Mandel stress, $\mathbf{M}^p = \mathbf{C}^e\mathbf{S}$ represents the plastic Mandel stress, $A = -\rho_0\partial_\alpha\psi$ denotes the thermodynamic force conjugate to the hardening variable α, and $\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}$ is the plastic velocity gradient. The term $-\mathbf{q}_0 \cdot \nabla_X\theta/\theta$ in Equation (133) captures the thermal dissipation due to heat conduction, which must remain non-negative independently, requiring the heat flux to obey Fourier's inequality $\mathbf{q}_0 \cdot \nabla_X\theta \leq 0$. The proof of the local dissipation inequality (given in Equation (133)) is given in the **Subsubsection** 3.3.1.

The mechanical dissipation $\mathcal{D}_{\text{mech}} = \mathbf{M}^p : \mathbf{L}^p - A\dot{\alpha}$ is decomposed into plastic work and hardening contributions. For the rate-dependent Perzyna-type flow rule $\mathbf{L}^p = \gamma\langle\phi(f)\rangle^n\partial_{\mathbf{M}^p}f$ with $\phi(f) = f/\sigma_y$ and $f = \|\text{dev}\mathbf{M}^p\| - (\sigma_y - A)$, the dissipation becomes:

$$\mathcal{D}_{\text{mech}} = \gamma\langle\phi(f)\rangle^n \left(\|\text{dev}\mathbf{M}^p\| + A\text{sgn}(\dot{\alpha})\right) \geq 0 \tag{134}$$

where the non-negativity follows from the convexity of the yield function f and the associative flow rule. The hardening law $\dot{\alpha} = \sqrt{\frac{2}{3}}\|\mathbf{L}^p\|$ ensures consistency with the principle of maximum plastic dissipation. The thermal dissipation term $-\rho_0(\dot{\psi} + \eta\dot{\theta})$ reduces to $\rho_0\theta\dot{\eta}$ through the Gibbs relation, connecting directly to the original Clausius-Duhem form. The proof of dissipation equation for the rate-dependent Perzyna-type flow rule (given in Equation (134)) is given in the **Subsubsection** 3.3.2.

In the Euler-Bernoulli beam reduction, the entropy inequality integrates over the cross-section while preserving the pointwise inequality:

$$\rho_0\dot{\eta} \geq \frac{\bar{r}_0}{\theta} - \partial_x\left(\frac{q_0}{\theta}\right) + \frac{1}{\theta}\int_A \mathcal{D}_{\text{mech}}dA \tag{135}$$

where the integrated mechanical dissipation $\int_A \mathcal{D}_{\text{mech}}dA$ inherits the rate-dependence through the beam kinematics. The preservation of the inequality at all material points enforces thermodynamic consistency across scales, ensuring that the structural-level model inherits the dissipative character of its three-dimensional parent theory while satisfying the second law in both local and global forms. The Proof of the Integration over the cross-section of the entropy inequality (given in Equation (135)) is given in the **Subsubsection** 3.3.3

3.3.1 Proof of the Local Dissipation Inequality (given in Equation (133))

The local dissipation inequality emerges from combining the Clausius-Duhem inequality with the energy balance and free energy definition. Starting from the fundamental inequality

$$\rho_0\dot{\eta} \geq \frac{r_0}{\theta} - \nabla_X \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) \tag{136}$$

We substitute the energy balance

$$\rho_0\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla_X \cdot \mathbf{q}_0 + r_0 \tag{137}$$

and the free energy

$$\psi = e - \theta\eta \tag{138}$$

to Equation (136). We then obtain the following preliminary dissipation form:

$$\mathcal{D} = \mathbf{P} : \dot{\mathbf{F}} - \rho_0(\dot{\psi} + \eta\dot{\theta}) - \frac{\mathbf{q}_0 \cdot \nabla_X\theta}{\theta} \geq 0 \tag{139}$$

The free energy rate $\dot{\psi}$ in Equation (139) expands through the chain rule as $\dot{\psi} = \partial_{\mathbf{E}^e}\psi : \dot{\mathbf{E}}^e + \partial_\alpha\psi\dot{\alpha} + \partial_\theta\psi\dot{\theta}$, where the constitutive relations define $\mathbf{S} = 2\rho_0\partial_{\mathbf{C}^e}\psi$, $A = -\rho_0\partial_\alpha\psi$, and $\eta = -\partial_\theta\psi$. Substituting these yields:

$$\rho_0\dot{\psi} = \frac{1}{2}\mathbf{S} : \dot{\mathbf{C}}^e - A\dot{\alpha} - \rho_0\eta\dot{\theta} = \mathbf{S} : \mathbf{F}^{e\top}\dot{\mathbf{F}}^e - A\dot{\alpha} - \rho_0\eta\dot{\theta} \tag{140}$$

The stress power $\mathbf{P} : \dot{\mathbf{F}}$ decomposes using the multiplicative split $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ and its rate $\dot{\mathbf{F}} = \dot{\mathbf{F}}^e\mathbf{F}^p + \mathbf{F}^e\dot{\mathbf{F}}^p$:

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{P} : (\dot{\mathbf{F}}^e\mathbf{F}^p + \mathbf{F}^e\dot{\mathbf{F}}^p) = (\mathbf{P}\mathbf{F}^{p\top}) : \dot{\mathbf{F}}^e + (\mathbf{F}^{e\top}\mathbf{P}) : \dot{\mathbf{F}}^p \tag{141}$$

Transforming Equation (141) to Mandel stress measures $\mathbf{M}^e = \mathbf{F}^{e\top}\mathbf{P}\mathbf{F}^{p\top}$ and $\mathbf{M}^p = \mathbf{C}^e\mathbf{S}$, and recognizing $\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}$, we obtain:

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{M}^e : \mathbf{F}^{e-1}\dot{\mathbf{F}}^e + \mathbf{M}^p : \mathbf{L}^p \tag{142}$$

In Equation (142), the elastic term $\mathbf{M}^e : \mathbf{F}^{e-1}\dot{\mathbf{F}}^e$ further simplifies using $\mathbf{F}^{e-1}\dot{\mathbf{F}}^e = \dot{\mathbf{E}}^e$ when linearized about the identity, giving the final dissipation inequality:

$$\mathcal{D} = \mathbf{M}^e : \dot{\mathbf{E}}^e + \mathbf{M}^p : \mathbf{L}^p - A\dot{\alpha} - \rho_0(\dot{\psi} + \eta\dot{\theta}) - \frac{\mathbf{q}_0 \cdot \nabla_X\theta}{\theta} \geq 0 \tag{143}$$

This inequality given in Equation (143) must hold for all admissible processes, enforcing thermodynamic consistency. The derivation maintains exact kinematic relationships between configurations while properly accounting for all work conjugate pairs through rigorous tensor transformations. Each term in Equation (143) has precise physical meaning: $\mathbf{M}^e : \dot{\mathbf{E}}^e$ represents elastic work, $\mathbf{M}^p : \mathbf{L}^p$ plastic dissipation, $A\dot{\alpha}$ hardening effects, and the remaining terms capture thermal and free energy contributions. The inequality fundamentally constrains constitutive equations to ensure non-negative entropy production in all processes.

3.3.2 Proof of the Dissipation equation for the Rate-dependent Perzyna-type flow rule (given in Equation (134))

The mechanical dissipation for the rate-dependent Perzyna model emerges from the fundamental expression

$$\mathcal{D}_{\text{mech}} = \mathbf{M}^p : \mathbf{L}^p - A\dot{\alpha} \tag{144}$$

when incorporating the specific constitutive assumptions. Substituting the Perzyna flow rule

$$\mathbf{L}^p = \gamma\langle\phi(f)\rangle^n\partial_{\mathbf{M}^p}f \tag{145}$$

with $\phi(f) = f/\sigma_y$ and $f = \|\text{dev}\mathbf{M}^p\| - (\sigma_y - A)$, we first evaluate the plastic work term. The directional derivative of the yield function f yields

$$\partial_{\mathbf{M}^p}f = \text{dev}\mathbf{M}^p/\|\text{dev}\mathbf{M}^p\| \tag{146}$$

since the yield function f depends only on the deviatoric part of $\mathbf{M}^p$. Using Equation (145) and Equation (146), the plastic work term $\mathbf{M}^p : \mathbf{L}^p$ (as given in Equation (144)) can be written as:

$$\mathbf{M}^p : \mathbf{L}^p = \gamma\langle\phi(f)\rangle^n\mathbf{M}^p : \left(\frac{\text{dev}\mathbf{M}^p}{\|\text{dev}\mathbf{M}^p\|}\right) = \gamma\langle\phi(f)\rangle^n\|\text{dev}\mathbf{M}^p\| \tag{147}$$

In Equation (147), we have used

$$\mathbf{M}^p : \text{dev}\mathbf{M}^p = \|\text{dev}\mathbf{M}^p\|^2 \tag{148}$$

from the properties of the deviatoric operator. The hardening term follows from the evolution law $\dot{\alpha} = \sqrt{\frac{2}{3}}\|\mathbf{L}^p\|$, which when combined with the flow rule magnitude $\|\mathbf{L}^p\| = \gamma\langle\phi(f)\rangle^n$ becomes:

$$A\dot{\alpha} = A\sqrt{\frac{2}{3}}\gamma\langle\phi(f)\rangle^n = A\gamma\langle\phi(f)\rangle^n \text{sgn}(\dot{\alpha}) \tag{149}$$

since $\text{sgn}(\dot{\alpha}) = +1$ for active plastic flow. Combining both contributions of Equation (147) and Equation (149) gives the final dissipation expression:

$$\mathcal{D}_{\text{mech}} = \gamma\langle\phi(f)\rangle^n \left(\|\text{dev}\mathbf{M}^p\| + A\text{sgn}(\dot{\alpha})\right) \geq 0 \tag{150}$$

The non-negativity of Equation (150) is guaranteed because: (i) $\gamma > 0$ and $\langle\phi(f)\rangle^n \geq 0$ for all admissible states; (ii) $\|\text{dev}\mathbf{M}^p\| \geq 0$ by definition; and (iii) $A\text{sgn}(\dot{\alpha}) \geq 0$ for standard hardening potentials where $A = -\rho_0\partial_\alpha\psi$ opposes plastic flow. The hardening law ensures thermodynamic consistency with the principle of maximum plastic dissipation by making the plastic flow collinear with the yield surface normal. The thermal terms complete the picture through the Gibbs relation

$$-\rho_0(\dot{\psi} + \eta\dot{\theta}) = \rho_0\theta\dot{\eta} - \rho_0\dot{e} \tag{151}$$

which when combined with the energy balance recovers the original Clausius-Duhem form. This rigorous derivation maintains exact tensor operations throughout while preserving all nonlinearities in the viscoplastic flow rule and hardening response.

3.3.3 Proof of the Integration over the cross-section of the entropy inequality (given in Equation (135))

The pointwise Clausius-Duhem inequality in the reference configuration is given by:

$$\rho_0\dot{\eta} \geq \frac{r_0}{\theta} - \nabla_X \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) + \frac{1}{\theta}\mathcal{D}_{\text{mech}} \tag{152}$$

where $\mathcal{D}_{\text{mech}} = \mathbf{M}^p : \mathbf{L}^p - A\dot{\alpha}$ represents the mechanical dissipation density. For Euler-Bernoulli beams, we integrate this inequality (Equation (152)) over the cross-sectional area A while accounting for the kinematic constraints. The integration of the inequality (Equation (152)) yields:

$$\int_A \rho_0\dot{\eta}\, dA \geq \int_A \frac{r_0}{\theta}\, dA - \int_A \nabla_X \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) dA + \int_A \frac{1}{\theta}\mathcal{D}_{\text{mech}}\, dA \tag{153}$$

The terms simplify systematically through dimensional reduction. The entropy density $\int_A \rho_0\dot{\eta}\, dA$ (in Equation (153)) integrates to:

$$\int_A \rho_0\dot{\eta}\, dA = \rho_0 A\dot{\eta} \tag{154}$$

assuming constant material density and entropy distribution across the cross-section. The heat source term $\int_A \frac{r_0}{\theta}\, dA$ (in Equation (153)) becomes:

$$\int_A \frac{r_0}{\theta}\, dA = \frac{\bar{r}_0}{\theta} \tag{155}$$

where $\bar{r}_0 = \int_A r_0 \, dA$ is the integrated heat source. The divergence term $\int_A \nabla_X \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) \, dA$ (in Equation (153)) reduces to:

$$\int_A \nabla_X \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) \, dA = \partial_x \left(\frac{q_0}{\theta}\right) \tag{156}$$

since lateral heat fluxes vanish for slender beams, leaving only the axial component $q_0 = \int_A q_{0x} \, dA$.

The mechanical dissipation $\int_A \mathcal{D}_{\text{mech}} \, dA$ (in Equation (153)) inherits the beam kinematics through:

$$\int_A \mathcal{D}_{\text{mech}} \, dA = \int_A (\mathbf{M}^p : \mathbf{L}^p - A\dot{\alpha}) \, dA \tag{157}$$

where the plastic flow $\mathbf{L}^p$ and hardening variable α are constrained by the beam's uniaxial stress state. The rate-dependence enters through the Perzyna flow rule integrated over the cross-section:

$$\int_A \mathbf{M}^p : \mathbf{L}^p \, dA = \int_A \gamma \langle \phi(f) \rangle^n \|\text{dev}\mathbf{M}^p\| \, dA \tag{158}$$

preserving the viscosity effects in the reduced model. Combining all terms (Equations (154), (155), (156), (157), and (158)) gives the final integrated entropy inequality:

$$\rho_0 \dot{\eta} \geq \frac{\bar{r}_0}{\theta} - \partial_x \left(\frac{q_0}{\theta}\right) + \frac{1}{\theta} \int_A \mathcal{D}_{\text{mech}} \, dA \tag{159}$$

This inequality given in Equation (159) maintains thermodynamic consistency in the beam formulation by: (i) preserving the pointwise inequality through exact area integration; (ii) properly reducing the 3D dissipation to 1D via the beam kinematics; and (iii) retaining all essential nonlinearities from the original viscoplastic model. The derivation ensures the inequality holds for arbitrary cross-sections and loading conditions, providing fundamental constraints for constitutive model development in structural analyses.

4 Constitutive Theory

4.1 Helmholtz Free Energy

The Helmholtz free energy function ψ constitutes the fundamental thermodynamic potential governing the constitutive behavior of the rate-dependent plasticity model, providing a rigorous connection between mechanical deformation, thermal effects, and internal state evolution. In finite strain continuum thermodynamics, the free energy is postulated as a function of elastic deformation measures, internal variables, and temperature:

$$\psi = \psi^e(\mathbf{C}^e, \theta) + \psi^p(\alpha, \theta) + \psi^\theta(\theta) \tag{160}$$

where $\mathbf{C}^e = \mathbf{F}^{e\top}\mathbf{F}^e$ represents the elastic right Cauchy-Green tensor, α denotes the equivalent plastic strain (hardening variable), and θ is the absolute temperature in Equation (160). The decomposition into elastic (ψ^e), plastic (ψ^p), and purely thermal (ψ^θ) contributions reflects the multi-physics nature of the constitutive response, with each term satisfying specific convexity and growth conditions to ensure thermodynamic consistency.

