A Proof of Existence of Symmetric Equilibrium for Quadratically Symmetric bi-matrix games

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Abstract

We provide a proof of existence of symmetric equilibrium for quadratically similar symmetric bi-matrix games. We prove that any solution to a certain quadratic programming problem, is a symmetric equilibrium for the associated symmetric bi-matrix game. We use no more than the continuity of real-valued multi-variable quadratic functions and the mean value theorem for real-valued quadratic functions of a single variable. This proof can be easily understood by anyone who is familiar with a beginner's course on real analysis.

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1. Symmetric games: For a positive integer ℓ, let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$ and $E^{(\ell)}$ the ℓ-dimensional column vector all whose coordinates are equal to 1.

For a positive integer n, let A be a n×n matrices.

The pair (A, A^T) is said to be a **symmetric bi-matrix game**.

General results for symmetric bi-matrix games are discussed in Lahiri (2025).

$x^* \in \Delta^{n-1}$ is said to be a **symmetric equilibrium** of (A, A^T) if $x^{*T}Ax^* \geq x^TAx^*$ for all $x \in \Delta^{n-1}$.

A symmetric bi-matrix game (A, A^T) is said to be a **quadratically similar symmetric bi-matrix game** if for all $x, y \in \Delta^{n-1}$: $y^TAx > x^TAx$ implies $x^TAy > x^TAx$.

2. The main result: We now prove the main result presented here.

Theorem 1: Let (A, A^T) be a quadratically similar symmetric bi-matrix game. Then:

(i) The quadratic maximization problem [Maximize x^TAx, subject to $x \in \Delta^{n-1}$] has a solution.
(ii) Every solution of this problem is a symmetric equilibrium for (A, A^T).

Proof: Consider the function $f:\mathbb{R}^n \rightarrow \mathbb{R}$ defined as follows: for all $x \in \mathbb{R}^n$, $f(x) = x^TAx$. f is continuous and twice continuously differentiable continuous on $\mathbb{R}^n$, with $Df(x) = x^T(A+A^T)$ for all $x \in \mathbb{R}^n$. Since Δ^{n-1} is a closed and bounded subset of $\mathbb{R}^n$, and f is continuous, by

Weirstrass's theorem (Corollary 5-2.1 on page 95 of Smith and Albrecht (1987)), $\underset{x \in \Delta^{n-1}}{\text{argmax}}\, f(x) \neq \phi$.

Let x^* solve the quadratic programming problem: Maximize f(x), subject to $x \in \Delta^{n-1}$ (i.e., x^* solves: Maximize $x^T A^T x$, subject to $x \in \Delta^{n-1}$.

Consider the linear programming problem: Maximize $x^{*T}(A+A^T)x$, subject to $x \in \Delta^{n-1}$.

Suppose towards a contradiction x^* does not solve the linear programming problem.

Then there exists $x \in \Delta^{n-1}$ such that $x^{*T}(A+A^T)x > x^{*T}(A+A^T)x^*$ and hence $Df(x^*)(x-x^*) = x^{*T}(A+A^T)(x - x^*) > 0$.

Note that $x^* + t(x-x^*) \in \Delta^{n-1}$ for all $t \in [0, 1]$.

By the Mean Value Theorem (see theorem 5-6.2 on page 107 of Smith and Albrecht (1987)), for all t in (0, 1), $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $Df(x^*)(x-x^*) > 0$, by the continuity of the function $y \mapsto Df(y)(x-x^*) = y^T(A+A^T)(x - x^*)$ on $\mathbb{R}^n$, there exists $s \in (0, 1)$ such that for all $0 < r \leq s$, $Df(x^* + r(x-x^*)(x - x^*) > 0$.

Let t belong to (0, s). Then, $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $0 < \xi < t < s$, it must be that $Df(x^* + \xi(x-x^*))(x-x^*) > 0$, whence $tDf(x^* + \xi(x-x^*))(x-x^*) > 0$.

Thus, $f(x^* + t(x-x^*)) - f(x^*) > 0$, contradicting x^* solves the quadratic programming problem.

Thus, it must be the case that $Df(x^*)(x-x^*) \leq 0$ for all $x \in \Delta^{n-1}$, (i.e., $x^{*T}(A+A^T)(x - x^*) \leq 0$ for all $x \in \Delta^{n-1}$), whence $x^{*T}(A+A^T)x \leq x^{*T}(A+A^T)x^*$ for all $x \in \Delta^{n-1}$.

Thus, $x^{*T}Ax + x^TAx^* \leq 2x^{*T}Ax^*$ for all $x \in \Delta^{n-1}$.

Towards a contradiction suppose $y^TAx^* > x^{*T}Ax^*$ for some $y \in \Delta^{n-1}$.

Then, since (A, A^T) is quadratically similar, it must be the case that $x^{*T}Ay > x^{*T}Ax^*$.

Thus, $x^{*T}Ay + y^TAx^* > 2x^{*T}Ax^*$, contradicting the requirement that $x^{*T}Ax + x^TAx^* \leq 2x^{*T}Ax^*$ for all $x \in \Delta^{n-1}$.

Thus, $x^TAx^* \leq x^{*T}Ax^*$ for all $x \in \Delta^{n-1}$. Q.E.D.

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References

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