A Simple Proof of Existence of Symmetric Equilibrium for Symmetric Bi-matrix games: A Quadratic Programming Approach

By

Somdeb Lahiri (Email: somdeb.lahiri@gmail.com)

ORCID: https://orcid.org/0000-0002-5247-3497

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Abstract

We provide a proof of existence of symmetric equilibrium for symmetric bi-matrix games, a result implied by a more general result that was proved by John Nash. Our proof, unlike the original proof due to Nash, does not appeal to the Brouwer fixed point theorem. We prove that any solution to a certain specific quadratic programming problem, is a symmetric equilibrium for the associated symmetric bi-matrix game.

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1. Symmetric bi-matrix games: For a positive integer ℓ, let $\Delta^{\ell-1} = \{x \in \mathbb{R}^{\ell}_{+} | \sum_{j=1}^{\ell} x_j = 1\}$.

For a positive integer n, let A be an n×n matrix.

The pair (A, A^T) is said to be a **symmetric bi-matrix game**.

$x^* \in \Delta^{n-1}$ is said to be a **symmetric equilibrium** of (A, A^T) if $x^{*T}Ax^* \geq x^TAx^*$ for all $x \in \Delta^{n-1}$.

The set $\{x^* | x^*$ is a symmetric equilibrium of $(A, A^T)\}$ is **the set of all symmetric equilibria of** (A, A^T).

2. The main result: We now prove the main result presented here. The implications of the theorem we prove are discussed in Lahiri (2025).

Theorem (Nash (1951)): There exists a symmetric equilibrium for (A, A^T).

Proof: Consider the function $f: \mathbb{R}^n \to \mathbb{R}$ defined as follows: for all $x \in \mathbb{R}^n$, $f(x) = x^TA^Tx$. f is continuous and twice continuously differentiable continuous on $\mathbb{R}^n$, with $Dg(x) = x^TA^T$ for all $x \in \mathbb{R}^n$. Since Δ^{n-1} is a closed and bounded subset of $\mathbb{R}^n$, and f is continuous, by Weirstrass's theorem (Corollary 5-2.1 on page 95 of Smith and Albrecht (1987)), $\underset{x \in \Delta^{n-1}}{\operatorname{argmax}} f(x) \neq \phi$.

Let $x^* \in \underset{x \in \Delta^{n-1}}{\operatorname{argmax}} f(x)$

Consider the quadratic maximization problem: Maximize f(x), subject to $x \in \Delta^{n-1}$.

Clearly x^* solves the quadratic programming problem.

We need to show that x^* solves: Maximize $x^{*T}A^Tx$, subject to $x \in \Delta^{n-1}$.

Suppose towards a contradiction x^* does not solve the linear programming problem.

Then there exists $x \in \Delta^{n-1}$ such that $Df(x^*)(x-x^*) > 0$.

Note that $x^* + t(x-x^*) \in \Delta^{n-1}$ for all $t \in [0, 1]$.

By the Mean Value Theorem (see theorem 5-6.2 on page 107 of Smith and Albrecht (1987)), for all t in (0, 1), $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $Dg(x^*)(x-x^*) > 0$, by the continuity of the function $y \mapsto Df(y)(x-x^*) = y^TA^T(x-x^*)$ on $\mathbb{R}^n$, there exists $s \in (0, 1)$ such that for all $0 < r \leq s$, $Df(x^* + r(x-x^*)(x -x^*) > 0$.

Let t belong to (0, s). Then, $f(x^* + t(x-x^*)) - f(x^*) = tDf(x^* + \xi(x-x^*))(x-x^*)$ for some $\xi \in (0, t)$, possibly depending on t.

Since $0 < \xi < t < s$, it must be that $Df(x^* + \xi(x-x^*))(x-x^*) > 0$, whence $tDf(x^* + \xi(x-x^*))(x-x^*) > 0$.

Thus, $f(x^* + t(x-x^*)) - f(x^*) > 0$, contradicting x^* solves the quadratic programming problem.

Thus, it must be the case that $Dg(x^*)(x-x^*) \leq 0$ for all $x \in \Delta^{n-1}$.

This establishes the desired implication.

Thus, $x^{*T}A^Tx^* \geq x^{*T}A^Tx$ for all $x \in \Delta^{n-1}$.

Hence, $x^{*T}Ax \geq x^TAx^*$ for all $x \in \Delta^{n-1}$, i.e., x^* is a symmetric equilibrium for (A, A^T). Q.E.D.

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