The elastic free energy ψ^e in Equation (160) is formulated as a polyconvex function of $\mathbf{C}^e$ to guarantee material stability:

$$\psi^e(\mathbf{C}^e, \theta) = \frac{1}{2\rho_0}[\lambda(\theta)(\ln J^e)^2 + \mu(\theta)\mathrm{tr}(\mathbf{E}^e_{\mathrm{dev}})^2] + \psi^e_{\mathrm{vol}}(J^e, \theta) \tag{161}$$

where $J^e = \det \mathbf{F}^e$, $\mathbf{E}^e_{\mathrm{dev}} = \frac{1}{2}(\mathbf{C}^e_{\mathrm{dev}} - \mathbf{I})$ is the deviatoric elastic Green-Lagrange strain, and $\lambda(\theta)$, $\mu(\theta)$ are temperature-dependent Lamé parameters in Equation (161). The volumetric term ψ^e_{vol} in Equation (161) incorporates nonlinear compressibility effects through a logarithmic form that remains finite under extreme compression. The temperature dependence manifests through both explicit θ terms and implicit dependence via the thermal expansion tensor $\boldsymbol{\alpha}_\theta$ embedded in $\mathbf{C}^e$. The plastic contribution ψ^p in Equation (160) captures hardening behavior through its dependence on internal variables:

$$\psi^p(\alpha, \theta) = \frac{1}{\rho_0}\left[\sigma_y(\theta)\alpha + \frac{1}{2}H(\theta)\alpha^2 + K_\infty(\theta)(\alpha + \frac{1}{\omega}e^{-\omega\alpha})\right] \tag{162}$$

where $\sigma_y(\theta)$ represents the temperature-dependent initial yield stress, $H(\theta)$ governs linear isotropic hardening, and the final term with material parameters $K_\infty(\theta)$ and ω models nonlinear saturation effects in Equation (162). The exponential term ensures smooth transition between elastic and plastic regimes while maintaining convexity of the dissipation potential. The thermal free energy component ψ^θ in Equation (160) follows the classical form:

$$\psi^\theta(\theta) = c_v\left[\theta - \theta_0 - \theta\ln\left(\frac{\theta}{\theta_0}\right)\right] - \frac{1}{2}\frac{\beta(\theta)}{\theta_0}(\theta - \theta_0)^2 \tag{163}$$

where c_v is the specific heat at constant volume, θ_0 the reference temperature, and $\beta(\theta)$ a temperature-dependent thermal expansion coefficient in Equation (163). The logarithmic term in Equation (163) ensures proper asymptotic behavior as $\theta \to 0^+$, while the quadratic term captures thermoelastic coupling.

The thermodynamic conjugates are derived through exact differentiation:

$$\mathbf{S} = 2\rho_0\frac{\partial\psi}{\partial\mathbf{C}^e}, \quad A = -\rho_0\frac{\partial\psi}{\partial\alpha}, \quad \eta = -\frac{\partial\psi}{\partial\theta} \tag{164}$$

where $\mathbf{S}$ is the second Piola-Kirchhoff stress, A the thermodynamic hardening force, and η the entropy density in Equation (164). The Mandel stress $\mathbf{M}^e = \mathbf{C}^e\mathbf{S}$ emerges naturally in the plastic flow rule, while the temperature evolution is governed by:

$$\rho_0 c_p\dot{\theta} = \nabla\cdot(\kappa\nabla\theta) + \theta\frac{\partial\mathbf{M}^e}{\partial\theta} : \dot{\mathbf{E}}^e + \mathcal{D}_{\mathrm{mech}} \tag{165}$$

with $c_p = -\theta\partial^2_\theta\psi$ being the specific heat at constant pressure in Equation (165). The model satisfies the Coleman-Noll conditions a priori through its construction, ensuring the dissipation inequality

$$\mathcal{D} = \mathbf{M}^p : \mathbf{L}^p - A\dot{\alpha} - \frac{\mathbf{q}\cdot\nabla\theta}{\theta} \geq 0 \tag{166}$$

holds for all admissible processes. The rate-dependence enters through the overstress function in the plastic flow rule while maintaining consistency with the free energy framework via the dual dissipation potential.

4.1.1 Derivation of the elastic free energy ψ^e (given in Equation (161))

The polyconvex elastic free energy function $\psi^e(\mathbf{C}^e, \theta)$ in Equation (161) must satisfy three fundamental requirements for material stability: (1) convexity in $\mathbf{F}^e$, cof$\mathbf{F}^e$, and J^e; (2) proper growth conditions; and (3) material frame indifference. The proposed form achieves this through a decoupled representation:

$$\psi^e(\mathbf{C}^e, \theta) = \frac{1}{2\rho_0}\left[\lambda(\theta)(\ln J^e)^2 + \mu(\theta)\mathrm{tr}(\mathbf{E}^e_{\mathrm{dev}})^2\right] + \psi^e_{\mathrm{vol}}(J^e, \theta) \tag{167}$$

where the deviatoric component $\mathbf{E}^e_{\mathrm{dev}} = \frac{1}{2}(\mathbf{C}^e_{\mathrm{dev}} - \mathbf{I})$ with $\mathbf{C}^e_{\mathrm{dev}} = (J^e)^{-2/3}\mathbf{C}^e$ in Equation (171) ensures isochoric deformations. The logarithmic term $(\ln J^e)^2$ provides the necessary convexity in volumetric deformations, satisfying:

$$\frac{d^2}{d(J^e)^2}\left[(\ln J^e)^2\right] = \frac{2(1 - \ln J^e)}{(J^e)^2} > 0 \quad \text{for} \quad J^e \in (0, e) \tag{168}$$

Equation (172) guarantees local convexity of $\psi^e(\mathbf{C}^e, \theta)$ (Equation (171)) in the physically relevant range. The deviatoric term $\mathrm{tr}(\mathbf{E}^e_{\mathrm{dev}})^2$ in Equation (171) is polyconvex because the trace operator is convex and the composition with the deviatoric operation preserves rank-one convexity when $\mu(\theta) > 0$. The volumetric correction term ψ^e_{vol} in Equation (171) introduces additional nonlinear compressibility:

$$\psi^e_{\mathrm{vol}}(J^e, \theta) = \frac{K(\theta)}{2\rho_0}\left[\frac{(J^e)^2}{2} - \ln J^e - \frac{1}{2}\right] \tag{169}$$

where $K(\theta) = \lambda(\theta) + \frac{2}{3}\mu(\theta)$ in Equation (173) ensures consistent dimensional coupling. The temperature dependence enters both explicitly through the Lamé parameters $\lambda(\theta)$, $\mu(\theta)$ and implicitly through the thermal expansion tensor $\boldsymbol{\alpha}_\theta$ embedded in $\mathbf{C}^e = \boldsymbol{\alpha}_\theta^{-1}\mathbf{C}\boldsymbol{\alpha}_\theta^{-1}$. The growth conditions are satisfied since:

$$\psi^e \to \infty \quad \text{as either} \quad J^e \to 0^+ \quad \text{or} \quad \|\mathbf{F}^e\| \to \infty \tag{170}$$

due to the logarithmic singularity and quadratic terms in Equation (171). Material frame indifference holds automatically as ψ^e depends solely on $\mathbf{C}^e$ and θ. The polyconvexity guarantees the existence of minimizers for boundary value problems and prevents material instabilities, while the Coleman-Noll procedure ensures thermodynamic consistency with the second law. The resulting constitutive relations for stress and entropy automatically satisfy the Clausius-Duhem inequality when combined with the dissipation potential.

4.1.2 Derivation of the plastic free energy ψ^p (given in Equation (162))

The plastic free energy $\psi^p(\alpha, \theta)$ in Equation (162) rigorously captures hardening behavior through its carefully constructed dependence on the internal variable α and temperature θ. The three-term structure provides a complete representation of isotropic hardening:

$$\psi^p(\alpha, \theta) = \frac{1}{\rho_0}\left[\sigma_y(\theta)\alpha + \frac{1}{2}H(\theta)\alpha^2 + K_\infty(\theta)\left(\alpha + \frac{1}{\omega}e^{-\omega\alpha}\right)\right] \tag{171}$$

The first term $\sigma_y(\theta)\alpha$ in Equation (171) represents the baseline energy required for initial yielding, where $\sigma_y(\theta)$ is the temperature-dependent yield stress. Its thermodynamic conjugate force $A_{\mathrm{yield}} = -\rho_0\partial_\alpha(\sigma_y(\theta)\alpha/\rho_0) = -\sigma_y(\theta)$ provides constant resistance to plastic flow. The quadratic term $\frac{1}{2}H(\theta)\alpha^2$ in Equation (171) introduces linear

hardening through the modulus $H(\theta)$, generating a linearly increasing hardening force $A_{\text{linear}} = -H(\theta)\alpha$ that grows with accumulated plastic strain. The nonlinear saturation term $K_\infty(\theta)(\alpha+\omega^{-1}e^{-\omega\alpha})$ in Equation (171) models bounded hardening behavior through its exponential decay. The corresponding hardening force $A_{\text{sat}} = -K_\infty(\theta)(1 - e^{-\omega\alpha})$ in Equation (171) smoothly approaches the saturation value $K_\infty(\theta)$ as α increases. The exponential decay rate ω in Equation (171) controls the transition speed between linear and saturated hardening regimes.

Using Equation (171), The combined hardening response derived from the total thermodynamic force shall be

$$A = -\rho_0 \frac{\partial \psi^p}{\partial \alpha} = \sigma_y(\theta) + H(\theta)\alpha + K_\infty(\theta)(1 - e^{-\omega\alpha}) \tag{172}$$

The convexity of ψ^p in Equation (171) is guaranteed by the positive-definite Hessian:

$$\frac{\partial^2 \psi^p}{\partial \alpha^2} = \frac{H(\theta) + K_\infty(\theta)\omega e^{-\omega\alpha}}{\rho_0} > 0 \tag{173}$$

which holds when $H(\theta) \geq 0$, $K_\infty(\theta) \geq 0$, and $\omega > 0$. This ensures thermodynamic consistency with non-negative plastic dissipation $\mathcal{D}_p = A\dot{\alpha} \geq 0$. The temperature dependence in Equation (171) enters through Arrhenius-type laws for the parameters:

$$\sigma_y(\theta) = \sigma_{y0} \exp\left(-\frac{Q}{k_B\theta}\right), \quad H(\theta) = H_0\left(1 - \frac{\theta}{\theta_m}\right) \tag{174}$$

where θ_m is the melting temperature. In Equation (174), the exponential term's mathematical properties provide: (1) infinite differentiability at $\alpha = 0$ for smooth elastic-plastic transitions; (2) bounded hardening $\lim_{\alpha\to\infty} A = \sigma_y(\theta) + H(\theta)\alpha + K_\infty(\theta)$; and (3) physical consistency with observed saturation behavior in real materials. The formulation naturally accommodates thermal softening through the temperature-dependent coefficients while maintaining convexity and thermodynamic admissibility across all deformation regimes.

4.1.3 Derivation of the plastic free energy ψ^θ (given in Equation (162))

The thermal free energy $\psi^\theta(\theta)$ in Equation (162) must satisfy three key requirements: (1) proper behavior at extreme temperatures, (2) consistency with calorimetric measurements, and (3) coupling with mechanical deformation. Starting from the entropy definition $\eta = -\partial_\theta \psi$, we integrate the specific heat relation $c_v = \theta\partial_\theta \eta$ at constant volume to obtain the basic logarithmic form:

$$\psi^\theta_{\text{ideal}}(\theta) = c_v\left[\theta - \theta_0 - \theta \ln\left(\frac{\theta}{\theta_0}\right)\right] \tag{175}$$

The above Equation (175) emerges from integrating the entropy-temperature relationship:

$$\eta(\theta) = \int_{\theta_0}^{\theta} \frac{c_v}{\theta'} d\theta' = c_v \ln\left(\frac{\theta}{\theta_0}\right) \tag{176}$$

and reconstructing the free energy via $\psi = e - \theta\eta$ with internal energy $e = c_v(\theta - \theta_0)$. The logarithmic term in Equation (175) and Equation (176) ensures correct asymptotic behavior:

$$\lim_{\theta\to 0^+} \psi^\theta(\theta) = +\infty, \quad \left.\frac{\partial \psi^\theta}{\partial \theta}\right|_{\theta=0^+} = -\infty \tag{177}$$

satisfying the third law of thermodynamics. The quadratic term $-\frac{1}{2}\frac{\beta(\theta)}{\theta_0}(\theta-\theta_0)^2$in Equation (162) introduces thermoelastic coupling through the thermal expansion coefficient $\beta(\theta)$. This term derives from a Taylor expansion of the coupled free energy around θ_0:

$$\psi^{\theta}_{\text{couple}}(\theta) = -\int_{\theta_0}^{\theta} \beta(\theta')(\theta'-\theta_0)d\theta' \tag{178}$$

where the integrand represents the thermal stress work. For small temperature variations, the above Equation (178) yields the quadratic form when $\beta(\theta)$ varies slowly. The complete expression:

$$\psi^{\theta}(\theta) = c_v\left[\theta-\theta_0-\theta\ln\left(\frac{\theta}{\theta_0}\right)\right] - \frac{1}{2}\frac{\beta(\theta)}{\theta_0}(\theta-\theta_0)^2 \tag{179}$$

Equation (179) satisfies all thermodynamic constraints: (1) convexity in θ when $\beta(\theta) < c_v\theta_0/(\theta-\theta_0)$; (2) proper entropy production via $\eta = -\partial_\theta\psi = c_v\ln(\theta/\theta_0)+\beta(\theta)(\theta-\theta_0)/\theta_0$; and (3) consistent specific heat $c_v = -\theta\partial^2_\theta\psi$ at $\theta=\theta_0$. The temperature-dependent $\beta(\theta)$ accommodates real material behavior while maintaining thermodynamic consistency. The logarithmic term in Equation (179) dominates near absolute zero, while the quadratic term in Equation (179) governs moderate temperature variations, together providing a complete description of thermal effects across all temperature regimes.

4.2 Coleman-Noll Procedure

The Clausius-Duhem inequality represents the local form of the second law of thermodynamics for irreversible processes. In material coordinates, it states:

$$\rho_0\dot{\eta} \geq \frac{r_0}{\theta} - \nabla_X\cdot\left(\frac{\mathbf{q}_0}{\theta}\right) \tag{180}$$

Substituting the energy balance law $\rho_0\dot{e} = \mathbf{P}:\dot{\mathbf{F}} - \nabla_X\cdot\mathbf{q}_0 + r_0$ and the free energy definition $\psi = e-\theta\eta$ in Equation (180) yields the expanded dissipation inequality:

$$\mathcal{D} = \mathbf{P}:\dot{\mathbf{F}} - \rho_0(\dot{\psi}+\eta\dot{\theta}) - \frac{\mathbf{q}_0\cdot\nabla_X\theta}{\theta} \geq 0 \tag{181}$$

The Coleman-Noll procedure systematically enforces this dissipation inequality (Equation (181)) by first expanding $\dot{\psi}$ via the chain rule for $\psi = \psi(\mathbf{F},\theta,\boldsymbol{\xi})$ (where $\boldsymbol{\xi}$ represents internal variables):

$$\dot{\psi} = \frac{\partial\psi}{\partial\mathbf{F}}:\dot{\mathbf{F}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}} \tag{182}$$

Substitution of Equation (182) into the dissipation inequality (Equation (181)) gives:

$$\mathcal{D} = \left(\mathbf{P}-\rho_0\frac{\partial\psi}{\partial\mathbf{F}}\right):\dot{\mathbf{F}} - \rho_0\left(\eta+\frac{\partial\psi}{\partial\theta}\right)\dot{\theta} - \rho_0\frac{\partial\psi}{\partial\boldsymbol{\xi}}\cdot\dot{\boldsymbol{\xi}} - \frac{\mathbf{q}_0\cdot\nabla_X\theta}{\theta} \geq 0 \tag{183}$$

Thermodynamic consistency requires this inequality to hold for all admissible processes, which demands:

1. The coefficients of $\dot{\mathbf{F}}$ and $\dot{\theta}$ in Equation (183) must vanish identically (Coleman-Noll relations):

$$\mathbf{P} = \rho_0\frac{\partial\psi}{\partial\mathbf{F}}, \quad \eta = -\frac{\partial\psi}{\partial\theta} \tag{184}$$

2. The remaining terms in Equation (183) must satisfy individual inequalities:

$$-\rho_0 \frac{\partial \psi}{\partial \boldsymbol{\xi}} \cdot \dot{\boldsymbol{\xi}} \geq 0 \quad \text{(internal dissipation)} \tag{185}$$

$$-\frac{\mathbf{q}_0 \cdot \nabla_X \theta}{\theta} \geq 0 \quad \text{(thermal dissipation)} \tag{186}$$

The first condition defines the stress and entropy as thermodynamic conjugates to deformation and temperature. The second condition constrains the evolution equations for internal variables $\dot{\boldsymbol{\xi}}$ and heat flux $\mathbf{q}_0$ to ensure non-negative dissipation. For Fourier-type heat conduction, the thermal inequality reduces to $\mathbf{q}_0 = -\kappa \nabla_X \theta$ with $\kappa \geq 0$, while the internal dissipation governs plastic flow evolution. This systematic decomposition ensures the model satisfies the second law in all processes while providing the fundamental structure for constitutive equations. The procedure maintains mathematical rigor by: (1) exact treatment of all tensor operations; (2) careful separation of reversible and irreversible processes; and (3) strict enforcement of inequality constraints through convex analysis.

4.3 Rate-Dependent Plastic Flow

4.3.1 Yield Criterion

The yield function $f(\mathbf{M}, A, \theta)$ defines the elastic domain boundary in stress space through the inequality $f \leq 0$. Its specific form:

$$f(\mathbf{M}, A, \theta) = \|\text{dev}\mathbf{M}\| - (\sigma_y(\theta) - A) \tag{187}$$

derives from three fundamental principles: (1) pressure independence (J_2 plasticity), (2) isotropic hardening, and (3) thermal softening. In Equation (187), The Mandel stress $\mathbf{M} = \mathbf{C}^e \mathbf{S}$ appears naturally when pulling back the Kirchhoff stress to the intermediate configuration, where $\mathbf{C}^e = \mathbf{F}^{e\top} \mathbf{F}^e$ is the elastic right Cauchy-Green tensor and $\mathbf{S} = 2\rho_0 \partial_{\mathbf{C}^e} \psi^e$ the second Piola-Kirchhoff stress. The deviatoric operator $\text{dev}(\cdot)$ in Equation (187) enforces pressure independence:

$$\|\text{dev}\mathbf{M}\| = \sqrt{\mathbf{M} : \mathbf{M} - \frac{1}{3}(\text{tr}\mathbf{M})^2} \tag{188}$$

The hardening force $A = -\rho_0 \partial_\alpha \psi^p$ in Equation (187) emerges from the plastic free energy $\psi^p(\alpha, \theta)$, where α is the equivalent plastic strain. For the specific form $\psi^p = \frac{1}{\rho_0}[\sigma_y(\theta)\alpha + \frac{1}{2} H \alpha^2]$, this yields:

$$A = -\sigma_y(\theta) - H\alpha \tag{189}$$

The temperature-dependent yield stress $\sigma_y(\theta)$ in Equation (189) follows an Arrhenius-type decay:

$$\sigma_y(\theta) = \sigma_{y0} \exp\left(-\frac{Q}{k_B \theta}\right) \tag{190}$$

where Q is the activation energy. The criterion's mathematical properties ensure:

1. **Convexity**: f is convex in $\mathbf{M}$ since $\|\text{dev}(\cdot)\|$ is a norm
2. **Smoothness**: Differentiable everywhere except $\mathbf{M} = 0$

3. **Frame-indifference**: Invariant under rigid rotations of the intermediate configuration

4. **Thermodynamic consistency**: f depends only on invariant stress measures and internal variables

The yield condition $f \leq 0$ given in Equation (187) partitions the stress space into elastic ($f < 0$) and plastic ($f = 0$) domains. When combined with the associative flow rule $\mathbf{L}^p = \dot{\gamma}\partial_{\mathbf{M}} f$, it generates plastic strain rates normal to the yield surface in Mandel stress space. The hardening modulus $H = -\partial_A f = 1$ ensures positive dissipation during yielding:

$$\mathcal{D}_p = \mathbf{M} : \mathbf{L}^p - A\dot{\alpha} \geq 0 \tag{191}$$

This construction given in Equation (191) satisfies the second law of thermodynamics while capturing the essential features of metal plasticity: pressure independence, isotropic hardening, and thermal softening. The criterion naturally extends to finite deformations through its geometric formulation in the intermediate configuration.

4.3.2 Perzyna-Type Flow Rule

The Perzyna flow rule defines the plastic velocity gradient $\mathbf{L}^p = \dot{\mathbf{F}}^p \mathbf{F}^{p-1}$ in the intermediate configuration $\bar{\mathcal{B}}$ through a constitutive relation that couples rate-dependent plasticity with thermodynamic driving forces. Its precise form derives from three fundamental principles:

1. **Overstress Theory**: The plastic flow occurs only when the stress exceeds the static yield surface, quantified by the viscous overstress function:

$$\left\langle \frac{f}{\sigma_y} \right\rangle^n := \begin{cases} \left(\frac{\|\mathrm{dev}\mathbf{M}\| - (\sigma_y - A)}{\sigma_y} \right)^n & \text{if } f > 0 \\ 0 & \text{if } f \leq 0 \end{cases} \tag{192}$$

   where f is the yield function and $\langle \cdot \rangle$ denotes the Macaulay brackets. The exponent $n \in [1, \infty)$ controls rate sensitivity, recovering rate-independent plasticity as $n \to \infty$. The overstress function $\left\langle \frac{f}{\sigma_y} \right\rangle^n$ in Equation (192) rigorously characterizes the departure from the static yield condition in rate-dependent plasticity. Its mathematical definition involves a nonsmooth constitutive operator that activates only when the yield function

$$f(\mathbf{M}, A, \theta) = \|\mathrm{dev}\mathbf{M}\| - (\sigma_y(\theta) - A) \tag{193}$$

   becomes positive, indicating stress states beyond the equilibrium yield surface. The Macaulay bracket operation $\langle x \rangle = \max(0, x)$ enforces this threshold behavior, creating a function that belongs to the Sobolev space $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ with generalized derivative given by the Heaviside function $H(x)$. The exponent $n \in [1, \infty)$ in Equation (192) controls the nonlinear rate sensitivity, appearing in the viscous overstress term as a power-law relation between the normalized overstress $\frac{f}{\sigma_y}$ and the resulting plastic flow rate.

The thermodynamic foundations require the overstress function to satisfy the second law of thermodynamics, which is verified through the plastic dissipation inequality

$$\mathcal{D} = \mathbf{M} : \mathbf{L}^p = \gamma \left\langle \frac{f}{\sigma_y} \right\rangle^n \|\text{dev}\mathbf{M}\| \geq 0 \tag{194}$$

This inequality Equation (194) holds because all components - the viscosity coefficient $\gamma > 0$, the Macaulay bracket term, and the stress norm $\|\text{dev}\mathbf{M}\|$ - are non-negative quantities. The function exhibits several key mathematical properties: Lipschitz continuity in stress space $\mathbb{R}^{3\times3}$, convexity with respect to $\mathbf{M}$ for $n \geq 1$, and homogeneity of degree n when active. The convergence to rate-independent plasticity as $n \to \infty$ can be formally understood through Gamma-convergence of the associated dissipation potential

$$\Phi(\mathbf{L}^p) = \frac{\gamma}{n+1} \left\langle \frac{f}{\sigma_y} \right\rangle^{n+1} \tag{195}$$

to the indicator function of the elastic domain. The physical interpretation emerges from considering the kinetic theory of thermally activated dislocation motion, where the power-law form approximates the Arrhenius-type rate equation for dislocation slip. The exponent n in Equation (195) directly relates to the material's strain-rate sensitivity measured experimentally through the relation

$$n = \partial \ln \dot{\varepsilon}^p / \partial \ln(\sigma/\sigma_y) \tag{196}$$

The viscosity parameter γ in Equation (195) carries dimensions of inverse time and connects to microscopic relaxation timescales through

$$\gamma = \tau^{-1}(\sigma_y/E)^n \tag{197}$$

where τ represents the characteristic atomic vibration period and E the elastic modulus. The overstress formulation naturally regularizes the rate-independent limit by providing a smooth transition across the yield surface while maintaining thermodynamic consistency through the convexity of the dissipation potential and proper work conjugacy between stresses and strains.

The kinematic hardening component A modifies the effective yield threshold through its appearance in the overstress function, creating a dynamic yield condition $\sigma_y(\theta) - A$ that evolves with plastic deformation. This introduces a coupling between the rate-dependent flow rule and the material's hardening behavior, captured through the thermodynamic force

$$A = -\rho_0 \partial \psi^p / \partial \alpha \tag{198}$$

The temperature dependence of the reference yield stress $\sigma_y(\theta)$ in Equation (196) and (197) typically follows an Arrhenius decay

$$\sigma_y(\theta) = \sigma_{y0} \exp(-Q/k_B\theta) \tag{199}$$

introducing additional thermo-viscoplastic coupling that affects both the overstress magnitude and the rate sensitivity. The complete formulation maintains material frame indifference through proper tensor transformations of $\mathbf{M}$ under rigid rotations and satisfies all fundamental continuum mechanics requirements for objectivity and thermodynamic admissibility.

2. **Normality Condition**: The flow direction derives from the associative flow rule:

$$\frac{\text{dev}\mathbf{M}}{\|\text{dev}\mathbf{M}\|} = \frac{\partial f}{\partial \mathbf{M}} \tag{200}$$

which ensures maximum plastic dissipation. This emerges from the convex analysis of the dissipation potential $\Phi(\mathbf{L}^p)$:

$$\Phi(\mathbf{L}^p) = \frac{\gamma}{n+1}\left\langle \frac{f}{\sigma_y} \right\rangle^{n+1} + I_{\mathcal{K}}(\mathbf{M}) \tag{201}$$

where $I_{\mathcal{K}}$ in Equation (201) is the indicator function of the elastic domain $\mathcal{K} = \{\mathbf{M} | f(\mathbf{M}) \leq 0\}$. The normality condition in plasticity theory establishes a fundamental connection between the yield criterion and the direction of plastic flow through rigorous convex analysis. The associative flow rule $\frac{\text{dev}\mathbf{M}}{\|\text{dev}\mathbf{M}\|} = \frac{\partial f}{\partial \mathbf{M}}$ in Equation (200) emerges from applying the maximum plastic dissipation principle to the yield function $f(\mathbf{M})$, where the gradient $\frac{\partial f}{\partial \mathbf{M}}$ defines the outward normal vector to the yield surface in stress space. For the von Mises yield criterion

$$f(\mathbf{M}) = \|\text{dev}\mathbf{M}\| - (\sigma_y - A) \tag{202}$$

The gradient of $f(\mathbf{M})$ (Equation (202)) evaluates explicitly to $\frac{\partial f}{\partial \mathbf{M}} = \frac{\text{dev}\mathbf{M}}{\|\text{dev}\mathbf{M}\|}$ (as given in Equation (200)) through direct differentiation of the Frobenius norm

$$\|\text{dev}\mathbf{M}\| = \sqrt{\text{dev}\mathbf{M} : \text{dev}\mathbf{M}} \tag{203}$$

where the deviatoric operator $\text{dev}\mathbf{M} = \mathbf{M} - \frac{1}{3}(\text{tr}\mathbf{M})\mathbf{I}$ in Equation (203) projects the Mandel stress onto the space of traceless tensors. The dissipation potential $\Phi(\mathbf{L}^p)$ encodes the thermodynamic constraints on plastic flow through its dual relationship with the yield function. The indicator function $I_{\mathcal{K}}(\mathbf{M})$ enforces the elastic domain constraint by taking infinite values outside $\mathcal{K}$ and zero within, making the dissipation potential proper, convex, and lower semicontinuous. The Legendre-Fenchel transform connects this potential to the plastic flow rule through the subdifferential inclusion $\mathbf{L}^p \in \partial\Phi^*(\mathbf{M})$, where Φ^* denotes the convex conjugate. For the specific form $\Phi(\mathbf{L}^p) = \frac{\gamma}{n+1}\left\langle \frac{f}{\sigma_y} \right\rangle^{n+1} + I_{\mathcal{K}}(\mathbf{M})$ (as given in Equation (201)), the flow rule derives from the optimality condition $\mathbf{M} \in \partial\Phi(\mathbf{L}^p)$, which when evaluated gives exactly the associative flow direction

$$\mathbf{L}^p = \dot{\gamma}\frac{\partial f}{\partial \mathbf{M}} \text{ with } \dot{\gamma} = \gamma\left\langle \frac{f}{\sigma_y} \right\rangle^{n} \tag{204}$$

The mathematical foundations rely on key results from convex analysis: the yield surface $\partial\mathcal{K}$ forms a C^1-smooth hypersurface almost everywhere in $\mathbb{R}^{3\times3}$, and the normal cone $N_{\mathcal{K}}(\mathbf{M})$ to the elastic domain $\mathcal{K}$ at point $\mathbf{M}$ contains exactly the subdifferential $\partial I_{\mathcal{K}}(\mathbf{M})$. The dissipation inequality $\mathcal{D} = \mathbf{M} : \mathbf{L}^p \geq 0$ follows automatically from this construction because the convexity of Φ ensures

$$\Phi(\mathbf{L}^p) + \Phi^*(\mathbf{M}) \geq \mathbf{M} : \mathbf{L}^p \text{ and } \Phi^*(\mathbf{M}) = 0 \tag{205}$$

for admissible stresses $\mathbf{M} \in \mathcal{K}$. The associative flow rule thus represents the only thermodynamically admissible evolution law that maximizes the plastic dissipation

rate $\mathcal{D}$ while respecting the yield constraint $f(\mathbf{M}) \leq 0$. This geometric interpretation holds for any convex yield function f, with the specific von Mises case producing deviatoric flow due to the spherical symmetry of $\|\text{dev}\mathbf{M}\|$ in stress space. The resulting plastic strain rate tensor $\mathbf{L}^p$ consequently inherits the deviatoric property $\text{tr}(\mathbf{L}^p) = 0$ enforcing isochoric plastic deformation as required by crystal plasticity considerations.

3. **Material Frame Indifference**: The flow rule is invariant under rigid rotations $\mathbf{Q}(t)$ of the current configuration:

$$\mathbf{L}^p \mapsto \mathbf{Q}\mathbf{L}^p\mathbf{Q}^\top \quad \text{under} \quad \mathbf{F} \mapsto \mathbf{Q}\mathbf{F} \tag{206}$$

The principle of material frame indifference imposes fundamental constraints on the form of the plastic flow rule through rigorous application of objectivity requirements under Euclidean transformations. For any time-dependent rigid rotation $\mathbf{Q}(t) \in SO(3)$ applied to the current configuration, the deformation gradient transforms as $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$, while the plastic deformation gradient must transform as $\mathbf{F}^{p*} = \mathbf{Q}\mathbf{F}^p$ to maintain kinematic consistency. This transformation rule emerges from the multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ and the requirement that elastic distortions $\mathbf{F}^e$ remain unaffected by rigid motions. The plastic velocity gradient $\mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}$ consequently obeys the transformation law:

$$\mathbf{L}^{p*} = \dot{\mathbf{F}}^{p*}\mathbf{F}^{p*-1} = (\mathbf{Q}\dot{\mathbf{F}}^p + \dot{\mathbf{Q}}\mathbf{F}^p)(\mathbf{F}^{p-1}\mathbf{Q}^\top) = \mathbf{Q}\mathbf{L}^p\mathbf{Q}^\top + \dot{\mathbf{Q}}\mathbf{Q}^\top \tag{207}$$

The second term $\dot{\mathbf{Q}}\mathbf{Q}^\top$ in Equation (207) represents the spin of the rotating frame and must vanish for frame-indifferent constitutive equations thus yielding Equation (206). The Perzyna flow rule as given in Equation (192) automatically satisfies this requirement because the Mandel stress $\mathbf{M} = \mathbf{C}^e\mathbf{S}$ in Equation (192) transforms objectively as $\mathbf{M}^* = \mathbf{Q}\mathbf{M}\mathbf{Q}^\top$ under rigid rotations, and the deviatoric operator $\text{dev}(\mathbf{M})$ in Equation (192) commutes with orthogonal transformations

$$\text{dev}(\mathbf{Q}\mathbf{M}\mathbf{Q}^\top) = \mathbf{Q}\text{dev}(\mathbf{M})\mathbf{Q}^\top \tag{208}$$

The yield function $f(\mathbf{M})$ in Equation (193) consequently remains invariant since $\|\text{dev}\mathbf{M}^*\| = \|\text{dev}\mathbf{M}\|$, ensuring the overstress function $\langle f/\sigma_y \rangle^n$ and plastic multiplier $\dot{\gamma}$ are properly frame-indifferent scalars.

The geometric foundations of this invariance property stem from the fact that the intermediate configuration $\bar{\mathcal{B}}$ remains unaltered by rigid motions of the current configuration - a consequence of the plastic deformation gradient $\mathbf{F}^p$ being defined purely in terms of material rearrangements rather than overall rotations. The Lie derivative

$$\mathcal{L}_v\mathbf{C}^p = -2\mathbf{C}^p\mathbf{L}^p \tag{209}$$

further confirms this invariance since it measures the rate of plastic deformation without reference to the observer's frame. The dissipation rate $\mathcal{D} = \mathbf{M} : \mathbf{L}^p$ maintains its value under transformations because both $\mathbf{M}$ and $\mathbf{L}^p$ transform covariantly, making their contraction an objective scalar. This transformation behavior holds not just for rotations but for general superposed rigid body motions $\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t)$, with the translation term $\mathbf{c}(t)$ automatically dropping out in rate-based formulations. The frame-indifference of the flow rule thus ensures that

material response depends only on intrinsic deformation measures rather than arbitrary observer kinematics.

The complete flow rule in tensor notation is:

$$\mathbf{L}^p = \dot{\mathbf{F}}^p \mathbf{F}^{p-1} = \gamma \left\langle \frac{f}{\sigma_y} \right\rangle^n \frac{\mathrm{dev}\mathbf{M}}{\|\mathrm{dev}\mathbf{M}\|} \tag{210}$$

where the viscosity parameter $\gamma > 0$ has units of inverse time. The kinematic decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ induces the following geometric constraints:

1. **Plastic Incompressibility**: $\det \mathbf{F}^p = 1$ implies $\mathrm{tr}\mathbf{L}^p = 0$, automatically satisfied since $\mathrm{dev}\mathbf{M}$ is traceless.

2. **Lie Transport**: The plastic flow evolves the intermediate configuration via:

$$\mathcal{L}_v \mathbf{C}^p = -2\mathbf{C}^p \mathbf{L}^p \tag{211}$$

   where $\mathcal{L}_v$ is the Lie derivative along the material velocity field.

The thermodynamic consistency of the viscoplastic formulation is rigorously ensured through the mechanical dissipation inequality $\mathcal{D}_{\mathrm{mech}} = \mathbf{M} : \mathbf{L}^p - A\dot{\alpha} \geq 0$, which represents the local form of the second law of thermodynamics for irreversible processes. Substituting the Perzyna flow rule $\mathbf{L}^p = \gamma \langle f/\sigma_y \rangle^n \mathrm{dev}\mathbf{M}/\|\mathrm{dev}\mathbf{M}\|$ and the hardening law $\dot{\alpha} = \sqrt{2/3}\|\mathbf{L}^p\|$ yields the explicit dissipation expression:

$$\mathcal{D}_{\mathrm{mech}} = \gamma \left\langle \frac{f}{\sigma_y} \right\rangle^n (\|\mathrm{dev}\mathbf{M}\| + A) \geq 0 \tag{212}$$

where the non-negativity follows from three essential conditions: the viscosity coefficient satisfies $\gamma > 0$, the Macaulay bracket ensures $\langle f/\sigma_y \rangle^n \geq 0$, and the term $(\|\mathrm{dev}\mathbf{M}\| + A)$ in Equation (212) remains non-negative through the yield condition $f = \|\mathrm{dev}\mathbf{M}\| - (\sigma_y - A) \leq 0$. The thermodynamic force $A = -\rho_0 \partial \psi^p / \partial \alpha$ derives from the plastic potential $\psi^p(\alpha, \theta)$, ensuring the hardening response remains energetically consistent. At the yield surface $f = 0$, the identity $\|\mathrm{dev}\mathbf{M}\| = \sigma_y - A$ guarantees exact balance between driving stresses and resistance.

Numerical implementation preserves the geometric structure of finite plasticity through the exponential map integration:

$$\mathbf{F}^p_{n+1} = \exp\big(\Delta t \mathbf{L}^p_{n+1}\big) \mathbf{F}^p_n \tag{213}$$

where the matrix exponential in Equation (213) maintains plastic incompressibility $\det \mathbf{F}^p = 1$ by ensuring $\exp\big(\Delta t \mathbf{L}^p_{n+1}\big) \in \mathrm{SL}(3)$ for any $\mathbf{L}^p_{n+1} \in \mathfrak{sl}(3)$, the Lie algebra of traceless matrices. This exact geometric integrator avoids spurious volumetric plastic deformation that would violate the isochoric nature of crystal slip mechanisms. The update formula as given in Equation (213) derives from the solution to the differential equation $\dot{\mathbf{F}}^p = \mathbf{L}^p \mathbf{F}^p$ with constant $\mathbf{L}^p$ over the time step Δt, preserving both the manifold structure and the multiplicative decomposition geometry.

The viscous parameter γ bridges continuum and atomistic scales through the relation:

$$\gamma = \frac{1}{\tau} \left(\frac{\sigma_y}{E} \right)^n \tag{214}$$

where τ represents the characteristic timescale of thermally activated dislocation motion, and E the elastic modulus provides stress normalization. This expression emerges from dimensional analysis of Arrhenius-type rate equations for dislocation glide, where the ratio σ_y/E in Equation (214) characterizes the activation barrier and τ^{-1} in Equation (214) sets the attempt frequency. The power-law exponent n in Equation (214) modifies this relationship to capture nonlinear rate sensitivity observed in crystalline materials. The complete formulation thus maintains consistency across scales: atomistic processes dictate the relaxation time τ, while continuum thermodynamics governs the macroscopic stress response through $\mathbf{M}$ and A. The numerical implementation consequently respects both the physical origins of viscoplasticity and the geometric constraints of finite deformation kinematics.

4.3.3 Hardening Evolution

The hardening evolution law

$$\dot{\alpha} = \sqrt{\frac{2}{3}}\|\mathbf{L}^p\| \tag{215}$$

with $\mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}$ constitutes a fundamental kinematic constraint that rigorously connects the rate of accumulated plastic strain α to the metric properties of plastic deformation. The specific form derives from three principal considerations: the geometric definition of equivalent plastic strain in finite deformation, the isochoric nature of plastic flow in metallic crystals, and the thermodynamic conjugacy between α and its associated hardening force $A = -\rho_0 \partial \psi^p / \partial \alpha$. The factor $\sqrt{2/3}$ emerges from consistent normalization with the classical infinitesimal theory, ensuring that under small deformations $\|\mathbf{L}^p\|$ reduces to $\sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p}$, where $\dot{\boldsymbol{\varepsilon}}^p$ is the symmetric plastic strain rate tensor.

The norm $\|\mathbf{L}^p\|$ is precisely defined through the Frobenius inner product

$$\|\mathbf{L}^p\| = \sqrt{\mathbf{L}^p : \mathbf{L}^p} = \sqrt{\mathrm{tr}(\mathbf{L}^{p\top}\mathbf{L}^p)} \tag{216}$$

which measures the magnitude of plastic distortion independent of material frame. For isochoric plastic flow ($\det \mathbf{F}^p = 1$), the plastic velocity gradient satisfies $\mathrm{tr}(\mathbf{L}^p) = 0$, making $\mathbf{L}^p$ a deviatoric tensor in $\mathfrak{sl}(3)$, the Lie algebra of special linear transformations. The evolution equation preserves the tensorial character of finite strain plasticity while maintaining consistency with the second law of thermodynamics, as evidenced by the non-negative mechanical dissipation $\mathcal{D}_{\mathrm{mech}} = A\dot{\alpha} \geq 0$ for positive hardening moduli.

The differential form $\dot{\alpha} = \sqrt{\frac{2}{3}}\|\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}\|$ integrates to give the finite accumulated plastic strain

$$\alpha(t) = \sqrt{\frac{2}{3}}\int_0^t \|\mathbf{L}^p(s)\| ds \tag{217}$$

representing the metric length of the plastic deformation path in the space of unimodular tensors. This geometric interpretation becomes exact when viewed on the manifold $\mathrm{SL}(3)$, where $\alpha(t)$ measures the intrinsic plastic deformation accumulated along the trajectory $\mathbf{F}^p(t)$. The numerical implementation requires careful treatment to preserve this geometric structure, typically achieved through exponential mapping algorithms that maintain the unimodular constraint and ensure objective integration of the plastic flow. The hardening evolution thus provides a complete kinematic description that is consistent across

scales - from the microscale dislocation dynamics that govern $\mathbf{L}^p$ to the macroscale yield surface evolution described by $\alpha(t)$.

5 Beam-Specific Reductions

5.1 Stress Resultants

The reduction of three-dimensional continuum mechanics to Euler-Bernoulli beam theory involves precise dimensional reduction of stress quantities through thickness integration. The axial force N represents the resultant of the first Piola-Kirchhoff stress component $\mathbf{P}_{11}$ (aligned with the beam axis) integrated over the cross-sectional area A:

$$N = \int_A \mathbf{P}_{11}\, dA \tag{218}$$

where $\mathbf{P}_{11}$ in Equation (218) is the axial component of the nominal stress tensor $\mathbf{P} = \rho_0 \partial\psi/\partial\mathbf{F}$ evaluated at the beam's neutral axis. The bending moment M emerges as the first moment of this axial stress about the neutral axis, with the negative sign ensuring conventional sign conventions in structural mechanics:

$$M = -\int_A y\mathbf{P}_{11}\, dA \tag{219}$$

Here y in Equation (219) denotes the vertical distance from the neutral axis in the reference configuration. These integrals assume the Euler-Bernoulli kinematic constraints, which enforce plane sections remaining plane and normal to the deformed midline, causing the axial strain to vary linearly as $\varepsilon_{11} = u' - yw''$ where u is axial displacement and w transverse deflection. The stress component $\mathbf{P}_{11}$ consequently inherits this linear variation through the constitutive law

$$\mathbf{P}_{11} = E(u' - yw'') \tag{220}$$

in the linear elastic case, where E in Equation (220) is Young's modulus. For nonlinear material behavior, the integrals must account for the full nonlinear stress-strain relationship and finite deformation effects captured in $\mathbf{P} = \rho_0 \partial\psi/\partial\mathbf{F}$. The resultant definitions maintain objectivity under finite rotations while correctly reducing to classical beam theory under infinitesimal deformations. The integrals implicitly assume stress components $\mathbf{P}_{12}$ and $\mathbf{P}_{13}$ (transverse shear stresses) are negligible compared to $\mathbf{P}_{11}$, consistent with the Euler-Bernoulli hypothesis of negligible shear deformation. These stress resultants N and M become the fundamental mechanical quantities governing the one-dimensional beam equations of motion through the balance laws of linear and angular momentum.

5.2 Weak Form for Beam Equilibrium

The weak form of equilibrium for an Euler-Bernoulli beam derives from a rigorous dimensional reduction of the three-dimensional virtual work principle through thickness integration and enforcement of kinematic constraints. The fundamental expression:

$$\int_0^L (N\delta u' - M\delta w'')\, dx = \text{External Virtual Work} \tag{221}$$

emerges from applying the principle of virtual power to the beam's one-dimensional manifold. N and M in Equation (221) are the energetically conjugate stress resultants to the virtual strain measures $\delta u'$ (axial stretching) and $-\delta w''$ (bending curvature). In Equation (221), the axial displacement variation δu and transverse displacement variation δw are test functions in the Sobolev spaces $\delta u \in H^1(0, L)$ and $\delta w \in H^2(0, L)$, satisfying essential boundary conditions. The term $N\delta u'$ represents virtual work from axial deformation, obtained by integrating the three-dimensional virtual power density $\mathbf{P} : \delta\dot{\mathbf{F}}$ over the cross-section and enforcing the beam kinematics

$$\delta F_{11} = \delta u' - y\delta w'' \tag{222}$$

where only the $\delta u'$ term contributes to axial force work due to antisymmetry in y. The bending term $-M\delta w''$ in Equation (221) arises from the moment-curvature relationship, where the negative sign ensures consistent orientation with the adopted sign convention for moments. The right-hand side represents the **virtual work of external forces**, including distributed loads $q(x)$ and concentrated forces/moments at boundaries, expressed formally through the duality pairing between test functions and applied loads. This weak formulation satisfies the **Lax-Milgram theorem conditions** when the material constitutive laws are elliptic, ensuring existence and uniqueness of solutions. The derivation maintains exact consistency with the three-dimensional parent theory by preserving all energetic conjugacy relationships through the careful integration over cross-sectional coordinates (y, z), while eliminating these dimensions through static condensation onto the beam midline. The resulting formulation is geometrically exact for finite rotations but assumes small strains consistent with Euler-Bernoulli beam theory. The integrals implicitly satisfy the stress boundary conditions $\mathbf{P}_{12} = \mathbf{P}_{13} = 0$ on the lateral surfaces through the fundamental lemma of calculus of variations applied to the dimensionally reduced system.

5.3 Thermo-Mechanical Coupling

The thermo-mechanical coupling in the model arises from the interplay between mechanical deformation and thermal effects, governed by the heat equation. The heat equation accounts for the evolution of temperature θ due to conduction and mechanical dissipation. The general form of the heat equation is given by:

$$\rho_0 c\dot{\theta} = \nabla \cdot (\kappa\nabla\theta) + \mathcal{D}_{\text{mech}} \tag{223}$$

where ρ_0 is the reference density, c is the specific heat capacity, and κ is the thermal conductivity. In Equation (223), the term $\nabla \cdot (\kappa\nabla\theta)$ represents heat conduction, while $\mathcal{D}_{\text{mech}}$ captures the mechanical dissipation, which acts as a heat source due to irreversible processes like plasticity. The mechanical dissipation $\mathcal{D}_{\text{mech}}$ is defined as the difference between the stress power and the rate of change of the Helmholtz free energy:

$$\mathcal{D}_{\text{mech}} = \mathbf{P} : \dot{\mathbf{F}} - \rho_0\dot{\psi} \tag{224}$$

Here, $\mathbf{P}$ is the first Piola-Kirchhoff stress, $\dot{\mathbf{F}}$ is the rate of the deformation gradient, and ψ is the Helmholtz free energy. The dissipation term $\mathcal{D}_{\text{mech}}$ in Equation (224) ensures thermodynamic consistency, as it must satisfy the Clausius-Duhem inequality $\mathcal{D}_{\text{mech}} \geq 0$.

The free energy ψ in the RHS of Equation (224) decomposes into elastic and plastic contributions:

$$\psi = \psi^e(\mathbf{E}^e, \theta) + \psi^p(\alpha, \theta) \tag{225}$$

where $\mathbf{E}^e$ is the elastic strain and α is the hardening variable. The temperature dependence of ψ introduces thermoelastic coupling, while the plastic dissipation contributes to heat generation. The rate of change of ψ (for the given definition of ψ Equation (225)) is:

$$\dot{\psi} = \frac{\partial \psi^e}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \psi^e}{\partial \theta}\dot{\theta} + \frac{\partial \psi^p}{\partial \alpha}\dot{\alpha} + \frac{\partial \psi^p}{\partial \theta}\dot{\theta} \tag{226}$$

Substituting Equation (226) into the dissipation expression in Equation (224) and using the Coleman-Noll procedure yields the constitutive relations for stress and entropy, as well as the restrictions on the dissipation mechanisms. The plastic dissipation dominates in viscoplasticity, where the Perzyna flow rule generates heat proportional to the plastic strain rate:

$$\mathcal{D}_{\text{mech}}^{\text{plastic}} = \mathbf{M} : \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} - A\dot{\alpha} \tag{227}$$

where $\mathbf{M}$ is the Mandel stress and A is the hardening force. This term $\mathcal{D}_{\text{mech}}^{\text{plastic}}$ in Equation (227) is always non-negative, ensuring adherence to the second law of thermodynamics. The coupled thermo-mechanical problem is closed by boundary and initial conditions for temperature and displacement fields, leading to a nonlinear system of partial differential equations.

5.3.1 Proof of the General form of the Heat Equation

We derive the general form of the heat equation rigorously by starting from the First Law of Thermodynamics (Energy Balance) and the Clausius-Duhem Inequality (Second Law) in the reference configuration $\mathcal{B}_0$. The balance of energy (First Law of Thermodynamics) in Lagrangian form states:

$$\rho_0 \dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla \cdot \mathbf{q}_0 + \rho_0 r \tag{228}$$

where e = internal energy per unit mass, $\mathbf{P}$ = first Piola-Kirchhoff stress, $\mathbf{q}_0$ = referential heat flux (related to spatial heat flux $\mathbf{q}$ by $\mathbf{q}_0 = J\mathbf{F}^{-1}\mathbf{q}$), and r = external heat source per unit mass. The Helmholtz free energy ψ (per unit mass) is defined as:

$$\psi = e - \eta\theta \tag{229}$$

where η is the entropy per unit mass and θ is the absolute temperature. Taking the time derivative of Helmholtz free energy ψ (per unit mass) as defined in Equation (229), we get:

$$\dot{e} = \dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta} \tag{230}$$

Substituting Equation (230) the energy balance (Equation (228)), we get:

$$\rho_0(\dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta}) = \mathbf{P} : \dot{\mathbf{F}} - \nabla \cdot \mathbf{q}_0 + \rho_0 r \tag{231}$$

The local form of the entropy inequality is:

$$\rho_0 \dot{\eta} \geq -\nabla \cdot \left(\frac{\mathbf{q}_0}{\theta}\right) + \frac{\rho_0 r}{\theta} \tag{232}$$

Expanding the divergence term of Equation (232), we get:

$$\rho_0 \dot{\eta} \geq -\frac{1}{\theta} \nabla \cdot \mathbf{q}_0 + \frac{\mathbf{q}_0 \cdot \nabla \theta}{\theta^2} + \frac{\rho_0 r}{\theta} \tag{233}$$

Multiplying by θ on both sides of Equation (233) and rearranging, we get:

$$\rho_0 \theta \dot{\eta} \geq -\nabla \cdot \mathbf{q}_0 + \frac{\mathbf{q}_0 \cdot \nabla \theta}{\theta} + \rho_0 r \tag{234}$$

We shall now combine the Energy Balance (Equation (231)) and Entropy Inequality (Equation (234)). Substituting the Energy Balance (Equation (231)) into the Entropy Inequality (Equation (234)), we get:

$$\rho_0(\dot{\psi} + \eta\dot{\theta}) + \mathbf{P} : \dot{\mathbf{F}} - \rho_0 \dot{e} \leq \frac{\mathbf{q}_0 \cdot \nabla \theta}{\theta} \tag{235}$$

But from $\dot{e} = \dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta}$ (Equation (230)), the above Equation (235) can be written as:

$$\rho_0(\dot{\psi} + \eta\dot{\theta}) + \mathbf{P} : \dot{\mathbf{F}} - \rho_0(\dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta}) \leq \frac{\mathbf{q}_0 \cdot \nabla \theta}{\theta} \tag{236}$$

Equation (236) simplifies to:

$$\mathbf{P} : \dot{\mathbf{F}} - \rho_0 \dot{\eta}\theta \leq \frac{\mathbf{q}_0 \cdot \nabla \theta}{\theta} \tag{237}$$

Let's now use the Coleman-Noll Procedure. Assuming the free energy $\psi = \psi(\mathbf{F}, \theta, \xi)$ (where ξ are internal variables), its time derivative is:

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} \tag{238}$$

From standard thermodynamic arguments, we obtain:

$$\mathbf{P} = \rho_0 \frac{\partial \psi}{\partial \mathbf{F}}, \quad \eta = -\frac{\partial \psi}{\partial \theta} \tag{239}$$

Substituting Equation (238) into the dissipation inequality (Equation (237)):

$$\left(\mathbf{P} - \rho_0 \frac{\partial \psi}{\partial \mathbf{F}}\right) : \dot{\mathbf{F}} - \rho_0 \left(\eta + \frac{\partial \psi}{\partial \theta}\right) \dot{\theta} - \rho_0 \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} \leq \frac{\mathbf{q}_0 \cdot \nabla \theta}{\theta} \tag{240}$$

Since this must hold for all processes, the terms in parentheses must vanish, yielding the constitutive relations, and the remaining dissipation inequality Equation (240) becomes:

$$\mathcal{D} = -\rho_0 \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} - \frac{\mathbf{q}_0 \cdot \nabla \theta}{\theta} \geq 0 \tag{241}$$

The mechanical dissipation $\mathcal{D}_{\text{mech}}$ in Equation (241) is defined as:

$$\mathcal{D}_{\text{mech}} = -\rho_0 \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} \tag{242}$$

which includes plastic dissipation, viscous effects, and other irreversible processes. Substituting $\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi}$ (Equation (238)) into the Energy Balance (Equation (231)) and using Equation (239), we get:

$$\rho_0 \left(\frac{\partial \psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} + \dot{\eta}\theta + \eta\dot{\theta} \right) = \mathbf{P} : \dot{\mathbf{F}} - \nabla \cdot \mathbf{q}_0 + \rho_0 r$$

$$\Rightarrow \rho_0 \frac{\partial \psi}{\partial \mathbf{F}} : \dot{\mathbf{F}} + \rho_0 \left(\frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} + \dot{\eta}\theta + \eta\dot{\theta} \right) = \mathbf{P} : \dot{\mathbf{F}} - \nabla \cdot \mathbf{q}_0 + \rho_0 r$$

$$\Rightarrow \mathbf{P} : \dot{\mathbf{F}} + \rho_0 \left(\frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} + \dot{\eta}\theta + \eta\dot{\theta} \right) = \mathbf{P} : \dot{\mathbf{F}} - \nabla \cdot \mathbf{q}_0 + \rho_0 r$$

$$\Rightarrow \rho_0 \left(\frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} + \dot{\eta}\theta + \eta\dot{\theta} \right) = -\nabla \cdot \mathbf{q}_0 + \rho_0 r \tag{243}$$

Using $\eta = -\frac{\partial \psi}{\partial \theta}$ (Equation (239)), the $\dot{\theta}$ terms cancel in Equation (243), leaving:

$$\rho_0 \left(\cancel{-\eta\dot{\theta}} + \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} + \dot{\eta}\theta + \cancel{\eta\dot{\theta}} \right) = -\nabla \cdot \mathbf{q}_0 + \rho_0 r$$

$$\Rightarrow \rho_0 \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} + \rho_0 \dot{\eta}\theta = -\nabla \cdot \mathbf{q}_0 + \rho_0 r$$

$$\Rightarrow \rho_0 \dot{\eta}\theta = -\nabla \cdot \mathbf{q}_0 - \rho_0 \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi} + \rho_0 r \tag{244}$$

Recognizing that $\mathcal{D}_{\text{mech}} = -\rho_0 \frac{\partial \psi}{\partial \xi} \cdot \dot{\xi}$ and using Fourier's law $\mathbf{q}_0 = -\kappa \nabla \theta$, we obtain:

$$\rho_0 c \dot{\theta} = \nabla \cdot (\kappa \nabla \theta) + \mathcal{D}_{\text{mech}} + \rho_0 r \tag{245}$$

where c is the specific heat capacity. If external heat sources are neglected ($r = 0$), we arrive at the general heat equation:

$$\rho_0 c \dot{\theta} = \nabla \cdot (\kappa \nabla \theta) + \mathcal{D}_{\text{mech}} \tag{246}$$

This completes the rigorous derivation. The term $\mathcal{D}_{\text{mech}}$ ensures that all mechanical dissipation (plastic work, viscous heating, etc.) contributes to temperature evolution in a thermodynamically consistent manner.

5.3.2 Derivation of the Plastic dissipation for Perzyna flow rule

The plastic dissipation $\mathcal{D}_{\text{mech}}^{\text{plastic}}$ arises from the irreversible work associated with plastic deformation and hardening. To rigorously derive its form, we begin with the **Clausius-Duhem inequality** in the reference configuration, which for purely mechanical dissipation (isothermal or adiabatic conditions) reduces to:

$$\mathcal{D} = \mathbf{P} : \dot{\mathbf{F}} - \rho_0 \dot{\psi} \geq 0 \tag{247}$$

where $\mathbf{P}$ is the first Piola-Kirchhoff stress, $\dot{\mathbf{F}}$ is the rate of the deformation gradient, and ψ is the Helmholtz free energy per unit mass in Equation (247). Under the **multiplicative decomposition** $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$, the plastic dissipation is isolated by expressing the stress power in terms of elastic and plastic contributions. The time derivative of $\mathbf{F}$ yields:

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e \mathbf{F}^p + \mathbf{F}^e \dot{\mathbf{F}}^p \tag{248}$$

Equation (248) when substituted into the stress power (first term in the RHS of in Equation (247)), gives:

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{P} : (\dot{\mathbf{F}}^e \mathbf{F}^p) + \mathbf{P} : (\mathbf{F}^e \dot{\mathbf{F}}^p) \tag{249}$$

Using the relation $\mathbf{P} = \mathbf{F}^e \mathbf{S} (\mathbf{F}^p)^{-T}$, where $\mathbf{S}$ is the second Piola-Kirchhoff stress, the first term $\mathbf{P} : (\dot{\mathbf{F}}^e \mathbf{F}^p)$ in RHS of the Equation (249) becomes purely elastic:

$$\mathbf{P} : (\dot{\mathbf{F}}^e \mathbf{F}^p) = \mathbf{S} : \dot{\mathbf{E}}^e \tag{250}$$

where $\mathbf{E}^e = \frac{1}{2}((\mathbf{F}^e)^T \mathbf{F}^e - \mathbf{I})$ is the elastic Green-Lagrange strain. The second term $\mathbf{P} : (\mathbf{F}^e \dot{\mathbf{F}}^p)$ in RHS of the Equation (249), representing plastic work, is rewritten using the Mandel stress $\mathbf{M} = \mathbf{C}^e \mathbf{S}$, where $\mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e$:

$$\mathbf{P} : (\mathbf{F}^e \dot{\mathbf{F}}^p) = \mathbf{S} : (\mathbf{F}^e)^T \mathbf{F}^e \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} = \mathbf{M} : \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} \tag{251}$$

The free energy ψ is decomposed into elastic and plastic parts:

$$\psi = \psi^e(\mathbf{E}^e) + \psi^p(\alpha) \tag{252}$$

where α is an internal hardening variable. The time derivative of free energy ψ (Equation (252)) shall be:

$$\dot{\psi} = \frac{\partial \psi^e}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \psi^p}{\partial \alpha} \dot{\alpha} \tag{253}$$

Substituting $\mathbf{S} = \rho_0 \frac{\partial \psi^e}{\partial \mathbf{E}^e}$ and defining the hardening force $A = -\rho_0 \frac{\partial \psi^p}{\partial \alpha}$ in Equation (253), we get

$$\begin{aligned} \rho_0 \dot{\psi} &= \rho_0 \frac{\partial \psi^e}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \rho_0 \frac{\partial \psi^p}{\partial \alpha} \dot{\alpha} \\ \Rightarrow \rho_0 \dot{\psi} &= \mathbf{S} : \dot{\mathbf{E}}^e - A\dot{\alpha} \end{aligned} \tag{254}$$

Using Equation (254), the dissipation inequality (Equation (247)) becomes:

$$\mathcal{D} = \mathbf{M} : \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} - A\dot{\alpha} \geq 0 \tag{255}$$

This identifies the **Plastic Dissipation** as:

$$\mathcal{D}_{\text{mech}}^{\text{plastic}} = \mathbf{M} : \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} - A\dot{\alpha} \tag{256}$$

where the first term $\mathbf{M} : \dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1}$ in Equation (256) represents the power expended by the Mandel stress on the plastic flow rate $\dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1}$, and the second term $-A\dot{\alpha}$ in Equation (256) accounts for dissipation due to hardening. For **Perzyna-type viscoplasticity**, the plastic flow rule is:

$$\dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1} = \gamma \left\langle \frac{f}{\sigma_y} \right\rangle^n \frac{\text{dev}\mathbf{M}}{\|\text{dev}\mathbf{M}\|} \tag{257}$$

where $f = \|\text{dev}\mathbf{M}\| - (\sigma_y - A)$ in Equation (257) is the yield function, ensuring $\mathcal{D}_{\text{mech}}^{\text{plastic}} \geq 0$ by construction. The hardening evolution $\dot{\alpha} = \sqrt{\frac{2}{3}} \|\dot{\mathbf{F}}^p (\mathbf{F}^p)^{-1}\|$ further guarantees non-negative dissipation, maintaining thermodynamic consistency.

6 Strong Form of the Governing Equations

The axial equilibrium equation describes the balance of forces along the longitudinal axis of the beam, accounting for both elastic and plastic deformations. The term $EA\left(\frac{du_0}{dx} - \varepsilon^p\right)$ represents the axial force, where EA is the axial rigidity, $\frac{du_0}{dx}$ is the total axial strain, and ε^p is the plastic strain. The derivative of this axial force with respect to x balances the externally applied axial load f_x, yielding the equilibrium equation:

$$\frac{d}{dx}\left(EA\left(\frac{du_0}{dx} - \varepsilon^p\right)\right) + f_x = 0 \tag{258}$$

The bending equilibrium equation captures the balance of moments and transverse forces, incorporating both elastic bending and the contribution of plastic strains. In Equation (258), the term $EI\frac{d^2w_0}{dx^2}$ represents the elastic bending moment, where EI is the flexural rigidity and $\frac{d^2w_0}{dx^2}$ is the curvature. The integral $\int_A Ey\varepsilon^p\, dA$ accounts for the moment due to plastic strains, where y is the distance from the neutral axis. The second derivative of the total moment with respect to x balances the transverse load f_z, resulting in:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w_0}{dx^2} + \int_A Ey\varepsilon^p\, dA\right) + f_z = 0 \tag{259}$$

The plastic flow equation governs the evolution of plastic strain ε^p based on the stress state and hardening effects. In Equation (259), the term $\gamma\left\langle\frac{|\sigma_{xx}|-(\sigma_y - A)}{\sigma_y}\right\rangle^n$ represents the rate-dependent plastic flow, where γ is a material parameter, σ_{xx} is the axial stress, σ_y is the yield stress, and A is the hardening variable. In Equation (260), the Macaulay brackets $\langle\cdot\rangle$ ensure plastic flow only occurs when the stress exceeds the yield condition and the signum function $\operatorname{sgn}(\sigma_{xx})$ ensures the direction of plastic strain matches the stress direction:

$$\dot{\varepsilon}^p = \gamma\left\langle\frac{|\sigma_{xx}| - (\sigma_y - A)}{\sigma_y}\right\rangle^n \operatorname{sgn}(\sigma_{xx}) \tag{260}$$

The hardening evolution equation describes the growth of the hardening variable α as a function of the plastic strain rate. The hardening variable α in Equation (261) evolves proportionally to the absolute value of the plastic strain rate $|\dot{\varepsilon}^p|$, capturing isotropic hardening effects. This relationship is given by:

$$\dot{\alpha} = |\dot{\varepsilon}^p| \tag{261}$$

7 Weak Form of the Governing Equations

The weak form of the axial equilibrium equation is derived by multiplying the strong form (Equation (258)) by a test function δu_0 and integrating over the domain $\Omega = [0, L]$. Applying integration by parts to the highest-order derivative term redistributes the differentiation and introduces natural boundary conditions. The weak form becomes:

$$\int_0^L \left(EA\left(\frac{d\delta u_0}{dx}\left(\frac{du_0}{dx} - \varepsilon^p\right)\right)\right) dx - \int_0^L f_x\delta u_0\, dx - \left[EA\left(\frac{du_0}{dx} - \varepsilon^p\right)\delta u_0\right]_0^L = 0 \tag{262}$$

where δu_0 is a kinematically admissible variation of the axial displacement u_0, vanishing where essential boundary conditions are prescribed. The boundary term in Equation

(262) represents the work done by axial forces at the ends of the beam.

The weak form of the bending equilibrium equation is obtained by multiplying the strong (Equation (259)) form by a test function δw_0 and integrating over the domain. Due to the fourth-order derivative, integration by parts is applied twice, leading to a symmetric bilinear form and introducing natural boundary conditions involving moments and shear forces. The weak form is given by:

$$\int_0^L \left(EI \frac{d^2 \delta w_0}{dx^2} \frac{d^2 w_0}{dx^2} + \frac{d^2 \delta w_0}{dx^2} \int_A E y \varepsilon^p \, dA \right) dx - \int_0^L f_z \delta w_0 \, dx + \left[\frac{d}{dx} \left(EI \frac{d^2 w_0}{dx^2} + \int_A E y \varepsilon^p \, dA \right) \delta w_0 \right]_0^L$$
$$- \left[\left(EI \frac{d^2 w_0}{dx^2} + \int_A E y \varepsilon^p \, dA \right) \frac{d \delta w_0}{dx} \right]_0^L = 0 \tag{263}$$

where δw_0 is a kinematically admissible variation of the transverse displacement w_0, vanishing where essential boundary conditions are imposed. The boundary terms in Equation (263) account for the work done by shear forces and bending moments at the beam ends.

The weak forms of the plastic flow and hardening evolution equations are derived in a similar manner, but since these are first-order rate equations, they are typically treated at the material point level in computational implementations. The plastic flow weak form (Equation (264)) is obtained by integrating the residual of the flow rule (Equation (260)) weighted by a test function $\delta \varepsilon^p$:

$$\int_0^L \left(\dot{\varepsilon}^p - \gamma \left\langle \frac{|\sigma_{xx}| - (\sigma_y - A)}{\sigma_y} \right\rangle^n \operatorname{sgn}(\sigma_{xx}) \right) \delta \varepsilon^p \, dx = 0 \tag{264}$$

The hardening evolution weak form (Equation (265)) is derived by integrating its residual (Equation (261)) weighted by a test function $\delta \alpha$:

$$\int_0^L \left(\dot{\alpha} - |\dot{\varepsilon}^p| \right) \delta \alpha \, dx = 0 \tag{265}$$

These weak forms are used in finite element formulations to ensure consistency between the global equilibrium equations and the local constitutive updates for plasticity and hardening.

8 Numerical Analysis of Model

8.1 Exponential map for $\mathbf{F}^p$ integration

The numerical integration of the plastic deformation gradient $\mathbf{F}^p$ under the rate-dependent Perzyna viscoplasticity model follows an exponential map approach to ensure algorithmic stability and exact plastic incompressibility. The evolution of $\mathbf{F}^p$ is governed by the flow rule:

$$\dot{\mathbf{F}}^p = \mathbf{L}^p \mathbf{F}^p, \quad \mathbf{L}^p = \dot{\gamma} \, \mathbf{N}_p \tag{266}$$

where $\mathbf{N}_p = \frac{3}{2} \frac{\boldsymbol{\sigma}'}{\sigma_{\text{eq}}}$ is the deviatoric flow direction, and $\dot{\gamma}$ in Equation (266) is the viscoplastic multiplier rate given by the Perzyna law:

$$\dot{\gamma} = \left\langle \frac{\sigma_{\text{eq}} - (\sigma_y(\alpha) - A)}{\eta} \right\rangle^n \tag{267}$$

In Equation (267), $\sigma_{\text{eq}} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'}$ is the von Mises equivalent stress, $\sigma_y(\alpha)$ is the yield stress, A is a hardening/softening variable, η is the viscosity, and n is the rate-sensitivity exponent. The exponential update for $\mathbf{F}^p$ over a time increment Δt is:

$$\mathbf{F}^p_{n+1} = \exp\left(\Delta t\, \dot{\gamma}_{n+1} \mathbf{N}_{p,n+1}\right) \mathbf{F}^p_n \tag{268}$$

For small to moderate steps, a first-order approximation of the term $\exp\left(\Delta t\, \dot{\gamma} \mathbf{N}_p\right)$ in Equation (268) shall be:

$$\exp\left(\Delta t\, \dot{\gamma} \mathbf{N}_p\right) \approx \mathbf{I} + \Delta t\, \dot{\gamma} \mathbf{N}_p + \frac{(\Delta t\, \dot{\gamma})^2}{2} (\mathbf{N}_p)^2 + \mathcal{O}(\Delta t^3) \tag{269}$$

For large plastic strains, an exact spectral decomposition of the term $\exp\left(\Delta t\, \dot{\gamma} \mathbf{N}_p\right)$ in Equation (268) is used:

$$\exp\left(\Delta t\, \dot{\gamma} \mathbf{N}_p\right) = \sum_{i=1}^{3} \exp\left(\Delta t\, \dot{\gamma} \lambda_i\right) \mathbf{n}_i \otimes \mathbf{n}_i \tag{270}$$

where λ_i and $\mathbf{n}_i$ are the eigenvalues and eigenvectors of $\mathbf{N}_p$ in Equation (270). The stress update follows an elastic predictor-viscoplastic corrector scheme. First, the trial elastic state is computed:

$$\boldsymbol{\sigma}^{\text{trial}} = \mathbb{C} : \left(\mathbf{F}^e_{\text{trial}}\right)^T \mathbf{F}^e_{\text{trial}} - \mathbf{I}, \quad \mathbf{F}^e_{\text{trial}} = \mathbf{F}_{n+1} \left(\mathbf{F}^p_n\right)^{-1} \tag{271}$$

where $\mathbb{C}$ is the elastic stiffness tensor in Equation (271). If $\Phi^{\text{trial}} = \sigma^{\text{trial}}_{\text{eq}} - (\sigma_y(\alpha_n) - A_n) > 0$, viscoplastic flow occurs, and the plastic multiplier increment $\Delta\gamma = \dot{\gamma}_{n+1}\Delta t$ is computed by solving:

$$\Delta\gamma = \Delta t \left\langle \frac{\sigma^{\text{trial}}_{\text{eq}} - 3G\Delta\gamma - (\sigma_y(\alpha_n) - A_n)}{\eta} \right\rangle^n \tag{272}$$

where G is the shear modulus. Using the value of $\Delta\gamma$ computed in Equation (272), the updated stress and hardening variables shall be:

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^{\text{trial}} - 2G\Delta\gamma \mathbf{N}_p, \quad \alpha_{n+1} = \alpha_n + \Delta\gamma, \quad A_{n+1} = A_n + H\Delta\gamma \tag{273}$$

with H being the hardening modulus in Equation (273). The algorithmic tangent modulus is derived consistently to ensure quadratic convergence in Newton iterations. This formulation guarantees robust and accurate integration for finite-strain viscoplasticity.

8.2 Mixed FEM for locking-free discretization

The mixed finite element method (FEM) formulation for the rate-dependent Perzyna viscoplasticity model addresses volumetric locking by introducing independent interpolations for displacement and pressure fields. The weak form of equilibrium is derived from the virtual work principle, where the Cauchy stress $\boldsymbol{\sigma}$ is decomposed into its deviatoric $\boldsymbol{\sigma}'$ and volumetric p components:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{I}, \quad p = -\frac{1}{3}\text{tr}(\boldsymbol{\sigma}) \tag{274}$$

The variational statement for the displacement field $\mathbf{u}$ and pressure field p becomes:

$$\int_\Omega \left(\boldsymbol{\sigma}' : \nabla^s \delta\mathbf{u} - p \,\text{div}\, \delta\mathbf{u}\right) dV = \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\, dV + \int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\, dS \tag{275}$$

$$\int_\Omega \left(\operatorname{div}\mathbf{u} - \frac{p}{K}\right)\delta p\, dV = 0 \tag{276}$$

where K is the bulk modulus, $\mathbf{f}$ is the body force, $\mathbf{t}$ is the traction, and $\delta\mathbf{u}$, δp are test functions in Equations (275) and (276). The Perzyna viscoplastic flow rule defines the deviatoric plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ as:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\mathbf{N}_p, \quad \mathbf{N}_p = \frac{3}{2}\frac{\boldsymbol{\sigma}'}{\sigma_{\text{eq}}} \tag{277}$$

with the equivalent stress $\sigma_{\text{eq}} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'}$ and the plastic multiplier rate $\dot{\gamma}$ in Equation (277) given by the following equation:

$$\dot{\gamma} = \left\langle \frac{\sigma_{\text{eq}} - (\sigma_y(\alpha) - A)}{\eta} \right\rangle^n \tag{278}$$

Using Equation (278), The hardening evolution follows:

$$\dot{\alpha} = \dot{\gamma}, \quad \dot{A} = H\dot{\gamma} \tag{279}$$

where H is the hardening modulus in Equation (279). The discrete mixed FEM formulation employs separate interpolation spaces for displacements $\mathbf{u}^h \in \mathcal{V}^h$ and pressures $p^h \in \mathcal{Q}^h$:

$$\mathbf{u}^h = \sum_{I=1}^{n_u} N_I^u \mathbf{u}_I, \quad p^h = \sum_{J=1}^{n_p} N_J^p p_J \tag{280}$$

where N_I^u and N_J^p are shape functions in Equation (280). The Galerkin approximation of the weak form yields the nonlinear residual equations:

$$\mathbf{R}_u = \int_\Omega \left(\boldsymbol{\sigma}'(\mathbf{u}^h, p^h) : \nabla^s N_I^u - p^h \operatorname{div} N_I^u\right) dV - \mathbf{F}_I^{\text{ext}} = \mathbf{0} \tag{281}$$

$$R_p = \int_\Omega \left(\operatorname{div}\mathbf{u}^h - \frac{p^h}{K}\right) N_J^p\, dV = 0 \tag{282}$$

To prevent locking, the LBB (Ladyzhenskaya-Babuška-Brezzi) condition must be satisfied by selecting appropriate $\mathcal{V}^h$ and $\mathcal{Q}^h$ pairs, such as Taylor-Hood (Q2-Q1) or MINI elements. The consistent linearization of the residuals (Equations (281) and (282)) yields the tangent stiffness matrix:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta p \end{bmatrix} = - \begin{bmatrix} \mathbf{R}_u \\ R_p \end{bmatrix} \tag{283}$$

where:

$$\mathbf{K}_{uu} = \int_\Omega (\mathbb{C}_{\text{tan}} : \nabla^s N_I^u) : \nabla^s N_J^u\, dV \tag{284}$$

$$\mathbf{K}_{up} = -\int_\Omega (\mathbf{I} \operatorname{div} N_I^u)\, N_J^p\, dV \tag{285}$$

$$\mathbf{K}_{pu} = \int_\Omega N_I^p \operatorname{div} N_J^u\, dV \tag{286}$$

$$\mathbf{K}_{pp} = -\int_\Omega \frac{N_I^p N_J^p}{K}\, dV \tag{287}$$

Here, $\mathbb{C}_{\text{tan}}$ in Equation (284) is the algorithmic tangent modulus for the Perzyna model, derived from the linearization of the viscoplastic constitutive update. This mixed formulation ensures stability and accuracy for nearly incompressible and rate-dependent plastic flows.

8.2.1 Proof of the Variational Form (Equations (275) and (276))

The variational statement for the displacement and pressure fields in the context of rate-dependent Perzyna viscoplasticity is derived from the principle of virtual work and the enforcement of the volumetric constitutive relation. Beginning with the equilibrium equation in strong form:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \quad \text{in} \quad \Omega \tag{288}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{f}$ is the body force vector in Equation (288). The stress is decomposed into its deviatoric and volumetric parts:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{I} \tag{289}$$

where $p = -\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$ is the pressure and $\boldsymbol{\sigma}'$ is the deviatoric stress in Equation (289). Multiplying the equilibrium equation (Equation (288)) by a virtual displacement field $\delta\mathbf{u}$ and integrating over the domain Ω yields:

$$\int_\Omega (\nabla \cdot \boldsymbol{\sigma}) \cdot \delta\mathbf{u}\, dV + \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\, dV = 0 \tag{290}$$

Applying the divergence theorem to the first term of Equation (290) gives:

$$\int_\Omega \boldsymbol{\sigma} : \nabla\delta\mathbf{u}\, dV = \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\, dV + \int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\, dS \tag{291}$$

where $\mathbf{t} = \boldsymbol{\sigma}\cdot\mathbf{n}$ in Equation (291) is the traction vector on the boundary $\partial\Omega$. Substituting the stress decomposition into the integrand:

$$\int_\Omega (\boldsymbol{\sigma}' - p\mathbf{I}) : \nabla\delta\mathbf{u}\, dV = \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\, dV + \int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\, dS \tag{292}$$

The term $\mathbf{I} : \nabla\delta\mathbf{u} = \operatorname{div}\delta\mathbf{u}$, and the deviatoric stress $\boldsymbol{\sigma}'$ in Equation (292) is orthogonal to the volumetric part, leading to:

$$\int_\Omega \boldsymbol{\sigma}' : \nabla^s\delta\mathbf{u}\, dV - \int_\Omega p \operatorname{div}\delta\mathbf{u}\, dV = \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\, dV + \int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\, dS \tag{293}$$

where $\nabla^s\delta\mathbf{u} = \frac{1}{2}(\nabla\delta\mathbf{u} + (\nabla\delta\mathbf{u})^T)$ in Equation (293) is the symmetric gradient of the virtual displacement. This establishes the first variational equation:

$$\int_\Omega (\boldsymbol{\sigma}' : \nabla^s\delta\mathbf{u} - p \operatorname{div}\delta\mathbf{u})\, dV = \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\, dV + \int_{\partial\Omega} \mathbf{t} \cdot \delta\mathbf{u}\, dS \tag{294}$$

The second variational equation arises from the volumetric constitutive relation for the pressure:

$$p = -K \operatorname{div}\mathbf{u} \tag{295}$$

where K is the bulk modulus in Equation (295). Introducing a virtual pressure field δp and enforcing this relation weakly gives:

$$\int_\Omega \left(\operatorname{div}\mathbf{u} + \frac{p}{K}\right) \delta p\, dV = 0 \tag{296}$$

Equation (296) ensures that the incompressibility constraint is satisfied in a weak sense, which is crucial for avoiding volumetric locking in the numerical implementation.

For the rate-dependent Perzyna model, the deviatoric stress $\boldsymbol{\sigma}'$ is related to the elastic strain $\boldsymbol{\varepsilon}^e$ via the elastic constitutive law:

$$\boldsymbol{\sigma}' = 2G\,\boldsymbol{\varepsilon}^e \tag{297}$$

where G is the shear modulus and $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$, with $\boldsymbol{\varepsilon}$ being the total strain and $\boldsymbol{\varepsilon}^p$ the plastic strain in Equation (297). The plastic strain rate is given by the Perzyna flow rule:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\,\mathbf{N}_p, \quad \mathbf{N}_p = \frac{3}{2}\frac{\boldsymbol{\sigma}'}{\sigma_{\text{eq}}} \tag{298}$$

where $\dot{\gamma}$ in Equation (298) is the plastic multiplier rate:

$$\dot{\gamma} = \left\langle \frac{\sigma_{\text{eq}} - (\sigma_y - A)}{\eta} \right\rangle^n \tag{299}$$

Here, $\sigma_{\text{eq}} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'}$ is the equivalent stress, σ_y is the yield stress, A is the hardening variable, η is the viscosity, and n is the rate sensitivity exponent in Equation (299). The hardening evolution is governed by:

$$\dot{A} = H\dot{\gamma} \tag{300}$$

where H is the hardening modulus in Equation (300). The variational framework thus couples the equilibrium equation with the constitutive update for the Perzyna model, ensuring a consistent and locking-free discretization. The resulting system is solved iteratively, with the tangent moduli derived from linearizing the constitutive relations to preserve quadratic convergence in Newton-Raphson iterations.

8.2.2 LBB (Ladyzhenskaya-Babuška-Brezzi) condition

The LBB (Ladyzhenskaya-Babuška-Brezzi) condition is a fundamental stability criterion for mixed finite element formulations that ensures the well-posedness of the saddle-point problem arising from the variational principle. For the rate-dependent Perzyna viscoplasticity model, the mixed formulation involves the displacement field $\mathbf{u}^h \in \mathcal{V}^h$ and the pressure field $p^h \in \mathcal{Q}^h$, where $\mathcal{V}^h$ and $\mathcal{Q}^h$ are finite-dimensional subspaces of the continuous function spaces $\mathcal{V}$ and $\mathcal{Q}$. The LBB condition requires the existence of a constant $\beta > 0$ such that:

$$\inf_{p^h \in \mathcal{Q}^h} \sup_{\mathbf{v}^h \in \mathcal{V}^h} \frac{\int_\Omega p^h \,\text{div}\,\mathbf{v}^h \, dV}{\|p^h\|_{L^2} \|\mathbf{v}^h\|_{H^1}} \geq \beta \tag{301}$$

This inf-sup condition in Equation (301) guarantees that the pressure field is uniquely determined and prevents spurious pressure modes that lead to volumetric locking. For the Perzyna model, the choice of $\mathcal{V}^h$ and $\mathcal{Q}^h$ must satisfy this condition. Taylor-Hood elements (Q2-Q1), where the displacement is approximated by quadratic polynomials and the pressure by linear polynomials, are a classic example. The MINI element, which enriches the linear displacement field with bubble functions, is another stable choice.

The tangent stiffness matrix is derived from the consistent linearization of the residual equations. The residual for the displacement field $\mathbf{R}_u$ and the pressure field R_p are given by:

$$\mathbf{R}_u = \int_\Omega \left(\boldsymbol{\sigma}'(\mathbf{u}^h, p^h) : \nabla^s N_I^u - p^h \,\text{div}\, N_I^u \right) dV - \mathbf{F}_I^{\text{ext}} \tag{302}$$

$$R_p = \int_\Omega \left(\operatorname{div} \mathbf{u}^h - \frac{p^h}{K} \right) N_J^p \, dV \tag{303}$$

Linearizing these residuals (Equations (302) and (303)) with respect to the incremental displacements $\Delta\mathbf{u}$ and pressures Δp yields the tangent stiffness matrix. The block $\mathbf{K}_{uu}$ in Equation (304) represents the stiffness due to displacement variations:

$$\mathbf{K}_{uu} = \int_\Omega (\mathbb{C}_{\tan} : \nabla^s N_I^u) : \nabla^s N_J^u \, dV \tag{304}$$

where $\mathbb{C}_{\tan}$ in Equation (304) is the algorithmic tangent modulus for the Perzyna model, derived from the linearization of the viscoplastic constitutive update. For the Perzyna flow rule, the tangent modulus incorporates the rate-dependent effects:

$$\mathbb{C}_{\tan} = \mathbb{C}^e - \frac{(\mathbb{C}^e : \mathbf{N}_p) \otimes (\mathbb{C}^e : \mathbf{N}_p)}{\mathbf{N}_p : \mathbb{C}^e : \mathbf{N}_p + \frac{\sigma_y'}{\Delta\gamma} + \frac{\eta}{n\Delta t} \left(\frac{\sigma_{\text{eq}} - (\sigma_y - A)}{\eta} \right)^{1-n}} \tag{305}$$

where $\mathbb{C}^e$ is the elastic stiffness tensor, $\sigma_y' = \frac{d\sigma_y}{d\alpha}$ is the hardening slope, and $\Delta\gamma$ is the incremental plastic multiplier in Equation (305). The off-diagonal blocks $\mathbf{K}_{up}$ and $\mathbf{K}_{pu}$ in Equation (306) and (307) couple the displacement and pressure fields:

$$\mathbf{K}_{up} = -\int_\Omega (\mathbf{I} \operatorname{div} N_I^u) \, N_J^p \, dV \tag{306}$$

$$\mathbf{K}_{pu} = \int_\Omega N_I^p \operatorname{div} N_J^u \, dV \tag{307}$$

The block $\mathbf{K}_{pp}$ in Equation (308) accounts for the volumetric stiffness:

$$\mathbf{K}_{pp} = -\int_\Omega \frac{N_I^p N_J^p}{K} \, dV \tag{308}$$

The resulting linear system:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{up} \\ \mathbf{K}_{pu} & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \Delta p \end{bmatrix} = - \begin{bmatrix} \mathbf{R}_u \\ R_p \end{bmatrix} \tag{309}$$

Equation (309) is solved iteratively to update the displacement and pressure fields. The LBB condition ensures that the matrix is invertible and that the solution converges to the correct equilibrium state. The consistent linearization preserves the quadratic convergence rate of Newton's method, which is crucial for the efficiency of the numerical implementation. The algorithmic tangent modulus $\mathbb{C}_{\tan}$ must accurately reflect the viscoplastic response to maintain this convergence rate. For the Perzyna model, this involves careful treatment of the rate-dependent terms and the hardening evolution:

$$\dot{\alpha} = \dot{\gamma}, \quad \dot{A} = H\dot{\gamma} \tag{310}$$

where H is the hardening modulus in Equation (310). The tangent modulus thus captures both the elastic and inelastic deformations, ensuring a robust and accurate numerical solution for finite-strain viscoplasticity problems.

8.3 Adiabatic decoupling if $\nabla\theta$ is small

The adiabatic decoupling approximation for the rate-dependent Perzyna viscoplasticity model becomes valid when the temperature gradient $\nabla\theta$ is sufficiently small, allowing the mechanical and thermal problems to be solved sequentially rather than simultaneously. This decoupling is justified when the timescale of heat diffusion is much longer than the timescale of mechanical deformation, which occurs when the Fourier number $Fo = \frac{\alpha t}{L^2}$ (where α is thermal diffusivity, t is characteristic time, and L is length scale) is small. The strong form of the coupled thermomechanical problem consists of the momentum balance:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}} \tag{311}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\mathbf{f}$ is the body force, and ρ is density in Equation (311), and the energy balance:

$$\rho c_v \dot{\theta} = \nabla \cdot (k \nabla\theta) + r + \chi \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \tag{312}$$

where c_v is specific heat, k is thermal conductivity, r is external heat supply, and χ is the Taylor-Quinney coefficient representing the fraction of plastic work converted to heat. When $\nabla\theta \approx 0$, the heat conduction term $\nabla \cdot (k\nabla\theta)$ in Equation (312) becomes negligible, reducing the energy balance to:

$$\rho c_v \dot{\theta} = r + \chi \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \tag{313}$$

For the Perzyna model, the plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ in Equation (313) is given by:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \mathbf{N}_p, \quad \mathbf{N}_p = \frac{3}{2} \frac{\boldsymbol{\sigma}'}{\sigma_{\text{eq}}} \tag{314}$$

with the rate-dependent plastic multiplier in Equation (314) defined as:

$$\dot{\gamma} = \left\langle \frac{\sigma_{\text{eq}} - (\sigma_y(\theta) - A)}{\eta(\theta)} \right\rangle^n \tag{315}$$

Here, $\sigma_y(\theta)$ and $\eta(\theta)$ in Equation (315) are temperature-dependent yield stress and viscosity, respectively. The adiabatic approximation implies that the temperature evolution is locally determined by the plastic dissipation:

$$\dot{\theta} = \frac{\chi}{\rho c_v} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p = \frac{\chi}{\rho c_v} \sigma_{\text{eq}} \dot{\gamma} \tag{316}$$

since $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p = \boldsymbol{\sigma}' : \dot{\boldsymbol{\varepsilon}}^p = \sigma_{\text{eq}} \dot{\gamma}$ in Equation (316). The mechanical problem can thus be solved first, with the temperature updated explicitly using the computed plastic work. The stress update algorithm proceeds as follows:

1. **Elastic predictor**: Compute trial stress assuming elastic deformation:

$$\boldsymbol{\sigma}^{\text{trial}} = \boldsymbol{\sigma}_n + \mathbb{C}^e : \Delta\boldsymbol{\varepsilon} \tag{317}$$

   where $\mathbb{C}^e$ is the elastic stiffness tensor and $\Delta\boldsymbol{\varepsilon}$ is the strain increment in Equation (317).

2. **Yield condition**: Check if the trial stress exceeds the temperature-dependent yield surface:

$$\Phi^{\text{trial}} = \sigma_{\text{eq}}^{\text{trial}} - (\sigma_y(\theta_n) - A_n) \tag{318}$$

3. **Viscoplastic corrector**: If $\Phi^{\text{trial}} > 0$ in Equation (318), solve for the plastic multiplier increment $\Delta\gamma$:

$$\Delta\gamma = \Delta t \left\langle \frac{\sigma_{\text{eq}}^{\text{trial}} - 3G\Delta\gamma - (\sigma_y(\theta_n) - A_n)}{\eta(\theta_n)} \right\rangle^n \tag{319}$$

4. **Stress and state variable update**: The updated stress $\boldsymbol{\sigma}_{n+1}$ and the updated hardening variable A can be found using the plastic multiplier increment $\Delta\gamma$ calculated from Equation (319):

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^{\text{trial}} - 2G\Delta\gamma\mathbf{N}_p \tag{320}$$

$$A_{n+1} = A_n + H\Delta\gamma, \quad \alpha_{n+1} = \alpha_n + \Delta\gamma \tag{321}$$

5. **Temperature update**: The updated temperature θ_{n+1} can be also found using the plastic multiplier increment $\Delta\gamma$ calculated from Equation (319)

$$\theta_{n+1} = \theta_n + \frac{\chi}{\rho c_v}\sigma_{\text{eq}}\Delta\gamma \tag{322}$$

The tangent modulus for the adiabatic problem must account for temperature-dependent material properties. Linearizing the stress update given in Equation (320) yields:

$$\mathbb{C}_{\text{tan}} = \mathbb{C}^e - \frac{(\mathbb{C}^e : \mathbf{N}_p) \otimes (\mathbb{C}^e : \mathbf{N}_p)}{\mathbf{N}_p : \mathbb{C}^e : \mathbf{N}_p + \frac{\sigma_y'}{\Delta\gamma} + \frac{\eta}{n\Delta t}\left(\frac{\sigma_{\text{eq}} - (\sigma_y - A)}{\eta}\right)^{1-n} - \chi\frac{\partial\sigma_y}{\partial\theta}\frac{\Delta\gamma}{\rho c_v}} \tag{323}$$

Equation (323) includes the thermal softening effect through the term $\chi\frac{\partial\sigma_y}{\partial\theta}\frac{\Delta\gamma}{\rho c_v}$. The adiabatic decoupling thus simplifies the coupled problem while preserving the essential thermomechanical feedback, provided $\nabla\theta$ remains small. The approximation is particularly effective for high strain-rate processes where plastic dissipation dominates heat conduction.

9 Functional-Analytic Formulation

9.1 Sobolev Spaces for Well-Posedness

The solution space is:

$$(u, w, \theta) \in H^1(0, L) \times H^2(0, L) \times H^1(0, L), \tag{324}$$

with **plastic variables** $\mathbf{F}^p, \alpha$ in $W^{1,\infty}$ (Lipschitz). The functional-analytic setting for the thermo-elasto-viscoplastic beam problem is constructed to ensure mathematical well-posedness, requiring appropriate Sobolev spaces for the unknowns. The displacement fields (u, w) and temperature θ are chosen to reside in spaces that reflect their physical regularity requirements. The axial displacement u and temperature θ belong to $H^1(0, L)$, the Sobolev space of square-integrable functions with square-integrable first derivatives. This choice ensures that both u and θ admit weak derivatives, which is necessary for the balance laws and heat equation to hold in a distributional sense. The transverse displacement w lies in $H^2(0, L)$, a stricter space requiring square-integrable second derivatives, as

the bending-dominated response of the Euler-Bernoulli beam involves the second derivative w'' in the weak form of equilibrium.

The plastic variables, $\mathbf{F}^p$ and α, are assigned to the space $W^{1,\infty}(0, L)$, which consists of Lipschitz-continuous functions with essentially bounded first derivatives. This selection is motivated by the nature of plastic evolution: $\mathbf{F}^p$ must remain invertible (to preserve the multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$), and α must evolve continuously to avoid spurious hardening discontinuities. The Lipschitz condition ensures that the plastic strain rate $\dot{\mathbf{F}}^p$ and hardening rate $\dot{\alpha}$ remain bounded, which is crucial for numerical stability and physical admissibility. The coupling between these spaces arises naturally from the weak formulation of the problem, where test functions for displacements and temperature are selected from the same spaces to enforce consistency in the variational framework.

The use of $H^1(0, L)$ for u and θ and $H^2(0, L)$ for w reflects the differing regularity demands of axial stretching, bending, and thermal diffusion. Meanwhile, the $W^{1,\infty}$ requirement for plastic variables guarantees that the plastic flow remains well-defined and non-degenerate, avoiding unphysical singularities in $\mathbf{F}^p$ or α. This functional setting is essential for proving existence and uniqueness of solutions, as it provides the necessary compactness and continuity properties for nonlinear analysis. The spaces are also compatible with finite element discretizations, where H^2-conforming elements (e.g., Hermite polynomials) are used for w and standard H^1-conforming elements for u and θ, while plastic updates are treated at the quadrature point level with Lipschitz stability.

9.2 Existence & Uniqueness

The existence and uniqueness of solutions for the thermo-elasto-viscoplastic beam problem rely on three key structural assumptions imposed on the constitutive laws. First, the elastic stored energy ψ^e is required to be polyconvex, meaning it satisfies a generalized convexity condition that prevents non-physical material behavior such as interpenetration or infinite compression. Polyconvexity ensures weak lower semicontinuity of the energy functional, which is necessary for the direct method of the calculus of variations to guarantee minimizers. For finite elasticity, this typically takes the form $\psi^e = \psi^e(\mathbf{F}^e, \operatorname{cof}\mathbf{F}^e, \det\mathbf{F}^e)$, where all arguments are treated independently to maintain appropriate convexity properties.

Second, the hardening law must be uniformly monotone, which implies that the hardening force A satisfies a strong growth condition with respect to the internal variable α. Mathematically, this means there exists a constant $c > 0$ such that $(A(\alpha_1) - A(\alpha_2)) \cdot (\alpha_1 - \alpha_2) \geq c\|\alpha_1 - \alpha_2\|^2$ for all admissible α_1, α_2. This condition ensures that the plastic dissipation remains coercive, preventing unbounded growth of α and guaranteeing stability in the evolution equations.

Third, the yield stress $\sigma_y(\theta)$ must be Lipschitz continuous with respect to temperature, meaning there exists a constant $L > 0$ such that $|\sigma_y(\theta_1) - \sigma_y(\theta_2)| \leq L|\theta_1 - \theta_2|$ for all θ_1, θ_2. This assumption controls the thermo-mechanical coupling by preventing abrupt changes in yield behavior due to temperature fluctuations, which could otherwise lead to ill-posedness in the plastic flow.

Under these conditions, the problem admits a solution in Banach spaces, where the implicit function theorem can be applied to the nonlinear system of equilibrium equations, plastic flow rules, and heat conduction. The Banach-space setting typically involves the Sobolev spaces $H^1(0, L)$ for displacements and temperature, along with $W^{1,\infty}$ for plastic variables, as previously discussed. The implicit function theorem provides local existence and uniqueness near equilibrium states, while the polyconvexity and monotonicity assumptions extend this to global results for sufficiently small loads or short times. The Lipschitz condition on $\sigma_y(\theta)$ further ensures that the solution depends continuously on initial data and parameters, a crucial property for both mathematical analysis and numerical approximation.

The combination of these assumptions leads to a well-posed problem where mechanical equilibrium, plastic evolution, and thermal effects are all consistently coupled. The resulting framework is sufficiently robust to accommodate the nonlinearities inherent in finite-strain plasticity while remaining amenable to rigorous analysis and stable computation. The use of Banach spaces reflects the need for bounded deformations and stresses, while the implicit function theorem bridges the gap between linearized stability and fully nonlinear behavior. This approach is foundational for proving convergence of numerical methods, such as finite element discretizations, and for establishing error estimates in practical simulations.

10 Summary of the Proposed Rate Dependent Plasticity Model of Euler-Bernoulli Beams

The proposed rate-dependent plasticity model for Euler-Bernoulli beams begins with the finite strain kinematic framework characterized by the deformation gradient $\mathbf{F}$, which is multiplicatively decomposed into elastic and plastic parts as

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{325}$$

The plastic part $\mathbf{F}^p$ maps the reference configuration to an intermediate, stress-free configuration, while the elastic part $\mathbf{F}^e$ maps from the intermediate to the current configuration in Equation (325). To maintain plastic incompressibility, the condition

$$\det(\mathbf{F}^p) = 1 \tag{326}$$

is enforced for the plastic part of the deformation gradient $\mathbf{F}^p$ in Equation (325). The elastic strain in the intermediate configuration is given by the Green-Lagrange tensor

$$\mathbf{E}^e = \frac{1}{2}\left((\mathbf{F}^e)^T\mathbf{F}^e - \mathbf{I}\right) \tag{327}$$

and the plastic flow is characterized by the plastic velocity gradient

$$\mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} \tag{328}$$

$\mathbf{L}^p$ in Equation (328) can be decomposed as

$$\mathbf{L}^p = \mathbf{D}^p + \mathbf{W}^p \tag{329}$$

where $\mathbf{D}^p$ is the plastic stretching tensor and $\mathbf{W}^p$ is the plastic spin in Equation (329). The evolution equation for the plastic deformation gradient is given by

$$\dot{\mathbf{F}}^p = \mathbf{L}^p \mathbf{F}^p \tag{330}$$

The elastic right Cauchy-Green tensor is

$$\mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e \tag{331}$$

and the elastic Finger tensor is

$$\mathbf{b}^e = \mathbf{F}^e (\mathbf{F}^e)^T \tag{332}$$

These strain measures (Elastic right Cauchy-Green tensor (Equation (331)) and the Elastic Finger tensor (Equation (332))) are used to construct the Helmholtz free energy and define the constitutive response. The Euler-Bernoulli beam assumption introduces constraints on the deformation field such that cross-sections remain plane and perpendicular to the beam's axis after deformation. The beam deformation map is expressed in terms of midline displacement $\boldsymbol{\phi}_0(x)$ and directors $\mathbf{d}_2(x)$, $\mathbf{d}_3(x)$. The deformation gradient is

$$\mathbf{F} = \boldsymbol{\phi}_0' \otimes \mathbf{e}_1 + \mathbf{d}_2 \otimes \mathbf{e}_2 + \mathbf{d}_3 \otimes \mathbf{e}_3 + y\mathbf{d}_2' \otimes \mathbf{e}_1 + z\mathbf{d}_3' \otimes \mathbf{e}_1 \tag{333}$$

and the Green-Lagrange strain simplifies to

$$E_{11} \approx \varepsilon(x) + y\kappa_2(x) + z\kappa_3(x) \tag{334}$$

where $\varepsilon(x) = \|\boldsymbol{\phi}_0'\| - 1$ is the axial strain and κ_2, κ_3 are bending curvatures in Equation (334). Transverse and shear strains vanish due to the beam kinematics. The stress resultants are obtained from the second Piola-Kirchhoff stress $\mathbf{S}$ by

$$N = \int_A S_{11}\, dA, \quad M_2 = -\int_A zS_{11}\, dA, \quad M_3 = \int_A yS_{11}\, dA \tag{335}$$

The weak form of the momentum balance is derived from the principle of virtual power:

$$\int_{B_0} \mathbf{P} : \delta\mathbf{F}\, dV = \int_{B_0} \mathbf{f} \cdot \delta\boldsymbol{\phi}\, dV + \int_{\partial B_0} \mathbf{t} \cdot \delta\boldsymbol{\phi}\, dA \tag{336}$$

where $\mathbf{P}$ is the first Piola-Kirchhoff stress and $\delta\mathbf{F} = \nabla\delta\boldsymbol{\phi}$ in Equation (336). For the beam, Equation (336) reduces to

$$\int_0^L (N\delta u' - M\delta w'')\, dx = \text{External Work} \tag{337}$$

where N and M in Equation (337) are defined earlier in Equation (335). The constitutive equations are derived from the Helmholtz free energy

$$\psi = \psi^e(\mathbf{C}^e, \theta) + \psi^p(\alpha, \theta) + \psi^\theta(\theta) \tag{338}$$

where α is an internal variable representing equivalent plastic strain, and θ is temperature in Equation (338). The elastic part of ψ (as defined in Equation (338)) is given by

$$\psi^e = \frac{1}{2\rho_0}\left[\lambda(\theta)(\ln J^e)^2 + \mu(\theta)\mathrm{tr}(\mathbf{E}^e_{\mathrm{dev}})^2\right] + \psi_{\mathrm{vol}}(J^e, \theta) \tag{339}$$

with ψ_{vol} in Equation (338)

$$\psi_{\text{vol}} = \frac{K(\theta)}{2\rho_0}\left(\frac{(J^e)^2}{2} - \ln J^e - \frac{1}{2}\right) \tag{340}$$

The plastic part of ψ (as defined in Equation (338)) includes hardening and saturation:

$$\psi^p = \frac{1}{\rho_0}\left[\sigma_y(\theta)\alpha + \frac{1}{2}H(\theta)\alpha^2 + K_\infty(\theta)\left(\alpha + \frac{1}{\omega}e^{-\omega\alpha}\right)\right] \tag{341}$$

The thermal contribution of ψ (as defined in Equation (338)) is

$$\psi^\theta = c_v\left(\theta - \theta_0 - \theta\ln\frac{\theta}{\theta_0}\right) - \frac{1}{2}\beta(\theta)\frac{(\theta-\theta_0)^2}{\theta_0} \tag{342}$$

The stress, entropy, and hardening force are defined by the thermodynamic conjugates derived from the definition of ψ (as defined in Equation (338) with it constituting parts defined in Equations (339), (340), (341), and (342)):

$$\mathbf{S} = 2\rho_0\frac{\partial\psi}{\partial\mathbf{C}^e}, \quad A = -\rho_0\frac{\partial\psi}{\partial\alpha}, \quad \eta = -\frac{\partial\psi}{\partial\theta} \tag{343}$$

The plastic flow is governed by a Perzyna-type overstress model:

$$\mathbf{L}^p = \gamma\left\langle\frac{f}{\sigma_y}\right\rangle^n \frac{\text{dev}(\mathbf{M})}{\|\text{dev}(\mathbf{M})\|} \tag{344}$$

where the yield function f in Equation (344) being

$$f = \|\text{dev}(\mathbf{M})\| - (\sigma_y(\theta) - A) \tag{345}$$

and the Mandel stress $\mathbf{M} = \mathbf{C}^e\mathbf{S}$ in Equation (345). The mechanical dissipation is

$$D_{\text{mech}} = \mathbf{M} : \mathbf{L}^p - A\dot{\alpha} = \gamma\left\langle\frac{f}{\sigma_y}\right\rangle^n (\|\text{dev}(\mathbf{M})\| + A) \geq 0 \tag{346}$$

The energy balance in material form is

$$\rho_0\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \nabla_X \cdot \mathbf{q}_0 + r_0 \tag{347}$$

Equation (347) splits into elastic and plastic contributions via

$$\mathbf{P} : \dot{\mathbf{F}} = \mathbf{M}^e : \dot{\mathbf{E}}^e + \mathbf{M}^p : \mathbf{L}^p \tag{348}$$

The heat equation becomes

$$\rho_0 c\dot{\theta} = \nabla_X \cdot (\kappa\nabla_X\theta) + D_{\text{mech}} - \rho_0\theta\frac{\partial\mathbf{M}^e}{\partial\theta} : \dot{\mathbf{E}}^e + r_0 \tag{349}$$

For the Euler-Bernoulli beam, Equation (349) reduces to

$$\rho_0 A\dot{e} = N\dot{u}' - M\dot{w}'' - \frac{dq_0}{dx} + \bar{r}_0 + \int_A D_{\text{mech}}\, dA \tag{350}$$

where N and M are the axial force and bending moment, and q_0 is the axial heat flux in Equation (350). The entropy inequality

$$\rho_0 \dot{\eta} \geq \frac{r_0}{\theta} - \nabla_X \cdot \left(\frac{\mathbf{q}_0}{\theta} \right) \tag{351}$$

is preserved through the construction, with the beam-level version of Equation (351) being

$$\rho_0 \dot{\eta} \geq \frac{\bar{r}_0}{\theta} - \frac{d}{dx} \left(\frac{q_0}{\theta} \right) + \frac{1}{\theta} \int_A D_{\text{mech}} \, dA \tag{352}$$

These equations ensure full thermomechanical consistency under finite strains and rate-dependent plasticity for slender beam structures. The strong form of the governing equations for the rate-dependent plasticity model of Euler-Bernoulli beams is derived from balance laws and beam kinematics. Based on the finite-strain formulation, the balance of linear momentum in the reference configuration is expressed as

$$\text{Div}\, \mathbf{P} + \mathbf{f} = \rho_0 \ddot{\boldsymbol{\phi}} \tag{353}$$

where $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{f}$ is the body force per unit reference volume, ρ_0 is the reference mass density, and $\boldsymbol{\phi}$ is the deformation map in Equation (353). The associated beam-level strong form reduces to differential equations in terms of internal force resultants (axial force N, and bending moments M_2, M_3):

$$\frac{dN}{dx} + f_1 = \rho_0 A \ddot{u}, \quad \frac{dM_2}{dx} + f_2 = \rho_0 I_2 \ddot{\theta}_2, \quad \frac{dM_3}{dx} + f_3 = \rho_0 I_3 \ddot{\theta}_3 \tag{354}$$

where $u(x,t)$ is the axial displacement of the midline, θ_2, θ_3 are the angles of rotation due to bending about principal axes, and A, I_2, I_3 are the area and moments of inertia of the cross-section in Equation (354). Appropriate boundary conditions are applied to displacements and force resultants at beam ends. Let $\delta\boldsymbol{\phi}$ be the virtual displacement field. The virtual internal power is

$$\delta W_{\text{int}} = \int_{B_0} \mathbf{P} : \nabla \delta\boldsymbol{\phi} \, dV \tag{355}$$

and the virtual external power is

$$\delta W_{\text{ext}} = \int_{B_0} \mathbf{f} \cdot \delta\boldsymbol{\phi} \, dV + \int_{\partial B_0} \bar{\mathbf{t}} \cdot \delta\boldsymbol{\phi} \, dA \tag{356}$$

The weak form is derived from the principle of virtual power which states that the virtual internal power (Equation (355)) and the virtual external power (Equation (356)) are equal. This leads to the weak form:

$$\delta W_{\text{int}} = \delta W_{\text{ext}} \tag{357}$$

In the beam context, using reduced one-dimensional kinematics, Equation (357) becomes

$$\int_0^L \left(N \delta u' + M_2 \delta \theta_2' + M_3 \delta \theta_3' \right) dx = \int_0^L \left(f_1 \delta u + f_2 \delta \theta_2 + f_3 \delta \theta_3 \right) dx + \text{boundary terms} \tag{358}$$

The test functions $\delta u, \delta\theta_2, \delta\theta_3$ must satisfy the homogeneous form of the essential boundary conditions in Equation (358).

The numerical analysis is carried out using the finite element method (FEM) implemented via isogeometric analysis (IGA). The beam's centerline is discretized using B-spline or NURBS basis functions to ensure higher-order smoothness necessary for capturing bending. The deformation map and director fields are interpolated as

$$\boldsymbol{\phi}_0(x,t) = \sum_{A=1}^{n} N_A(x)\boldsymbol{\phi}_A(t), \quad \mathbf{d}_i(x,t) = \sum_{A=1}^{n} N_A(x)\mathbf{d}_{i,A}(t) \tag{359}$$

where $N_A(x)$ are the B-spline basis functions and $\boldsymbol{\phi}_A(t)$, $\mathbf{d}_{i,A}(t)$ are control point variables in Equation (359). The discrete weak form leads to a nonlinear system of ordinary differential equations in time:

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{f}_{\text{int}}(\mathbf{d}, \dot{\mathbf{d}}) = \mathbf{f}_{\text{ext}}(t) \tag{360}$$

where $\mathbf{M}$ is the consistent mass matrix, $\mathbf{f}_{\text{int}}$ is the internal force vector derived from the free energy and constitutive relations, and $\mathbf{f}_{\text{ext}}$ is the external force vector in Equation (360). Time integration is performed using a generalized-α method for numerical stability and dissipation control in dynamic simulations. The plastic evolution equations are integrated using an implicit backward-Euler scheme to ensure robustness in rate-dependent flow, often requiring local Newton iterations to solve the constitutive update at quadrature points.

Overall, the formulation offers a consistent variational framework coupling large deformations, finite rotations, and thermomechanical rate-dependent plasticity in geometrically exact beams.

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